# Categorifying Induction: From Modules to Birepresentations

*with an application to extended affine type A Soergel bimodules*

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## The central question

An algebra homomorphism

$$\varphi : S \longrightarrow R$$

gives restriction of scalars

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**Question**

What is the correct categorified analogue of induction?

# Example: parabolic induction for symmetric groups

Let $n = k + m$. There is an embedding

$$\mathbb{C}[S_k] \otimes \mathbb{C}[S_m] \longrightarrow \mathbb{C}[S_n].$$

On generators,

$$s_i \otimes 1 \longmapsto s_i, \qquad 1 \otimes s_j \longmapsto s_{k+j}.$$

This gives the induction functor

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This is the classical pattern we want to understand categorically.

## From representations to birepresentations

A representation is an algebra homomorphism

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A birepresentation is a monoidal $\mathbb{k}$-linear functor

$$\mathcal{A} \longrightarrow \mathrm{END}_{\mathbb{k}}(\mathcal{M}).$$

Examples include additive, abelian and triangulated birepresentations.

# The dictionary

| Classical representation theory | Categorified representation theory |
|---|---|
| algebra $R$ | monoidal category $\mathcal{A}$ |
| vector space or module $M$ | category $\mathcal{M}$ |
| $R \longrightarrow \mathrm{End}_{\mathbb{k}}(M)$ | $\mathcal{A} \longrightarrow \mathrm{END}_{\mathbb{k}}(\mathcal{M})$ |
| representation | birepresentation |
| $R \otimes_S -$ | categorical induction |

Part of the goal is to understand the last line.

# The plan

1. Replace algebras and modules by monoidal categories and birepresentations.
2. How to induce birepresentations ?
3. Parabolic induction in extended affine type $A$ Soergel theory.

# How to induce categorically?

A monoidal functor

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Algebra objects provide the guiding mechanism.

# Algebra objects

An algebra object is $X \in \mathcal{C}$ with

$$\mu : X \otimes X \longrightarrow X, \qquad \eta : \mathbb{1} \longrightarrow X,$$

satisfying associativity and unitality.

Thus $X$ behaves like an algebra internal to $\mathcal{C}$.

# Three examples of algebra objects

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2. In $\mathrm{Rep}(G)$, an algebra object is an algebra equipped with a $G$-action by algebra automorphisms; equivalently, $\mu$ and $\eta$ are $G$-equivariant.

3. In the monoidal category of $(R, R)$-bimodules, with tensor product $\otimes_R$, the regular bimodule $R$ is an algebra object:

$$R \otimes_R R \longrightarrow R, \qquad r \otimes r' \longmapsto rr'.$$

# Module objects

Let $X \in \mathcal{C}$ be an algebra object.

A right $X$-module object is $M \in \mathcal{C}$ with an action

$$M \otimes X \longrightarrow M$$

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Write $\mathrm{mod}_{\mathcal{C}}(X)$ for the category of right $X$-module objects.

It has a natural left $\mathcal{C}$-action:

$$F \cdot M = F \otimes M.$$

# Birepresentations from algebra objects

The construction gives

$$X \quad \rightsquigarrow \quad \mathrm{mod}_{\mathcal{C}}(X) : \text{ a birepresentation of } \mathcal{C}.$$

Thus algebra objects construct birepresentations and, under suitable hypotheses, encode them.

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**Caveat**

There is no unrestricted correspondence. One needs hypotheses such as a generator and an internal endomorphism object, possibly after completion.

# Categorical induction

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**Status**

At this level of generality, we do not claim that this construction is a left adjoint to restriction along $\Psi$.

# What is actually known?

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Induction via algebra objects is a guiding mechanism, not yet a fully developed general theory.

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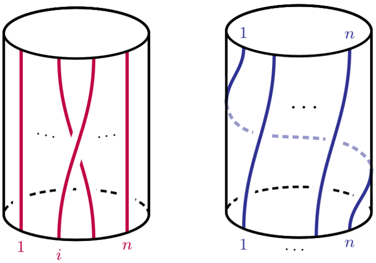
**The rest of the talk is devoted to**

maximal parabolic induction for extended affine type $A$ Soergel categories.

- The monoidal parabolic embedding is constructed in the maximal parabolic case.
- The induced birepresentation is worked out explicitly for $n = 2$, $k = 1$.

Extended affine Hecke algebra $\widehat{H}_n^{\mathrm{ext}}$:

- Generators $t_i$ and rotation $\rho^{\pm 1}$.
- Relations $\rho t_i \rho^{-1} = t_{i+1}$, $(t_i + q)(t_i - q^{-1}) = 0, \ldots$
- $H_n \subset \widehat{H}_n^{\mathrm{ext}}$ is the finite-dimensional subalgebra generated by $t_1, \ldots, t_{n-1}$.



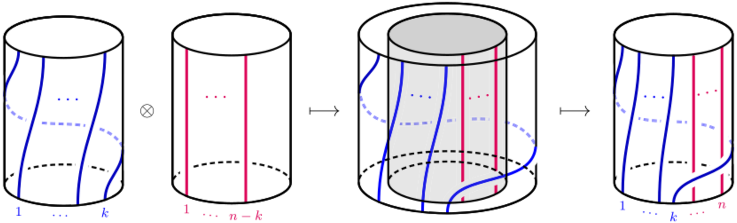

$t_i$

$(1 \leq i \leq n)$

indices mod $n$

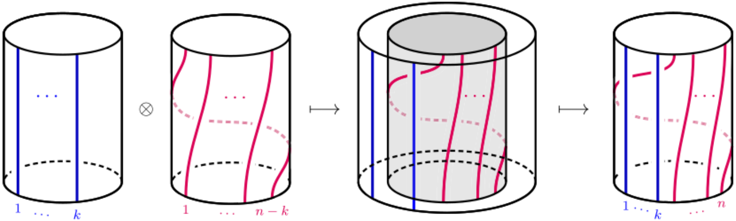
$\rho$

# Parabolic induction: cylinders inside one another




$$b_i \otimes 1_{n-k} \mapsto b_i$$

$$\rho \otimes 1_{n-k} \mapsto \rho t_{n-1} \dots t_k$$




$$1_k \otimes \rho \mapsto t_k^{-1} \dots t_1^{-1} \rho$$

$$1_k \otimes b_j \mapsto b_{j+k}$$

The rotations involve the standard generators $t_j$.

## Classical parabolic induction in extended affine type $A$

A decomposition $n = k + (n - k)$ gives the maximal parabolic embedding

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For modules $M_1, M_2$, the Zelevinsky tensor product is

$$M_1 \odot M_2 := \mathrm{Ind}(M_1 \otimes M_2).$$

$$\begin{tikzcd}
\boxed{M_1 \text{ over } \widehat{H}_k^{\mathrm{ext}}} \arrow[dr] & & \boxed{M_2 \text{ over } \widehat{H}_{n-k}^{\mathrm{ext}}} \arrow[dl] \\
& \boxed{M_1 \odot M_2 \text{ over } \widehat{H}_n^{\mathrm{ext}}} &
\end{tikzcd}$$

No prior knowledge of Soergel bimodules is required.

We now study a concrete realization of the induction method for maximal parabolic induction in extended affine type $A$.

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The algebra-object mechanism is independent of this example.

# The affine and homotopy hurdles

## New feature

In the extended affine type $A$ example, the construction naturally leads to homotopy and triangulated categories.

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- The relevant monoidal categories have infinitely many indecomposable objects, so they are not finitary.
- The relevant functor is not simply $\mathcal{B} \to \mathcal{A}$, but

$$\mathcal{B} \longrightarrow K^b(\mathcal{A}).$$

- The algebra objects may be countably infinite coproducts of indecomposable objects.
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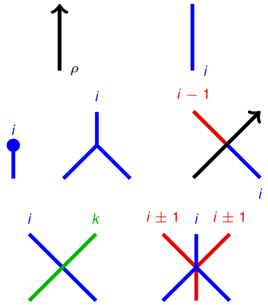
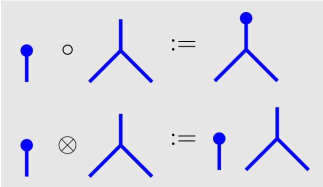
Nevertheless, in extended affine type $A$, the construction can be made to work.

- Soergel categories categorify Hecke algebras:
  - They are $\mathbb{k}$-linear, additive and monoidal (one-object bicategories).
  - $K_0(\mathcal{S}(W)) \cong H(W)$ for a Coxeter group $W$.
  - The Kazhdan–Lusztig generators are $b_i := t_i + q$.
  - $\mathrm{B}_i$ categorifies $b_i$, $\mathrm{B}_\rho$ categorifies $\rho$.




- We use the extended affine category $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$.
- Diagrammatic calculus encodes morphisms and relations.



## Why homotopy and triangulated categories?

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$$\mathsf{T}_i = \left[\mathsf{B}_i \xrightarrow{i} \mathbb{1}\langle 1\rangle\right], \qquad \mathsf{T}_i^{-1} = \left[\mathbb{1}\langle -1\rangle \longrightarrow \mathsf{B}_i\right],$$

up to conventions. They live in

$$K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}}),$$

not in $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$ itself.

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up to conventions. They live in

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not in $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$ itself.

The correct framework is triangulated rather than merely additive.

# Maximal parabolic setup

Let $n = k + (n-k)$. We seek a monoidal functor

$$\widehat{\mathcal{S}}_k^{\mathrm{ext}} \boxtimes \widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}} \longrightarrow K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$$

categorifying

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It is built from two categorical embeddings, one for each parabolic factor.

There are monoidal functors

$$\Psi_L : \widehat{\mathcal{S}}_k^{\mathrm{ext}} \longrightarrow K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}}), \qquad \Psi_R : \widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}} \longrightarrow K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}}),$$

with

$$\Psi_L(\mathrm{B}_i) = \mathrm{B}_i, \qquad \Psi_R(\mathrm{B}_j) = \mathrm{B}_{j+k},$$
$$\Psi_L(\mathrm{B}_\rho) = \mathrm{B}_\rho \, \mathrm{T}_{n-1} \cdots \mathrm{T}_k, \qquad \Psi_R(\mathrm{B}_\rho) = \mathrm{T}_k^{-1} \cdots \mathrm{T}_1^{-1} \, \mathrm{B}_\rho \, .$$

Finite generators embed as expected; rotations require Rouquier complexes.

# Why the two embeddings combine

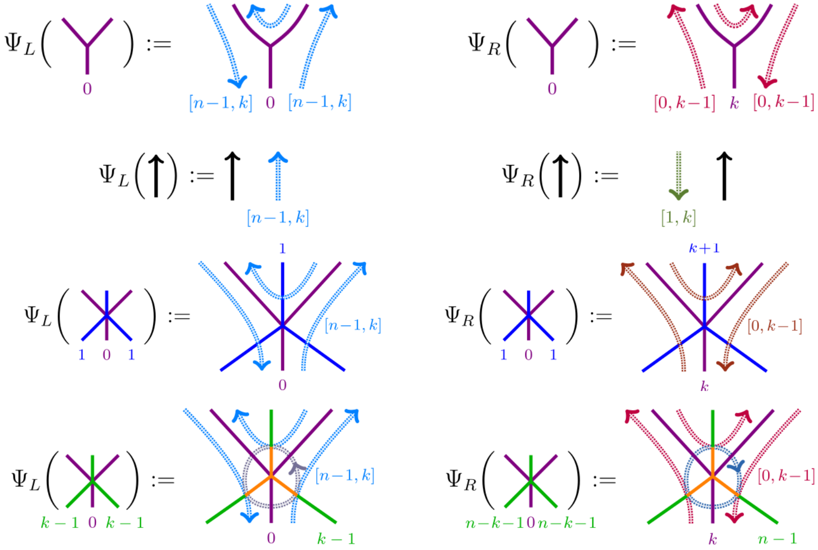
The images commute up to coherent isomorphism:

$$\Psi_L(X)\Psi_R(Y) \xrightarrow{\sim} \Psi_R(Y)\Psi_L(X).$$

**Remark**

The Rouquier calculus in this setting is only partially developed, but the relations available so far suffice to define $\Psi_L$ and $\Psi_R$ and to construct the required coherent isomorphisms.

# The categorical embeddings on morphisms



# The categorified parabolic embedding

**Theorem (Mackaay–Miemietz–Vaz)**

There is a monoidal functor

$$\Psi_{k,n-k} : \widehat{\mathcal{S}}^{\mathrm{ext}}_{k} \boxtimes \widehat{\mathcal{S}}^{\mathrm{ext}}_{n-k} \longrightarrow K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_{n})$$

categorifying the maximal parabolic embedding.

On objects,

$$\Psi_{k,n-k}(X \boxtimes Y) = \Psi_L(X)\Psi_R(Y).$$

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On objects,

$$\Psi_{k,n-k}(X \boxtimes Y) = \Psi_L(X)\Psi_R(Y).$$

This categorifies the algebra map along which the Zelevinsky tensor product is formed.

## Inducing birepresentations

If $X$ encodes a birepresentation of

$$\widehat{\mathcal{S}}_k^{\mathrm{ext}} \boxtimes \widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}},$$

then

$$\Psi_{k,n-k}(X)$$

is an algebra object in a suitable completion of $K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$.

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The induced birepresentation is built from

$$\mathrm{mod}_{K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_{n})}\big(\Psi_{k,n-k}(X)\big).$$

# Example: $n = 2,\ k = 1$

Let V be the trivial birepresentation of

$$\widehat{\mathcal{S}}_1^{\mathrm{ext}} = \langle \mathrm{B}_\rho \rangle.$$

- Algebra object for V: $\mathrm{X} \in \widehat{\mathcal{S}}_1^{\mathrm{ext},\diamond}$ is given by

$$\mathrm{X} := \coprod_{r \in \mathbb{Z}} \mathrm{B}_\rho^r, \qquad \mathrm{B}_\rho \mathrm{X} \cong \mathrm{X}.$$

$$\begin{tikzcd} V \arrow[dr] & & V \arrow[dl] \\ & V \odot V & \end{tikzcd}$$

- Induction:

$$\mathrm{Y} := \Psi_{1,1}\big(\mathrm{X} \boxtimes \mathrm{X}\big) \in K^b\big(\widehat{\mathcal{S}}_2^{\mathrm{ext},\diamond}\big),$$

where

$$\mathrm{Y} \cong \coprod_{r \in \mathbb{Z}} \coprod_{s \in \mathbb{Z}} (\mathrm{B}_\rho \mathrm{T}_1)^r (\mathrm{T}_1^{-1} \mathrm{B}_\rho)^s.$$

# The induced triangulated birepresentation

Starting from $Y$, consider

$$\mathrm{add}\{FY \mid F \in K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})\} \subset K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext},\diamond}).$$

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A general construction of Fan–Keller–Qiu provides the relevant triangulated closure.

**Remark**

Even induction from the trivial rank-one birepresentation requires infinite coproducts, a completion, Rouquier complexes and a triangulated closure.

# Answer to the central question

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## Categorical induction via algebra objects

Given a monoidal functor $\Psi : \mathcal{B} \longrightarrow \mathcal{A}$ and $X \in \mathrm{Alg}(\mathcal{B})$, the categorical mechanism is

$$\boxed{\mathrm{mod}_{\mathcal{B}}(X) \longmapsto \mathrm{mod}_{\mathcal{A}}\big(\Psi(X)\big)}.$$

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## Application: Extended affine type $A$

$$\Psi_{k,n-k} : \widehat{\mathcal{S}}^{\mathrm{ext}}_k \boxtimes \widehat{\mathcal{S}}^{\mathrm{ext}}_{n-k} \longrightarrow K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_n)$$

categorifies the maximal parabolic embedding along which induction is performed.

# Take-home messages

- Algebra objects provide a method for inducing certain birepresentations along a monoidal functor.
- The method is conditional: the appropriate algebra objects, module categories and completions must exist.
- $\Psi_{k,n-k}$ categorifies the maximal parabolic embedding in extended affine type $A$.
- Triangulated birepresentations arise in this affine example; they are not part of a general theory developed here.

Thank you!

# When do birepresentations come from algebra objects?

A Morita-type reconstruction is available when the birepresentation is generated by an object $m$, the internal endomorphism object exists, and the ambient category has the required additive and idempotent-complete properties. Then,

schematically,

$$X = \underline{\mathrm{End}}(m), \qquad \mathcal{M} \simeq \mathrm{mod}_{\mathcal{C}}(X).$$

In affine settings, $X$ may exist only after completion.

# A diagram in Rouquier–Soergel calculus