On linear exactness properties

Pierre-Alain Jacqmin[a] and Zurab Janelidze[b]

[a] *Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, Chemin du Cyclotron 2, B 1348 Louvain-la-Neuve, Belgium*
[b] *Mathematics Division, Department of Mathematical Sciences, Stellenbosch University, Private Bag X1 Matieland 7602, South Africa*

Abstract

We study those exactness properties of a regular category $\mathbb{C}$ that can be expressed in the following form: for any diagram of a given 'finite shape' in $\mathbb{C}$, a given canonical morphism between finite limits built from this diagram is a regular epimorphism. The main goal of the paper is to characterize essentially algebraic categories which satisfy this property via (essential versions of) linear Mal'tsev conditions, which are known to correspond to the so-called matrix properties. We then apply this characterization, along with our earlier work on preservation of exactness properties by pro-completions, to prove that these exactness properties can be reduced to matrix properties already in the general setting of regular categories.

2020 Mathematics Subject Classification: 18A30, 18E13, 08B05, 18E08, 03G30, 08A55 (primary); 18C30, 18A40, 18C10, 18C35, 18C05, 08C05 (secondary).

Keywords: approximate operation, essentially algebraic category, exactness property, linear Mal'tsev condition, matrix property, regular category.

Introduction

A significant part of research in category theory concerns with the study of properties of general categories, which abstract fundamental features of some specific categories of structures. It will not be too imprecise to say that it all started with investigating properties of the category of abelian groups. Abstracting basic features of this category gives rise to properties defining *abelian categories*, a collection of categories which includes categories of modules, as well as sheaves of modules. The principal defining properties of abelian categories deal with 'exact sequences' (in the sense of homological algebra), which are defined in terms of particular types of limits and colimits: kernels and cokernels. The strong influence that the notion of an abelian category had on subsequent developments in category theory is marked by the fact that all properties *expressed* in terms of limits and colimits were came to be called 'exactness properties'. Here we are not saying what is meant by 'expressed' because, in fact, the notion of an exactness property has not yet obtained, in its full generality, a formal definition.

In our recent work, we proposed a formal approach to exactness properties, described in the next few lines. Treat a category as a 'universe' in which one could consider models of a given theory. This viewpoint is, of course, a very old one in categorical logic. There, a 'theory' is often understood as a 'sketch'. A sketch can be defined as a graph (i.e., a category without composition and identity morphisms) with a data of diagrams, cones and cocones, which in a model are supposed to be turned into commutative diagrams, limits and colimits, respectively. A sketch may admit a model in any category. In the case of the category of sets, models of sketches are models of a particular type of (infinitary) first-order theories. Now, according to our approach, an exactness property of a category states that every model of a certain sketch $\mathcal{A}$ can be extended to a model of another sketch $\mathcal{B}$, along a given sketch morphism $\mathcal{A} \to \mathcal{B}$. This seemingly simple framework turns out to encompass a surprisingly large class of exactness properties that arise in the literature. It also enables one to apply insights from the theory of sketches in the investigation of general questions concerning exactness properties.

A wide selection of exactness properties can be found in the overlap of the subjects of categorical and universal algebra. Many of the so-called 'Mal'tsev conditions' on algebraic categories can be reformulated as exactness properties. A Mal'tsev condition asks for the algebraic theory (of an algebraic category) to possess certain terms satisfying certain term identities. A general category does not have an algebraic theory associated to it, but in many cases it is possible to find an exactness property that is equivalent, for algebraic categories, to the given Mal'tsev condition. It is an open question whether this is always possible. The reverse problem is also interesting: for a given exactness property to find, if possible, a Mal'tsev condition that would be equivalent to it in the case of algebraic categories. Our initial problem that led to this paper was to identify a wide class of exactness properties, which would be equivalent to Mal'tsev conditions, and moreover, for which it would be possible to write down explicitly the term identities in the corresponding Mal'tsev conditions. We ended up with a class of exactness properties given by a pair of sketches $\mathcal{A}$ and $\mathcal{B}$, where: $\mathcal{A}$ is an underlying sketch of a finite category (free of distinguished cones or cocones), while $\mathcal{B}$ is an extension of $\mathcal{A}$ declaring that a certain morphism made from canonical morphisms between (iterated) finite limits of diagrams in $\mathcal{A}$ is a regular epimorphism. These exactness properties are meaningful not only in algebraic categories, but more generally in arbitrary regular categories. The next question was whether exactness properties of this kind have already been encountered in the literature. We will refer to exactness properties of this kind as *linear exactness properties* for the reasons apparent in what follows.

A so-called 'matrix property' of a regular category is an exactness property, where $\mathcal{A}$ sketches a first-order theory with a single n-ary relation symbol ϱ with no axioms, while $\mathcal{B}$ extends this theory with a single axiom of the form

$$\forall_{x_1,\dots,x_l}\left[\left[\bigwedge_{j\in\{1,\dots,m\}}\varrho(x_{1j},\dots,x_{nj})\right]\implies\exists_{x_{l+1},\dots,x_k}\left[\bigwedge_{j\in\{m+1,\dots,m+m'\}}\varrho(x_{1j},\dots,x_{nj})\right]\right]$$

where $x_{ij}\in\{x_1,\dots,x_l\}$ when $j\leqslant m$ and $x_{ij}\in\{x_1,\dots,x_k\}$ when $j>m$. The name refers to the fact that such an exactness property can be encoded in an extended matrix

$$\left[\begin{array}{ccc|ccc} x_{11} & \cdots & x_{1m} & x_{1,m+1} & \cdots & x_{1,m+m'} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n1} & \cdots & x_{nm} & x_{n,m+1} & \cdots & x_{n,m+m'} \end{array}\right]$$

of variables. Note that such a pair $(\mathcal{A},\mathcal{B})$ is not of the kind used in the description of a linear exactness property given above, since the $\mathcal{A}$ here cannot be presented as an underlying sketch of a category (it requires a distinguished cone to account for the projections of the relation ϱ to be jointly monomorphic). However, it is possible to give another presentation of a matrix property of a regular category in terms of sketches. Instead of consisting of a single n-ary relation symbol ϱ, let $\mathcal{A}$ consist of n functional symbols $\varrho_1,\dots,\varrho_n$ representing the projections of ϱ, with a source sort R and a separate target sort A. So in a term $\varrho_i(x)$, the variable x must have sort R and the term $\varrho_i(x)$ itself has sort A. The sketch $\mathcal{B}$ is still an extension of $\mathcal{A}$, where the formula above has been added as an axiom, after replacing each atomic formula $\varrho(x_{1j}, ..., x_{nj})$ in it with the formula

$$\exists_{z_j} \left[\bigwedge_{i\in\{1,\dots,n\}} x_{ij} = \varrho_i(z_j) \right]$$

where the variables $z_1, \dots, z_{m+m'}$ are all distinct from each other and are also distinct from $x_1, \dots, x_k$. The pair $(\mathcal{A}, \mathcal{B})$ in this presentation is of the kind required for a linear exactness property, and so we get that every matrix property of a regular category is a linear exactness property. What about the converse?

Matrix properties correspond in universal algebra to 'linear' Mal'tsev conditions, where term identities are either of the form $p(x_1, \dots, x_n) = y$ or the form $p(x_1, \dots, x_n) = q(y_1, \dots, y_m)$ for not necessarily distinct variables $x_1, \dots, x_n, y, y_1, \dots, y_m$. Here we can assume without loss of generality that the arity of the terms is the same for all term identities, except when some of the terms are constants. When this happens, we can split up the Mal'tsev condition as a conjunction of a linear Mal'tsev condition where no constant terms are involved and the existence of constants. On the side of matrix properties, this forces one to consider a conjunction of a matrix property with another one given by the matrix $[\,|\,x_1]$, which, as a property on an algebraic category, is equivalent to the existence of constants. In this paper, we show that the Mal'tsev conditions corresponding to linear exactness properties are in fact the same as linear Mal'tsev conditions. Moreover, it is possible to explicitly extract the term identities of the Mal'tsev condition from the presentation of the linear exactness property in terms of the pair of sketches $\mathcal{A}$ and $\mathcal{B}$. We obtain these results not only in the case of algebraic categories for single-sorted theories but for arbitrary essentially algebraic categories (up to equivalences, these happen to be the categories of models of sketches with limiting cones but no colimiting cocones). From this we can conclude that for general regular categories, linear exactness properties are equivalent to matrix properties (with the correction that on the matrix side, we can have either a single matrix or a conjunction of two matrices, one of which is the matrix $[\,|\,x_1]$). This is thanks to the stability properties of the pro-completion established in our earlier work [30].

For the readers convenience, we include here the list of the main results of this paper, which are the following four characterization theorems:

- Theorem 1.4, which characterizes, in the regular context [4], linear exactness properties as those exactness properties which in terms of sketches is given by an underlying sketch $\mathcal{A}$ of a finite category, and an extension $\mathcal{B}$ of $\mathcal{A}$ declaring that certain morphisms are regular epimorphisms, certain finite cones are limit cones and certain diagrams are commutative, considering only finitely many of these and each of them being canonically built from morphisms between (iterated) finite limits of diagrams in $\mathcal{A}$ (the difference with the definition of a linear exactness property is that there we limit the number of declared regular epimorphisms in $\mathcal{B}$ to be exactly one).

- Theorem 2.1, which gives a characterization of finitely cocomplete regular categories having a linear exactness property, in the style of the theory of approximate operations [10, 36, 26, 29]. The viewpoint of Kan extensions contained in this theorem is actually a new insight to this theory.

- The theorem mentioned above is a stepping stone to our main characterization theorem of essentially algebraic categories having a linear exactness property, in terms of a linear Mal'tsev condition. This is Theorem 3.3.

- Theorem 4.11, where the link between linear exactness properties and matrix properties is established in the context of regular categories.

Each of the theorems above appear in a separate section, with the first section devoted to a preliminary work on exactness properties that recalls required material from the literature, as well as gives the formal definition and an analysis of linear exactness properties.

Finally, here are some pointers to the literature, for the main topics mentioned in this Introduction:

- Abelian categories first appeared in the work of S. Mac Lane [38], and in their modern form in the works of D.A. Buchsbaum [11] (where they are called 'exact categories') and A. Grothendieck [22]. The book [19] by P. Freyd is entirely devoted to the concept of an abelian category and the category theory arising in the investigation of this concept. The book [7] by F. Borceux and D. Bourn gives an account of categories defined by exactness properties in the context of more recent research in categorical algebra. See [30] for a formal approach to exactness properties and preservation under the pro-completion.

- The study of Mal'tsev conditions takes its start from the work of A.I. Mal'tsev [39]. A formal definition of a Mal'tsev condition (in the single-sorted case) was given by P. Taylor in [45], which also contains illustrative examples of Mal'tsev conditions. See also [9, 23, 25, 31, 37, 42, 46] for some of those and other examples. The book [17] by I. Chajda, G. Eigenthaler and H. Länger has an encyclopedic account of Mal'tsev conditions researched in universal algebra before this millennium.

- The theory of sketches is due to C. Ehresmann [18]. See also the work of M. Makkai and R. Paré [40] for development of model theory based on sketches. Our presentation of sketches is similar to that of M. Barr and C. Wells [5].

- Essentially algebraic theories were introduced by P. Freyd in [20]. See the works of J. Adámek, H. Herrlich and J. Rosický [1, 2] for an elaborate treatment of essentially algebraic categories. Up to equivalences, essentially algebraic categories are exactly the locally presentable categories in the sense of P. Gabriel and P. Ulmer [21]. See also, e.g., the two part book [12, 43] of P. Burmeister and H. Reichel for applications in computer science of essentially algebraic categories.

- Matrix properties were introduced in [33, 35] and further studied in [36, 26, 28, 29]. The notion of a linear Mal'tsev condition (in the single-sorted algebraic case) is due to J.W. Snow [44]. Examples of collections of categories that can be defined by matrix properties include Mal'tsev categories [14, 15], n-permutable categories [13], majority categories [24] and, in the context of exact categories with coequalizers, arithmetical categories [41].

- Mal'tsev conditions for essentially algebraic categories were first considered by the first author in [26, 27, 29] to characterize matrix properties in that context.

Acknowledgments

The first author would like to warmly thank Stellenbosch University for its kind hospitality during his visit in January 2020, when part of the project was realized. He is also grateful to the Belgian FNRS for its generous support. The second author is grateful to the South African NRF for its support. Both authors are grateful for the useful remarks and suggestions by the anonymous referee on the presentation of the original version of this paper.

1 Sketching linear exactness properties

We introduced in [30] the notion of an exactness sequent as a way of encoding exactness properties. We recall here the necessary material, see [30] for more details. For a graph $\mathcal{G}$ (i.e., a diagram $d, c\colon E \rightrightarrows V$ in the category **Set** of sets), a *commutativity condition* in $\mathcal{G}$ is a pair of paths

$$((A_0, f_1, A_1, \dots, f_n, A_n), (B_0, g_1, B_1, \dots, g_m, B_m))$$

in $\mathcal{G}$ such that $A_0 = B_0$ and $A_n = B_m$. We represent it by

$$f_n \cdots f_1 = g_m \cdots g_1$$

or by

$$f_n \cdots f_1 = 1_{B_0}$$

if $m = 0$ (and similarly if $n = 0$). A *finite diagram* in $\mathcal{G}$ is given by a finite graph $\mathcal{H}$ together with a morphism of graphs $D\colon \mathcal{H} \to \mathcal{G}$. A *finite limit condition* (respectively, a *finite colimit condition*) in $\mathcal{G}$ is an equivalence class of 4-tuples $(\mathcal{H}, D, C, (c_H)_{H\in\mathcal{H}})$ where $D\colon \mathcal{H} \to \mathcal{G}$ is a finite diagram, C is an object in $\mathcal{G}$ and for each object H in $\mathcal{H}$, $c_H\colon C \to D(H)$ (respectively, $c_H\colon D(H) \to C$) is an arrow in $\mathcal{G}$. Two such 4-tuples $(\mathcal{H}, D, C, (c_H)_{H\in\mathcal{H}})$ and $(\mathcal{H}', D', C', (c'_{H'})_{H'\in\mathcal{H}'})$ are considered to be equivalent if $C = C'$ and if there exists an isomorphism of graphs $I\colon \mathcal{H} \to \mathcal{H}'$ such that $D'I = D$ and $c_H = c'_{I(H)}$ for any $H \in \mathcal{H}$. Such a condition $[(\mathcal{H}, D, C, (c_H)_{H\in\mathcal{H}})]$ is represented by

$$(C, (c_H)_H) = \mathsf{limit}(\mathcal{H}, D) \quad \text{(respectively by } (C, (c_H)_H) = \mathsf{colimit}(\mathcal{H}, D)).$$

Finite limit conditions and finite colimit conditions are called *convergence conditions.* A *sketch* is then a finite graph equipped with a set of commutativity conditions and a set of convergence conditions. A *morphism of sketches* is a morphism $\mu\colon \mathcal{G} \to \mathcal{G}'$ of underlying graphs of sketches which carries each commutativity condition on $\mathcal{G}$ to a commutativity condition on $\mathcal{G}'$ and each convergence condition on $\mathcal{G}$ to a convergence condition on $\mathcal{G}'$. A *subsketch* of a sketch $\mathcal{B}$ is a subgraph $\mathcal{A}$ of the underlying graph of $\mathcal{B}$ equipped with a sketch structure that turns the inclusion of graphs $\mathcal{A} \to \mathcal{B}$ into a sketch morphism. We call such morphisms *subsketch inclusions.*

Given a sketch $\mathcal{A}$ and a category $\mathbb{C}$, an $\mathcal{A}$*-structure* in $\mathbb{C}$ is a morphism $F\colon \mathcal{A} \to \mathbb{C}$ of underlying graphs which carries each commutativity condition of $\mathcal{A}$ to an actual commutative diagram in $\mathbb{C}$, and each finite limit/colimit condition of $\mathcal{A}$ to an actual limit/colimit in $\mathbb{C}$. The category of $\mathcal{A}$-structures in $\mathbb{C}$ (and natural transformations as morphisms) is denoted by $\mathcal{A}\mathbb{C}$. Every morphism $\beta\colon \mathcal{A} \to \mathcal{B}$ of sketches gives rise to a functor

$$\beta_{\mathbb{C}}\colon \mathcal{B}\mathbb{C} \to \mathcal{A}\mathbb{C}$$

by 'pre-composing' with β.

The following notion is a particular case of the one introduced in [30]. An *exactness sequent* (called an $\varnothing$-sequent in [30]) is a subsketch inclusion $\beta\colon \mathcal{A} \to \mathcal{B}$ considered as a sequence of subsketch inclusions

$$\varnothing \xrightarrow{\alpha} \mathcal{A} \xrightarrow{\beta} \mathcal{B}$$

starting with the empty sketch $\varnothing$. We denote this by $\mathcal{A} \vdash \mathcal{B}$, or, as in [30], by $\alpha \vdash \beta$ as the pairs $(\mathcal{A}, \mathcal{B})$ and (α, β) uniquely determined each other. As detailed further below, we will be interested only in particular exactness sequents where β is 'constructible'. In that case, $\beta_{\mathbb{C}}$ is fully faithful for any category $\mathbb{C}$, and we write $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ (or $\alpha \vdash_{\mathbb{C}} \beta$) when $\beta_{\mathbb{C}}$ is an equivalence of categories (see Lemma 1.4 in [30]). The more general notion of an exactness sequent given in [30] has a general sketch $\mathcal{X}$ in the place of the empty sketch $\varnothing$. We will not make use of this more general notion in the present paper.

Each finite category $\mathbb{A}$ induces a sketch, called the *underlying sketch of* $\mathbb{A}$, and denoted by $\mathcal{U}(\mathbb{A})$. This sketch has as underlying graph the underlying graph of the category $\mathbb{A}$ and as commutativity conditions:

- $((A, f, B, g, C), (A, g \circ f, C))$ for any pair of composable arrows $f\colon A \to B$ and $g\colon B \to C$ in $\mathbb{A}$;
- $((A, 1_A, A), (A))$ for any object A in $\mathbb{A}$.

The sketch $\mathcal{U}(\mathbb{A})$ is not equipped with any convergence conditions (even though some cones/cocones in $\mathbb{A}$ may be limits/colimits). A more appropriate name for 'underlying sketch' would have been 'underlying commutativity sketch'. For the sake of brevity and coherence with [30], we drop 'commutativity' in the name. A $\mathcal{U}(\mathbb{A})$-structure in a category $\mathbb{C}$ corresponds to a functor $\mathbb{A} \to \mathbb{C}$.

If $\mathcal{A}$ is a sketch with underlying graph $\mathcal{G}$ and if $f_n \cdots f_1 = g_m \cdots g_1$ is a commutativity condition on the graph $\mathcal{G}$ (one that is not necessarily included in $\mathcal{A}$), we write

$$f_n \cdots f_1 \equiv_{\mathcal{A}} g_m \cdots g_1 \tag{1}$$

if, for any $\mathcal{A}$-structure F in any category $\mathbb{C}$, the equality

$$F(f_n) \circ \cdots \circ F(f_1) = F(g_m) \circ \cdots \circ F(g_1)$$

holds. Similarly, if $D\colon \mathcal{H} \to \mathcal{G}$ is a finite diagram in $\mathcal{G}$ and if $(p_H)_{H \in \mathcal{H}}$ is a family of paths $p_H\colon C \to D(H)$ in $\mathcal{G}$ indexed by the objects of $\mathcal{H}$, we write

$$(C, (p_H)_H) \equiv_{\mathcal{A}} \mathsf{limit}(\mathcal{H}, D) \tag{2}$$

if, for any $\mathcal{A}$-structure F in any category $\mathbb{C}$, the cone $(F(C), (F(p_H))_H)$ is an actual limit of $F \circ D$ in $\mathbb{C}$. As usual, $F(p_H)$ represents the composite in $\mathbb{C}$ of the actual images under F of the arrows constituting the path p_H.

We will impose some conditions on the exactness sequents $\mathcal{A} \vdash \mathcal{B}$ we consider in order for the property $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ to be expressible in terms of finite limits and regular epimorphisms, and so to be suitable for the context of regular categories (in the sense of [4]). However, in this paper, we are not interested in all those sequents where the exactness property is so expressible, but only some. For instance, we require that the following axiom holds:

Ax. 1. $\mathcal{A}$ is the underlying sketch $\mathcal{A} = \mathcal{U}(\mathbb{A})$ of a finite category $\mathbb{A}$ (i.e., $\alpha\colon \varnothing \to \mathcal{A}$ is 'unconditional of finite kind' in the sense of [30]).

We furthermore require that the subsketch inclusion $\beta\colon \mathcal{A} \to \mathcal{B}$ decomposes as a finite sequence

$$\begin{tikzcd} \mathcal{A} = \mathcal{B}_0 \arrow[r] \arrow[rrrrr, bend right=10, "\beta_{\mathsf{L}}"'] & \cdots \arrow[r] & \mathcal{B}_i \arrow[r] & \mathcal{B}_{i+1} \arrow[r] & \cdots \arrow[r] & \mathcal{B}_l = \mathcal{B}_{\mathsf{L}} \arrow[r, "\beta_{\mathsf{R}}"'] & \mathcal{B}_{l+1} = \mathcal{B} \end{tikzcd}$$

of subsketch inclusions (with β_{L} denoting the composite up to $\mathcal{B}_{\mathsf{L}}$), where each $\mathcal{B}_{i+1}$ is obtained from the previous sketch $\mathcal{B}_i$ in a certain way. We distinguish between exactness sequents of three different types, depending on what this 'certain way' is. We introduce them one-by-one and, along the way, prove that they all actually determine equivalent exactness properties in the context of regular categories, the 'linear exactness properties' from the title and the Introduction.

1.1 Sequents of Type I

An exactness sequent of *Type I* is an exactness sequent $\mathcal{A} \vdash \mathcal{B}$ for which Ax. 1 holds along with there being a decomposition of the subsketch inclusion $\beta\colon \mathcal{A} \to \mathcal{B}$

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha"] & \mathcal{A} = \mathcal{U}(\mathbb{A}) \arrow[r, "\beta_{\mathsf{L}}"] \arrow[rr, bend right, "\beta"'] & \mathcal{B}_{\mathsf{L}} \arrow[r, "\beta_{\mathsf{R}}"] & \mathcal{B} \end{tikzcd}, \quad \beta = \beta_{\mathsf{R}} \circ \beta_{\mathsf{L}},$$

such that the following axioms hold (the first of these is an axiom on β_{L}, while the second is an axiom on β_{R}):

Ax. 2. β_{L} is the composite of a finite sequence

$$\mathcal{A} = \mathcal{B}_0 \longrightarrow \cdots \longrightarrow \mathcal{B}_i \longrightarrow \mathcal{B}_{i+1} \longrightarrow \cdots \longrightarrow \mathcal{B}_l = \mathcal{B}_{\mathsf{L}}$$

of subsketch inclusions, where every next subsketch $\mathcal{B}_{i+1}$ of $\mathcal{B}_{\mathsf{L}}$ is obtained from the previous subsketch $\mathcal{B}_i$ by any one of the following procedures:

- Proc. A. Add to $\mathcal{B}_i$ a new object C together with a family of pairwise distinct arrows $(c_H)_H$ (indexed by H, an object in a finite graph $\mathcal{H}$) and a condition $(C, (c_H)_H) = \mathsf{limit}(\mathcal{H}, D)$, where D is a finite diagram in $\mathcal{B}_i$.
- Proc. B. Given in $\mathcal{B}_i$ a condition $(C, (c_H)_H) = \mathsf{limit}(\mathcal{H}, D)$, an object A, a family of arrows $(a_H\colon A \to D(H))_{H\in\mathcal{H}}$ and, for each arrow $h\colon H \to H'$ in $\mathcal{H}$, the commutativity condition $D(h)\cdot a_H = a_{H'}$, add to $\mathcal{B}_i$ a new arrow $f\colon A \to C$ and commutativity conditions $c_H \cdot f = a_H$ for each object $H \in \mathcal{H}$.
- Proc. C. Add some commutativity conditions $f_n \cdots f_1 = g_m \cdots g_1$ to $\mathcal{B}_i$ expressed using objects and arrows which belong to $\mathcal{B}_i$ and for which $f_n \cdots f_1 \equiv_{\mathcal{B}_i} g_m \cdots g_1$ (in other words, add some redundant commutativity conditions).
- Proc. D. Add a new arrow f and a commutativity condition $f = g_n \cdots g_1$ in $\mathcal{B}_i$, for $n \geqslant 0$ and existing arrows $g_1, \ldots, g_n$ in $\mathcal{B}_i$.

Ax. 3. β_{R} is constructed from a given arrow $\mathsf{q}\colon \mathsf{X} \to \mathsf{Y}$ in $\mathcal{B}_{\mathsf{L}}$ via the following two steps:

- Add to $\mathcal{B}_{\mathsf{L}}$ a new object R and pairwise distinct arrows $r_1, r_2\colon R \to \mathsf{X}$ and $h\colon R \to \mathsf{Y}$, together with the condition $(R, (r_1, r_2, h)) = \mathsf{limit}(\mathcal{H}_{\mathsf{Pb}}, D_{\mathsf{Pb}}^{\mathsf{q};\mathsf{q}})$ where $\mathcal{H}_{\mathsf{Pb}}$ is the graph

$$\begin{tikzcd} & V_2 \arrow[d, "v_2"] \\ V_1 \arrow[r, "v_1"'] & V_3 \end{tikzcd}$$

with three distinct objects V_1, V_2, V_3, and the diagram $D_{\mathsf{Pb}}^{\mathsf{q};\mathsf{q}}\colon \mathcal{H}_{\mathsf{Pb}} \to \mathcal{B}_{\mathsf{L}}$ is defined via $D_{\mathsf{Pb}}^{\mathsf{q};\mathsf{q}}(v_1) = \mathsf{q} = D_{\mathsf{Pb}}^{\mathsf{q};\mathsf{q}}(v_2)$ (in other words, (r_1, r_2) represents the kernel pair of q).

- Then add the condition $(\mathsf{Y}, (h, \mathsf{q})) = \mathsf{colimit}(\mathcal{H}_{\mathsf{Coeq}}, D^{r_1,r_2}_{\mathsf{Coeq}})$ where $\mathcal{H}_{\mathsf{Coeq}}$ is the graph
$$W_1 \underset{w_2}{\overset{w_1}{\rightrightarrows}} W_2$$
with two distinct objects W_1, W_2 and two distinct arrows w_1, w_2, while the diagram $D^{r_1,r_2}_{\mathsf{Coeq}} \colon \mathcal{H}_{\mathsf{Coeq}} \to \mathcal{B}$ is defined via $D^{r_1,r_2}_{\mathsf{Coeq}}(w_1) = r_1$ and $D^{r_1,r_2}_{\mathsf{Coeq}}(w_2) = r_2$ (in other words, q represents the coequalizer of r_1 and r_2).

Note that we have used a special font for the arrow $\mathsf{q} \colon \mathsf{X} \to \mathsf{Y}$ in Ax. 3 since we refer to this arrow often when we work with an exactness sequent of Type I. Further notation that will be reused in the paper with a similar meaning as in the formulation of the axioms above are: α, β, β_{L}, β_{R}, $\mathcal{B}_{\mathsf{L}}$, $\mathbb{A}$, $\mathcal{H}_{\mathsf{Pb}}$, $\mathcal{H}_{\mathsf{Coeq}}$, $D^{x,y}_{\mathsf{Pb}}$, $D^{x,y}_{\mathsf{Coeq}}$. The first six of these depend on being given an exactness sequent $\mathcal{A} \vdash \mathcal{B}$ of Type I.

Exactness sequents $\mathcal{A} \vdash \mathcal{B}$ of Type I have therefore the property to have α being 'unconditional of finite kind' and β 'constructible' in the sense of [30]. A finitely complete category $\mathbb{C}$ has the property $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ exactly when, for each functor $F \colon \mathbb{A} \to \mathbb{C}$, its unique (up to isomorphism) extension to a $\mathcal{B}_{\mathsf{L}}$-structure in $\mathbb{C}$, which we denote by F_{L}, is such that $F_{\mathsf{L}}(\mathsf{q})$ is a regular epimorphism (i.e., the coequalizer of its kernel pair). The extension F_{L} is obtained by constructing finite limits, induced morphisms to those finite limits and composite morphisms as prescribed by the subsketch inclusion $\beta_{\mathsf{L}} \colon \mathcal{U}(\mathbb{A}) \to \mathcal{B}_{\mathsf{L}}$.

Our first result is that, in the regular context, the class of exactness properties induced by exactness sequents of Type I is closed under finite conjunctions. Regular categories are used here since, in those categories, regular epimorphisms are closed under finite products.

Theorem 1.1. *Let $a \geqslant 0$ be a natural number and let $(\mathcal{A}_i \vdash \mathcal{B}_i)_{1 \leqslant i \leqslant a}$ be a (finite) family of exactness sequents of Type I. There exists an exactness sequent $\mathcal{A} \vdash \mathcal{B}$ of Type I such that, for any regular category $\mathbb{C}$, one has $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ if and only if $\mathcal{A}_i \vdash_{\mathbb{C}} \mathcal{B}_i$ holds for all $i \in \{1, \dots, a\}$.*

Proof. For each $i \in \{1, \dots, a\}$, let
$$\begin{tikzcd} \varnothing \arrow[r, "\alpha_i"] & \mathcal{A}_i = \mathcal{U}(\mathbb{A}_i) \arrow[r, "\beta_{i,\mathsf{L}}"] \arrow[rr, bend right, "\beta_i"'] & \mathcal{B}_{i,\mathsf{L}} \arrow[r, "\beta_{i,\mathsf{R}}"] & \mathcal{B}_i \end{tikzcd}$$
be the presentation of $\mathcal{A}_i \vdash \mathcal{B}_i$ as in the definition of an exactness sequent of Type I. We are going to construct an exactness sequent $\mathcal{A} \vdash \mathcal{B}$, presented as
$$\begin{tikzcd} \varnothing \arrow[r, "\alpha"] & \mathcal{A} = \mathcal{U}(\mathbb{A}) \arrow[r, "\beta_{\mathsf{L}}"] \arrow[rr, bend right, "\beta"'] & \mathcal{B}_{\mathsf{L}} \arrow[r, "\beta_{\mathsf{R}}"] & \mathcal{B}, \end{tikzcd}$$
fulfilling the requirement of the theorem. Let $\mathbb{A}$ be the disjoint union of the categories $\mathbb{A}_i$'s, i.e., $\mathbb{A} = \amalg_{i \in \{1,\dots,a\}} \mathbb{A}_i$. Let the sketch $\mathcal{C}$ be the disjoint union of the sketches $\mathcal{B}_{i,\mathsf{L}}$'s, i.e., $\mathcal{C} = \amalg_{i \in \{1,\dots,a\}} \mathcal{B}_{i,\mathsf{L}}$. This means, we form the disjoint union of the underlying finite graphs and equip it with the commutativity and convergence conditions from the $\mathcal{B}_{i,\mathsf{L}}$'s. For $i \in \{1, \dots, a\}$, we write the objects of $\mathcal{C}$ coming from the i-th term $\mathcal{B}_{i,\mathsf{L}}$ in the disjoint union as pairs (i, B) where $B \in \mathcal{B}_{i,\mathsf{L}}$, and similarly for the arrows. We denote by $\mathsf{q}_i \colon \mathsf{X}_i \to \mathsf{Y}_i$ the arrow q from Ax. 3 applied to the exactness sequent $\mathcal{A}_i \vdash \mathcal{B}_i$. We then form a sketch $\mathcal{C}'$ following Proc. A by adding to $\mathcal{C}$

- a new object X,

- a new arrow $p_i^{\mathsf{X}}\colon \mathsf{X} \to (i, \mathsf{X}_i)$ for each $i \in \{1, \dots, a\}$,
- and the convergence condition $(\mathsf{X}, (p_i^{\mathsf{X}})_i) = \mathsf{limit}(\mathcal{H}, D)$, where $\mathcal{H}$ is the graph with a objects $1, \dots, a$ and no arrows, and D is defined via $D(i) = (i, \mathsf{X}_i)$.

Note that $(\mathsf{X}, (p_i^{\mathsf{X}})_i)$ represents a product of the family $(\mathsf{X}_i)_{i \in \{1,\dots,a\}}$. In a similar way, we extend the sketch $\mathcal{C}'$ to a sketch $\mathcal{C}''$ where a representing pair $(\mathsf{Y}, (p_i^{\mathsf{Y}})_i)$ for the product of the family $(\mathsf{Y}_i)_{i \in \{1,\dots,a\}}$ has been added. Then we form a sketch $\mathcal{C}'''$ following Proc. D (a many times) by adding to $\mathcal{C}''$, for each $i \in \{1, \dots, a\}$, a new arrow $f_i\colon \mathsf{X} \to (i, \mathsf{Y}_i)$ together with the commutativity conditions $f_i = (i, \mathsf{q}_i) \cdot p_i^{\mathsf{X}}$. Finally, we form the sketch $\mathcal{B}_{\mathsf{L}}$ following Proc. B by adding to $\mathcal{C}'''$ a new arrow $\mathsf{q}\colon \mathsf{X} \to \mathsf{Y}$ and the commutativity condition $p_i^{\mathsf{Y}} \cdot \mathsf{q} = f_i$ for each $i \in \{1, \dots, a\}$. We thus have a subsketch inclusion $\beta_{\mathsf{L}}\colon \mathcal{U}(\mathbb{A}) \to \mathcal{B}_{\mathsf{L}}$ which can be decomposed as in Ax. 2. We form the sketch $\mathcal{B}$ from $\mathcal{B}_{\mathsf{L}}$ and q, as in Ax. 3. We then get an exactness sequent $\mathcal{A} \vdash \mathcal{B}$ of Type I.

Now let $\mathbb{C}$ be a fixed regular category. We notice that giving a functor $F\colon \mathbb{A} \to \mathbb{C}$ is the same as giving a family $(F_i\colon \mathbb{A}_i \to \mathbb{C})_{1 \leqslant i \leqslant a}$ of functors. We denote by F_{L} the extension of such a functor F as a $\mathcal{B}_{\mathsf{L}}$-structure in $\mathbb{C}$ and by $F_{i,\mathsf{L}}$ the extension of F_i as a $\mathcal{B}_{i,\mathsf{L}}$-structure. The category $\mathbb{C}$ satisfies $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ if and only if, for any functor $F\colon \mathbb{A} \to \mathbb{C}$, the morphism $F_{\mathsf{L}}(\mathsf{q})$ is a regular epimorphism; that is, if and only if the product

$$\prod_{1 \leqslant i \leqslant a} F_{i,\mathsf{L}}(\mathsf{q}_i)\colon \prod_{1 \leqslant i \leqslant a} F_{i,\mathsf{L}}(\mathsf{X}_i) \to \prod_{1 \leqslant i \leqslant a} F_{i,\mathsf{L}}(\mathsf{Y}_i)$$

of the morphisms $F_{\mathsf{L}}(i, \mathsf{q}_i) = F_{i,\mathsf{L}}(\mathsf{q}_i)$ is a regular epimorphism. Therefore, if $\mathcal{A}_i \vdash_{\mathbb{C}} \mathcal{B}_i$ holds for all $i \in \{1, \dots, a\}$, all $F_{i,\mathsf{L}}(\mathsf{q}_i)$ are regular epimorphisms and thus so is their product since $\mathbb{C}$ is a regular category. Conversely, suppose $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ and let us fix $j \in \{1, \dots, a\}$ and a functor $F_j\colon \mathbb{A}_j \to \mathbb{C}$. For any $i \neq j$, we define the functor $F_i\colon \mathbb{A}_i \to \mathbb{C}$ to be the constant functor mapping each object of $\mathbb{A}_i$ to the terminal object 1 of $\mathbb{C}$. In view of the step-by-step construction of $\mathcal{B}_{i,\mathsf{L}}$ described in Ax. 2, the induced $\mathcal{B}_{i,\mathsf{L}}$-structure $F_{i,\mathsf{L}}$ also maps any object of $\mathcal{B}_{i,\mathsf{L}}$ to 1 for all $i \neq j$. Considering the functor $F\colon \mathbb{A} \to \mathbb{C}$ induced by these F_i's and F_j, the morphism $F_{\mathsf{L}}(\mathsf{q})$ is isomorphic to the morphism $F_{j,\mathsf{L}}(\mathsf{q}_j)$ (seen as objects in the morphism category). Since $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$, this proves that $F_{j,\mathsf{L}}(\mathsf{q}_j)$ is a regular epimorphism and thus $\mathcal{A}_j \vdash_{\mathbb{C}} \mathcal{B}_j$. $\square$

We now prove a result needed to establish a link between exactness sequents of Type I and those of Type II introduced in the next subsection.

Lemma 1.2. *Let $\beta_{\mathsf{L}}\colon \mathcal{U}(\mathbb{A}) \to \mathcal{B}_{\mathsf{L}}$ be a subsketch inclusion from the underlying sketch of a finite category and which can be decomposed as in Ax. 2. Then,*

- *for any object $Z \in \mathcal{B}_{\mathsf{L}}$, there exists a finite diagram $D_Z\colon \mathcal{H}_Z \to \mathcal{U}(\mathbb{A})$ and a family of paths $(p_H^Z\colon Z \to D_Z(H))_{H \in \mathcal{H}_Z}$ in $\mathcal{B}_{\mathsf{L}}$ such that*

$$(Z, (p_H^Z)_H) \equiv_{\mathcal{B}_{\mathsf{L}}} \mathsf{limit}(\mathcal{H}_Z, \beta_{\mathsf{L}} \circ D_Z)$$

  and satisfying the following condition;

- *for any arrow $z\colon Z \to Z'$ in $\mathcal{B}_{\mathsf{L}}$ and for any object $H \in \mathcal{H}_{Z'}$, there exists an object $K_H^z \in \mathcal{H}_Z$ and a morphism $c_H^z\colon D_Z(K_H^z) \to D_{Z'}(H)$ in $\mathbb{A}$ such that*

$$p_H^{Z'} \cdot z \equiv_{\mathcal{B}_{\mathsf{L}}} c_H^z \cdot p_{K_H^z}^Z.$$

Proof. We are going to prove both assertions simultaneously by induction. If $\mathcal{B}_{\mathsf{L}} = \mathcal{U}(\mathbb{A})$, we define, for each object $Z \in \mathcal{U}(\mathbb{A})$, $\mathcal{H}_Z = \{*_Z\}$ to be a one object graph with no arrows, $D_Z(*_Z) = Z$ and $p^Z_{*_Z}$ to be the empty path on Z. For $z\colon Z \to Z'$ in $\mathcal{U}(\mathbb{A})$, we set $K^z_{*_{Z'}} = *_Z$ and $c^z_{*_{Z'}} = z$.

We now suppose these statements hold for the subsketch inclusion $\mathcal{U}(\mathbb{A}) \to \mathcal{B}_i$ and we are going to prove they also hold for the subsketch inclusion $\mathcal{U}(\mathbb{A}) \to \mathcal{B}_{i+1}$ where $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ are consecutive steps in the construction

$$\mathcal{U}(\mathbb{A}) = \mathcal{B}_0 \longrightarrow \cdots \longrightarrow \mathcal{B}_i \longrightarrow \mathcal{B}_{i+1} \longrightarrow \cdots \longrightarrow \mathcal{B}_l = \mathcal{B}_{\mathsf{L}}$$

of $\mathcal{B}_{\mathsf{L}}$. We distinguish four cases, corresponding to the four different kinds of possible procedures in Ax. 2. Notice that, whichever procedure is considered, for objects $Z \in \mathcal{B}_i$ and arrows $z \in \mathcal{B}_i$ belonging to the previous subsketch, the same $\mathcal{H}_Z$, D_Z, K^z_H's and c^z_H's can be chosen. We thus only need to take the new objects and the new arrows into account. We use the notation from the description of the different procedures contained in Ax. 2.

Proc. A. We construct the graph $\mathcal{H}_C$ as follows. We first consider the disjoint union $\amalg_{H\in\mathcal{H}}\mathcal{H}_{D(H)}$ of the graphs $\mathcal{H}_{D(H)}$, i.e., its set of objects is

$$\{(H,J)\,|\,H \in \mathcal{H},\ J \in \mathcal{H}_{D(H)}\}$$

and its set of arrows is

$$\{(H,j)\colon (H,J) \to (H,J')\,|\,H \in \mathcal{H},\ j\colon J \to J' \in \mathcal{H}_{D(H)}\}.$$

To obtain $\mathcal{H}_C$, we add to this graph, for each arrow $h\colon H \to H'$ in $\mathcal{H}$ and each object $J \in \mathcal{H}_{D(H')}$, an arrow

$$(h,J)\colon (H, K^{D(h)}_J) \to (H',J).$$

We now define the diagram $D_C\colon \mathcal{H}_C \to \mathcal{U}(\mathbb{A})$ via the equalities

$$\begin{aligned} D_C(H,J) &= D_{D(H)}(J) && \text{for } H \in \mathcal{H} \text{ and } J \in \mathcal{H}_{D(H)};\\ D_C(H,j) &= D_{D(H)}(j) && \text{for } H \in \mathcal{H} \text{ and } j\colon J \to J' \in \mathcal{H}_{D(H)};\\ D_C(h,J) &= c^{D(h)}_J && \text{for } h\colon H \to H' \in \mathcal{H} \text{ and } J \in \mathcal{H}_{D(H')}. \end{aligned}$$

For each object $(H,J) \in \mathcal{H}_C$, we let $p^C_{(H,J)}$ be the composite path $p^C_{(H,J)} = p^{D(H)}_J \cdot c_H$.

$$C \xrightarrow{c_H} D(H) \xrightarrow{p^{D(H)}_J} D_{D(H)}(J)$$

Finally, for any object $H \in \mathcal{H}$ and any object $J \in \mathcal{H}_{D(H)}$, we define $K^{c_H}_J$ as $(H,J) \in \mathcal{H}_C$ and $c^{c_H}_J$ as the identity morphism on $D_{D(H)}(J)$ in $\mathbb{A}$. It is then routine verification to check that $(C,(p^C_{(H,J)})_{(H,J)}) \equiv_{\mathcal{B}_{i+1}} \mathsf{limit}(\mathcal{H}_C, D_C)$ and $p^{D(H)}_J \cdot c_H \equiv_{\mathcal{B}_{i+1}} c^{c_H}_J \cdot p^C_{K^{c_H}_J}$ for any $H \in \mathcal{H}$ and any $J \in \mathcal{H}_{D(H)}$.

Proc. B. Since $(C,(c_H)_H) = \mathsf{limit}(\mathcal{H}, D)$ is supposed to be a convergence condition in $\mathcal{B}_i$, it has to come from a Proc. A. Therefore, we know $\mathcal{H}_C$ has been constructed as above. Let $(H,J) \in \mathcal{H}_C$, i.e., $H \in \mathcal{H}$ and $J \in \mathcal{H}_{D(H)}$. We set $K^f_{(H,J)} = K^{a_H}_J \in \mathcal{H}_A$ and

$$c^f_{(H,J)} = c^{a_H}_J\colon D_A(K^f_{(H,J)}) = D_A(K^{a_H}_J) \to D_C(H,J) = D_{D(H)}(J).$$

We can compute

$$p^C_{(H,J)} \cdot f = p^{D(H)}_J \cdot c_H \cdot f \equiv_{\mathcal{B}_{i+1}} p^{D(H)}_J \cdot a_H \equiv_{\mathcal{B}_{i+1}} c^{a_H}_J \cdot p^A_{K^{a_H}_J} = c^f_{(H,J)} \cdot p^A_{K^f_{(H,J)}}.$$

Proc. C. There is nothing to prove here since there is no new objects or new arrows.

Proc. D. We use here an induction on the number $n \geqslant 0$. If $n = 0$, this procedure adds an arrow $f\colon Z \to Z$ together with the condition $f = 1_Z$. For any $H \in \mathcal{H}_Z$, we choose $K_H^f = H$ and c_H^f to be the identity morphism on $D_Z(H)$. The required condition is then trivially satisfied.

Let us suppose that the procedure adds $f\colon Z_1 \to Z_{n+1}$ and the condition $f = g_n \cdots g_1$ where each $g_j\colon Z_j \to Z_{j+1}$ is in $\mathcal{B}_i$. Suppose moreover that for each $H \in \mathcal{H}_{Z_{n+1}}$, there exists $K_H^{g_n \cdots g_2} \in \mathcal{H}_{Z_2}$ and $c_H^{g_n \cdots g_2}\colon D_{Z_2}(K_H^{g_n \cdots g_2}) \to D_{Z_{n+1}}(H)$ in $\mathbb{A}$ such that

$$p_H^{Z_{n+1}} \cdot g_n \cdots g_2 \equiv_{\mathcal{B}_{i+1}} c_H^{g_n \cdots g_2} \cdot p^{Z_2}_{K_H^{g_n \cdots g_2}}.$$

We then set, for each $H \in \mathcal{H}_{Z_{n+1}}$, the object $K_H^f = K_H^{g_n \cdots g_1}$ to be $K^{g_1}_{K_H^{g_n \cdots g_2}}$ and $c_H^f = c_H^{g_n \cdots g_1}$ to be the composite $c_H^{g_n \cdots g_2} \circ c^{g_1}_{K_H^{g_n \cdots g_2}}$ in $\mathbb{A}$.

$$D_{Z_1}(K_H^f) = D_{Z_1}(K^{g_1}_{K_H^{g_n \cdots g_2}}) \xrightarrow{c^{g_1}_{K_H^{g_n \cdots g_2}}} D_{Z_2}(K_H^{g_n \cdots g_2}) \xrightarrow{c_H^{g_n \cdots g_2}} D_{Z_{n+1}}(H)$$

We can then compute

$$\begin{aligned}
p_H^{Z_{n+1}} \cdot f &\equiv_{\mathcal{B}_{i+1}} p_H^{Z_{n+1}} \cdot g_n \cdots g_1 \\
&\equiv_{\mathcal{B}_{i+1}} c_H^{g_n \cdots g_2} \cdot p^{Z_2}_{K_H^{g_n \cdots g_2}} \cdot g_1 \\
&\equiv_{\mathcal{B}_{i+1}} c_H^{g_n \cdots g_2} \cdot c^{g_1}_{K_H^{g_n \cdots g_2}} \cdot p^{Z_1}_{K^{g_1}_{K_H^{g_n \cdots g_2}}} \\
&\equiv_{\mathcal{B}_{i+1}} c_H^{g_n \cdots g_1} \cdot p^{Z_1}_{K_H^{g_n \cdots g_1}} \\
&= c_H^f \cdot p^{Z_1}_{K_H^f}
\end{aligned}$$

concluding the induction. □

1.2 Sequents of Type II

As a consequence of Lemma 1.2, one can strengthen condition Ax. 2 on the considered exactness sequents without changing the class of exactness properties on finitely complete categories one obtains from them. That is, we consider those exactness sequents $\mathcal{A} \vdash \mathcal{B}$, called of *Type II*, that can be decomposed into the subsketch inclusions

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha"] & \mathcal{A} \arrow[r, "\beta_{\mathsf{L}}"] \arrow[rr, bend right, "\beta"'] & \mathcal{B}_{\mathsf{L}} \arrow[r, "\beta_{\mathsf{R}}"] & \mathcal{B} \end{tikzcd}, \quad \beta = \beta_{\mathsf{R}} \circ \beta_{\mathsf{L}},$$

and who satisfy conditions Ax. 1, Ax. 2′ and Ax. 3 where Ax. 2′ is now:

Ax. 2′. β_{L} is the composite of a finite sequence

$$\mathcal{A} = \mathcal{B}_0 \longrightarrow \mathcal{B}_1 \longrightarrow \mathcal{B}_2 \longrightarrow \cdots \longrightarrow \mathcal{B}_{2+b} \longrightarrow \mathcal{B}_{3+b} \longrightarrow \mathcal{B}_{4+b} = \mathcal{B}_{\mathsf{L}}$$

of subsketch inclusions (with $b \geqslant 0$), where every next subsketch $\mathcal{B}_{i+1}$ of $\mathcal{B}_{\mathsf{L}}$ is obtained from the previous subsketch $\mathcal{B}_i$ following one of Proc. A, Proc. B, Proc. C or Proc. D as in Ax. 2, but in the following order: $\mathcal{B}_1$ is obtained from $\mathcal{B}_0$ following Proc. A, $\mathcal{B}_2$ is obtained from $\mathcal{B}_1$ also following Proc. A, $\mathcal{B}_{2+b}$ is obtained from $\mathcal{B}_2$ following Proc. D a finite number b of times, $\mathcal{B}_{3+b}$ is obtained from $\mathcal{B}_{2+b}$ following Proc. C and finally $\mathcal{B}_{4+b}$ is obtained from $\mathcal{B}_{3+b}$ following Proc. B. Moreover, we require that the diagram D considered in Proc. A to form $\mathcal{B}_2$ lies entirely in $\mathcal{A}$.

Each exactness sequent $\mathcal{A} \vdash \mathcal{B}$ of Type II is in particular of Type I. In view of Lemma 1.2, one can prove the second part of the following theorem which provides a converse of this fact on the level of the induced exactness properties on finitely complete categories.

Theorem 1.3. *Each exactness sequent of Type II is in particular of Type I. Conversely, let $\mathcal{A} \vdash \mathcal{B}$ be an exactness sequent of Type I. There exists an exactness sequent $\mathcal{A} \vdash \mathcal{B}'$ of Type II such that, for any finitely complete category $\mathbb{C}$, one has $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ if and only if $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}'$.*

Proof. The first part of the statement being obvious, let us prove the second part. Let

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha"] & \mathcal{A} = \mathcal{U}(\mathbb{A}) \arrow[r, "\beta_{\mathsf{L}}"] \arrow[rr, bend right, "\beta"'] & \mathcal{B}_{\mathsf{L}} \arrow[r, "\beta_{\mathsf{R}}"] & \mathcal{B} \end{tikzcd}$$

be the presentation of $\mathcal{A} \vdash \mathcal{B}$ as in the definition of an exactness sequent of Type I. Let $\mathsf{q}\colon \mathsf{X} \to \mathsf{Y}$ be the arrow in $\mathcal{B}_{\mathsf{L}}$ with respect to which $\mathcal{B}$ is constructed as in Ax. 3. Let us consider the finite diagrams $D_{\mathsf{X}}\colon \mathcal{H}_{\mathsf{X}} \to \mathcal{U}(\mathbb{A})$ and $D_{\mathsf{Y}}\colon \mathcal{H}_{\mathsf{Y}} \to \mathcal{U}(\mathbb{A})$, the families of paths $(p^{\mathsf{X}}_K\colon \mathsf{X} \to D_{\mathsf{X}}(K))_{K \in \mathcal{H}_{\mathsf{X}}}$ and $(p^{\mathsf{Y}}_H\colon \mathsf{Y} \to D_{\mathsf{Y}}(H))_{H \in \mathcal{H}_{\mathsf{Y}}}$ in $\mathcal{B}_{\mathsf{L}}$ and, for each object $H \in \mathcal{H}_{\mathsf{Y}}$, the object K^{q}_H in $\mathcal{H}_{\mathsf{X}}$ and the morphism $c^{\mathsf{q}}_H\colon D_{\mathsf{X}}(K^{\mathsf{q}}_H) \to D_{\mathsf{Y}}(H)$ in $\mathbb{A}$ given by Lemma 1.2. We are going to construct an exactness sequent $\mathcal{A} \vdash \mathcal{B}'$ of Type II, presented as

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha"] & \mathcal{A} = \mathcal{U}(\mathbb{A}) \arrow[r, "\beta'_{\mathsf{L}}"] \arrow[rr, bend right, "\beta'"'] & \mathcal{B}'_{\mathsf{L}} \arrow[r, "\beta'_{\mathsf{R}}"] & \mathcal{B}', \end{tikzcd}$$

fulfilling the requirement of the theorem. We first form a sketch $\mathcal{B}'_1$ following Proc. A by adding to $\mathcal{U}(\mathbb{A})$ a new object C_1, a new arrow $c^1_K\colon C_1 \to D_{\mathsf{X}}(K)$ for each $K \in \mathcal{H}_{\mathsf{X}}$ and the convergence condition $(C_1, (c^1_K)_K) = \mathsf{limit}(\mathcal{H}_{\mathsf{X}}, D_{\mathsf{X}})$. We then form a sketch $\mathcal{B}'_2$ following Proc. A by adding to $\mathcal{B}'_1$ a new object C_2, a new arrow $c^2_H\colon C_2 \to D_{\mathsf{Y}}(H)$ for each $H \in \mathcal{H}_{\mathsf{Y}}$ and the convergence condition $(C_2, (c^2_H)_H) = \mathsf{limit}(\mathcal{H}_{\mathsf{Y}}, D_{\mathsf{Y}})$. Let b be the number of objects in $\mathcal{H}_{\mathsf{Y}}$. We then form a sketch $\mathcal{B}'_{2+b}$ following Proc. D (b many times) by adding to $\mathcal{B}'_2$, for each $H \in \mathcal{H}_{\mathsf{Y}}$, a new arrow $f_H\colon C_1 \to D_{\mathsf{Y}}(H)$ and the commutativity condition $f_H = c^{\mathsf{q}}_H \cdot c^1_{K^{\mathsf{q}}_H}$.

We now would like to show that for each arrow $h\colon H \to H'$ in $\mathcal{H}_{\mathsf{Y}}$, the identity $D_{\mathsf{Y}}(h) \cdot f_H \equiv_{\mathcal{B}'_{2+b}} f_{H'}$ holds. In order to do so, let $\mathbb{C}$ be any category and G be any $\mathcal{B}'_{2+b}$-structure in $\mathbb{C}$. Let $\mathbb{D}$ be the full subcategory of $\mathbb{C}$ generated by all objects of the form $G(B)$ for some object B in $\mathcal{B}'_{2+b}$. Let us denote by $I\colon \mathbb{D} \hookrightarrow \mathbb{C}$ the full embedding of $\mathbb{D}$ into $\mathbb{C}$. The $\mathcal{B}'_{2+b}$-structure G in $\mathbb{C}$ induces a unique $\mathcal{B}'_{2+b}$-structure G' in $\mathbb{D}$ such that $I \circ G' = G$. Since $\mathbb{D}$ is a small category, we can consider the Yoneda embedding $Y\colon \mathbb{D} \to \mathbf{Set}^{\mathbb{D}^{\mathsf{op}}}$ to the category of functors from the dual category $\mathbb{D}^{\mathsf{op}}$ of $\mathbb{D}$ to the category $\mathbf{Set}$ of sets. Since Y preserves finite limits and since $\mathcal{B}'_{2+b}$ does not contain any finite colimit condition, the composite $Y \circ G'$ is a $\mathcal{B}'_{2+b}$-structure in $\mathbf{Set}^{\mathbb{D}^{\mathsf{op}}}$. Denoting by β'_{2+b} the inclusion $\mathcal{U}(\mathbb{A}) \to \mathcal{B}'_{2+b}$, we have a $\mathcal{U}(\mathbb{A})$-structure $Y \circ G' \circ \beta'_{2+b}$ in $\mathbf{Set}^{\mathbb{D}^{\mathsf{op}}}$ that we abbreviate by F. Since $\mathbf{Set}^{\mathbb{D}^{\mathsf{op}}}$ is complete, this structure extends as a $\mathcal{B}_{\mathsf{L}}$-structure F_{L}. Therefore, in view of the properties given by Lemma 1.2, we have, for each arrow $h\colon H \to H'$ in $\mathcal{H}_{\mathsf{Y}}$,

$$\begin{aligned}
YG'\left(D_{\mathsf{Y}}(h)\right) \circ YG'\left(c^{\mathsf{q}}_{H}\right) \circ F_{\mathsf{L}}\left(p^{\mathsf{X}}_{K^{\mathsf{q}}_{H}}\right) &= F_{\mathsf{L}}\left(D_{\mathsf{Y}}(h)\right) \circ F_{\mathsf{L}}\left(c^{\mathsf{q}}_{H}\right) \circ F_{\mathsf{L}}\left(p^{\mathsf{X}}_{K^{\mathsf{q}}_{H}}\right) \\
&= F_{\mathsf{L}}\left(D_{\mathsf{Y}}(h)\right) \circ F_{\mathsf{L}}\left(p^{\mathsf{Y}}_{H}\right) \circ F_{\mathsf{L}}\left(\mathsf{q}\right) \\
&= F_{\mathsf{L}}\left(p^{\mathsf{Y}}_{H'}\right) \circ F_{\mathsf{L}}\left(\mathsf{q}\right) \\
&= F_{\mathsf{L}}\left(c^{\mathsf{q}}_{H'}\right) \circ F_{\mathsf{L}}\left(p^{\mathsf{X}}_{K^{\mathsf{q}}_{H'}}\right) \\
&= YG'\left(c^{\mathsf{q}}_{H'}\right) \circ F_{\mathsf{L}}\left(p^{\mathsf{X}}_{K^{\mathsf{q}}_{H'}}\right).
\end{aligned}$$

Moreover, both $(F_{\mathsf{L}}(\mathsf{X}), (F_{\mathsf{L}}(p^{\mathsf{X}}_{K}))_K)$ and $(YG'(C_1), (YG'(c^1_K))_K)$ are limits of $F \circ D_{\mathsf{X}}$. Hence, we know that

$$\begin{aligned}
YG'\left(D_{\mathsf{Y}}(h)\right) \circ YG'\left(f_H\right) &= YG'\left(D_{\mathsf{Y}}(h)\right) \circ YG'\left(c^{\mathsf{q}}_{H}\right) \circ YG'\left(c^{1}_{K^{\mathsf{q}}_{H}}\right) \\
&= YG'\left(c^{\mathsf{q}}_{H'}\right) \circ YG'\left(c^{1}_{K^{\mathsf{q}}_{H'}}\right) \\
&= YG'\left(f_{H'}\right).
\end{aligned}$$

Since Y is faithful, we thus know that $G'(D_{\mathsf{Y}}(h)) \circ G'(f_H) = G'(f_{H'})$ which implies $G(D_{\mathsf{Y}}(h)) \circ G(f_H) = G(f_{H'})$. This proves that $D_{\mathsf{Y}}(h) \cdot f_H \equiv_{\mathcal{B}'_{2+b}} f_{H'}$ for any arrow $h\colon H \to H'$ in $\mathcal{H}_{\mathsf{Y}}$. We can thus construct a sketch $\mathcal{B}'_{3+b}$ following Proc. C by adding to $\mathcal{B}'_{2+b}$ the commutativity conditions $D_{\mathsf{Y}}(h) \cdot f_H = f_{H'}$ for all $h\colon H \to H'$ in $\mathcal{H}_{\mathsf{Y}}$. We then construct $\mathcal{B}'_{\mathsf{L}}$ following Proc. B by adding to $\mathcal{B}'_{3+b}$ a new arrow $f\colon C_1 \to C_2$ and, for each object $H \in \mathcal{H}_{\mathsf{Y}}$, the commutativity condition $c^2_H \cdot f = f_H$. We finally construct $\mathcal{B}'$ from $\mathcal{B}'_{\mathsf{L}}$ as prescribed by Ax. 3 with respect to the arrow $f\colon C_1 \to C_2$. This concludes the construction of the exactness sequent $\mathcal{A} \vdash \mathcal{B}'$ of Type II.

Given a finitely complete category $\mathbb{C}$, it remains to prove that $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ holds if and only if $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}'$ holds. Given any $\mathcal{U}(\mathbb{A})$-structure F in $\mathbb{C}$, we denote by F_{L} its extension as a $\mathcal{B}_{\mathsf{L}}$-structure and by F'_{L} its extension as a $\mathcal{B}'_{\mathsf{L}}$-structure. In view of the properties obtained from Lemma 1.2 and of the construction of $\mathcal{B}'_{\mathsf{L}}$, we know that $F_{\mathsf{L}}(\mathsf{q})$ is isomorphic to $F'_{\mathsf{L}}(f)$. One is thus a regular epimorphism if and only if the other is; proving the required equivalence. $\square$

1.3 Sequents of Type III

We have seen that the class of exactness sequents of Type I can be reduced to the class of exactness sequents of Type II without changing the class of induced exactness properties on finitely complete categories. We now establish a result in the opposite direction, where we enlarge the class of exactness sequents we consider without changing the class of induced exactness properties on regular categories. The regular context is needed here in order to use Theorem 1.1.

We now consider those exactness sequents $\mathcal{A} \vdash \mathcal{B}$, called of *Type III*, that can be decomposed into the subsketch inclusions

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha"] & \mathcal{A} \arrow[r, "\beta_{\mathsf{L}}"] \arrow[rr, bend right, "\beta"'] & \mathcal{B}_{\mathsf{L}} \arrow[r, "\beta_{\mathsf{R}}"] & \mathcal{B} \end{tikzcd}, \quad \beta = \beta_{\mathsf{R}} \circ \beta_{\mathsf{L}},$$

and who satisfy conditions Ax. 1, Ax. 2 and Ax. 3$'$ where Ax. 3$'$ is now:

Ax. 3′. $\beta_{\mathsf{R}} \colon \mathcal{B}_{\mathsf{L}} \to \mathcal{B}$ is constructed from $\mathcal{B}_{\mathsf{L}}$ by adding to it:

- for each q_i in a finite family $(\mathsf{q}_i \colon \mathsf{X}_i \to \mathsf{Y}_i)_{i\in\{1,\dots,a\}}$ of arrows in $\mathcal{B}_{\mathsf{L}}$, an object R_i (not already in $\mathcal{B}_{\mathsf{L}}$), arrows $r_1^i, r_2^i \colon R_i \to \mathsf{X}_i$ and $h^i \colon R_i \to \mathsf{Y}_i$, the finite limit condition $(R_i, (r_1^i, r_2^i, h^i)) = \mathsf{limit}(\mathcal{H}_{\mathsf{Pb}}, D_{\mathsf{Pb}}^{\mathsf{q}_i,\mathsf{q}_i})$ and the finite colimit condition $(\mathsf{Y}_i, (h^i, \mathsf{q}_i)) = \mathsf{colimit}(\mathcal{H}_{\mathsf{Coeq}}, D_{\mathsf{Coeq}}^{r_1^i, r_2^i})$ as described in Ax. 3 (with all the objects R_i's pairwise distinct and all the added arrows also pairwise distinct);
- for each i in a finite set $\{1, \dots, b\}$, a finite limit condition $(C_i, (c_H^i)_H) = \mathsf{limit}(\mathcal{H}_i, D_i)$ where each D_i, C_i and c_H^i belongs to $\mathcal{B}_{\mathsf{L}}$ and
- for each i in a finite set $\{1, \dots, c\}$, a commutativity condition $f_{n_i}^i \cdots f_1^i = g_{m_i}^i \cdots g_1^i$ where each f_j^i and each g_j^i belongs to $\mathcal{B}_{\mathsf{L}}$.

A finitely complete category $\mathbb{C}$ has the property $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ for such an exactness sequent $\mathcal{A} \vdash \mathcal{B}$ if and only if, for any $\mathcal{U}(\mathbb{A})$-structure F in $\mathbb{C}$ (i.e., for any functor $\mathbb{A} \to \mathbb{C}$ where the category $\mathbb{A}$ is given by Ax. 1), its unique (up to isomorphism) extension F_{L} as a $\mathcal{B}_{\mathsf{L}}$-structure satisfies

- $F_{\mathsf{L}}(\mathsf{q}_i)$ is a regular epimorphism for all $i \in \{1, \dots, a\}$;
- $(F_{\mathsf{L}}(C_i), (F_{\mathsf{L}}(c_H^i))_H)$ is the limit of $F_{\mathsf{L}} \circ D_i$ for all $i \in \{1, \dots, b\}$ and
- $F_{\mathsf{L}}(f_{n_i}^i) \circ \cdots \circ F_{\mathsf{L}}(f_1^i) = F_{\mathsf{L}}(g_{m_i}^i) \circ \cdots \circ F_{\mathsf{L}}(g_1^i)$ for all $i \in \{1, \dots, c\}$.

Each exactness sequent $\mathcal{A} \vdash \mathcal{B}$ of Type I is in particular of Type III with $(a, b, c) = (1, 0, 0)$. The second part of the following theorem provides a converse of this fact on the level of the induced exactness properties in the regular context.

Theorem 1.4. *Each exactness sequent of Type I is in particular of Type III. Conversely, let $\mathcal{A} \vdash \mathcal{B}$ be an exactness sequent of Type III. There exists an exactness sequent $\mathcal{A}' \vdash \mathcal{B}'$ of Type I such that, for any regular category $\mathbb{C}$, one has $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ if and only if $\mathcal{A}' \vdash_{\mathbb{C}} \mathcal{B}'$.*

Proof. The first part of the statement being obvious, let us prove the second part. Let

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha"] & \mathcal{A} = \mathcal{U}(\mathbb{A}) \arrow[r, "\beta_{\mathsf{L}}"] \arrow[rr, bend right, "\beta"'] & \mathcal{B}_{\mathsf{L}} \arrow[r, "\beta_{\mathsf{R}}"] & \mathcal{B} \end{tikzcd}$$

be the presentation of $\mathcal{A} \vdash \mathcal{B}$ as in the definition of an exactness sequent of Type III. As in the description of Ax. 3′, let a be the number of arrows $\mathsf{q}_i \colon \mathsf{X}_i \to \mathsf{Y}_i$, b the number of finite limit conditions $(C_i, (c_H^i)_H) = \mathsf{limit}(\mathcal{H}_i, D_i)$ and c the number of commutativity conditions $f_{n_i}^i \cdots f_1^i = g_{m_i}^i \cdots g_1^i$ which are considered. In view of Theorem 1.1, it is enough to treat only the two cases $(a, b, c) = (0, 1, 0)$ and $(a, b, c) = (0, 0, 1)$. Let us first assume that $(a, b, c) = (0, 1, 0)$. We are going to construct an exactness sequent $\mathcal{A} \vdash \mathcal{B}''$ of Type III with $(a'', b'', c'') = (3, 0, 0)$ such that $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ holds for a regular category $\mathbb{C}$ if and only if $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}''$ holds. Again by Theorem 1.1, this will conclude the proof in that case. Starting from $\mathcal{B}_{\mathsf{L}}$, we first construct a sketch $\mathcal{C}_1$ following Proc. A by adding to $\mathcal{B}_{\mathsf{L}}$ a new object Z, for each $H \in \mathcal{H}_1$, a new arrow $z_H \colon Z \to D_1(H)$ and the condition $(Z, (z_H)_H) = \mathsf{limit}(\mathcal{H}_1, D_1)$. We then form a sketch $\mathcal{C}_2$ following Proc. D (a finite number of times) by adding to $\mathcal{C}_1$, for each arrow $h \colon H \to H'$ in $\mathcal{H}_1$, a new arrow $x_h \colon C_1 \to D_1(H')$ and the commutativity condition $x_h = D_1(h) \cdot c_H^1$. We now consider the graph $\mathcal{H}'$ which has

$$\{V\} \cup \{W_h \,|\, h \colon H \to H' \in \mathcal{H}_1\}$$

as set of objects and

$$\{v_h, w_h \colon V \to W_h \,|\, h \colon H \to H' \in \mathcal{H}_1\}$$

as set of arrows. We then define the finite diagram $D' \colon \mathcal{H}' \to \mathcal{C}_2$ via $D'(v_h) = c^1_{H'}$ and $D'(w_h) = x_h$ for each arrow $h \colon H \to H'$ in $\mathcal{H}_1$. We form a sketch $\mathcal{C}_3$ following Proc. A by adding to $\mathcal{C}_2$ a new object E, a new arrow $e \colon E \to C_1$, for each arrow $h \colon H \to H'$ in $\mathcal{H}_1$, a new arrow $e_h \colon E \to D_1(H')$ and the finite limit condition $(E, (e, (e_h)_h)) = \mathsf{limit}(\mathcal{H}', D')$. Next, we form a sketch $\mathcal{C}_4$ following Proc. D (a finite number of times) by adding to $\mathcal{C}_3$, for each object H of $\mathcal{H}_1$, a new arrow $y_H \colon E \to D_1(H)$ together with the commutativity condition $y_H = c^1_H \cdot e$. Notice that, for each arrow $h \colon H \to H'$ in $\mathcal{H}_1$, we have the identities

$$D_1(h) \cdot y_H \equiv_{\mathcal{C}_4} D_1(h) \cdot c^1_H \cdot e \equiv_{\mathcal{C}_4} x_h \cdot e \equiv_{\mathcal{C}_4} e_h \equiv_{\mathcal{C}_4} c^1_{H'} \cdot e \equiv_{\mathcal{C}_4} y_{H'}.$$

Therefore, following Proc. C, we can form a sketch $\mathcal{C}_5$ by adding to $\mathcal{C}_4$ the commutativity condition $D_1(h) \cdot y_H = y_{H'}$ for each arrow $h \colon H \to H'$ in $\mathcal{H}_1$. Following Proc. B, we can now form a sketch $\mathcal{C}_6$ by adding to $\mathcal{C}_5$ a new arrow $f \colon E \to Z$ and the commutativity condition $z_H \cdot f = y_H$ for each object H in $\mathcal{H}_1$. We then form a sketch $\mathcal{C}_7$ following Proc. A by adding to $\mathcal{C}_6$ a new object R, new arrows $r_1, r_2 \colon R \to E$ and $k \colon R \to Z$ and the finite limit condition $(R, (r_1, r_2, k)) = \mathsf{limit}(\mathcal{H}_{\mathsf{Pb}}, D^{f,f}_{\mathsf{Pb}})$ where $\mathcal{H}_{\mathsf{Pb}}$ and $D^{f,f}_{\mathsf{Pb}}$ are defined as in Ax. 3. We form the sketch $\mathcal{B}''_{\mathsf{L}}$ following Proc. A by adding to $\mathcal{C}_7$ a new object E', new arrows $e'_1 \colon E' \to R$ and $e'_2 \colon E' \to E$ and the finite limit condition $(E', (e'_1, e'_2)) = \mathsf{limit}(\mathcal{H}_{\mathsf{Eq}}, D^{r_1,r_2}_{\mathsf{Eq}})$ where $\mathcal{H}_{\mathsf{Eq}} = \mathcal{H}_{\mathsf{Coeq}}$ and $D^{r_1,r_2}_{\mathsf{Eq}} = D^{r_1,r_2}_{\mathsf{Coeq}}$ are defined as in Ax. 3. Finally, we form the sketch $\mathcal{B}''$ as prescribed by Ax. 3$'$ with $(a'', b'', c'') = (3, 0, 0)$ and where q_1 is the arrow $e \colon E \to C_1$, q_2 is the arrow $f \colon E \to Z$ and q_3 is the arrow $e'_1 \colon E' \to R$. This completes the definition of the exactness sequent $\mathcal{A} \vdash \mathcal{B}''$ which is of Type III.

It remains to prove that, for a regular category $\mathbb{C}$, one has $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ if and only if $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}''$. Given any $\mathcal{U}(\mathbb{A})$-structure F in $\mathbb{C}$ and considering its extension F_{L} as a $\mathcal{B}_{\mathsf{L}}$-structure and its further extension F''_{L} as a $\mathcal{B}''_{\mathsf{L}}$-structure in $\mathbb{C}$, its suffices to prove that F_{L} extends as a $\mathcal{B}$-structure if and only if F''_{L} extends as a $\mathcal{B}''$-structure. The former happens exactly when $(F_{\mathsf{L}}(C_1), (F_{\mathsf{L}}(c^1_H))_H)$ is the limit of $F_{\mathsf{L}} \circ D_1$. The latter happens exactly when $F''_{\mathsf{L}}(e)$, $F''_{\mathsf{L}}(f)$ and $F''_{\mathsf{L}}(e'_1)$ are regular epimorphisms. Since $F''_{\mathsf{L}}(e'_1)$ is the equalizer of the kernel pair of $F''_{\mathsf{L}}(f)$, we know that $F''_{\mathsf{L}}(e'_1)$ is a regular epimorphism exactly when $F''_{\mathsf{L}}(f)$ is a monomorphism. Therefore, $F''_{\mathsf{L}}(f)$ and $F''_{\mathsf{L}}(e'_1)$ are both regular epimorphisms if and only if $F''_{\mathsf{L}}(f)$ is an isomorphism. Moreover, $F''_{\mathsf{L}}(e)$ is the joint equalizer of the pairs $(F''_{\mathsf{L}}(c^1_{H'}), F''_{\mathsf{L}}(x_h))$ when $h \colon H \to H'$ runs through the arrows of $\mathcal{H}_1$. Therefore, $F''_{\mathsf{L}}(e)$ is a regular epimorphism if and only if $F''_{\mathsf{L}}(c^1_{H'}) = F''_{\mathsf{L}}(x_h)$ for all such h, that is, if and only if $(F''_{\mathsf{L}}(C_1), (F''_{\mathsf{L}}(c^1_H))_H)$ forms a cone over $F_{\mathsf{L}} \circ D_1$. This cone is a limit exactly when $F''_{\mathsf{L}}(f)$ is an isomorphism, concluding the proof in the case $(a, b, c) = (0, 1, 0)$.

We now treat the case $(a, b, c) = (0, 0, 1)$. We are going to construct an exactness sequent $\mathcal{A} \vdash \mathcal{B}'''$ of Type I such that $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ holds for a regular category $\mathbb{C}$ if and only if $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}'''$ holds. Let us suppose the two paths $f^1_{n_1} \cdots f^1_1$ and $g^1_{m_1} \cdots g^1_1$ are from A to B in $\mathcal{B}_{\mathsf{L}}$. Starting from $\mathcal{B}_{\mathsf{L}}$, we construct a sketch $\mathcal{C}$ following Proc. D twice by adding to $\mathcal{B}_{\mathsf{L}}$ two new arrows $f, g \colon A \to B$ and the commutativity conditions $f = f^1_{n_1} \cdots f^1_1$ and $g = g^1_{m_1} \cdots g^1_1$. We then form the sketch $\mathcal{B}'''_{\mathsf{L}}$ following Proc. A by adding to $\mathcal{C}$ a new object E, new arrows $e_1 \colon E \to A$ and $e_2 \colon E \to B$ and the finite limit condition $(E, (e_1, e_2)) = \mathsf{limit}(\mathcal{H}_{\mathsf{Eq}}, D^{f,g}_{\mathsf{Eq}})$ where $\mathcal{H}_{\mathsf{Eq}} = \mathcal{H}_{\mathsf{Coeq}}$ and $D^{f,g}_{\mathsf{Eq}} = D^{f,g}_{\mathsf{Coeq}}$ are defined as in Ax. 3. Finally, we form the sketch $\mathcal{B}'''$ as prescribed by Ax. 3 with respect to the arrow $e_1 \colon E \to A$. This completes the construction of the exactness sequent $\mathcal{A} \vdash \mathcal{B}'''$ which is of Type I. Given a $\mathcal{U}(\mathbb{A})$-structure F in a regular category $\mathbb{C}$ and considering its extension F_{L} as a $\mathcal{B}_{\mathsf{L}}$-structure, we know that F_{L} extends as a $\mathcal{B}$-structure if and only if $F_{\mathsf{L}}(f^1_{n_1}) \circ \cdots \circ F_{\mathsf{L}}(f^1_1) = F_{\mathsf{L}}(g^1_{m_1}) \circ \cdots \circ F_{\mathsf{L}}(g^1_1)$. Considering the extension F'''_{L} of F_{L} as a

$\mathcal{B}_{\mathsf{L}}'''$-structure, this happens if and only if $F_{\mathsf{L}}'''(f) = F'''(g)$, i.e., exactly when the equalizer $F_{\mathsf{L}}'''(e_1)$ of $F_{\mathsf{L}}'''(f)$ and $F_{\mathsf{L}}'''(g)$ is a regular epimorphism; or in other words when F_{L}''' extends as a $\mathcal{B}'''$-structure. This proves that a regular category $\mathbb{C}$ satisfies $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ if and only if it satisfies $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}'''$. $\square$

Example 1.5. Let us conclude this section with a concrete example. Mal'tsev categories were introduced in [14, 15] as finitely complete categories in which each reflexive binary relation is an equivalence relation. Equivalently, these are finitely complete categories in which every reflexive binary relation is symmetric. In the regular context, this is further equivalent to require that given any reflexive graph

$$\begin{tikzcd} \mathsf{Y} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[l, bend left=50, "s"] \end{tikzcd}$$

and considering the limit cone made of the dashed morphisms over the diagram made of the plain morphisms in

$$\begin{tikzcd} & & \mathsf{X} \arrow[lld, dotted, "p_1"'] \arrow[ld, dotted, "p_2"] \arrow[rd, dotted, "p_3"'] \arrow[rrd, dotted, "p_4"] & & \\ \mathsf{Y} \arrow[rd, "f"'] \arrow[rrrd, "g", near start] & & & & \mathsf{Y} \arrow[ld, "f"] \arrow[llld, "g"', near start] \\ & A & & A & \end{tikzcd}$$

the projection p_1 is a regular epimorphism. In order to express this exactness property via an exactness sequent $\mathcal{A} \vdash \mathcal{B}$ of Type I, we consider the underlying sketch $\mathcal{A}$ of the finite category $\mathbb{A}$ as displayed in

$$\begin{tikzcd} \mathsf{Y} \arrow[loop left, "1_{\mathsf{Y}}"] \arrow[loop above, "s\circ f"] \arrow[loop below, "s\circ g"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[l, bend left=50, "s"] \arrow[loop right, "1_A"] \end{tikzcd}$$

where $f \circ s = 1_A = g \circ s$. The sketch $\mathcal{B}_{\mathsf{L}}$ is obtained by adding to $\mathcal{A}$ a new object X together with the four arrows $p_{K_1}^{\mathsf{X}}, p_{K_2}^{\mathsf{X}}, p_{K_3}^{\mathsf{X}}, p_{K_4}^{\mathsf{X}}$ as in

$$\begin{tikzcd} \mathsf{Y} & \mathsf{X} \arrow[l, shift right, "p_{K_1}^{\mathsf{X}}"'] \arrow[l, shift left, "p_{K_4}^{\mathsf{X}}"] \arrow[r, shift left, "p_{K_2}^{\mathsf{X}}"] \arrow[r, shift right, "p_{K_3}^{\mathsf{X}}"'] & A \end{tikzcd}$$

and the convergence condition $(\mathsf{X}, (p_{K_1}^{\mathsf{X}}, p_{K_2}^{\mathsf{X}}, p_{K_3}^{\mathsf{X}}, p_{K_4}^{\mathsf{X}})) = \mathsf{limit}(\mathcal{H}_{\mathsf{X}}, D_{\mathsf{X}})$ where $\mathcal{H}_{\mathsf{X}}$ is in this case the graph

$$\begin{tikzcd} K_1 \arrow[rd, "k_1"'] \arrow[rrrd, "k_2", near start] & & & & K_4 \arrow[ld, "k_4"] \arrow[llld, "k_3"', near start] \\ & K_2 & & K_3 & \end{tikzcd}$$

with four objects and four arrows and D_{X} is defined via $D_{\mathsf{X}}(k_1) = D_{\mathsf{X}}(k_4) = f$ and $D_{\mathsf{X}}(k_2) = D_{\mathsf{X}}(k_3) = g$. The sketch $\mathcal{B}$ is then constructed from $\mathcal{B}_{\mathsf{L}}$ as prescribed by condition Ax. 3 for the arrow $\mathsf{q} = p_{K_1}^{\mathsf{X}} \colon \mathsf{X} \to \mathsf{Y}$. The exactness sequent $\mathcal{A} \vdash \mathcal{B}$ is of Type I (and thus of Type III, but, strictly speaking, not of Type II since Y is already in $\mathcal{A}$). For a regular category $\mathbb{C}$, one has $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ if and only if $\mathbb{C}$ is a Mal'tsev category.

2 The context of regular categories with finite colimits

In this section, we fix a regular category $\mathbb{C}$ with finite colimits. Moreover, we also fix an exactness sequent $\mathcal{A} \vdash \mathcal{B}$

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha"] & \mathcal{A} = \mathcal{U}(\mathbb{A}) \arrow[r, "\beta_{\mathsf{L}}"] \arrow[rr, bend right, "\beta"'] & \mathcal{B}_{\mathsf{L}} \arrow[r, "\beta_{\mathsf{R}}"] & \mathcal{B} \end{tikzcd}$$

of Type I. We denote by $\mathsf{q}\colon \mathsf{X} \to \mathsf{Y}$ the arrow in $\mathcal{B}_{\mathsf{L}}$ with respect to which β_{R} is constructed as in Ax. 3. By abuse of notation, we will denote a functor $F\colon \mathbb{A} \to \mathbb{C}$ and its corresponding $\mathcal{U}(\mathbb{A})$-structure in $\mathbb{C}$ in the same way. Moreover, since F has a unique (up to isomorphism) extension to a $\mathcal{B}_{\mathsf{L}}$-structure in $\mathbb{C}$, we denote this extension by $F_{\mathsf{L}}\colon \mathcal{B}_{\mathsf{L}} \to \mathbb{C}$. As already mentioned in Section 1, we have $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ if and only if, for any functor $F\colon \mathbb{A} \to \mathbb{C}$, the morphism $F_{\mathsf{L}}(\mathsf{q})$ is a regular epimorphism in $\mathbb{C}$.

We will consider the finite diagram $D_{\mathsf{Y}}\colon \mathcal{H}_{\mathsf{Y}} \to \mathcal{U}(\mathbb{A})$ and the family of paths $(p^{\mathsf{Y}}_H\colon \mathsf{Y} \to D_{\mathsf{Y}}(H))_{H \in \mathcal{H}_{\mathsf{Y}}}$ in $\mathcal{B}_{\mathsf{L}}$ given by Lemma 1.2 for Y (the codomain of q). The diagram D_{Y} induces by composition in $\mathbb{A}$ a functor

$$\mathsf{Path}(\mathcal{H}_{\mathsf{Y}}) \xrightarrow{\underline{D_{\mathsf{Y}}}} \mathbb{A}$$

where $\mathsf{Path}(\mathcal{H}_{\mathsf{Y}})$ is the path category of $\mathcal{H}_{\mathsf{Y}}$ (see e.g. [6]). Moreover, each diagram $E\colon \mathcal{H}_{\mathsf{Y}} \to \mathbb{C}$ also induces a functor $\underline{E}\colon \mathsf{Path}(\mathcal{H}_{\mathsf{Y}}) \to \mathbb{C}$. Since $\mathbb{C}$ is finitely cocomplete, we can consider the left Kan extension $\mathsf{Lan}_{\underline{D_{\mathsf{Y}}}}\, \underline{E}\colon \mathbb{A} \to \mathbb{C}$ of $\underline{E}$ along $\underline{D_{\mathsf{Y}}}$.

$$\begin{tikzcd} \mathsf{Path}(\mathcal{H}_{\mathsf{Y}}) \arrow[r, "\underline{D_{\mathsf{Y}}}"] \arrow[dr, "\underline{E}"'] & \mathbb{A} \arrow[d, "\mathsf{Lan}_{\underline{D_{\mathsf{Y}}}}\, \underline{E}"] \arrow[dl, phantom, "\Uparrow \lambda^{E}"] \\ & \mathbb{C} \end{tikzcd}$$

The universal natural transformation $\lambda^E\colon \underline{E} \Rightarrow \mathsf{Lan}_{\underline{D_{\mathsf{Y}}}}\, \underline{E} \circ \underline{D_{\mathsf{Y}}}$ corresponds to a morphism of diagrams $\lambda^E\colon E \to \mathsf{Lan}_{\underline{D_{\mathsf{Y}}}}\, \underline{E} \circ D_{\mathsf{Y}}$, which also satisfies the universal property of a left Kan extension of E along D_{Y}, namely, for any $\mathcal{U}(\mathbb{A})$-structure F in $\mathbb{C}$ and any morphism $\mu\colon E \to F \circ D_{\mathsf{Y}}$ of diagrams, there exists a unique morphism $\nu\colon \mathsf{Lan}_{\underline{D_{\mathsf{Y}}}}\, \underline{E} \to F$ of $\mathcal{U}(\mathbb{A})$-structures such that $(\nu \bullet D_{\mathsf{Y}}) \circ \lambda^E = \mu$. We thus write the $\mathcal{U}(\mathbb{A})$-structure $\mathsf{Lan}_{\underline{D_{\mathsf{Y}}}}\, \underline{E}$ simply by $\mathsf{Lan}_{D_{\mathsf{Y}}} E$.

$$\begin{tikzcd} \mathcal{H}_{\mathsf{Y}} \arrow[r, "D_{\mathsf{Y}}"] \arrow[dr, "E"'] & \mathcal{U}(\mathbb{A}) \arrow[d, "\mathsf{Lan}_{D_{\mathsf{Y}}} E"] \arrow[dl, phantom, "\Uparrow \lambda^{E}"] \\ & \mathbb{C} \end{tikzcd}$$

Viewing the graph $\mathcal{H}_{\mathsf{Y}}$ as a sketch with no conditions, this defines a left adjoint

$$\mathsf{Lan}_{D_{\mathsf{Y}}}\colon \mathcal{H}_{\mathsf{Y}}\mathbb{C} \to \mathcal{U}(\mathbb{A})\mathbb{C}$$

to the functor $(D_{\mathsf{Y}})_{\mathbb{C}}\colon \mathcal{U}(\mathbb{A})\mathbb{C} \to \mathcal{H}_{\mathsf{Y}}\mathbb{C}$ with unit $\lambda\colon 1_{\mathcal{H}_{\mathsf{Y}}\mathbb{C}} \to (D_{\mathsf{Y}})_{\mathbb{C}} \circ \mathsf{Lan}_{D_{\mathsf{Y}}}$.

For each object C in $\mathbb{C}$, we denote by $\Delta_C\colon \mathcal{H}_{\mathsf{Y}} \to \mathbb{C}$ the constant diagram mapping each object of $\mathcal{H}_{\mathsf{Y}}$ to C and each arrow of $\mathcal{H}_{\mathsf{Y}}$ to the identity on C. Since $(\mathsf{Y}, (p^{\mathsf{Y}}_H)_H) \equiv_{\mathcal{B}_{\mathsf{L}}} \mathsf{limit}(\mathcal{H}_{\mathsf{Y}}, \beta_{\mathsf{L}} \circ D_{\mathsf{Y}})$, we know that $((\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(\mathsf{Y}), ((\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(p^{\mathsf{Y}}_H))_H)$ is the limit of $(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}} \circ \beta_{\mathsf{L}} \circ D_{\mathsf{Y}} = \mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C \circ D_{\mathsf{Y}}$. Thus, the cone $\lambda^{\Delta_C}\colon \Delta_C \to \mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C \circ D_{\mathsf{Y}}$ induces a unique morphism $e_C\colon C \to (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(\mathsf{Y})$ in $\mathbb{C}$ such that

$$(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(p^{\mathsf{Y}}_H) \circ e_C = \lambda^{\Delta_C}_H$$

for all $H \in \mathcal{H}_{\mathsf{Y}}$. We will also need the pullback of e_C along $(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(\mathsf{q})$.

$$\begin{tikzcd}
\mathsf{Ap}(C) \arrow[r, "\delta_C"] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(\mathsf{X}) \arrow[d, "(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(\mathsf{q})"] \\
C \arrow[r, "e_C"'] & (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(\mathsf{Y})
\end{tikzcd}$$

Every morphism $f\colon C \to C'$ in $\mathbb{C}$ induces a morphism of diagrams $\Delta_f\colon \Delta_C \to \Delta_{C'}$ defined for each $H \in \mathcal{H}_{\mathsf{Y}}$ by $(\Delta_f)_H = f$. It thus induces a morphism of $\mathcal{U}(\mathbb{A})$-structures

$$\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_f\colon\ \mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C \to \mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{C'}.$$

Since β_{L} is constructible, the functor $\beta_{\mathsf{L}}\mathbb{C}$ is fully faithful. Hence, there exists a unique morphism

$$(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_f)_{\mathsf{L}}\colon (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}} \to (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{C'})_{\mathsf{L}}$$

in $\mathcal{B}_{\mathsf{L}}\mathbb{C}$ such that $(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_f)_{\mathsf{L}} \bullet \beta_{\mathsf{L}} = \mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_f$. Since for all $H \in \mathcal{H}_{\mathsf{Y}}$, the following identities

$$\begin{aligned}
&(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{C'})_{\mathsf{L}}(p_H^{\mathsf{Y}}) \circ e_{C'} \circ f \circ \gamma_C \\
&= \lambda_H^{\Delta_{C'}} \circ f \circ \gamma_C \\
&= (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_f)_{D_{\mathsf{Y}}(H)} \circ \lambda_H^{\Delta_C} \circ \gamma_C \\
&= ((\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_f)_{\mathsf{L}})_{D_{\mathsf{Y}}(H)} \circ (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(p_H^{\mathsf{Y}}) \circ e_C \circ \gamma_C \\
&= (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{C'})_{\mathsf{L}}(p_H^{\mathsf{Y}}) \circ ((\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_f)_{\mathsf{L}})_{\mathsf{Y}} \circ (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(\mathsf{q}) \circ \delta_C \\
&= (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{C'})_{\mathsf{L}}(p_H^{\mathsf{Y}}) \circ (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{C'})_{\mathsf{L}}(\mathsf{q}) \circ ((\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_f)_{\mathsf{L}})_{\mathsf{X}} \circ \delta_C
\end{aligned}$$

hold and since the family $((\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{C'})_{\mathsf{L}}(p_H^{\mathsf{Y}}))_{H \in \mathcal{H}_{\mathsf{Y}}}$ is jointly monomorphic, we have

$$e_{C'} \circ f \circ \gamma_C = (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{C'})_{\mathsf{L}}(\mathsf{q}) \circ ((\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_f)_{\mathsf{L}})_{\mathsf{X}} \circ \delta_C.$$

By the universal property of the pullback, there is a unique morphism $\mathsf{Ap}(f)\colon \mathsf{Ap}(C) \to \mathsf{Ap}(C')$ in $\mathbb{C}$ such that $\gamma_{C'} \circ \mathsf{Ap}(f) = f \circ \gamma_C$ and $\delta_{C'} \circ \mathsf{Ap}(f) = ((\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_f)_{\mathsf{L}})_{\mathsf{X}} \circ \delta_C$. This induces an endofunctor $\mathsf{Ap}\colon \mathbb{C} \to \mathbb{C}$ and a natural transformation $\gamma\colon \mathsf{Ap} \to 1_{\mathbb{C}}$.

Theorem 2.1. *Let $\mathbb{C}$ be a finitely cocomplete regular category and let $\mathcal{A} \vdash \mathcal{B}$ be an exactness sequent of Type I. Using the above notation, the following statements are equivalent:*

(i) $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$;

(ii) for any diagram $E\colon \mathcal{H}_{\mathsf{Y}} \to \mathbb{C}$, the morphism $(\mathsf{Lan}_{D_{\mathsf{Y}}} E)_{\mathsf{L}}(\mathsf{q})$ is a regular epimorphism in $\mathbb{C}$;

(iii) for any object C in $\mathbb{C}$, the morphism $(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(\mathsf{q})$ is a regular epimorphism in $\mathbb{C}$;

(iv) for any object C in $\mathbb{C}$, the morphism $\gamma_C\colon \mathsf{Ap}(C) \to C$ is a regular epimorphism in $\mathbb{C}$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial. The implication (iii) $\Rightarrow$ (iv) follows directly from the fact that regular epimorphisms are stable under pullbacks in any regular category. It remains to prove (iv) $\Rightarrow$ (i). Let F be any functor $\mathbb{A} \to \mathbb{C}$. We must show that $F_{\mathsf{L}}(\mathsf{q})$ is a regular epimorphism. The projections $\mu_H = F_{\mathsf{L}}(p_H^{\mathsf{Y}})\colon F_{\mathsf{L}}(\mathsf{Y}) \to F(D_{\mathsf{Y}}(H))$ give rise to a morphism of diagrams $\mu\colon \Delta_{F_{\mathsf{L}}(\mathsf{Y})} \to F \circ D_{\mathsf{Y}}$. By the universal property of the Kan extension, there exists a unique morphism $\nu\colon \mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{F_{\mathsf{L}}(\mathsf{Y})} \to F$ such that

$(\nu \bullet D_{\mathsf{Y}}) \circ \lambda^{\Delta_{F_{\mathsf{L}}(\mathsf{Y})}} = \mu$. Since β_{L} is constructible, the functor $\beta_{\mathsf{L}}\mathbb{C}$ is fully faithful. Hence, there exists a unique morphism $\xi \colon (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{F_{\mathsf{L}}(\mathsf{Y})})_{\mathsf{L}} \to F_{\mathsf{L}}$ in $\mathcal{B}_{\mathsf{L}}\mathbb{C}$ such that $\xi \bullet \beta_{\mathsf{L}} = \nu$. Since, for any $H \in \mathcal{H}_{\mathsf{Y}}$, we have

$$\begin{aligned} F_{\mathsf{L}}(p^{\mathsf{Y}}_H) \circ \xi_{\mathsf{Y}} \circ e_{F_{\mathsf{L}}(\mathsf{Y})} &= \xi_{D_{\mathsf{Y}}(H)} \circ (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{F_{\mathsf{L}}(\mathsf{Y})})_{\mathsf{L}}(p^{\mathsf{Y}}_H) \circ e_{F_{\mathsf{L}}(\mathsf{Y})} \\ &= \nu_{D_{\mathsf{Y}}(H)} \circ (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{F_{\mathsf{L}}(\mathsf{Y})})_{\mathsf{L}}(p^{\mathsf{Y}}_H) \circ e_{F_{\mathsf{L}}(\mathsf{Y})} \\ &= \nu_{D_{\mathsf{Y}}(H)} \circ \lambda_H^{\Delta_{F_{\mathsf{L}}(\mathsf{Y})}} \\ &= \mu_H \\ &= F_{\mathsf{L}}(p^{\mathsf{Y}}_H) \end{aligned}$$

and since the family $(F_{\mathsf{L}}(p^{\mathsf{Y}}_H))_{H \in \mathcal{H}_{\mathsf{Y}}}$ is jointly monomorphic (being the legs of a limit), we know that $\xi_{\mathsf{Y}} \circ e_{F_{\mathsf{L}}(\mathsf{Y})} = 1_{F_{\mathsf{L}}(\mathsf{Y})}$. Therefore, we also have that

$$\begin{aligned} F_{\mathsf{L}}(\mathsf{q}) \circ \xi_{\mathsf{X}} \circ \delta_{F_{\mathsf{L}}(\mathsf{Y})} &= \xi_{\mathsf{Y}} \circ (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{F_{\mathsf{L}}(\mathsf{Y})})_{\mathsf{L}}(\mathsf{q}) \circ \delta_{F_{\mathsf{L}}(\mathsf{Y})} \\ &= \xi_{\mathsf{Y}} \circ e_{F_{\mathsf{L}}(\mathsf{Y})} \circ \gamma_{F_{\mathsf{L}}(\mathsf{Y})} \\ &= \gamma_{F_{\mathsf{L}}(\mathsf{Y})} \end{aligned}$$

proving that $F_{\mathsf{L}}(\mathsf{q})$ is a regular epimorphism since $\gamma_{F_{\mathsf{L}}(\mathsf{Y})}$ is. □

Note that the equivalence (i) ⇔ (iv) in the particular case of the matrix exactness properties from [33, 35] (see Section 4) gives the approximate co-operation characterizations of them, introduced in the Mal'tsev case in [10], and generalized in [36, 26, 29]. See Example 2.2 below in the Mal'tsev case and Remark 4.10 in the general case for more details.

Example 2.2. Let us come back to Example 1.5 about Mal'tsev categories and let us use the notation introduce there. We denote by $\mathcal{H}_{\mathsf{Y}}$ the graph consisting of a single object H and without any arrow and we denote by D_{Y} the finite diagram $\mathcal{H}_{\mathsf{Y}} \to \mathcal{U}(\mathbb{A})$ defined by $D_{\mathsf{Y}}(H) = \mathsf{Y}$. Writing p^{Y}_H for the arrow 1_{Y} in $\mathcal{A}$, one obviously has

$$(\mathsf{Y}, (p^{\mathsf{Y}}_H)) \equiv_{\mathcal{A}} \mathsf{limit}(\mathcal{H}_{\mathsf{Y}}, D_{\mathsf{Y}})$$

as in Lemma 1.2. The finite diagram $D_{\mathsf{X}} \colon \mathcal{H}_{\mathsf{X}} \to \mathcal{U}(\mathbb{A})$ and the arrows $p^{\mathsf{X}}_{K_1}, p^{\mathsf{X}}_{K_2}, p^{\mathsf{X}}_{K_3}, p^{\mathsf{X}}_{K_4}$ have already been described in Example 1.5. Still in regards of the notation of Lemma 1.2, we also write K^{q}_H for the object K_1 in $\mathcal{H}_{\mathsf{X}}$ and c^{q}_H for the morphism 1_{Y} in $\mathbb{A}$. Let $\mathbb{C}$ be a finitely cocomplete regular category. For an object C in $\mathbb{C}$, the pointwise Kan extension formula gives us that $\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C$ is described by the reflexive graph

$$\begin{tikzcd} 3C \arrow[r, shift left, "\left(\begin{smallmatrix}\iota_1\\ \iota_1\\ \iota_2\end{smallmatrix}\right)"] \arrow[r, shift right, "\left(\begin{smallmatrix}\iota_1\\ \iota_2\\ \iota_2\end{smallmatrix}\right)"'] & 2C \arrow[l, bend left=60, "\left(\begin{smallmatrix}\iota'_1\\ \iota'_3\end{smallmatrix}\right)"] \end{tikzcd}$$

in $\mathbb{C}$, where $2C$ is the cosquare of C with coproduct injections $\iota_1, \iota_2 \colon C \to 2C$ and $3C$ is the third copower of C with coproduct injections $\iota'_1, \iota'_2, \iota'_3 \colon C \to 3C$. The morphism $\lambda^{\Delta_C}_H$ is given by $\iota'_2 \colon C \to 3C$. The morphism $(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(\mathsf{q})$ is then constructed via the limit of

$\mathsf{Lan}_{D_\mathsf{Y}} \Delta_C \circ D_\mathsf{X}$, or equivalently, via the right pullback rectangle in the following diagram.

$$\begin{tikzcd}[column sep=huge]
\mathsf{Ap}(C) \arrow[r, "\delta_C"] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathsf{Lan}_{D_\mathsf{Y}} \Delta_C)_\mathsf{L}(\mathsf{X}) \arrow[r, "(\mathsf{Lan}_{D_\mathsf{Y}} \Delta_C)_\mathsf{L}(p^\mathsf{X}_{K_4})"] \arrow[d, "(\mathsf{Lan}_{D_\mathsf{Y}} \Delta_C)_\mathsf{L}(\mathsf{q})" description] \arrow[dr, phantom, "\lrcorner", very near start] & 3C \arrow[d, "\left(\begin{smallmatrix} \iota_1 & \iota_1 \\ \iota_2 & \iota_1 \\ \iota_2 & \iota_2 \end{smallmatrix}\right)"] \\
C \arrow[r, "e_C=\iota'_2"'] \arrow[rr, bend right, "(\,\iota_1\ \iota_2\,)"'] & 3C \arrow[r, "\left(\begin{smallmatrix} \iota_1 & \iota_1 \\ \iota_1 & \iota_2 \\ \iota_2 & \iota_2 \end{smallmatrix}\right)"'] & (2C)^2
\end{tikzcd}$$

This diagram also shows the construction of $\gamma_C\colon \mathsf{Ap}(C) \to C$ in this case. Theorem 2.1 thus says in this particular case that a finitely cocomplete regular category $\mathbb{C}$ is a Mal'tsev category if and only if, for every object C in $\mathbb{C}$, the morphism $(\mathsf{Lan}_{D_\mathsf{Y}} \Delta_C)_\mathsf{L}(\mathsf{q})$ so defined is a regular epimorphism; or equivalently, if and only if, for every object C in $\mathbb{C}$, the morphism γ_C as above is a regular epimorphism. This last characterization is the characterization of Mal'tsev categories via approximate Mal'tsev co-operations obtained in [10].

3 The context of essentially algebraic categories

For an exactness sequent $\mathcal{A} \vdash \mathcal{B}$ of Type I, we can extract from Theorem 2.1 a Mal'tsev condition characterizing the validity of $\mathcal{A} \vdash_\mathbb{C} \mathcal{B}$ in the case where $\mathbb{C}$ is a regular locally presentable category [21]. We recall that locally presentable categories are exactly the (many-sorted) essentially algebraic categories [1, 2]. These are described by *essentially algebraic theories*, i.e., quintuples $\Gamma = (S, \Sigma, E, \Sigma_t, \mathsf{Def})$ where

- S is a set of sorts;
- Σ is an S-sorted signature of algebras, i.e., a set of operation symbols σ with prescribed arity $\sigma\colon \prod_{u\in U} s_u \to s$ where U is a set, $s_u \in S$ for each $u \in U$ and $s \in S$;
- E is a set of Σ-equations;
- Σ_t is a subset of Σ called the set of *total operation symbols*;
- Def is a function assigning to each operation symbol $\sigma\colon \prod_{u\in U} s_u \to s$ in $\Sigma \setminus \Sigma_t$ a set $\mathsf{Def}(\sigma)$ of Σ_t-equations in the variables from $\mathcal{X}$, the S-sorted set defined by $\mathcal{X}_{s'} = \{x_u \,|\, u \in U, s_u = s'\}$ for each $s' \in S$.

A Γ-*model* is an S-sorted set $A = (A_s)_{s\in S}$ together with, for each operation symbol $\sigma\colon \prod_{u\in U} s_u \to s$ in Σ, a partial function $\sigma^A\colon \prod_{u\in U} A_{s_u} \to A_s$ such that:

(1) for each $\sigma \in \Sigma_t$, σ^A is defined everywhere;

(2) given $\sigma\colon \prod_{u\in U} s_u \to s$ in $\Sigma\backslash\Sigma_t$ and a family $(a_u \in A_{s_u})_{u\in U}$ of elements, $\sigma^A((a_u)_{u\in U})$ is defined if and only if the elements a_u's satisfy all equations of $\mathsf{Def}(\sigma)$ in A;

(3) A satisfies the equations of E wherever they are defined.

A *homomorphism* $f\colon A \to B$ of Γ-models is an S-sorted function $(f_s\colon A_s \to B_s)_{s\in S}$ such that, given $\sigma\colon \prod_{u\in U} s_u \to s$ in Σ and a family $(a_u \in A_{s_u})_{u\in U}$ such that $\sigma^A((a_u)_{u\in U})$ is defined in A, then $\sigma^B((f_{s_u}(a_u))_{u\in U})$ is defined in B and the identity

$$f_s(\sigma^A((a_u)_{u\in U})) = \sigma^B((f_{s_u}(a_u))_{u\in U})$$

holds. The Γ-models and their homomorphisms form the category $\mathsf{Mod}(\Gamma)$. A category which is equivalent to a category $\mathsf{Mod}(\Gamma)$ for some essentially algebraic theory Γ is called *essentially algebraic*. These are exactly the locally presentable categories.

Let us now recall the description of the left adjoint $\mathsf{Fr}_\Gamma = \mathsf{Fr}\colon \mathbf{Set}^S \to \mathsf{Mod}(\Gamma)$ to the forgetful functor $U_\Gamma\colon \mathsf{Mod}(\Gamma) \to \mathbf{Set}^S$ from [26, 27]. Given two Σ-terms $t, t'\colon \prod_{u\in U} s_u \to s$, we say that $t = t'$ is a *theorem* of Γ if for any Γ-model A and any family $(a_u \in A_{s_u})_{u\in U}$ such that both $t((a_u)_{u\in U})$ and $t'((a_u)_{u\in U})$ are defined, then $t((a_u)_{u\in U}) = t'((a_u)_{u\in U})$. We say that a Σ-term t in the variables from an S-sorted set $\mathcal{X}$ is *everywhere-defined* if it belongs to the smallest S-subset $\mathcal{Y}$ of the S-sorted set of Σ-terms in the variables from $\mathcal{X}$ satisfying the following conditions:

- for each $s \in S$, $\mathcal{X}_s \subseteq \mathcal{Y}_s$ (i.e., variables are everywhere-defined);
- if $\sigma\colon \prod_{u\in U} s_u \to s$ is in Σ and if $(t_u \in \mathcal{Y}_{s_u})_{u\in U}$ is a family of everywhere-defined terms such that either $\sigma \in \Sigma_t$ or $\sigma \in \Sigma \setminus \Sigma_t$ and for each equation $r_1 = r_2 \in \mathsf{Def}(\sigma)$, $r_1((t_u)_{u\in U}) = r_2((t_u)_{u\in U})$ is a theorem of Γ, then the term $\sigma((t_u)_{u\in U})$ belongs to $\mathcal{Y}_s$.

Intuitively, everywhere-defined terms are terms which are defined everywhere in any Γ-model. We can now describe the free Γ-models as follows. Given an S-sorted set $\mathcal{X}$ let, for each $s \in S$, $\mathsf{Fr}_\Gamma(\mathcal{X})_s = \mathsf{Fr}(\mathcal{X})_s$ be the set of equivalence classes of everywhere-defined Σ-terms of sort s in the variables from $\mathcal{X}$, where we identify the two terms t_1 and t_2 if and only if $t_1 = t_2$ is a theorem of Γ.

Each essentially algebraic category $\mathsf{Mod}(\Gamma)$ has a (strong epimorphism, monomorphism)-factorization system. Given a homomorphism $f\colon A \to B$ of Γ-models, its image is the smallest submodel $\mathsf{Im}(f)$ of B which contains $f_s(a)$ for each $s \in S$ and each $a \in A_s$. In order words, $\mathsf{Im}(f)$ can be described for each sort $s \in S$ as

$$\mathsf{Im}(f)_s = \{t((f_{s_u}(a_u))_{u\in U}) \,|\, t\colon \prod_{u\in U} s_u \to s \text{ is a } \Sigma\text{-term and } (a_u \in A_{s_u})_{u\in U} \text{ is a family such that } t((f_{s_u}(a_u))_{u\in U}) \text{ is defined in } B\}.$$

Finally, let us recall from [26, 27] a syntactic characterization of regular essentially algebraic categories (note that it is not the same as a similar theorem in [16], where the term ‘syntactic’ is used in a different sense).

Theorem 3.1 ([26, 27])**.** *Let $\Gamma = (S, \Sigma, E, \Sigma_t, \mathsf{Def})$ be an essentially algebraic theory. Then $\mathsf{Mod}(\Gamma)$ is a regular category if and only if, for each Σ-term $t\colon \prod_{u\in U} s_u \to s$, there exists in Γ*

- *a term $\pi\colon \prod_{v\in V} s'_v \to s$,*
- *for each $v \in V$, an everywhere-defined term $\rho_v\colon s \to s'_v$,*
- *for each $v \in V$, an everywhere-defined term $\tau_v\colon \prod_{u\in U} s_u \to s'_v$*

such that

(a) $\pi((\rho_v(x))_{v\in V})\colon s \to s$ *is an everywhere-defined term,*

(b) $\pi((\rho_v(x))_{v\in V}) = x$ *is a theorem of* Γ*,*

(c) $\rho_v(t((x_u)_{u\in U})) = \tau_v((x_u)_{u\in U})$ *is a theorem of* Γ *for each* $v \in V$*.*

We will need a stronger version of this theorem where a finite number of terms $t_1, \ldots, t_n$ (instead of just one) are taken into account.

Theorem 3.2. *Let* $\Gamma = (S, \Sigma, E, \Sigma_t, \mathsf{Def})$ *be an essentially algebraic theory. Then* $\mathsf{Mod}(\Gamma)$ *is a regular category if and only if, for any integer* $n \geqslant 1$ *and any finite family* $(t_i \colon \prod_{u \in U_i} s_{i,u} \to s)_{i \in \{1,\dots,n\}}$ *of* Σ*-terms, there exists in* Γ

- *a term* $\pi \colon \prod_{v \in V} s'_v \to s$*,*
- *for each* $v \in V$*, an everywhere-defined term* $\rho_v \colon s \to s'_v$*,*
- *for each* $v \in V$ *and each* $1 \leqslant i \leqslant n$*, an everywhere-defined term* $\tau_{i,v} \colon \prod_{u \in U_i} s_{i,u} \to s'_v$

such that

(a) $\pi((\rho_v(x))_{v \in V}) \colon s \to s$ *is an everywhere-defined term,*

(b) $\pi((\rho_v(x))_{v \in V}) = x$ *is a theorem of* Γ*,*

(c) $\rho_v(t_i((x_u)_{u \in U_i})) = \tau_{i,v}((x_u)_{u \in U_i})$ *is a theorem of* Γ *for each* $v \in V$ *and each* $1 \leqslant i \leqslant n$*.*

Proof. The 'if part' follows immediately from Theorem 3.1. Let us prove the 'only if part' by induction on n. We suppose $\mathsf{Mod}(\Gamma)$ is regular. If $n = 1$, the property follows immediately from Theorem 3.1. So let us suppose the property holds for $n-1$ terms and let us prove it holds for $n \geqslant 2$ terms $t_1, \dots, t_n$ as in the statement. Consider the terms $\pi \colon \prod_{v \in V} s'_v \to s$, $\rho_v \colon s \to s'_v$ and $\tau_{i,v} \colon \prod_{u \in U_i} s_{i,u} \to s'_v$ obtained via our inductive hypothesis on $t_1, \dots, t_{n-1}$. For each $v \in V$, we apply again Theorem 3.1 with the term $\rho_v(t_n) \colon \prod_{u \in U_n} s_{n,u} \to s'_v$. We get in that way a term $\pi^v \colon \prod_{w \in W_v} s''_w \to s'_v$ and everywhere-defined terms $\rho^v_w \colon s'_v \to s''_w$ and $\tau^v_w \colon \prod_{u \in U_n} s_{n,u} \to s''_w$ for each $w \in W_v$. We now get the desired term $\pi' \colon \prod_{v \in V} \prod_{w \in W_v} s''_w \to s$ as $\pi' = \pi((\pi^v)_{v \in V})$; for each pair (v,w) with $v \in V$ and $w \in W_v$, we define $\rho'_{(v,w)} \colon s \to s''_w$ as $\rho'_{(v,w)} = \rho^v_w(\rho_v)$; for each such pair (v,w) and each $1 \leqslant i \leqslant n-1$, we define $\tau'_{i,(v,w)} \colon \prod_{u \in U_i} s_{i,u} \to s''_w$ as $\tau'_{i,(v,w)} = \rho^v_w(\tau_{i,v})$; and for each such pair (v,w), we define $\tau'_{n,(v,w)} \colon \prod_{u \in U_n} s_{n,u} \to s''_w$ as $\tau'_{n,(v,w)} = \tau^v_w$. To check that these terms satisfy the required property is just routine verifications. $\square$

We can now strengthen Theorem 2.1 in the case where $\mathbb{C} = \mathsf{Mod}(\Gamma)$ is a regular locally presentable category. Let us fix an exactness sequent $\mathcal{A} \vdash \mathcal{B}$

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha"] & \mathcal{A} = \mathcal{U}(\mathbb{A}) \arrow[r, "\beta_{\mathsf{L}}"] \arrow[rr, bend right, "\beta"'] & \mathcal{B}_{\mathsf{L}} \arrow[r, "\beta_{\mathsf{R}}"] & \mathcal{B} \end{tikzcd}$$

of Type I. For each $A \in \mathbb{A}$, we will need the quotient $\mathcal{X}_A$ of the disjoint union

$$\amalg_{H \in \mathcal{H}_{\mathsf{Y}}} \mathbb{A}(D_{\mathsf{Y}}(H), A)$$

by the smallest equivalence relation which identifies the pair $(H', f \colon D_{\mathsf{Y}}(H') \to A)$ with the pair $(H, f \circ D_{\mathsf{Y}}(h) \colon D_{\mathsf{Y}}(H) \to A)$ for each $h \colon H \to H'$ in $\mathcal{H}_{\mathsf{Y}}$ and each $f \colon D_{\mathsf{Y}}(H') \to A$ in $\mathbb{A}$.

$$\mathcal{X}_A = \frac{\amalg_{H \in \mathcal{H}_{\mathsf{Y}}} \mathbb{A}(D_{\mathsf{Y}}(H), A)}{\{(H', f) = (H, f \circ D_{\mathsf{Y}}(h)) \,|\, h \colon H \to H' \in \mathcal{H}_{\mathsf{Y}},\ f \in \mathbb{A}(D_{\mathsf{Y}}(H'), A)\}}$$

Since both $\mathcal{H}_{\mathsf{Y}}$ and $\mathbb{A}$ are finite, the set $\mathcal{X}_A$ is also finite. For each sort $s \in S$, we will denote by $\mathcal{X}^s_A$ the S-sorted set defined by $(\mathcal{X}^s_A)_s = \mathcal{X}_A$ and $(\mathcal{X}^s_A)_{s'} = \varnothing$ for each $s' \neq s \in S$. We also denote by $\{\star_s\}$ the S-sorted set with a single element $\star_s$ in the sort s and nothing in the other sorts.

Theorem 3.3. *Let* $\Gamma = (S, \Sigma, E, \Sigma_t, \mathsf{Def})$ *be an essentially algebraic theory such that* $\mathsf{Mod}(\Gamma)$ *is a regular category. Let also* $\mathcal{A} \vdash \mathcal{B}$ *be an exactness sequent*

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha"] & \mathcal{A} = \mathcal{U}(\mathbb{A}) \arrow[r, "\beta_{\mathsf{L}}"] \arrow[rr, bend right, "\beta"'] & \mathcal{B}_{\mathsf{L}} \arrow[r, "\beta_{\mathsf{R}}"] & \mathcal{B} \end{tikzcd}$$

of Type I (with β_{R} *constructed via* $\mathsf{q} \colon \mathsf{X} \to \mathsf{Y}$ *in* $\mathcal{B}_{\mathsf{L}}$*). Using the above notation and that of Lemma 1.2, the following statements are equivalent:*

(i) $\mathcal{A} \vdash_{\mathsf{Mod}(\Gamma)} \mathcal{B}$*;*

(ii) for any diagram $E \colon \mathcal{H}_{\mathsf{Y}} \to \mathsf{Mod}(\Gamma)$*, the homomorphism* $(\mathsf{Lan}_{D_{\mathsf{Y}}} E)_{\mathsf{L}}(\mathsf{q})$ *is a regular epimorphism in* $\mathsf{Mod}(\Gamma)$*;*

(iii) for any Γ*-model* A*, the homomorphism* $(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_A)_{\mathsf{L}}(\mathsf{q})$ *is a regular epimorphism in* $\mathsf{Mod}(\Gamma)$*;*

(iv) for any Γ*-model* A*, the homomorphism* $\gamma_A \colon \mathsf{Ap}(A) \to A$ *is a regular epimorphism in* $\mathsf{Mod}(\Gamma)$*;*

(v) for any $s \in S$*, the homomorphism* $(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{\mathsf{Fr}(\{\star_s\})})_{\mathsf{L}}(\mathsf{q})$ *is a regular epimorphism in* $\mathsf{Mod}(\Gamma)$*;*

(vi) for any $s \in S$*, the homomorphism* $\gamma_{\mathsf{Fr}(\{\star_s\})} \colon \mathsf{Ap}(\mathsf{Fr}(\{\star_s\})) \to \mathsf{Fr}(\{\star_s\})$ *is a regular epimorphism in* $\mathsf{Mod}(\Gamma)$*;*

(vii) for any $s \in S$*, the element* $\star_s$ *is in the image of* $\gamma_{\mathsf{Fr}(\{\star_s\})} \colon \mathsf{Ap}(\mathsf{Fr}(\{\star_s\})) \to \mathsf{Fr}(\{\star_s\})$*;*

(viii) for each $s \in S$*, there exists in* Γ

- *a term* $\pi^s \colon \prod_{u \in U_s} s_u \to s$*,*
- *for each* $u \in U_s$ *and each* $K \in \mathcal{H}_{\mathsf{X}}$*, an everywhere-defined term* $t_K^{s,u}$ *of sort* s_u *in the variables from* $\mathcal{X}^s_{D_{\mathsf{X}}(K)}$*, i.e.,* $t_K^{s,u} \colon s^{\#\mathcal{X}_{D_{\mathsf{X}}(K)}} \to s_u$*,*

satisfying

(a) for each $u \in U_s$ *and each* $k \colon K \to K' \in \mathcal{H}_{\mathsf{X}}$*,*

$$t_K^{s,u}\left(([(H, D_{\mathsf{X}}(k) \circ f)])_{[(H,f)] \in \mathcal{X}_{D_{\mathsf{X}}(K)}}\right) = t_{K'}^{s,u}\left(([(H', f')])_{[(H',f')] \in \mathcal{X}_{D_{\mathsf{X}}(K')}}\right)$$

is a theorem of Γ *in the variables from* $\mathcal{X}^s_{D_{\mathsf{X}}(K')}$*,*

(b) for each $H \in \mathcal{H}_{\mathsf{Y}}$*, the term*

$$\pi^s\left((t^{s,u}_{K^{\mathsf{q}}_H}(([(H', c^{\mathsf{q}}_H \circ f)])_{[(H',f)] \in \mathcal{X}_{D_{\mathsf{X}}(K^{\mathsf{q}}_H)}}))_{u \in U_s}\right) : s^{\#\mathcal{X}_{D_{\mathsf{Y}}(H)}} \to s$$

is everywhere-defined,

(c) for each $H \in \mathcal{H}_{\mathsf{Y}}$*,*

$$\pi^s\left((t^{s,u}_{K^{\mathsf{q}}_H}(([(H', c^{\mathsf{q}}_H \circ f)])_{[(H',f)] \in \mathcal{X}_{D_{\mathsf{X}}(K^{\mathsf{q}}_H)}}))_{u \in U_s}\right) = [(H, 1_{D_{\mathsf{Y}}(H)})]$$

is a theorem of Γ *in the variables from* $\mathcal{X}^s_{D_{\mathsf{Y}}(H)}$*.*

Proof. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follow immediately from Theorem 2.1. The implication (iii) $\Rightarrow$ (v) is trivial. The implication (v) $\Rightarrow$ (vi) follows from the fact that regular epimorphisms are stable under pullbacks in any regular category. The implication (vi) $\Rightarrow$ (vii) is trivial. Let us prove (vii) $\Rightarrow$ (iv). Given a Γ-model A, a sort $s \in S$ and any element $a \in A_s$, we must show that a is in the image of γ_A. We consider the unique homomorphism of Γ-models $f\colon \mathsf{Fr}(\{\star_s\}) \to A$ such that $f_s(\star_s) = a$. Since the following diagram commutes by naturality of γ,

$$\begin{tikzcd} \mathsf{Ap}(\mathsf{Fr}(\{\star_s\})) \arrow[r, "\mathsf{Ap}(f)"] \arrow[d, "\gamma_{\mathsf{Fr}(\{\star_s\})}"'] & \mathsf{Ap}(A) \arrow[d, "\gamma_A"] \\ \mathsf{Fr}(\{\star_s\}) \arrow[r, "f"'] & A \end{tikzcd}$$

and since $\star_s$ is in the image of $\gamma_{\mathsf{Fr}(\{\star_s\})}$ by assumption, we know that a is in the image of $\gamma_A \circ \mathsf{Ap}(f)$. It is thus in the image of γ_A.

It remains to prove that the explicit description of (vii) is the Mal'tsev condition appearing in (viii). Let us fix a sort $s \in S$. Since the morphism $\gamma_{\mathsf{Fr}(\{\star_s\})}$ is the pullback of $(\mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})})_\mathsf{L}(\mathsf{q})$ along $e_{\mathsf{Fr}(\{\star_s\})}$, we know that $\star_s$ is in the image of $\gamma_{\mathsf{Fr}(\{\star_s\})}$ if and only if $(e_{\mathsf{Fr}(\{\star_s\})})_s(\star_s)$ is in the image of $(\mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})})_\mathsf{L}(\mathsf{q})$. Since $\mathsf{Mod}(\Gamma)$ is cocomplete, the functor $\mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})}\colon \mathbb{A} \to \mathsf{Mod}(\Gamma)$ can be described via the pointwise Kan extension formula. Particularized to this situation, this formula gives us, for each object $A \in \mathbb{A}$,

$$\mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})}(A) = \mathsf{Fr}(\mathcal{X}^s_A)$$

and for each morphism $a\colon A \to A'$ in $\mathbb{A}$ and each $[(H,f)] \in \mathcal{X}^s_A$,

$$\mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})}(a)_s([(H,f)]) = [(H, a \circ f)].$$

Moreover, the universal morphism $\lambda^{\Delta_{\mathsf{Fr}(\{\star_s\})}}\colon \Delta_{\mathsf{Fr}(\{\star_s\})} \to \mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})} \circ D_\mathsf{Y}$ is uniquely defined by

$$\left(\lambda_H^{\Delta_{\mathsf{Fr}(\{\star_s\})}}\right)_s(\star_s) = [(H, 1_{D_\mathsf{Y}(H)})]$$

for each object $H \in \mathcal{H}_\mathsf{Y}$. By the description of images in $\mathsf{Mod}(\Gamma)$, $(e_{\mathsf{Fr}(\{\star_s\})})_s(\star_s)$ is in the image of $(\mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})})_\mathsf{L}(\mathsf{q})$ if and only if there exists a term $\pi^s\colon \prod_{u \in U_s} s_u \to s$ and a family $(t^{s,u} \in (\mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})})_\mathsf{L}(\mathsf{X})_{s_u})_{u \in U_s}$ such that

$$\pi^s(((\mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})})_\mathsf{L}(\mathsf{q})_{s_u}(t^{s,u}))_{u \in U_s})$$

is defined in $(\mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})})_\mathsf{L}(\mathsf{Y})$ and equal to $(e_{\mathsf{Fr}(\{\star_s\})})_s(\star_s)$. Since $(\mathsf{X}, (p^\mathsf{X}_K)_K) \equiv_{\mathcal{B}_\mathsf{L}} \mathsf{limit}(\mathcal{H}_\mathsf{X}, \beta_\mathsf{L} \circ D_\mathsf{X})$ and since small limits in $\mathsf{Mod}(\Gamma)$ are computed as in $\mathbf{Set}^S$ (i.e., componentwise as in $\mathbf{Set}$), for each $u \in U_s$, giving an element $t^{s,u}$ in $(\mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})})_\mathsf{L}(\mathsf{X})_{s_u}$ is equivalent to give a family

$$\left(t^{s,u}_K \in \mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})}(D_\mathsf{X}(K))_{s_u}\right)_{K \in \mathcal{H}_\mathsf{X}}$$

such that for each arrow $k\colon K \to K'$ in $\mathcal{H}_\mathsf{X}$, we have

$$\mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})}(D_\mathsf{X}(k))_{s_u}(t^{s,u}_K) = t^{s,u}_{K'}. \tag{3}$$

Since $\mathsf{Lan}_{D_\mathsf{Y}} \Delta_{\mathsf{Fr}(\{\star_s\})}(D_\mathsf{X}(K)) = \mathsf{Fr}(\mathcal{X}^s_{D_\mathsf{X}(K)})$, these $t^{s,u}_K$ are everywhere-defined terms of sort s_u in the variables from $\mathcal{X}^s_{D_\mathsf{X}(K)}$. The equalities in (3) exactly correspond to condition (a) in (viii). Now, since for each $H \in \mathcal{H}_\mathsf{Y}$, we have $p^\mathsf{Y}_H \cdot \mathsf{q} \equiv_{\mathcal{B}_\mathsf{L}} c^\mathsf{q}_H \cdot p^\mathsf{X}_{K^\mathsf{q}_H}$, we know that for each $u \in U_s$ and each $H \in \mathcal{H}_{\mathsf{Y}}$,

$$\begin{aligned}(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{\mathsf{Fr}(\{\star_s\})})_{\mathsf{L}}(p^{\mathsf{Y}}_H \circ \mathsf{q})_{s_u}(t^{s,u}) &= (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{\mathsf{Fr}(\{\star_s\})})_{\mathsf{L}}(c^{\mathsf{q}}_H \circ p^{\mathsf{X}}_{K^{\mathsf{q}}_H})_{s_u}(t^{s,u}) \\ &= (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{\mathsf{Fr}(\{\star_s\})})_{\mathsf{L}}(c^{\mathsf{q}}_H)_{s_u}\left(t^{s,u}_{K^{\mathsf{q}}_H}\right).\end{aligned}$$

But $\pi^s(((\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{\mathsf{Fr}(\{\star_s\})})_{\mathsf{L}}(\mathsf{q})_{s_u}(t^{s,u}))_{u \in U_s})$ is defined in $(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{\mathsf{Fr}(\{\star_s\})})_{\mathsf{L}}(\mathsf{Y})$ if and only if $\pi^s(((\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{\mathsf{Fr}(\{\star_s\})})_{\mathsf{L}}(p^{\mathsf{Y}}_H \circ \mathsf{q})_{s_u}(t^{s,u}))_{u \in U_s})$ is defined in $\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{\mathsf{Fr}(\{\star_s\})}(D_{\mathsf{Y}}(H))$ for each $H \in \mathcal{H}_{\mathsf{Y}}$. Therefore, this condition corresponds exactly to condition (b). Finally, the condition

$$\pi^s\left(((\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{\mathsf{Fr}(\{\star_s\})})_{\mathsf{L}}(\mathsf{q})_{s_u}(t^{s,u}))_{u \in U_s}\right) = \left(e_{\mathsf{Fr}(\{\star_s\})}\right)_s(\star_s)$$

interpreted componentwise becomes the identity

$$\begin{aligned}\pi^s\left(((\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{\mathsf{Fr}(\{\star_s\})})_{\mathsf{L}}(c^{\mathsf{q}}_H)_{s_u}(t^{s,u}_{K^{\mathsf{q}}_H}))_{u \in U_s}\right) &= \left(\lambda^{\Delta_{\mathsf{Fr}(\{\star_s\})}}_H\right)_s(\star_s) \\ &= [(H, 1_{D_{\mathsf{Y}}(H)})]\end{aligned}$$

for each $H \in \mathcal{H}_{\mathsf{Y}}$, which is exactly condition (c). This proves the equivalence (vii) $\Leftrightarrow$ (viii). □

If we particularize this theorem to the case of a variety of universal algebras, we get the following corollary, where $\{\star\}$ is a singleton set, i.e., $\{\star_s\}$ for the unique sort s in the variety.

Corollary 3.4. *Let $\mathbb{V}$ be a single-sorted variety of universal algebras. Let also $\mathcal{A} \vdash \mathcal{B}$ be an exactness sequent*

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha"] & \mathcal{A} = \mathcal{U}(\mathbb{A}) \arrow[r, "\beta_{\mathsf{L}}"] \arrow[rr, bend right, "\beta"'] & \mathcal{B}_{\mathsf{L}} \arrow[r, "\beta_{\mathsf{R}}"] & \mathcal{B} \end{tikzcd}$$

of Type I (with β_{R} constructed via $\mathsf{q}\colon \mathsf{X} \to \mathsf{Y}$ in $\mathcal{B}_{\mathsf{L}}$). Using the above notation and that of Lemma 1.2, the following statements are equivalent:

(i) $\mathcal{A} \vdash_{\mathbb{V}} \mathcal{B}$*;*

(ii) for any diagram $E\colon \mathcal{H}_{\mathsf{Y}} \to \mathbb{V}$, the homomorphism $(\mathsf{Lan}_{D_{\mathsf{Y}}} E)_{\mathsf{L}}(\mathsf{q})$ is surjective;

(iii) for any $\mathbb{V}$-algebra A, the homomorphism $(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_A)_{\mathsf{L}}(\mathsf{q})$ is surjective;

(iv) for any $\mathbb{V}$-algebra A, the homomorphism $\gamma_A\colon \mathsf{Ap}(A) \to A$ is surjective;

(v) the homomorphism $(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_{\mathsf{Fr}(\{\star\})})_{\mathsf{L}}(\mathsf{q})$ is surjective;

(vi) the homomorphism $\gamma_{\mathsf{Fr}(\{\star\})}\colon \mathsf{Ap}(\mathsf{Fr}(\{\star\})) \to \mathsf{Fr}(\{\star\})$ is surjective;

(vii) the element $\star$ is in the image of $\gamma_{\mathsf{Fr}(\{\star\})}\colon \mathsf{Ap}(\mathsf{Fr}(\{\star\})) \to \mathsf{Fr}(\{\star\})$;

(viii) for each $K \in \mathcal{H}_{\mathsf{X}}$, there exists in the theory of $\mathbb{V}$ a $(\#\mathcal{X}_{D_{\mathsf{X}}(K)})$-ary term t_K (in the variables from $\mathcal{X}_{D_{\mathsf{X}}(K)}$) satisfying

(a) for each $k\colon K \to K' \in \mathcal{H}_{\mathsf{X}}$,

$$t_K\left(([(H, D_{\mathsf{X}}(k) \circ f)])_{[(H,f)] \in \mathcal{X}_{D_{\mathsf{X}}(K)}}\right) = t_{K'}\left(([(H', f')])_{[(H',f')] \in \mathcal{X}_{D_{\mathsf{X}}(K')}}\right)$$

is a theorem of $\mathbb{V}$ (in the variables from $\mathcal{X}_{D_{\mathsf{X}}(K')}$), *(b) for each $H \in \mathcal{H}_{\mathsf{Y}}$,*

$$t_{K_H^{\mathsf{q}}}\left(([(H', c_H^{\mathsf{q}} \circ f)])_{[(H',f)] \in \mathcal{X}_{D_{\mathsf{X}}(K_H^{\mathsf{q}})}}\right) = [(H, 1_{D_{\mathsf{Y}}(H)})]$$

is a theorem of $\mathbb{V}$ (in the variables from $\mathcal{X}_{D_{\mathsf{Y}}(H)}$).

Proof. Statements (i) to (vii) are direct translations of the corresponding ones in Theorem 3.3. Condition (viii) can be obtained by describing condition (vii) in a similar way we did in the proof of Theorem 3.3. It can also be obtained by choosing t_K to be $\pi^s((t_K^{s,u})_{u \in U_s})$ where π^s and the $t_K^{s,u}$'s are the ones from condition (viii) of Theorem 3.3 for the unique sort s. □

Example 3.5. Let us come back to our running example of Mal'tsev categories. We use the notation of Examples 1.5 and 2.2. In this case, the set $\mathcal{X}_A$ has two elements denoted $x_1 = [(H, f)]$ and $x_2 = [(H, g)]$ and $\mathcal{X}_{\mathsf{Y}}$ has three elements denoted $y_1 = [(H, s \circ f)]$, $y_2 = [(H, 1_{\mathsf{Y}})]$ and $y_3 = [(H, s \circ g)]$. The equivalence (i) ⇔ (viii) of Corollary 3.4 thus says that a single-sorted variety $\mathbb{V}$ of universal algebras is a Mal'tsev category if and only if there exists in the theory of $\mathbb{V}$ two ternary terms $t_{K_1}(y_1, y_2, y_3)$ and $t_{K_4}(y_1, y_2, y_3)$ and two binary terms $t_{K_2}(x_1, x_2)$ and $t_{K_3}(x_1, x_2)$ such that

$$\begin{aligned}
&t_{K_1}(x_1, x_1, x_2) = t_{K_2}(x_1, x_2) && \text{(condition (viii)(a) for } k_1) \\
&t_{K_1}(x_1, x_2, x_2) = t_{K_3}(x_1, x_2) && \text{(condition (viii)(a) for } k_2) \\
&t_{K_4}(x_1, x_2, x_2) = t_{K_2}(x_1, x_2) && \text{(condition (viii)(a) for } k_3) \\
&t_{K_4}(x_1, x_1, x_2) = t_{K_3}(x_1, x_2) && \text{(condition (viii)(a) for } k_4) \\
&t_{K_1}(y_1, y_2, y_3) = y_2 && \text{(condition (viii)(b) for } H)
\end{aligned}$$

are theorems of $\mathbb{V}$ (recall that $K_H^{\mathsf{q}} = K_1$ and $c_H^{\mathsf{q}} = 1_{\mathsf{Y}}$). This can be easily simplified by saying that $\mathbb{V}$ is a Mal'tsev category if and only if there exists in the theory of $\mathbb{V}$ a ternary term $t_{K_4}(y_1, y_2, y_3)$ such that $t_{K_4}(x_1, x_2, x_2) = x_1$ and $t_{K_4}(x_1, x_1, x_2) = x_2$ are theorems of $\mathbb{V}$, which is the well-known Mal'tsev condition characterizing Mal'tsev single-sorted varieties of universal algebras from [39].

We recall from [21] that locally finitely presentable categories (i.e., finitary essentially algebraic categories) can be further characterized as the categories that are equivalent to the category $\mathsf{Lex}(\mathbb{T}, \mathbf{Set})$ of finite limit preserving functors from a small finitely complete category $\mathbb{T}$ to the category $\mathbf{Set}$. We will need the following theorem which follows from the main theorem of [30] (see the references therein for the parts of the theorem dealing with regularity, completeness and cocompleteness). We denote by $\mathbb{C}^{\mathsf{op}}$ the dual of a category $\mathbb{C}$.

Theorem 3.6 (adapted from [30]). *Let $\mathcal{A} \vdash \mathcal{B}$ be an exactness sequent of Type I. Let also $\mathbb{T}$ be a small regular category. Then, $\mathsf{Lex}(\mathbb{T}, \mathbf{Set})^{\mathsf{op}}$ is a regular complete and cocomplete category and*

$$\mathcal{A} \vdash_{\mathbb{T}} \mathcal{B} \iff \mathcal{A} \vdash_{\mathsf{Lex}(\mathbb{T}, \mathbf{Set})^{\mathsf{op}}} \mathcal{B}.$$

If $\mathbb{T}$ is moreover finitely cocomplete, then $\mathsf{Lex}(\mathbb{T}^{\mathsf{op}}, \mathbf{Set})$ is also regular, complete and cocomplete and

$$\mathcal{A} \vdash_{\mathbb{T}} \mathcal{B} \iff \mathcal{A} \vdash_{\mathsf{Lex}(\mathbb{T}^{\mathsf{op}}, \mathbf{Set})} \mathcal{B}.$$

4 Equivalence with matrix properties

Matrix properties have been introduced in [33, 34] to unify many examples of exactness properties such as being a Mal'tsev category [14, 15], being unital [8], strongly unital [8], subtractive [32], majority [24] and so forth. We will use here the more general type of matrices appearing in [35] and which brings in the additional example of being n-permutable [13], among many others. However, since we only considered in the previous sections regular statements for (**Set**-enriched) categories, we will also only consider matrices of terms in $\mathsf{Th}[\mathbf{Set}]$, the algebraic theory of **Set**.

An *extended matrix M of variables* (i.e., of terms in $\mathsf{Th}[\mathbf{Set}]$) is given by a matrix

$$M = \left[\begin{array}{ccc|ccc} t_{11} & \cdots & t_{1m} & u_{11} & \cdots & u_{1m'} \\ \vdots & & \vdots & \vdots & & \vdots \\ t_{n1} & \cdots & t_{nm} & u_{n1} & \cdots & u_{nm'} \end{array}\right] \tag{4}$$

where $n \geqslant 1$, $m \geqslant 0$, $m' \geqslant 0$ and where the t_{ij}'s and the u_{ij}'s are variables; the set of t_{ij}'s being denoted as

$$\{t_{ij} \,|\, 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\} = \{x_1, \dots, x_l\} \tag{5}$$

and the set of t_{ij}'s and u_{ij}'s as

$$\{t_{ij} \,|\, 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\} \cup \{u_{ij} \,|\, 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m'\} = \{x_1, \dots, x_k\} \tag{6}$$

with $0 \leqslant l \leqslant k$. Given an object A in a regular category $\mathbb{C}$, each variable t in $\{x_1, \dots, x_l\}$ gives rise to the corresponding projection $t^A\colon A^l \to A$ from the l-th power of A (and similarly, each variable t in $\{x_1, \dots, x_k\}$ gives rise to the corresponding projection $t^A\colon A^k \to A$).

Given such an extended matrix M, an n-ary relation $r\colon R \rightarrowtail A^n$ in a regular category $\mathbb{C}$ is said to be *M-closed* if, when we consider the pullbacks

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, tail, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & R^m \arrow[d, tail, "r^m"] \\ A^l \arrow[r, "\begin{pmatrix} t^A_{11} & \cdots & t^A_{1m} \\ \vdots & & \vdots \\ t^A_{n1} & \cdots & t^A_{nm} \end{pmatrix}"'] & (A^n)^m \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r, "g'"] \arrow[d, tail, "g"'] \arrow[dr, phantom, very near start, "\lrcorner"] & R^{m'} \arrow[d, tail, "r^{m'}"] \\ A^k \arrow[r, "\begin{pmatrix} u^A_{11} & \cdots & u^A_{1m'} \\ \vdots & & \vdots \\ u^A_{n1} & \cdots & u^A_{nm'} \end{pmatrix}"'] & (A^n)^{m'} \end{tikzcd}$$

and

$$\begin{tikzcd} T \arrow[r, tail, "h'"] \arrow[dd, "h"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Q \arrow[d, tail, "g"] \\ & A^k \cong A^l \times A^{k-l} \arrow[d, "\pi_1=(p_1,\dots,p_l)"] \\ P \arrow[r, tail, "f"'] & A^l \end{tikzcd}$$

then h is a regular epimorphism (or, in other words, f factors through the image of $\pi_1 g$). Here, $p_j\colon A^k \to A$ is the j-th projection for each $1 \leqslant j \leqslant k$. We say that the regular category $\mathbb{C}$ *has M-closed relations* if any n-ary relation $r \colon R \rightarrowtail A^n$ in $\mathbb{C}$ is M-closed. Clearly, this property is equivalent to the condition that for each diagram

$$R \underset{r_n}{\overset{r_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}}}} A$$

in $\mathbb{C}$, the morphism h constructed via the pullbacks P, Q and T as above is a regular epimorphism. Therefore, for each extended matrix of variables M, there exists an exactness sequent $\mathcal{A} \vdash \mathcal{B}$ of Type I such that a regular category $\mathbb{C}$ has M-closed relations if and only if $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$. This is detailed in Remark 4.10 which particularizes the main results of the previous sections to this exactness sequent.

In [35], a more general type of matrices is considered where 'ghost variables' are taken into account, i.e., the equalities (5) and (6) are replaced by inclusions $\subseteq$ which can impact the definition for a relation to be M-closed. For instance, the reflexivity of a binary relation can no longer be expressed by a matrix with no ghost variables, while the matrix

$$\left[\begin{array}{c|c} & x_1 \\ & x_1 \end{array}\right]$$

with $l = k = 1$ expresses the reflexivity of a binary relation. However, as an immediate corollary of Theorem 4.11, up to conjunctions, the two types of matrices describe the same properties on regular categories.

Let us now describe classical examples of matrix properties.

Example 4.1. Let M be the extended matrix given by:

$$M = \left[\begin{array}{ccc|c} x_1 & x_2 & x_2 & x_1 \\ x_1 & x_1 & x_2 & x_2 \end{array}\right]$$

A regular category has M-closed relations if and only if it is a Mal'tsev category as introduced in [14] (see Example 1.5).

Example 4.2. More generally, for any $n \geqslant 2$, let M be the extended matrix given by:

$$M = \left[\begin{array}{ccc|ccccc} x_1 & x_2 & x_2 & x_1 & x_3 & x_4 & \cdots & x_n \\ x_1 & x_1 & x_2 & x_3 & x_4 & \cdots & x_n & x_2 \end{array}\right]$$

A regular category has M-closed relations if and only if it is an n-permutable category as introduced in [13].

Example 4.3. Let M be the extended matrix given by:

$$M = \left[\begin{array}{ccc|c} x_1 & x_1 & x_2 & x_1 \\ x_1 & x_2 & x_1 & x_1 \\ x_2 & x_1 & x_1 & x_1 \end{array}\right]$$

A regular category has M-closed relations if and only if it is a majority category as introduced in [24].

Example 4.4. Let M be the extended matrix given by:

$$M = \left[\begin{array}{c|c} x_1 & x_1 \end{array}\right]$$

Then, any regular category has M-closed relations.

Example 4.5. Let M be the extended matrix given by:

$$M = \left[\begin{array}{c|c} & x_1 \end{array} \right]$$

A regular category $\mathbb{C}$ has M-closed relations if and only if every object X of $\mathbb{C}$ *has global support*, i.e., the unique morphism $X \to 1$ to the terminal object is a regular epimorphism.

Example 4.6. Let M be the extended matrix given by:

$$M = \left[\begin{array}{cc|c} x_1 & x_2 & x_1 \\ x_1 & x_2 & x_2 \end{array} \right]$$

A regular category $\mathbb{C}$ has M-closed relations if and only if every morphism is a monomorphism, i.e., if and only if $\mathbb{C}$ is a preorder.

The matrix presentation of these properties makes it very easy to deduce from it the Mal'tsev condition characterizing (essentially) algebraic categories satisfying the property. Given a sort $s \in S$ in an essentially algebraic theory Γ and a variable t in $\{x_1, \ldots, x_l\}$ (respectively in $\{x_1, \ldots, x_k\}$), we denote by $t^s \colon s^l \to s$ (respectively by $t^s \colon s^k \to s$) the everywhere-defined term given by the corresponding projection. If the theory Γ is single-sorted, we will simply write t instead of t^s for the unique sort s.

Theorem 4.7 ([35]). *Let M be an extended matrix of variables as in (4) and let $\mathbb{V}$ be a single-sorted variety of universal algebras. Then $\mathbb{V}$ has M-closed relations if and only if the theory of $\mathbb{V}$ admits m-ary terms $p_1, \ldots, p_{m'}$ and l-ary terms $q_{l+1}, \ldots, q_k$ such that*

$$p_j(t_{i1}(x_1, \ldots, x_l), \ldots, t_{im}(x_1, \ldots, x_l)) = u_{ij}(x_1, \ldots, x_l, q_{l+1}(x_1, \ldots, x_l), \ldots, q_k(x_1, \ldots, x_l))$$

is a theorem of the theory of $\mathbb{V}$ for each $i \in \{1, \ldots, n\}$ and each $j \in \{1, \ldots, m'\}$.

Theorem 4.8 ([26, 29]). *Let M be an extended matrix of variables as in (4) and let $\Gamma = (S, \Sigma, E, \Sigma_t, \mathsf{Def})$ be an essentially algebraic theory such that $\mathsf{Mod}(\Gamma)$ is a regular category. Then, $\mathsf{Mod}(\Gamma)$ has M-closed relations if and only if, for each $s \in S$, there exists in Γ*

- *a term $\pi^s \colon \prod_{u \in U_s} s_u \to s$,*
- *for each $v \in \{1, \ldots, k\}$ and each $u \in U_s$, an everywhere-defined term $q_v^{s,u} \colon s^l \to s_u$,*
- *for each $j \in \{1, \ldots, m'\}$ and each $u \in U_s$, a term $p_j^{s,u} \colon s^m \to s_u$*

such that

(a) *the term*
$$p_j^{s,u}(t_{i1}^s(x_1, \ldots, x_l), \ldots, t_{im}^s(x_1, \ldots, x_l)) \colon s^l \to s_u$$
is everywhere-defined for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m'\}$ and $u \in U_s$,

(b) *the theorem*
$$p_j^{s,u}(t_{i1}^s(x_1, \ldots, x_l), \ldots, t_{im}^s(x_1, \ldots, x_l)) = u_{ij}^{s_u}(q_1^{s,u}(x_1, \ldots, x_l), \ldots, q_k^{s,u}(x_1, \ldots, x_l))$$
holds in Γ for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m'\}$ and $u \in U_s$,

(c) *the term*
$$\pi^s((q_v^{s,u}(x_1, \ldots, x_l))_{u \in U_s}) \colon s^l \to s$$
is everywhere-defined for each $v \in \{1, \ldots, l\}$,

(d) *the theorem*

$$\pi^s((q_v^{s,u}(x_1,\ldots,x_l))_{u\in U_s}) = x_v$$

holds in Γ *for each* $v \in \{1,\ldots,l\}$.

We now prove that, without loss of generality, we can assume the terms $p_j^{s,u}$'s to be everywhere-defined.

Theorem 4.9. *Let* M *be an extended matrix of variables as in (4) and let* $\Gamma = (S,\Sigma,E,\Sigma_t,\mathsf{Def})$ *be an essentially algebraic theory such that* $\mathsf{Mod}(\Gamma)$ *is a regular category. Then,* $\mathsf{Mod}(\Gamma)$ *has* M*-closed relations if and only if, for each* $s \in S$*, there exists in* Γ

- *a term* $\pi^s \colon \prod_{u\in U_s} s_u \to s$,
- *for each* $v \in \{1,\ldots,k\}$ *and each* $u \in U_s$*, an everywhere-defined term* $q_v^{s,u} \colon s^l \to s_u$,
- *for each* $j \in \{1,\ldots,m'\}$ *and each* $u \in U_s$*, an everywhere-defined term* $p_j^{s,u} \colon s^m \to s_u$

such that

(a) *the theorem*

$$p_j^{s,u}(t_{i1}^s(x_1,\ldots,x_l),\ldots,t_{im}^s(x_1,\ldots,x_l)) = u_{ij}^{s_u}(q_1^{s,u}(x_1,\ldots,x_l),\ldots,q_k^{s,u}(x_1,\ldots,x_l))$$

holds in Γ *for all* $i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,m'\}$ *and* $u \in U_s$,

(b) *the term*

$$\pi^s((q_v^{s,u}(x_1,\ldots,x_l))_{u\in U_s}) \colon s^l \to s$$

is everywhere-defined for each $v \in \{1,\ldots,l\}$,

(c) *the theorem*

$$\pi^s((q_v^{s,u}(x_1,\ldots,x_l))_{u\in U_s}) = x_v$$

holds in Γ *for each* $v \in \{1,\ldots,l\}$.

Proof. In view of Theorem 4.8, the 'if part' of the statement is trivial. Conversely, let us assume $\mathsf{Mod}(\Gamma)$ has M-closed relations and consider, for a fixed sort $s \in S$, the terms π^s, $q_v^{s,u}$ and $p_j^{s,u}$ given by Theorem 4.8. By Theorem 3.2 applied to the terms $p_1^{s,u},\ldots,p_{m'}^{s,u}$, we know that for each $u \in U_s$, there exist a term $\pi_u^s \colon \prod_{w\in W_u} s'_w \to s_u$, a family of everywhere-defined terms

$$(\rho_{u,w}^s \colon s_u \to s'_w)_{w\in W_u}$$

and a family of everywhere-defined terms

$$(\tau_{u,j,w}^s \colon s^m \to s'_w)_{\substack{j\in\{1,\ldots,m'\}\\ w\in W_u}}$$

such that $\pi_u^s((\rho_{u,w}^s(x))_{w\in W_u}) \colon s_u \to s_u$ is an everywhere-defined term, $\pi_u^s((\rho_{u,w}^s(x))_{w\in W_u}) = x$ is a theorem of Γ and $\rho_{u,w}^s(p_j^{s,u}(x_1,\ldots,x_m)) = \tau_{u,j,w}^s(x_1,\ldots,x_m)$ is a theorem of Γ for each $w \in W_u$ and each $1 \leqslant j \leqslant m'$. It suffices now to consider the term

$$\pi'^s = \pi^s((\pi_u^s)_{u\in U_s}) \colon \prod_{u\in U_s}\prod_{w\in W_u} s'_w \to s,$$

for each $v \in \{1,\dots,k\}$ and each pair (u,w) with $u \in U_s$ and $w \in W_u$, the everywhere-defined term

$$q_v'^{s,(u,w)} = \rho^s_{u,w}(q_v^{s,u})\colon s^l \to s'_w$$

and for each $j \in \{1,\dots,m'\}$ and each pair (u,w) with $u \in U_s$ and $w \in W_u$, the everywhere-defined term

$$p_j'^{s,(u,w)} = \tau^s_{u,j,w}\colon s^m \to s'_w.$$

Given any $i \in \{1,\dots,n\}$, $j \in \{1,\dots,m'\}$, $u \in U_s$ and $w \in W_u$, we have the following theorems in Γ

$$\begin{aligned}
&p_j'^{s,(u,w)}(t^s_{i1}(x_1,\dots,x_l),\dots,t^s_{im}(x_1,\dots,x_l))\\
&= \tau^s_{u,j,w}(t^s_{i1}(x_1,\dots,x_l),\dots,t^s_{im}(x_1,\dots,x_l))\\
&= \rho^s_{u,w}(p_j^{s,u}(t^s_{i1}(x_1,\dots,x_l),\dots,t^s_{im}(x_1,\dots,x_l)))\\
&= \rho^s_{u,w}(u^{s_u}_{ij}(q_1^{s,u}(x_1,\dots,x_l),\dots,q_k^{s,u}(x_1,\dots,x_l)))\\
&= u^{s'_w}_{ij}(\rho^s_{u,w}(q_1^{s,u}(x_1,\dots,x_l)),\dots,\rho^s_{u,w}(q_k^{s,u}(x_1,\dots,x_l)))\\
&= u^{s'_w}_{ij}(q_1'^{s,(u,w)}(x_1,\dots,x_l),\dots,q_k'^{s,(u,w)}(x_1,\dots,x_l)).
\end{aligned}$$

Since every intermediate step is formed by an everywhere-defined term, this gives that

$$\begin{aligned}
&p_j'^{s,(u,w)}(t^s_{i1}(x_1,\dots,x_l),\dots,t^s_{im}(x_1,\dots,x_l))\\
&= u^{s'_w}_{ij}(q_1'^{s,(u,w)}(x_1,\dots,x_l),\dots,q_k'^{s,(u,w)}(x_1,\dots,x_l))
\end{aligned}$$

is a theorem of Γ. Given any $v \in \{1,\dots,l\}$ and any $u \in U_s$, the term

$$\pi^s_u((\rho^s_{u,w}(q_v^{s,u}(x_1,\dots,x_l)))_{w\in W_u})$$

is everywhere-defined and

$$\pi^s_u((\rho^s_{u,w}(q_v^{s,u}(x_1,\dots,x_l)))_{w\in W_u}) = q_v^{s,u}(x_1,\dots,x_l)$$

is a theorem of Γ. Therefore, by the properties (c) and (d) of Theorem 4.8, for any $v \in \{1,\dots,l\}$, the term

$$\pi'^s\left((q_v'^{s,(u,w)}(x_1,\dots,x_l))_{\substack{u\in U_s\\ w\in W_u}}\right) = \pi^s((\pi^s_u((\rho^s_{u,w}(q_v^{s,u}(x_1,\dots,x_l)))_{w\in W_u}))_{u\in U_s})$$

is everywhere-defined and

$$\pi'^s\left((q_v'^{s,(u,w)}(x_1,\dots,x_l))_{\substack{u\in U_s\\ w\in W_u}}\right) = x_v$$

is a theorem of Γ. $\square$

Remark 4.10. We already know that for an extended matrix of variables M as in (4), there exists an exactness sequent $\mathcal{A} \vdash \mathcal{B}$ of Type I such that a regular category $\mathbb{C}$ has M-closed relations if and only if $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$. The sketch $\mathcal{A}$ needed here is the underlying sketch of the category $\mathbb{A}$ displayed as:

$$\begin{tikzcd} R \arrow[loop left, "1_R"] \arrow[r, shift left=3, "r_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "r_n"'] & A \arrow[loop right, "1_A"] \end{tikzcd}$$

The sketch $\mathcal{B}_{\mathsf{L}}$ is then constructed following Proc. A–Proc. D as indicated by the pullbacks P, Q and T in the definition of M-closed relations recalled in the beginning of this section. The sketch $\mathcal{B}$ is constructed from $\mathcal{B}_{\mathsf{L}}$ as prescribed by condition Ax. 3 where the role of the arrow $\mathsf{q}\colon \mathsf{X} \to \mathsf{Y}$ is played by the arrow $h\colon T \to P$ (using the notation of the beginning of this section). Given an object C in a finitely cocomplete regular category $\mathbb{C}$, using the pointwise Kan extension formula, one can compute that the left Kan extension $\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C$ is described by

$$\begin{tikzcd} mC \arrow[loop left, "1_{mC}"] \arrow[r, shift left=2, "r_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "r_n"'] & lC \arrow[loop right, "1_{lC}"] \end{tikzcd}$$

where $r_i = \begin{pmatrix} \iota_{t_{i1}} \\ \vdots \\ \iota_{t_{im}} \end{pmatrix} : mC \to lC$ for each $i \in \{1, \dots, n\}$ and where $\iota_{x_1}, \dots, \iota_{x_l} \colon C \to lC$ are the coproduct injections. The morphism $\gamma_C\colon \mathsf{Ap}(C) \to C$ is thus given by the left morphism in the following diagram where both rectangles are pullbacks.

$$\begin{tikzcd} \mathsf{Ap}(C) \arrow[r, "\delta_C"] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(T) \arrow[r, "(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(h')"] \arrow[d, "(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(h)"] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(Q) \arrow[d, "(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(\pi_1) \circ (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(g)"] \\ C \arrow[r, "e_c"'] \arrow[rr, bend right, "(\iota_{x_1}, \dots, \iota_{x_l})"'] & (\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(P) \arrow[r, "(\mathsf{Lan}_{D_{\mathsf{Y}}} \Delta_C)_{\mathsf{L}}(f)"'] & (lC)^l \end{tikzcd}$$

The equivalence (i) $\Leftrightarrow$ (iv) of Theorem 2.1 is the characterization of regular finitely cocomplete categories with M-closed relations in terms of approximate co-operations from [26, 29], which generalizes results of [10, 36]. The equivalence (i) $\Leftrightarrow$ (viii) of Corollary 3.4 in this particular case gives exactly the characterization from [35] of varieties with M-closed relations recalled in Theorem 4.7. The equivalence (i) $\Leftrightarrow$ (viii) of Theorem 3.3 gives that, for an essentially algebraic theory $\Gamma = (S, \Sigma, E, \Sigma_t, \mathsf{Def})$ such that $\mathsf{Mod}(\Gamma)$ is a regular category, $\mathsf{Mod}(\Gamma)$ has M-closed relations if and only if, for each $s \in S$, there exists in Γ

- a term $\pi^s\colon \prod_{u \in U_s} s_u \to s$,
- for each $v \in \{1, \dots, k\}$ and each $u \in U_s$, an everywhere-defined term $q_v^{s,u}\colon s^l \to s_u$,
- for each $j \in \{1, \dots, m'\}$ and each $u \in U_s$, an everywhere-defined term $p_j^{s,u}\colon s^m \to s_u$,
- for each $j \in \{1, \dots, m\}$ and each $u \in U_s$, an everywhere-defined term $d_j^{s,u}\colon s^m \to s_u$

satisfying properties (a), (b) and (c) of Theorem 4.9 and satisfying in addition

(d) the theorem

$$d_j^{s,u}(t_{i1}^s(x_1, \dots, x_l), \dots, t_{im}^s(x_1, \dots, x_l)) = t_{ij}^{s_u}(q_1^{s,u}(x_1, \dots, x_l), \dots, q_l^{s,u}(x_1, \dots, x_l))$$

holds in Γ for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$ and $u \in U_s$,

(e) the term

$$\pi^s((d_j^{s,u}(x_1, \dots, x_m))_{u \in U_s})\colon s^m \to s$$

is everywhere-defined for each $j \in \{1, \dots, m\}$,

(f) the theorem
$$\pi^s((d_j^{s,u}(x_1,\dots,x_m))_{u\in U_s}) = x_j$$
holds in Γ for each $j \in \{1,\dots,m\}$.

Of course, in the regular context, this Mal'tsev condition is equivalent to the one of Theorem 4.9. To obtain this new Mal'tsev condition from the one of Theorem 4.9, one can use Theorem 3.1 with the term π^s of the latter Mal'tsev condition. We omit details here for the sake of brevity and since we will not need this remark in what follows.

We conclude by proving that the linear exactness properties in the regular context, as described in Section 1, are exactly the finite conjunctions of matrix properties. For that we will need the axiom of universes from [3]. It says that for any set x, there exists a universe $\mathfrak{U}$ such that $x \in \mathfrak{U}$. By a $\mathfrak{U}$-category (for a universe $\mathfrak{U}$), we mean a category $\mathbb{C}$ such that $\mathsf{Ob}(\mathbb{C}) \subseteq \mathfrak{U}$ where $\mathsf{Ob}(\mathbb{C})$ is the collection of objects of $\mathbb{C}$ and such that, for any pair (C, C') of objects, $\hom_{\mathbb{C}}(C, C') \in \mathfrak{U}$. A $\mathfrak{U}$-category is said to be $\mathfrak{U}$-small if moreover $\mathsf{Ob}(\mathbb{C}) \in \mathfrak{U}$. It is not difficult to prove that, in the definition of the notation $\equiv_{\mathcal{A}}$ for a sketch $\mathcal{A}$ in (1) and (2), one can restrict to consider only $\mathfrak{U}$-categories $\mathbb{C}$ for some universe $\mathfrak{U}$ containing the set of natural numbers. Therefore, the notion of an exactness sequent of Type I is independent of the base uncountable universe. In the following theorem, we denote by REG the collection of all regular $\mathfrak{U}$-categories for all universes $\mathfrak{U}$. Let us remark that, if one does not admit the axiom of universes, one could replace REG by the collection of all regular $\mathfrak{U}$-categories where $\mathfrak{U}$ is an uncountable universe for which there exist universes $\mathfrak{V}$ and $\mathfrak{W}$ such that $\mathfrak{U} \in \mathfrak{V} \in \mathfrak{W}$; Theorem 4.11 would still hold in that case.

Theorem 4.11. *Let $\mathcal{P} \subseteq \mathsf{REG}$ be a (potentially large) collection of regular categories. Then, the following statements are equivalent:*

(i) there exists an exactness sequent $\mathcal{A} \vdash \mathcal{B}$ of Type I such that
$$\mathbb{C} \in \mathcal{P} \iff \mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$$
for any regular category $\mathbb{C} \in \mathsf{REG}$;

(ii) there exists an integer $a \geqslant 0$ and extended matrices $M_1,\dots,M_a$ of variables such that
$$\mathbb{C} \in \mathcal{P} \iff \mathbb{C} \textit{ has } M_i\textit{-closed relations for all } i \in \{1,\dots,a\}$$
for any regular category $\mathbb{C} \in \mathsf{REG}$;

(iii) there exists an extended matrix M of variables such that either
$$\mathbb{C} \in \mathcal{P} \iff \mathbb{C} \textit{ has } M\textit{-closed relations}$$
for any regular category $\mathbb{C} \in \mathsf{REG}$ or
$$\mathbb{C} \in \mathcal{P} \iff \begin{cases} \mathbb{C} \textit{ has } M\textit{-closed relations} \\ \textit{and every object of } \mathbb{C} \textit{ has global support} \end{cases}$$
for any regular category $\mathbb{C} \in \mathsf{REG}$.

Proof. In view of Example 4.5, the implication (iii) $\Rightarrow$ (ii) is trivial. Let us now prove (ii) $\Rightarrow$ (i). For any $i \in \{1,\dots,a\}$, we already know there exists an exactness sequent $\mathcal{A}_i \vdash \mathcal{B}_i$ of Type I such that a regular category $\mathbb{C}$ has M_i-closed relations if and only if

$\mathcal{A}_i \vdash_{\mathbb{C}} \mathcal{B}_i$ (see Remark 4.10). We then conclude by Theorem 1.1. It remains to prove the implication (i) $\Rightarrow$ (iii). So let $\mathcal{A} \vdash \mathcal{B}$ be as in the statement. Firstly, let us assume we have constructed an extended matrix M such that either

$$\mathcal{A} \vdash_{\mathsf{Mod}(\Gamma)} \mathcal{B} \iff \mathsf{Mod}(\Gamma) \text{ has } M\text{-closed relations}$$

for any regular essentially algebraic category $\mathsf{Mod}(\Gamma)$ (with respect to any universe $\mathfrak{W}$) or

$$\mathcal{A} \vdash_{\mathsf{Mod}(\Gamma)} \mathcal{B} \iff \begin{cases} \mathsf{Mod}(\Gamma) \text{ has } M\text{-closed relations} \\ \text{and every object of } \mathsf{Mod}(\Gamma) \text{ has global support} \end{cases}$$

for any regular essentially algebraic category $\mathsf{Mod}(\Gamma)$ (with respect to any universe $\mathfrak{W}$). In that case, we can prove the extended matrix M satisfies the condition in (iii). Indeed, let $\mathbb{C}$ be a regular $\mathfrak{U}$-category for a universe $\mathfrak{U}$. By the axiom of universe, there exists a universe $\mathfrak{V}$ such that $\mathfrak{U} \in \mathfrak{V}$. This implies that $\mathbb{C}$ is a $\mathfrak{V}$-small regular category. We can thus form the $\mathfrak{V}$-category $\mathsf{Lex}(\mathbb{C}, \mathbf{Set}_{\mathfrak{V}})$ where $\mathbf{Set}_{\mathfrak{V}}$ is the category of sets x such that $x \in \mathfrak{V}$. By Theorem 3.6, $\mathsf{Lex}(\mathbb{C}, \mathbf{Set}_{\mathfrak{V}})^{\mathsf{op}}$ is a regular finitely cocomplete category. Again by the axiom of universes, there exists a universe $\mathfrak{W}$ such that $\mathfrak{V} \in \mathfrak{W}$. Hence, $\mathsf{Lex}(\mathbb{C}, \mathbf{Set}_{\mathfrak{V}})^{\mathsf{op}}$ is a $\mathfrak{W}$-small category and we can form the $\mathfrak{W}$-category $\mathsf{Lex}(\mathsf{Lex}(\mathbb{C}, \mathbf{Set}_{\mathfrak{V}}), \mathbf{Set}_{\mathfrak{W}})$ where $\mathbf{Set}_{\mathfrak{W}}$ is defined in an analogous way as $\mathbf{Set}_{\mathfrak{V}}$. Again by Theorem 3.6, this category $\mathsf{Lex}(\mathsf{Lex}(\mathbb{C}, \mathbf{Set}_{\mathfrak{V}}), \mathbf{Set}_{\mathfrak{W}})$ is regular and

$$\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B} \iff \mathcal{A} \vdash_{\mathsf{Lex}(\mathbb{C}, \mathbf{Set}_{\mathfrak{V}})^{\mathsf{op}}} \mathcal{B} \iff \mathcal{A} \vdash_{\mathsf{Lex}(\mathsf{Lex}(\mathbb{C}, \mathbf{Set}_{\mathfrak{V}}), \mathbf{Set}_{\mathfrak{W}})} \mathcal{B}.$$

Moreover, in view of the implication (ii) $\Rightarrow$ (i), we also know that $\mathbb{C}$ has M-closed relations if and only if $\mathsf{Lex}(\mathsf{Lex}(\mathbb{C}, \mathbf{Set}_{\mathfrak{V}}), \mathbf{Set}_{\mathfrak{W}})$ has M-closed relations. In view of Example 4.5, we also know that every object of $\mathbb{C}$ has global support if and only if every object of $\mathsf{Lex}(\mathsf{Lex}(\mathbb{C}, \mathbf{Set}_{\mathfrak{V}}), \mathbf{Set}_{\mathfrak{W}})$ has global support. Since $\mathsf{Lex}(\mathsf{Lex}(\mathbb{C}, \mathbf{Set}_{\mathfrak{V}}), \mathbf{Set}_{\mathfrak{W}})$ is a (finitary) essentially algebraic regular category (with respect to the universe $\mathfrak{W}$), this concludes the proof under our assumption. It thus remains to construct the extended matrix M with the required property for regular essentially algebraic categories $\mathsf{Mod}(\Gamma)$. Let

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha"] & \mathcal{A} = \mathcal{U}(\mathbb{A}) \arrow[r, "\beta_{\mathsf{L}}"] \arrow[rr, bend right, "\beta"'] & \mathcal{B}_{\mathsf{L}} \arrow[r, "\beta_{\mathsf{R}}"] & \mathcal{B} \end{tikzcd}$$

be the presentation of $\mathcal{A} \vdash \mathcal{B}$ as in the definition of an exactness sequent of Type I and let us denote by $\mathsf{q} \colon \mathsf{X} \to \mathsf{Y}$ the arrow in $\mathcal{B}_{\mathsf{L}}$ via which $\mathcal{B}$ is constructed. Using the notation previously introduced, we consider the union $\mathcal{K}$ of the connected components in $\mathcal{H}_{\mathsf{X}}$ of those objects $K \in \mathcal{H}_{\mathsf{X}}$ for which $\mathcal{X}_{D_{\mathsf{X}}(K)} = \varnothing$, i.e., for which $\mathbb{A}(D_{\mathsf{Y}}(H), D_{\mathsf{X}}(K)) = \varnothing$ for all $H \in \mathcal{H}_{\mathsf{Y}}$. We also consider, for an arbitrary essentially algebraic theory $\Gamma = (S, \Sigma, E, \Sigma_t, \mathsf{Def})$ such that $\mathsf{Mod}(\Gamma)$ is regular and $\mathcal{A} \vdash_{\mathsf{Mod}(\Gamma)} \mathcal{B}$ and for an arbitrary sort $s \in S$, the terms $\pi^s \colon \prod_{u \in U_s} s_u \to s$ and $t_K^{s,u} \colon s^{\# \mathcal{X}_{D_{\mathsf{X}}(K)}} \to s_u$ given by Theorem 3.3. Notice that for each $K \in \mathcal{H}_{\mathsf{X}}$ such that $\mathcal{X}_{D_{\mathsf{X}}(K)} = \varnothing$, the everywhere-defined terms $t_K^{s,u}$ are simply constants of sort s_u. Using the identities (viii)(a) of Theorem 3.3, we have, for any $k \colon K \to K' \in \mathcal{H}_{\mathsf{X}}$ and any $u \in U_s$, the following theorem in Γ in a single variable x of sort s:

$$t_K^{s,u}\left((x)_{[(H,f)] \in \mathcal{X}_{D_{\mathsf{X}}(K)}}\right) = t_{K'}^{s,u}\left((x)_{[(H',f')] \in \mathcal{X}_{D_{\mathsf{X}}(K')}}\right)$$

It is then straightforward to prove by induction on the length of the zigzag, that given any zigzag of paths in $\mathcal{H}_{\mathsf{X}}$ from K to K' where $\mathcal{X}_{D_{\mathsf{X}}(K)} = \varnothing$ and given any $u \in U_s$, we have the following theorem in Γ in a single variable x of sort s:

$$t_K^{s,u} = t_{K'}^{s,u}\left((x)_{[(H',f')] \in \mathcal{X}_{D_{\mathsf{X}}(K')}}\right) \tag{7}$$

Let us now treat the case where there exists $H \in \mathcal{H}_{\mathsf{Y}}$ such that $K^{\mathsf{q}}_H \in \mathcal{K}$. By the identity (viii)(c) of Theorem 3.3, we know that

$$\pi^s\left((t^{s,u}_{K^{\mathsf{q}}_H}((x)_{[(H',f)]\in\mathcal{X}_{D_{\mathsf{X}}(K^{\mathsf{q}}_H)}}))_{u\in U_s}\right) = x$$

is a theorem of Γ in a single variable x of sort s. In view of the theorem (7) above, we have that

$$\pi^s((t^{s,u}_K)_{u\in U_s}) = x$$

is also a theorem of Γ where K is any object of $\mathcal{H}_{\mathsf{X}}$ in the same connected component as K^{q}_H and such that $\mathcal{X}_{D_{\mathsf{X}}(K)} = \varnothing$. Moreover, by (viii)(b) of Theorem 3.3, the left hand side is an everywhere-defined term of Γ. We can thus rephrase this by saying that for any sort $s \in S$, there exists an everywhere-defined constant term c^s of sort s such that $c^s = x$ is a theorem of Γ in the single variable x of sort s. Therefore, for any Γ-model A, the unique homomorphism $A \to 1$ to the terminal object of $\mathsf{Mod}(\Gamma)$ is an isomorphism and so $\mathsf{Mod}(\Gamma)$ is equivalent to the terminal category. This condition is equivalent to the conjunction of the matrix properties from Examples 4.5 and 4.6. We can thus suppose now, without loss of generality, that $K^{\mathsf{q}}_H \notin \mathcal{K}$ for any $H \in \mathcal{H}_{\mathsf{Y}}$.

We will now treat the case where $\mathcal{K} \neq \varnothing$. In that case, we consider any $K \in \mathcal{H}_{\mathsf{X}}$ such that $\mathcal{X}_{D_{\mathsf{X}}(K)} = \varnothing$. As we have mentioned above, this K induces an everywhere-defined constant term $t^{s,u}_K$ of sort s_u for any $u \in U_s$. Since we are also given the term $\pi^s\colon \prod_{u\in U_s} s_u \to s$, this means Γ satisfies the conditions of Theorem 4.9 for the extended matrix of Example 4.5, and therefore every object of $\mathsf{Mod}(\Gamma)$ has global support. Moreover, as a restriction of the Mal'tsev condition (viii) of Theorem 3.3, it is obvious that for each sort $s \in S$, there exists in Γ a term $\pi^s\colon \prod_{u\in U_s} s_u \to s$ and for each $u \in U_s$ and each $K \in \mathcal{H}_{\mathsf{X}} \setminus \mathcal{K}$, there exists an everywhere-defined term $t^{s,u}_K\colon s^{\#\mathcal{X}_{D_{\mathsf{X}}(K)}} \to s_u$ such that the condition (viii)(a) restricted only to arrows $k\colon K \to K' \in \mathcal{H}_{\mathsf{X}}\setminus\mathcal{K}$, the condition (viii)(b) and the condition (viii)(c) of Theorem 3.3 are satisfied. Conversely, suppose that this restricted Mal'tsev condition holds in Γ and that every object of $\mathsf{Mod}(\Gamma)$ has global support. The latter condition implies, by Theorem 4.9 applied to Example 4.5, that for each $u \in U_s$, there exists a constant term c^{s_u} of sort s_u. Applying now Theorem 3.1 to this constant term, we know there exists in Γ a term $\pi^s_u\colon \prod_{v\in V_u} s'_v \to s_u$ and for each $v \in V_u$, an everywhere-defined term $\rho^s_{u,v}\colon s_u \to s'_v$ and an everywhere-defined constant term $\tau^s_{u,v}$ of sort s'_v such that $\pi^s_u((\rho^s_{u,v}(x))_{v\in V_u})\colon s_u \to s_u$ is an everywhere-defined term, $\pi^s_u((\rho^s_{u,v}(x))_{v\in V_u}) = x$ is a theorem of Γ and $\rho^s_{u,v}(c^{s_u}) = \tau^s_{u,v}$ is a theorem of Γ for each $v \in V_u$. Considering now the term

$$\pi^s((\pi^s_u)_{u\in U_s})\colon \prod_{u\in U_s}\prod_{v\in V_u} s'_v \to s,$$

and for each pair (u,v) with $u \in U_s$ and $v \in V_u$ and each $K \in \mathcal{H}_{\mathsf{X}}$, the everywhere-defined term

$$t^{s,(u,v)}_K\colon s^{\#\mathcal{X}_{D_{\mathsf{X}}(K)}} \to s'_v = \begin{cases} \tau^s_{u,v} & \text{if } K \in \mathcal{K} \\ \rho^s_{u,v}(t^{s,u}_K) & \text{if } K \in \mathcal{H}_{\mathsf{X}} \setminus \mathcal{K}, \end{cases}$$

we see that Γ satisfies the full Mal'tsev condition (viii) of Theorem 3.3. Therefore, an essentially algebraic theory Γ such that $\mathsf{Mod}(\Gamma)$ is regular satisfies $\mathcal{A} \vdash_{\mathsf{Mod}(\Gamma)} \mathcal{B}$ if and only if every object of $\mathsf{Mod}(\Gamma)$ has global support and Γ satisfies the restricted Mal'tsev condition described above. It remains thus to prove that there exists an extended matrix M such that such a Γ satisfies the above restricted Mal'tsev condition if and only if $\mathsf{Mod}(\Gamma)$ has M-closed relations.

In view of the above discussion, we can now assume that $\mathcal{K} = \varnothing$, i.e., $\mathcal{X}_{D_{\mathsf{X}}(K)} \neq \varnothing$ for all $K \in \mathcal{H}_{\mathsf{X}}$ and we shall prove that the Mal'tsev condition (viii) of Theorem 3.3 is equivalent, for some extended matrix M, to the Mal'tsev condition of Theorem 4.9. Firstly, if $\mathcal{H}_{\mathsf{X}} = \varnothing$, then, by Lemma 1.2, $\mathcal{H}_{\mathsf{Y}} = \varnothing$. Conversely, if $\mathcal{H}_{\mathsf{Y}} = \varnothing$, since $\mathcal{K} = \varnothing$, we must have $\mathcal{H}_{\mathsf{X}} = \varnothing$ by definition of the $\mathcal{X}_A$'s. Thus $\mathcal{H}_{\mathsf{X}} = \varnothing$ if and only if $\mathcal{H}_{\mathsf{Y}} = \varnothing$ and in this case, any Γ satisfies the Mal'tsev condition of Theorem 3.3. The result can then be obtained with the extended matrix of Example 4.4. We can thus also assume that $\mathcal{H}_{\mathsf{X}} \neq \varnothing$ and $\mathcal{H}_{\mathsf{Y}} \neq \varnothing$. In order to construct M, we denote the finite non-empty set of objects of $\mathcal{H}_{\mathsf{X}}$ as $\{K_1, \dots, K_{m'}\}$, the finite set of arrows of $\mathcal{H}_{\mathsf{X}}$ as $\{k_1, \dots, k_a\}$ and the finite non-empty set of objects of $\mathcal{H}_{\mathsf{Y}}$ as $\{H_1, \dots, H_b\}$. Moreover, for each $K_j \in \mathcal{H}_{\mathsf{X}}$, we denote the finite non-empty set $\mathcal{X}_{D_{\mathsf{X}}(K_j)}$ as $\{x^j_1, \dots, x^j_{l_j}\}$ and for each $H_i \in \mathcal{H}_{\mathsf{Y}}$, we denote the finite non-empty set $\mathcal{X}_{D_{\mathsf{Y}}(H_i)}$ as $\{x'^i_1, \dots, x'^i_{l'_i}\}$ where $x'^i_1 = [(H_i, 1_{D_{\mathsf{Y}}(H_i)})]$. We denote by $\varphi\colon \{1, \dots, b\} \to \{1, \dots, m'\}$ the function such that, for each $i \in \{1, \dots, b\}$, $K^{\mathsf{q}}_{H_i} = K_{\varphi(i)}$. For each $i \in \{1, \dots, a\}$ (i.e., for each arrow $k_i\colon K_j \to K_{j'} \in \mathcal{H}_{\mathsf{X}}$), the condition (viii)(a) of Theorem 3.3 gives, for each sort $s \in S$ and each $u \in U_s$, a theorem in Γ of the form

$$t^{s,u}_{K_j}\left(x^{j'}_{\theta_i(1)}, \dots, x^{j'}_{\theta_i(l_j)}\right) = t^{s,u}_{K_{j'}}\left(x^{j'}_1, \dots, x^{j'}_{l_{j'}}\right) \tag{8}$$

where $\theta_i\colon \{1, \dots, l_j\} \to \{1, \dots, l_{j'}\}$ is a function. In addition, for each $i \in \{1, \dots, b\}$ (i.e., for each object $H_i \in \mathcal{H}_{\mathsf{Y}}$), the condition (viii)(c) of Theorem 3.3 gives a theorem in Γ of the form

$$\pi^s\left((t^{s,u}_{K_{\varphi(i)}}(x'^i_{\chi_i(1)}, \dots, x'^i_{\chi_i(l_{\varphi(i)})}))_{u \in U_s}\right) = x'^i_1 \tag{9}$$

where $\chi_i\colon \{1, \dots, l_{\varphi(i)}\} \to \{1, \dots, l'_i\}$ is a function and where the left hand side is an everywhere-defined term $s^{l'_i} \to s$ by condition (viii)(b) of that theorem. If there exists an $i \in \{1, \dots, b\}$ such that 1 is not in the image of χ_i, the corresponding equation (9) implies that the equation $x = y$ holds for any pair of variables x, y of sort s, for each sort $s \in S$. Since the terms $t^{s,u}_{K_j}$ involved in the Mal'tsev condition (viii) of Theorem 3.3 are all not constant terms, this Mal'tsev condition is, in that case, actually equivalent to the condition that $x = y$ holds in any sort (to prove the converse implication, one can e.g. take π^s to be the identity term $s \to s$ and each $t^{s,u}_{K_j}$ to be the first projection). This latter Mal'tsev condition is in turn equivalent to the Mal'tsev condition of Theorem 4.9 for the extended matrix M of Example 4.6, which thus satisfies the required conditions. In view of this, we can also assume without loss of generality that 1 is in the image of χ_i for each $i \in \{1, \dots, b\}$. Hence, we can factorize χ_i as a surjective function $\chi'_i\colon \{1, \dots, l_{\varphi(i)}\} \to \{1, \dots, l''_i\}$ followed by an injection $\zeta_i\colon \{1, \dots, l''_i\} \to \{1, \dots, l'_i\}$ such that $\zeta_i(1) = 1$ and where $0 < l''_i \leqslant l'_i$. We set

$$n = 2a + b, \quad m = \max_{j \in \{1, \dots, m'\}} l_j,$$

$$l = \max\left(\max_{\substack{j' \in \{1, \dots, m'\} \\ \text{s.t. } \exists\, k_i\colon K_j \to K_{j'}}} l_{j'}\ ;\ \max_{i \in \{1, \dots, b\}} l''_i\right) \quad \text{and} \quad k = l + a(2m' - 1) + b(m' - 1).$$

The extended matrix M we are going to construct has n rows, m left columns, m' right columns, the set of variables in the left part is $\{x_1, \dots, x_l\}$ and the set of variables in the whole matrix is $\{x_1, \dots, x_k\}$. For each $i \in \{1, \dots, a\}$ (i.e., for each arrow $k_i\colon K_j \to K_{j'} \in \mathcal{H}_{\mathsf{X}}$), in view of the identity (8), we fix the $(2i-1)$-th row of M to be

$$M_{(2i-1)*} = \left[\begin{array}{ccccc|ccc} x_{\theta_i(1)} & x_{\theta_i(2)} & \cdots & x_{\theta_i(l_j)} & \cdots & x_{\theta_i(l_j)} & y_1 & \cdots & y_{m'} \end{array}\right]$$

where $y_r = x_{l+(i-1)(2m'-1)+r}$ for $r \in \{1, \ldots, m'\}$ and the $2i$-th row to be

$$M_{(2i)*} = \left[\begin{array}{cccccc|ccc} x_1 & x_2 & \cdots & x_{l_{j'}} & \cdots & x_{l_{j'}} & z_1 & \cdots & z_{m'} \end{array} \right]$$

where $z_r = x_{l+(i-1)(2m'-1)+m'+r}$ for $r \in \{1, \ldots, j'-1\}$, $z_{j'} = x_{l+(i-1)(2m'-1)+j}$ and $z_r = x_{l+(i-1)(2m'-1)+m'+r-1}$ for $r \in \{j'+1, \ldots, m'\}$. The idea is that the right hand side variables $y_1, \ldots, y_{m'}$ and $z_1, \ldots, z_{m'}$ are variables which have not been used in the previous rows and which are pairwise different except that $y_j = z_{j'}$. For each $i \in \{1, \ldots, b\}$ (i.e., for each object $H_i \in \mathcal{H}_{\mathsf{Y}}$), in view of the identity (9), we fix the $(2a+i)$-th row of M to be

$$M_{(2a+i)*} = \left[\begin{array}{cccccc|ccc} x_{\chi'_i(1)} & x_{\chi'_i(2)} & \cdots & x_{\chi'_i(l_{\varphi(i)})} & \cdots & x_{\chi'_i(l_{\varphi(i)})} & w_1 & \cdots & w_{m'} \end{array} \right]$$

where $w_r = x_{l+a(2m'-1)+(i-1)(m'-1)+r}$ for $r \in \{1, \ldots, \varphi(i)-1\}$, $w_{\varphi(i)} = x_1$ and $w_r = x_{l+a(2m'-1)+(i-1)(m'-1)+r-1}$ for $r \in \{\varphi(i)+1, \ldots, m'\}$. The idea is that the right hand side variables have not been used in the previous rows, except for $w_{\varphi(i)} = x_1$, and are pairwise distinct. This concludes the construction of the extended matrix M. It remains to prove that an essentially algebraic theory Γ for which $\mathsf{Mod}(\Gamma)$ is regular satisfies the Mal'tsev condition of Theorem 3.3 for $\mathcal{A} \vdash \mathcal{B}$ if and only if it satisfies the Mal'tsev condition of Theorem 4.9 for M.

Let us first suppose that $\mathsf{Mod}(\Gamma)$ has M-closed relations. By Theorem 4.9, we are given for each sort s of Γ, a term $\pi^s \colon \prod_{u \in U_s} s_u \to s$, for each $v \in \{1, \ldots, k\}$ and each $u \in U_s$ an everywhere-defined term $q_v^{s,u} \colon s^l \to s_u$ and for each $j \in \{1, \ldots, m'\}$ and each $u \in U_s$ an everywhere-defined term $p_j^{s,u} \colon s^m \to s_u$ satisfying conditions (a), (b) and (c) of that theorem. We now set, for such a j and such a u, the everywhere-defined term $t_{K_j}^{s,u} \colon s^{l_j} \to s_u$ as

$$t_{K_j}^{s,u}\left(x_1^j, x_2^j, \ldots, x_{l_j}^j\right) = p_j^{s,u}\left(x_1^j, x_2^j, \ldots, x_{l_j}^j, \ldots, x_{l_j}^j\right).$$

The condition (viii)(a) of Theorem 3.3, which has been re-written for each $k_i \colon K_j \to K_{j'} \in \mathcal{H}_{\mathsf{X}}$ in (8), is now satisfied since

$$\begin{aligned} t_{K_j}^{s,u}\left(x_{\theta_i(1)}, x_{\theta_i(2)}, \ldots, x_{\theta_i(l_j)}\right) &= p_j^{s,u}\left(x_{\theta_i(1)}, x_{\theta_i(2)}, \ldots, x_{\theta_i(l_j)}, \ldots, x_{\theta_i(l_j)}\right) \\ &= q_{l+(i-1)(2m'-1)+j}^{s,u}(x_1, \ldots, x_l) \\ &= p_{j'}^{s,u}\left(x_1, x_2, \ldots, x_{l_{j'}}, \ldots, x_{l_{j'}}\right) \\ &= t_{K_{j'}}^{s,u}\left(x_1, x_2, \ldots, x_{l_{j'}}\right) \end{aligned}$$

hold in Γ by applying the condition (a) of Theorem 4.9 with the $(2i-1)$-th and the $2i$-th row of M. Notice that these identities actually give the theorem

$$t_{K_j}^{s,u}\left(x_{\theta_i(1)}, x_{\theta_i(2)}, \ldots, x_{\theta_i(l_j)}\right) = t_{K_{j'}}^{s,u}\left(x_1, x_2, \ldots, x_{l_{j'}}\right)$$

involving terms $s^l \to s_u$; but since $l_{j'} \neq 0$, this also gives the same theorem where the terms are considered $s^{l_{j'}} \to s_u$. The term appearing in conditions (viii)(b) and (viii)(c) of Theorem 3.3 has been re-written for each $H_i \in \mathcal{H}_{\mathsf{Y}}$ in (9) as

$$\pi^s\left((t_{K_{\varphi(i)}}^{s,u}(x'^i_{\chi_i(1)}, x'^i_{\chi_i(2)}, \ldots, x'^i_{\chi_i(l_{\varphi(i)})}))_{u \in U_s}\right) \colon s^{l'_i} \to s$$

which, using the definition of $t_{K_{\varphi(i)}}^{s,u}$, is just

$$\pi^s\left((p_{\varphi(i)}^{s,u}(x'^i_{\chi_i(1)}, x'^i_{\chi_i(2)}, \ldots, x'^i_{\chi_i(l_{\varphi(i)})}, \ldots, x'^i_{\chi_i(l_{\varphi(i)})}))_{u \in U_s}\right).$$

By applying the condition (a) of Theorem 4.9 with the $(2a+i)$-th row of M, we know that

$$p^{s,u}_{\varphi(i)}\left(x_{\chi'_i(1)}, x_{\chi'_i(2)}, \dots, x_{\chi'_i(l_{\varphi(i)})}, \dots, x_{\chi'_i(l_{\varphi(i)})}\right) = q^{s,u}_1\,(x_1, \dots, x_l)$$

is a theorem of Γ for each $u \in U_s$. By conditions (b) and (c) of Theorem 4.9,

$$\pi^s\,((q^{s,u}_1(x_1, \dots, x_l))_{u \in U_s}) = x_1$$

is a theorem of Γ with the left hand side term being everywhere-defined. Therefore,

$$\pi^s\left((p^{s,u}_{\varphi(i)}(x_{\chi'_i(1)}, x_{\chi'_i(2)}, \dots, x_{\chi'_i(l_{\varphi(i)})}, \dots, x_{\chi'_i(l_{\varphi(i)})}))_{u \in U_s}\right) = x_1$$

is a theorem of Γ with the left hand side being an everywhere-defined term $s^l \to s$. Since $l''_i \neq 0$, the same holds if the left hand side is considered as a term $s^{l''_i} \to s$. Since $\chi_i = \zeta_i \circ \chi'_i$ and $\zeta_i(1) = 1$, replacing the variable x_c by $x'^i_{\zeta_i(c)}$ in the above theorem, we know that

$$\pi^s\left((p^{s,u}_{\varphi(i)}(x'^i_{\chi_i(1)}, x'^i_{\chi_i(2)}, \dots, x'^i_{\chi_i(l_{\varphi(i)})}, \dots, x'^i_{\chi_i(l_{\varphi(i)})}))_{u \in U_s}\right) = x'^i_1$$

is a theorem of Γ with the left hand side being an everywhere-defined term $s^{l'_i} \to s$, proving that the conditions (viii)(b) and (viii)(c) of Theorem 3.3 are satisfied.

Conversely, let us suppose that Γ satisfies the Mal'tsev condition of Theorem 3.3. We have thus, for each sort $s \in S$, a term $\pi^s\colon \prod_{u \in U_s} s_u \to s$ in Γ and for each $j \in \{1, \dots, m'\}$ and each $u \in U_s$, an everywhere-defined term $t^{s,u}_{K_j}\colon s^{l_j} \to s_u$ such that (8) is a theorem of Γ for each $u \in U_s$ and each $k_i\colon K_j \to K_{j'} \in \mathcal{H}_{\mathsf{X}}$ and the left hand side term of (9) is everywhere-defined and (9) is a theorem of Γ for each $H_i \in \mathcal{H}_{\mathsf{Y}}$. We consider the terms given by Theorem 3.1 for π^s, namely $\pi'^s\colon \prod_{u' \in U'_s} s'_{u'} \to s$ and for each $u' \in U'_s$, $\rho^s_{u'}\colon s \to s'_{u'}$ and $\tau^s_{u'}\colon \prod_{u \in U_s} s_u \to s'_{u'}$. We must show that Γ satisfies the Mal'tsev condition of Theorem 4.9. The first term in this Mal'tsev condition will be given, for an $s \in S$, by π'^s. Given $j \in \{1, \dots, m'\}$ and $u' \in U'_s$, we define the everywhere-defined term $p^{s,u'}_j\colon s^m \to s'_{u'}$ as

$$p^{s,u'}_j(x_1, \dots, x_{l_j}, \dots, x_m) = \tau^s_{u'}((t^{s,u}_{K_j}(x_1, \dots, x_{l_j}))_{u \in U_s}).$$

Given $v \in \{1, \ldots, k\}$ and $u' \in U'_s$, we define the everywhere-defined term $q_v^{s,u'} : s^l \to s'_{u'}$ as

$$q_v^{s,u'}(x_1, \ldots, x_l) = \begin{cases} \rho^s_{u'}(x_v) & \text{if } v \leqslant l, \\ p_r^{s,u'}(x_{\theta_i(1)}, x_{\theta_i(2)}, \ldots, x_{\theta_i(l_j)}, \ldots, x_{\theta_i(l_j)}) & \\ & \text{if } v = l + (i-1)(2m'-1) + r \\ & \text{for } i \in \{1, \ldots, a\},\ \mathsf{dom}(k_i) = K_j \\ & \text{and } r \in \{1, \ldots, m'\}, \\ p_r^{s,u'}(x_1, x_2, \ldots, x_{l_{j'}}, \ldots, x_{l_{j'}}) & \\ & \text{if } v = l + (i-1)(2m'-1) + m' + r \\ & \text{for } i \in \{1, \ldots, a\},\ \mathsf{codom}(k_i) = K_{j'} \\ & \text{and } r \in \{1, \ldots, j'-1\}, \\ p_r^{s,u'}(x_1, x_2, \ldots, x_{l_{j'}}, \ldots, x_{l_{j'}}) & \\ & \text{if } v = l + (i-1)(2m'-1) + m' + r - 1 \\ & \text{for } i \in \{1, \ldots, a\},\ \mathsf{codom}(k_i) = K_{j'} \\ & \text{and } r \in \{j'+1, \ldots, m'\}, \\ p_r^{s,u'}(x_{\chi'_i(1)}, x_{\chi'_i(2)}, \ldots, x_{\chi'_i(l_{\varphi(i)})}, \ldots, x_{\chi'_i(l_{\varphi(i)})}) & \\ & \text{if } v = l + a(2m'-1) + (i-1)(m'-1) + r \\ & \text{for } i \in \{1, \ldots, b\} \text{ and } r \in \{1, \ldots, \varphi(i)-1\}, \\ p_r^{s,u'}(x_{\chi'_i(1)}, x_{\chi'_i(2)}, \ldots, x_{\chi'_i(l_{\varphi(i)})}, \ldots, x_{\chi'_i(l_{\varphi(i)})}) & \\ & \text{if } v = l + a(2m'-1) + (i-1)(m'-1) + r - 1 \\ & \text{for } i \in \{1, \ldots, b\} \text{ and } r \in \{\varphi(i)+1, \ldots, m'\}. \end{cases}$$

The condition (a) of Theorem 4.9 is trivial for the row $M_{(2i-1)*}$ for each $i \in \{1, \ldots, a\}$. It is also trivial for the row $M_{(2i)*}$ except for the j'-th right column. In that case, we can prove it using (8) as follows:

$$\begin{aligned} p_{j'}^{s,u'}\left(x_1, x_2, \ldots, x_{l_{j'}}, \ldots, x_{l_{j'}}\right) &= \tau^s_{u'}\left((t^{s,u}_{K_{j'}}(x_1, \ldots, x_{l_{j'}}))_{u \in U_s}\right) \\ &= \tau^s_{u'}\left((t^{s,u}_{K_j}(x_{\theta_i(1)}, \ldots, x_{\theta_i(l_j)}))_{u \in U_s}\right) \\ &= p_j^{s,u'}\left(x_{\theta_i(1)}, x_{\theta_i(2)}, \ldots, x_{\theta_i(l_j)}, \ldots, x_{\theta_i(l_j)}\right) \\ &= q^{s,u'}_{l+(i-1)(2m'-1)+j}(x_1, \ldots, x_l) \end{aligned}$$

The condition (a) for the row $M_{(2a+i)*}$ for $i \in \{1, \ldots, b\}$ is also trivial except for the $\varphi(i)$-th right column. In that case, we prove it as follows where we use condition (c) of Theorem 3.1 and (9) with the variable $x'^i_{\zeta_i(c)}$ replaced by x_c, which is allowed since ζ_i is injective:

$$\begin{aligned} & p_{\varphi(i)}^{s,u'}\left(x_{\chi'_i(1)}, x_{\chi'_i(2)}, \ldots, x_{\chi'_i(l_{\varphi(i)})}, \ldots, x_{\chi'_i(l_{\varphi(i)})}\right) \\ &= \tau^s_{u'}\left((t^{s,u}_{K_{\varphi(i)}}(x_{\chi'_i(1)}, \ldots, x_{\chi'_i(l_{\varphi(i)})}))_{u \in U_s}\right) \\ &= \rho^s_{u'}\left(\pi^s\left((t^{s,u}_{K_{\varphi(i)}}(x_{\chi'_i(1)}, \ldots, x_{\chi'_i(l_{\varphi(i)})}))_{u \in U_s}\right)\right) \\ &= \rho^s_{u'}(x_1) \\ &= q_1^{s,u'}(x_1, \ldots, x_l) \end{aligned}$$

The conditions (b) and (c) of Theorem 4.9 immediately follow from the fact that, for each $v \in \{1,\ldots,l\}$, the term

$$\pi'^s\left((q_v^{s,u'}(x_1,\ldots,x_l))_{u'\in U'_s}\right) = \pi'^s\left((\rho^s_{u'}(x_v))_{u'\in U'_s}\right)$$

is everywhere-defined and equal to x_v by conditions (a) and (b) of Theorem 3.1. $\square$

Theorem 4.11 shows that any condition $\mathcal{A} \vdash_{\mathbb{C}} \mathcal{B}$ on a regular category $\mathbb{C}$ for $\mathcal{A} \vdash \mathcal{B}$ an exactness sequent of Type I can equivalently be stated as the conjunction of one or two matrix properties. To find the associated matrices, one can follow the following algorithm:

1. Identify the diagrams $D_{\mathsf{X}} \colon \mathcal{H}_{\mathsf{X}} \to \mathcal{U}(\mathbb{A})$ and $D_{\mathsf{Y}} \colon \mathcal{H}_{\mathsf{Y}} \to \mathcal{U}(\mathbb{A})$ together with the objects $K_H^{\mathsf{q}} \in \mathcal{H}_{\mathsf{X}}$ and the morphisms $c_H^{\mathsf{q}} \in \mathbb{A}$ as inductively constructed in Lemma 1.2.

2. If some K_H^{q} is in the same connected component in $\mathcal{H}_{\mathsf{X}}$ as an object K for which $\mathbb{A}(D_{\mathsf{Y}}(H), D_{\mathsf{X}}(K)) = \varnothing$ for all $H \in \mathcal{H}_{\mathsf{Y}}$, then we just need the two matrices

$$\left[\begin{array}{c|c} & x_1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} x_1 & x_2 & x_1 \\ x_1 & x_2 & x_2 \end{array}\right]$$

   of Examples 4.5 and 4.6.

3. Otherwise, write down the Mal'tsev condition associated to $\mathcal{A} \vdash \mathcal{B}$ as described in Theorem 3.3.

4. If there exists a $K \in \mathcal{H}_{\mathsf{X}}$ such that $\mathbb{A}(D_{\mathsf{Y}}(H), D_{\mathsf{X}}(K)) = \varnothing$ for all $H \in \mathcal{H}_{\mathsf{Y}}$, we need the matrix $\left[\begin{array}{c|c} & x_1 \end{array}\right]$ of Example 4.5. Then, remove in the Mal'tsev condition, all terms $t_K^{s,u}$ (and all identities involving them) for any K connected in $\mathcal{H}_{\mathsf{X}}$ to any K' such that $\mathbb{A}(D_{\mathsf{Y}}(H), D_{\mathsf{X}}(K')) = \varnothing$ for all $H \in \mathcal{H}_{\mathsf{Y}}$. In that case, we need a second matrix as described in the next step.

5. If no terms $t_K^{s,u}$ are left in the Mal'tsev condition, we need the matrix $\left[\begin{array}{c|c} x_1 & x_1 \end{array}\right]$ of Example 4.4. If the Mal'tsev condition contains an identity of the form

$$\pi^s\left((t^{s,u}_{K_H^{\mathsf{q}}}(x_{\chi(1)},\ldots,x_{\chi(\#\mathcal{X}_{D_{\mathsf{X}}(K_H^{\mathsf{q}})})}))_{u\in U_s}\right) = x_1$$

   where 1 is not in the image of χ, then we need the matrix

$$\left[\begin{array}{cc|c} x_1 & x_2 & x_1 \\ x_1 & x_2 & x_2 \end{array}\right]$$

   of Example 4.6. Otherwise, we need the matrix M as constructed in the proof of Theorem 4.11.

References

[1] J. Adámek, H. Herrlich and J. Rosický, Essentially equational categories, *Cah. Topol. Géom. Différ. Catég.* **29** (1988), 175–192.

[2] J. Adámek and J. Rosický, Locally presentable and accessible categories, *London Math. Soc.* **189** (1994).

[3] M. Artin, A. Grothendieck and J.L. Verdier, Séminaire de géométrie algébrique du Bois Marie 1963–1964, Théorie des topos et cohomologie étale des schémas (SGA4), *Springer Lect. Notes Math.* **269** (1972).

[4] M. Barr, P.A. Grillet and D.H. van Osdol, Exact categories and categories of sheaves, *Springer Lect. Notes Math.* **236** (1971).

[5] M. Barr and C. Wells, Toposes, triples and theories, *Grundlehren der Mathematischen Wissenschaften* **278**, Springer (1985).

[6] F. Borceux, Handbook of Categorical Algebra 1, *Encycl. Math. Appl.* **50**, Cambridge Uni. Press (1994).

[7] F. Borceux and D. Bourn, Mal'cev, Protomodular, Homological and Semi-Abelian Categories, *Math. Appl.* **566**, Kluwer (2004).

[8] D. Bourn, Mal'cev categories and fibration of pointed objects, *Appl. Categ. Struct.* **4** (1996), 307–327.

[9] D. Bourn and G. Janelidze, Characterization of protomodular varieties of universal algebras, *Theory Appl. Categ.* **11** (2003), 143–147.

[10] D. Bourn and Z. Janelidze, Approximate Mal'tsev operations, *Theory Appl. Categ.* **21** (2008), 152–171.

[11] D.A. Buchsbaum, Exact categories and duality, *Trans. Am. Math. Soc.* **80** (1955), 1–34.

[12] P. Burmeister, A model theoretic oriented approach to partial algebras; Part I of Introduction to theory and application of partial algebras, *Math. Res.* **32**, Akademie-Verlag (1986).

[13] A. Carboni, G.M. Kelly and M.C. Pedicchio, Some remarks on Maltsev and Goursat categories, *Appl. Categ. Struct.* **1** (1993), 385–421.

[14] A. Carboni, J. Lambek and M.C. Pedicchio, Diagram chasing in Mal'cev categories, *J. Pure Appl. Algebra* **69** (1990), 271–284.

[15] A. Carboni, M.C. Pedicchio and N. Pirovano, Internal graphs and internal groupoids in Mal'tsev categories, *Proc. Conf. Montreal 1991* (1992), 97–109.

[16] A. Carboni, M.C. Pedicchio and J. Rosický, Syntactic characterizations of various classes of locally presentable categories, *J. Pure Appl. Algebra* **161** (2001), 65–90.

[17] I. Chajda, G. Eigenthaler and H. Länger, Congruence Classes in Universal Algebra, *Research Expositions in Mathematics* **26**, Heldermann Verlag, Berlin (2003).

[18] C. Ehresmann, Esquisses et types des structures algébriques, *Bul. Inst. Politehn. Iasi* **14** (1968), 1–14.

[19] P. Freyd, Abelian categories, *Harper & Row*, New York (1964).

[20] P. Freyd, Aspects of topoi, *Bull. Austral. Math. Soc.* **7** (1972), 1–76.

[21] P. Gabriel and P. Ulmer, Lokal präsentierbare kategorien, *Springer Lect. Notes Math.* **221** (1971).

[22] A. Grothendieck, Sur quelques points d'algèbre homologique, *Tôhoku Math. J.* **9** (1957), 119–221.

[23] J. Hagemann and A. Mitschke, On n-permutable congruences, *Algebra Univers.* **3** (1973), 8–12.

[24] M.A. Hoefnagel, Majority categories, *Theory Appl. Categ.* **34** (2019), 249–268.

[25] M.A. Hoefnagel, Z. Janelidze and D. Rodelo, On difunctionality of class relations, *Algebra Univers.* **81** (2020), Paper No. 19.

[26] P.-A. Jacqmin, Embedding theorems in non-abelian categorical algebra, *PhD thesis, Université catholique de Louvain* (2016).

[27] P.-A. Jacqmin, An embedding theorem for regular Mal'tsev categories, *J. Pure Appl. Algebra* **222** (2018), 1049–1068.

[28] P.-A. Jacqmin, Partial algebras and embedding theorems for (weakly) Mal'tsev categories and matrix conditions, *Cah. Topol. Géom. Differ. Catég.* **60** (2019), 365–403.

[29] P.-A. Jacqmin, Embedding theorems for Janelidze's matrix conditions, *J. Pure Appl. Algebra* **224** (2020), 469–506.

[30] P.-A. Jacqmin and Z. Janelidze, On stability of exactness properties under the pro-completion, *Advances in Mathematics* **377** (2021), 107484.

[31] Z. Janelidze, Varieties of universal algebras with normal local projections, *Georgian Math. J.* **11** (2004), 93–98.

[32] Z. Janelidze, Subtractive categories, *Appl. Categ. Struct.* **13** (2005), 343–350.

[33] Z. Janelidze, Closedness properties of internal relations I: A unified approach to Mal'tsev, unital and subtractive categories, *Theory Appl. Categ.* **16** (2006), 236–261.

[34] Z. Janelidze, Closedness properties of internal relations II: Bourn localization, *Theory Appl. Categ.* **16** (2006), 262–282.

[35] Z. Janelidze, Closedness properties of internal relations V: Linear Mal'tsev conditions, *Algebra Univers.* **58** (2008), 105–117.

[36] Z. Janelidze, Closedness properties of internal relations VI: Approximate operations, *Cah. Topol. Géom. Differ. Catég.* **50** (2009), 298–319.

[37] B. Jónsson, Algebras whose congruence lattices are distributive, *Math. Scand.* **21** (1967), 110–121.

[38] S. Mac Lane, Duality for groups, *Bull. Am. Math. Soc* **56** (1950), 485–516.

[39] A.I. Mal'tsev, On the general theory of algebraic systems, *Mat. Sbornik* **35** (1954), 3–20 (in Russian); English translation: *Amer. Math. Soc. Trans.* **27** (1963), 125–142.

[40] M. Makkai and R. Paré, Accessible categories: the foundations of categorical model theory, *Contemporary Mathematics* **104** (1989), American Mathematical Society.

[41] M.C. Pedicchio, Arithmetical categories and commutator theory, *Appl. Categ. Structures* **4** (1996), 297–305.

[42] A.F. Pixley, Distributivity and permutability of congruence relations in equational classes of algebras, *Proc. Amer. Math. Soc.* **14** (1963), 105–109.

[43] H. Reichel, Structural induction on partial algebras; Part II of Introduction to theory and application of partial algebras, *Math. Res.* **18**, Akademie-Verlag (1984).

[44] J.W. Snow, Maltsev conditions and relations on algebras, *Algebra Univers.* **42** (1999), 299–309.

[45] W. Taylor, Characterizing Mal'cev conditions, *Algebra Univers.* **3** (1973), 351–397.

[46] A. Ursini, On subtractive varieties, I, *Algebra Univers.* **31** (1994), 204–222.

Email: pierre-alain.jacqmin@uclouvain.be
zurab@sun.ac.za