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Embedding Theorems in Non-Abelian Categorical Algebra

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Table of contents

Introduction

With their paper [37] in 1945, S. Eilenberg and S. Mac Lane brought a new way to understand algebra and mathematics in general. These were the early days of category theory. While classical mathematics studies sets equipped with structures (groups, monoids, sets, topological spaces, manifolds, Banach spaces, . . .) and their elementwise homomorphisms, category theory is concerned with abstract morphisms and their composition.

In this elementless approach to mathematics, one is no longer interested in explicit descriptions of basic constructions but aim to provide universal properties determining them. As a standard example, one can cite the product of two sets A and B. While classical mathematics describes $A \times B$ as the cartesian product $\{(a,b) \,|\, a \in A, b \in B\}$, category theory defines it as the unique (up to isomorphism) object equipped with morphisms

$$\begin{tikzcd}
 & A \times B \arrow[ld, "\pi_A"'] \arrow[rd, "\pi_B"] & \\
A & & B
\end{tikzcd}$$

such that any such span factorises uniquely through it. As simply as the above universal property, one can express much more evolved classical mathematical constructions. For instance, the free product of two groups G and H is categorically described as the unique (up to isomorphism) pair of group homomorphisms

$$\begin{tikzcd}
G \arrow[dr, "\iota_G"'] & & H \arrow[dl, "\iota_H"] \\
& G+H &
\end{tikzcd}$$

through which any such cospan factors uniquely, whereas its classical definition using elements is quite long and much more complicated.

The similarity between the two universal properties described above is striking. It is made precise by a crucial concept illuminated by category theory: duality. The simple fact that one can reverse arrows of a category to get a new category has powerful consequences. If one proves a statement for all categories, its dual happens to be also true for every category. As obvious as it seems to be, this duality principle would not come to someone's mind without a categorical way of thinking.

Moreover, the high level of generality offered by category theory enables one to unify concepts which apparently look very different. For example, there is now a way to think of the free product of two groups, the supremum of two elements in a poset, the disjoint union of two sets or the direct sum of two abelian groups as different instances of the same construction: these are all binary coproducts.

This elegant and fundamental way of doing mathematics has an apparent cost: proofs can be much longer in this language. These 'difficulties' can be overcome using the technique of generalised elements. If we fix an object X in a category $\mathcal{C}$, each object A of $\mathcal{C}$ can be thought to have $\mathcal{C}(X,A)$ as underlying set. Each morphism $f\colon A \to B$ now becomes a function $- \circ f\colon \mathcal{C}(X,A) \to \mathcal{C}(X,B)$. With this analogy, the categorical product $A \times B$ corresponds to the cartesian product $\mathcal{C}(X, A\times B) \cong \mathcal{C}(X,A)\times\mathcal{C}(X,B)$. This can be formalised via the Yoneda embedding. Each small category $\mathcal{C}$ admits a full and faithful embedding

$$\begin{aligned} Y\colon \mathcal{C} &\hookrightarrow \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \\ A &\longmapsto \mathcal{C}(-,A) \end{aligned}$$

where Set denotes the category of sets. Moreover, this embedding preserves limits. With that in mind, many categorical statements about small limits can be reduced to be shown in powers of Set to prove they hold in every category. Since small limits are computed componentwise in functor categories $\mathrm{Set}^{\mathcal{P}}$, it suffices to prove these statements in Set, which is much easier in view of the description of its small limits.

It would be a mistake to think that category theory then comes down to the study of Set. On one hand, the Yoneda embedding does not preserve colimits. Even if statements about colimits follow by duality from the corresponding properties of limits, there is no way to study interactions between limits and colimits restricting our attention to the category Set via the Yoneda embedding. On the other hand, this technique does not provide a satisfactory way to prove and understand (limit) categorical statements which hold only for a given class of categories.

Such an important class of categories has been introduced in the fifties [81, 25, 52] and is known nowadays as abelian categories. This notion captures the categorical properties of abelian groups (or more generally of modules over a ring) and is considered as one of the first categorical algebraic properties ever studied. It has been considered for a long time as the right level of generality in which to develop homological algebra. In addition of its own duality principle (the dual of any abelian category is abelian), finite limits are very close to finite colimits in abelian categories (e.g. the binary product $A \times B$ is isomorphic to the coproduct $A + B$). One can then hope having an embedding theorem for abelian categories preserving both finite limits and finite colimits. Moreover, the hom-sets $\mathcal{C}(A, B)$ in an abelian category are equipped with an abelian group structure, so that one can replace the category Set with the category Ab of abelian groups in Yoneda's embedding theorem. Using these ideas, Lubkin showed in [80] that each small abelian category $\mathcal{C}$ admits a faithful conservative embedding $\mathcal{C} \hookrightarrow \mathrm{Ab}$ which preserves finite limits and finite colimits. Besides, Mitchell constructed for a small abelian category $\mathcal{C}$, a ring R and a full and faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Mod}_R$ which preserves finite limits and finite colimits (where Mod_R is the category of right R-modules), see [93, 42].

On the logic side, Barr [9] introduced regular categories as finitely complete categories with coequalisers of kernel pairs and pullback stable regular epimorphisms. The idea behind this definition is that 'regular categories are categories in which regular epimorphisms and finite limits act as they do in Set'. This idea has been made precise in [10] where he proved that each small regular category admits a fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Set}^{\mathcal{P}}$ in a power of the category of sets which preserves finite limits and regular epimorphisms. One can thus reduce the proof of many statements about finite limits and regular epimorphisms in an arbitrary regular category to the particular case of Set. For such applications, one does not need fullness of such an embedding, but only its faithfulness and the fact it reflects isomorphisms (i.e., its conservativeness). For that reason, by an embedding, we mean here a faithful conservative functor. One can thus also use Z. Janelidze's variant of Barr's embedding theorem in which fullness is weakened to conservativeness but the category $\mathcal{P}$ is now known to be the discrete category (i.e., set) of subobjects of the terminal object in $\mathcal{C}$. In particular, it vanishes if the embedded category is pointed.

The notion of an abelian category separates the categories Ab and Mod_R from other algebraic categories as Set, Mon (the category of monoids) and Gp (the category of groups). One can then ask how to distinguish these latter categories. Or in other words, which categorical properties does Gp have that Mon or Set do not? This question, which goes back to [81], is the essence of non-abelian categorical algebra. In some sense, the notion of a regular category does not provide a satisfactory answer because every 'algebraic category' is regular. In the nineties, many such exactness properties were introduced, which brought a new life for non-abelian categorical algebra. We refer the reader to the introduction of [62] for more historical developments and references on the subject.

Mal'tsev categories were defined in [29] (as a generalisation of regular Mal'tsev categories from [28]) as finitely complete categories in which each binary relation is difunctional. It is equivalent to the condition that each reflexive relation is an equivalence relation. In a regular context, this is further equivalent to the condition that the composition of equivalence relations R and S on a same object is commutative: $R \circ S = S \circ R$. Their name comes from the mathematician Mal'tsev who characterised [87] (one-sorted finitary) algebraic categories in which this last property holds as the ones whose corresponding theory has a ternary operation $p(x, y, z)$ satisfying the axioms $p(x, y, y) = x = p(y, y, x)$. In addition to abelian categories, Gp is a Mal'tsev category in view of the term $p(x, y, z) = xy^{-1}z$, while Set and Mon are not.

As another example of a non-abelian categorical algebraic property, one can mention unital categories introduced by Bourn [18] as pointed finitely complete categories in which the cospan

$$\begin{tikzcd} X \arrow[r, "(1_X{,}0)"] & X \times Y & Y \arrow[l, "(0{,}1_Y)"'] \end{tikzcd}$$

is jointly strongly epimorphic for each pair of objects X and Y. Unital (one-sorted finitary) algebraic categories are characterised by the presence in the theory of a unique constant term 0 and a Jónsson-Tarski operation, i.e., a binary operation $u(x, y)$ such that $u(x, 0) = x = u(0, x)$. This property now separates Mon and Gp which are unital from the category of pointed sets Set_* that is pointed but not unital.

As a last example, let us cite protomodular categories, also introduced by Bourn [17] as categories in which a non-pointed version of the Split Short Five Lemma holds. Due to this notion, one can define homological (pointed regular protomodular [15]) and semi-abelian (homological exact with binary coproducts [62]) categories providing more general contexts than abelian categories in which to develop homological algebra.

Up to now, there were no embedding theorems for all those non-abelian algebraic categorical properties. The main aim of this thesis is to provide such embedding theorems.

Of course, to have a 'good' embedding theorem for a class of categories, one has to first find a 'representative' in that class and then embed each category belonging to this class in (a power of) this chosen representative. The embedding should moreover preserve and reflect the important objects involved in the definition of the property characterising those categories (like finite limits or regular epimorphisms). The first major problem one encounters to prove such a theorem, e.g., for Mal'tsev categories, is the fact that, for a Mal'tsev category $\mathcal{C}$, hom-sets

$\mathcal{C}(X,Y)$ are not equipped with a Mal'tsev operation $p(x,y,z)$. Since we use the Yoneda embedding to prove embedding theorems, it then seems hard to find a particular Mal'tsev category in which every other embeds. This problem was 'approximatively' solved by Bourn and Z. Janelidze in [21] in a regular context. More precisely, they showed that in a regular Mal'tsev category with binary coproducts, considering the pullback square below

$$\begin{tikzcd}
W(X) \arrow[r, "p^X"] \arrow[d, "d^X"'] \arrow[dr, phantom, very near start, "\lrcorner"] & 3X \arrow[d, "\left(\begin{smallmatrix} \iota_1 & \iota_1 \\ \iota_2 & \iota_1 \\ \iota_2 & \iota_2 \end{smallmatrix}\right)"] \\
X \arrow[r, "{(\iota_1,\iota_2)}"'] & (2X)^2
\end{tikzcd}$$

where $\iota_1, \iota_2 \colon X \to 2X$ are the coproduct injections, the morphism d^X is a regular epimorphism. This does not provide $\mathcal{C}(X,Y)$ with a Mal'tsev operation but only with an approximate one. Indeed, p^X induces an operation

$$\begin{aligned} \rho \colon \mathcal{C}(X,Y)^3 &\longrightarrow \mathcal{C}(W(X),Y) \\ (f,g,h) &\longmapsto \left(\begin{smallmatrix} f \\ g \\ h \end{smallmatrix}\right) p^X \end{aligned}$$

while d^X induces an injection (an approximation)

$$\begin{aligned} \alpha \colon \mathcal{C}(X,Y) &\hookrightarrow \mathcal{C}(W(X),Y) \\ f &\longmapsto f d^X. \end{aligned}$$

This operation ρ satisfies Mal'tsev axioms up to the approximation α, i.e., $\rho(x,y,y) = \alpha(x) = \rho(y,y,x)$.

This was an important step towards an embedding theorem for regular Mal'tsev categories. One can then think the representative category one has to choose is that of approximate Mal'tsev algebras, i.e., pairs of sets A, B together with an approximation $\alpha \colon A \hookrightarrow B$ and an operation $\rho \colon A^3 \to B$ satisfying the Mal'tsev axioms up to the approximation α. However, this category, containing Set, is not a Mal'tsev category. It can therefore not be considered as the 'representative regular Mal'tsev category'.

To solve this problem, we use in this thesis essentially algebraic categories from [4, 3]. As in the algebraic case, objects in these categories are S-sorted sets $A \in \mathrm{Set}^S$ (for a fixed set S) endowed with operations $\prod_{i \in I} A_{s_i} \to A_s$ satisfying some given equations. The difference is that some of these operations can be only partially defined (and defined exactly for I-tuples satisfying some totally defined equations). Since this is a central notion of this thesis, we devote Chapter 1 recalling it in detail. In addition, this chapter exposes different treatments of universal algebra as Birkhoff's approach [13], Lawvere theories [76], monadic categories [38], locally presentable categories [44] (which are nothing else but essentially algebraic categories). Contrary to algebraic categories, essentially algebraic categories are not regular in general and we give a syntactic characterisation of the ones which are. By 'syntactic', we mean here a characterisation in terms of operations and axioms. In the same way it is done for algebraic categories [40], we also describe $\mathcal{T}$-enrichments of an essentially algebraic category for a Lawvere theory $\mathcal{T}$.

Another issue with using approximate co-operations is the assumption about the existence of coproducts. Since we do not want to require such an assumption in our embedding theorem, we first need to embed each (small) regular Mal'tsev category in a regular Mal'tsev category with binary coproducts. This can be achieved using the free cofiltered limit completion of $\mathcal{C}$, given by the Yoneda embedding

$$i \colon \mathcal{C} \hookrightarrow \mathrm{Lex}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}} = \widetilde{\mathcal{C}}$$

where $\mathrm{Lex}(\mathcal{C}, \mathrm{Set})$ denotes the category of finite limit preserving functors $\mathcal{C} \to \mathrm{Set}$. This category $\widetilde{\mathcal{C}}$ is complete, cocomplete and i preserves colimits and finite limits. Moreover, we describe in Chapter 3 some exactness properties preserved under this completion, in the sense that if $\mathcal{C}$ satisfies them, then so does $\widetilde{\mathcal{C}}$. These properties are called 'unconditional exactness properties' since they are of the form: given a diagram of a fixed finite shape in $\mathcal{C}$, if we build some finite (co)limits from it, then some specified finite (co)cones are also (co)limits. As examples of such exactness properties, we have: being pointed, regular, normal, regular Mal'tsev, regular unital, linear, additive, semi-abelian, abelian and so forth. In this way, we thus have embedded each small regular Mal'tsev category $\mathcal{C}$ in the cocomplete regular Mal'tsev category $\widetilde{\mathcal{C}}$. Moreover, Barr showed as a preliminary step of his embedding theorem [10] that if $\mathcal{C}$ is a small regular category, then $\widetilde{\mathcal{C}}$ (which is also regular) admits a $\mathcal{C}$-projective covering. This is used to prove that the embedding preserves regular epimorphisms and we thus also need to use it. This result goes back to Grothendieck [52] in the abelian case. Grothendieck actually constructed a $\mathcal{C}$-projective covering in a functorial way which Barr did not. To complete this result we prove functoriality in the regular case in Section 4.2.

Preservation of unconditional exactness properties under the free cofiltered limit completion has another application. If one wants to show the statement $P \Rightarrow Q$ holds in a finitely complete category $\mathcal{C}$ where P is an unconditional exactness property, it is often allowed to suppose that $\mathcal{C}$ has some colimits, without loss of generality. Indeed, if $\mathcal{C}$ satisfies P, then so does $\widetilde{\mathcal{C}}$. We then prove that $\widetilde{\mathcal{C}}$ satisfies also Q which, depending on the nature of Q, often implies that $\mathcal{C}$ does. Such an application is given in detail in Chapter 3. In addition, to be able to use this, we first need to decide whether some category generated by a finite conditional graph is finite. An algorithm to do so is presented also in Chapter 3.

Once this preliminary work is done, we set out to prove an embedding theorem for regular Mal'tsev categories. We first need to construct our representative regular Mal'tsev category $\mathcal{M}$. As announced earlier, this is an essentially algebraic category. In Chapter 2, we characterise those categories which are Mal'tsev as the ones whose theory contains, for each sort $s \in S$, a term $p\colon s^3 \to s$ such that both terms $p(x,y,y)$ and $p(y,y,x)$ are defined in any model A for any $x,y \in A_s$ and satisfying the usual axioms $p(x,y,y) = x = p(y,y,x)$. In view of the syntactical characterisation of regularity, our representative regular Mal'tsev essentially algebraic category $\mathcal{M}$ is quite technical to construct. However, in order to use it in practise, the only thing one has to remember is that regular epimorphisms behave well, and for each sort s, we have totally defined operations ρ^s, α^s and a partial operation π^s as picture below

$$\begin{tikzcd}
s^3 \arrow[r, "\rho^s"] & (s,0) \arrow[dl, dotted, bend left, "\pi^s"] \\
s \arrow[ur, "\alpha^s"] &
\end{tikzcd}$$

such that $\pi^s(\alpha^s(x))$ is everywhere-defined, and satisfying the axioms $\rho^s(x,y,y) = \alpha^s(x) = \rho^s(y,y,x)$ and $\pi^s(\alpha^s(x)) = x$. In particular α^s is injective and it thus gives rise to an approximate Mal'tsev algebra. Of course, the sort $(s,0)$ also has its own operation $\rho^{(s,0)}$ which does not necessarily agree with ρ^s on s^3. This construction of our representative regular Mal'tsev category $\mathcal{M}$ is done in Chapter 4. Once it is constructed, we prove that each small regular Mal'tsev category $\mathcal{C}$ admits a faithful embedding $\mathcal{C} \hookrightarrow \mathcal{M}^{\mathrm{Sub}(1)}$ which preserves and reflects finite limits, isomorphisms and regular epimorphisms (Sub(1) being the set of subobjects of the terminal object of $\mathcal{C}$). Due to this embedding theorem, for many statements about finite limits and regular epimorphisms in regular Mal'tsev categories, it is now equivalent to prove them only in $\mathcal{M}$ using elements and approximate Mal'tsev operations. Moreover, in practise, a proof in the algebraic category of sets equipped with a Mal'tsev operation p can often be translated into a proof in $\mathcal{M}$.

The different techniques and the embedding theorem described above for regular Mal'tsev categories can be immediately transposed to the case of regular unital, regular strongly unital, regular subtractive and n-permutable categories. In order to prove them for all those properties at the same time, we use their general treatment using matrices developed by Z. Janelidze in [64, 67] and recalled in Chapter 2. There, for a fixed matrix condition, we syntactically characterise essentially algebraic categories satisfying it. We also extend the result concerning approximate co-operations (as in the Mal'tsev case explained above) from the simple matrix conditions proved in [68] to the more general matrices of the form developed in [67]. In Chapter 4, we construct our corresponding representative essentially algebraic category and prove its embedding theorem. As applications of those embedding theorems, concrete examples of proofs using elements and operations are given. We also show that these matrix conditions are preserved under the exact completion of a regular category [77, 98, 73], giving rise to analogous embedding theorems in the exact context.

A thesis on embedding theorems for non-abelian algebraic categorical properties could not be complete without a word on protomodularity [17]. For now, there are no matrix conditions inducing protomodularity, and we thus treat this case separately in Section 4.4. Similarly, we give a syntactic characterisation of protomodular essentially algebraic categories and construct our representing such. As it is done for matrix conditions, we also have a corresponding embedding theorem for regular protomodular categories and homological categories. Since these conditions are not known to be unconditional exactness properties as such, we need to further assume that the categories we want to embed have binary coproducts. Again, using the exact completion of a regular category, we get an embedding theorem for semi-abelian categories.

Weakly Mal'tsev categories have been introduced by Martins-Ferreira in [91]. While Mal'tsev categories are characterised [18] as finitely complete categories in which, for all pullbacks of split epimorphisms,

$$\begin{tikzcd}
P \arrow[r, two heads, shift right] \arrow[d, two heads, shift left] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, bend right, "r_Y"'] \arrow[d, two heads, shift right] \\
X \arrow[u, bend left, "l_X"] \arrow[r, two heads, shift left] & Z \arrow[l, bend left] \arrow[u, bend right]
\end{tikzcd}$$

the induced morphisms l_X and r_Y are jointly strongly epimorphic, weakly Mal'tsev categories are defined by the condition that l_X and r_Y are jointly epimorphic. As an example, we show in Section 4.5 that the category of sets equipped with a partial Mal'tsev operation $p(x, y, z)$ (i.e., at least $p(x, y, y)$ and $p(y, y, x)$ are defined and equal to x) is weakly Mal'tsev. We then prove that each small weakly Mal'tsev category admits a full and faithful embedding in a power of this category of partial Mal'tsev algebras which preserves and reflects finite limits. The main difference with the essentially algebraic regular Mal'tsev category $\mathcal{M}$ is that, now, monomorphisms do not reflect triples in which p is defined. This means that, if f is a monomorphism of partial Mal'tsev algebras, $p(f(x), f(y), f(z))$ might be defined whereas $p(x, y, z)$ is not. This phenomenon does not occur with essentially algebraic categories since the domain of definition of partial operations is the solution set of some totally defined equations. Again, in order to encompass more examples such as weakly unital categories [90], we do this in the context of simple matrix conditions.

As we said, we present in this work embedding theorems for regular Mal'tsev categories and weakly Mal'tsev categories. One can wonder if these techniques could lead to such a theorem for Mal'tsev categories which lie in between. In our opinion, it seems there is a major obstruction to this. Indeed, being regular Mal'tsev can be characterised, using coproducts, by the condition that $d^X \colon W(X) \to X$ defined above is a regular epimorphism. In other words, if an element $f \in \mathcal{C}(W(X), A)$ satisfies $fr_1 = fr_2$ where (r_1, r_2) is the kernel pair of d^X, there is a unique element $g \in \mathcal{C}(X, A)$ such that $gd^X = f$. Besides, weakly Mal'tsev categories are characterised by the condition that l_X and r_Y as above are jointly epimorphic. This means that for any elements $f, g \in \mathcal{C}(P, A)$, if $fl_X = gl_X$ and $fr_Y = gr_Y$, then $f = g$. These two properties look very 'algebraic' and can easily be expressed in terms of generalised elements. On the contrary, to characterise Mal'tsev categories we need that some morphism(s) are (jointly) strongly epimorphic. Since it requires to quantify over all monomorphisms in the category, this seems very hard to state via elements of hom-sets and does not look algebraic any more. Maybe in some sense 'being a Mal'tsev category' is not an algebraic property while 'being regular Mal'tsev' and 'being weakly Mal'tsev' are.

The last chapter of this thesis is not immediately related with embedding theorems. There, we describe the bicategory of fractions with respect to weak equivalences between internal groupoids in $\mathcal{C}^{\mathbb{T}}$ for a monad $\mathbb{T}$ on a regular category $\mathcal{C}$ where the Axiom of Choice holds. Weak equivalences between internal groupoids in a regular category $\mathcal{D}$ are essentially surjective full and faithful functors. If the Axiom of Choice holds (i.e., every regular epimorphism splits) in $\mathcal{D}$, these are exactly the equivalences. In general this is a weaker notion and the bicategory of fractions for them is the universal solution for the problem to find a pseudo-functor $\mathrm{Grpd}(\mathcal{D}) \to \mathcal{B}$ that sends weak equivalences to equivalences. In the case where $\mathcal{D} = \mathrm{Gp}$, it suffices to define $\mathcal{B}$ as the 2-category of internal groupoids in Gp and monoidal functors [102]. A similar description is also given in [102] for the case of Lie algebras. We generalise those two cases in Chapter 5 defining $\mathbb{T}$-monoidal functors between internal groupoids in the Eilenberg-Moore category $\mathcal{C}^{\mathbb{T}}$ where $\mathbb{T}$ is a monad on $\mathcal{C}$. If $\mathcal{C}$ is a regular category where the Axiom of Choice holds, this gives the bicategory of fractions for weak equivalences in Grpd($\mathcal{C}^{\mathbb{T}}$).

Chapter 1

Different approaches to universal algebra

Universal algebra can be seen as the study of categories of sets equipped with some operations satisfying some equations and maps preserving these operations. There are plenty of examples of such categories in the literature: the category Set_* of pointed sets, Mon of monoids, ComMon of commutative monoids, Gp of groups, Ab of abelian groups, Rng of rings, Vect_k of vector spaces over a field k, RGraph of reflexive graphs, $\bigvee$-Lat of complete lattices and sup-preserving maps or even TorsFreeAb of torsion free abelian groups or Cat of small categories. Many different approaches have been developed to unify those categories and we recall in this chapter five of them.

We start with Birkhoff's work [13] which is actually the first one to appear historically. His idea is somehow the most intuitive: to define a theory, we have to list the operations and the equations.

We then quickly describe Lawvere's general treatment of 'algebraic-like categories' [76], which, using category theory, is much more elegant: a theory is a small category with finite products and an algebra is a finite product preserving functor from the theory to the category Set of sets.

With the two previous approaches, if we want to have a large number of operations and equations, we will quickly have some set-theoretical sizes problems. A conceptually simple way to avoid these problems is to describe universal algebra using monads and monadic categories, as recalled in Section 1.5.

If we replace the words ‘finite products’ by ‘finite limits’ in Lawvere’s work, we get the notion of a locally finitely presentable category introduced by Gabriel and Ulmer in [44]. The main difference is that, now, operations can be partial. We treat this subject in Section 1.6.

The notion of a sketch introduced by Ehresmann [36] is a way to unify categories of finite product preserving functors (Lawvere’s work) and of finite limit preserving functors (Gabriel-Ulmer’s work). The idea is that a theory (called a sketch) is now a small category together with a specified set of cones (and cocones) which have to be sent to (co)limits by the models. This is the topic of our Section 1.7.

Since the main results in this thesis are about embedding theorems, we will be careful about sizes and set-theoretical issues from the beginning. We thus devote the first section of this chapter to quickly recall the Axiom of Universes.

1.1 Grothendieck universes

The system of axioms used in this work is the so called ‘ZFCU’, although almost every result also holds for other foundations. ZF stands for the axiomatic system of set theory named ‘Zermelo-Fraenkel’, see [41] for a detailed treatment of it. To obtain ZFC, we add the Axiom of Choice to ZF.

Axiom of Choice. Each surjective function $f\colon x \twoheadrightarrow y$ has a section, i.e., a function $s\colon y \to x$ such that $fs = 1_y$.

It may be stated in many other equivalent forms. For example it is equivalent to Zorn’s lemma, the well-ordering theorem or the comparability theorem (see again [41] and the references therein).

Finally, to obtain ZFCU, we need to add the Axiom of Universes.

Definition 1.1. [7] A *Grothendieck universe* (or *universe* in short) is a set $\mathcal{U}$ satisfying the following properties.

1. if $x \in y$ and $y \in \mathcal{U}$, then $x \in \mathcal{U}$,
2. if $x, y \in \mathcal{U}$, then $\{x, y\} \in \mathcal{U}$,

3. if $x \in \mathcal{U}$, the power set $\mathcal{P}(x) \in \mathcal{U}$,
4. if $I \in \mathcal{U}$ and $x_i \in \mathcal{U}$ for each $i \in I$, then $\bigcup_{i \in I} x_i \in \mathcal{U}$,
5. $\mathbb{N} \in \mathcal{U}$.

Axiom 5 was not in the original definition. We add it here to avoid too small universes. To fix notations, $\mathbb{N}$ denotes the set $\{0, 1, \dots\}$ of natural numbers.

The idea of a universe is that you can do all usual operations of set theory to its elements and still get an element of the universe. For example, we have the following proposition.

Proposition 1.2. [7] The following properties hold for a universe $\mathcal{U}$.

1. If $x \subseteq y$ and $y \in \mathcal{U}$, then $x \in \mathcal{U}$.
2. If $x, y \in \mathcal{U}$, then $x \cup y$, $x \times y$ and x^y are also in $\mathcal{U}$.
3. If $I \in \mathcal{U}$ and $x_i \in \mathcal{U}$ for each $i \in I$, then $\prod_{i \in I} x_i \in \mathcal{U}$.
4. If there exists a surjective function $f \colon x \twoheadrightarrow y$ with $x \in \mathcal{U}$ and $y \subseteq \mathcal{U}$, then $y \in \mathcal{U}$.

We now add to ZFC the Axiom of Universes in order to get the system of axioms ZFCU we will work with.

Axiom of Universes. For every set x, there exists a universe $\mathcal{U}$ such that $x \in \mathcal{U}$.

For a universe $\mathcal{U}$, we call an element $x \in \mathcal{U}$ a *$\mathcal{U}$-small set* and a subset $y \subseteq \mathcal{U}$ a *$\mathcal{U}$-class*. Accordingly, a *$\mathcal{U}$-group* is a group for which the underlying set is $\mathcal{U}$-small, and so on with other mathematical structures on a set. A *$\mathcal{U}$-category* is a category $\mathcal{C}$ for which each hom-set $\mathcal{C}(A, B)$ is $\mathcal{U}$-small and the objects form a $\mathcal{U}$-class $\mathrm{ob}(\mathcal{C})$. If moreover $\mathrm{ob}(\mathcal{C})$ is a $\mathcal{U}$-small set, we say that $\mathcal{C}$ is a *$\mathcal{U}$-small category*. We can therefore speak about $\mathcal{U}$-Set, the $\mathcal{U}$-category of $\mathcal{U}$-small sets. A particular instance of the Axiom of Universes implies the following classical proposition.

Proposition 1.3. Let $\mathcal{U}$ be a universe and $\mathcal{C}$ a $\mathcal{U}$-category. There exists a universe $\mathcal{V}$ such that $\mathcal{C}$ is a $\mathcal{V}$-small category.

Proof. By the Axiom of Universes, consider a universe $\mathcal{V}$ such that $\mathcal{U} \in \mathcal{V}$. It thus contains each hom-set $\mathcal{C}(A, B)$ and $\mathrm{ob}(\mathcal{C})$. $\square$

Convention 1.4. We now fix a universe $\mathcal{U}$ throughout this thesis. For the sake of brevity, we will call a $\mathcal{U}$-small set just a *set*, a $\mathcal{U}$-group a *group*, a $\mathcal{U}$-class a *class*, a $\mathcal{U}$-category a *category*, a $\mathcal{U}$-small category a *small category* and so forth. Notice that this implies that our categories are locally small. We will then write Set for the category of sets and Cat for the category of small categories as usual. The word *cardinal* stands then to mean the cardinal of a $\mathcal{U}$-small set. If we need to consider another universe, notations will be made explicit.

1.2 Regular and exact categories

Homomorphisms in 'algebraic-like categories' can often be factorised as a surjective homomorphism followed by an inclusion. This property can be stated in a categorical context and is a key property of regular and exact categories introduced by Barr in [9]. Before recalling some general treatments of universal algebra, we devote a section to these notions. The reader may consult for instance the second section of the second volume of [14] for a more detailed treatment of regular and exact categories.

Definition 1.5. A morphism q in a category is a *regular epimorphism* if it is the coequaliser of two parallel morphisms.

Definition 1.6. [9] A category $\mathcal{C}$ is said to be *regular* if it satisfies the following conditions:

1. it has finite limits,
2. every kernel pair has a coequaliser,
3. the pullback of a regular epimorphism along any morphism is again a regular epimorphism.

Example 1.7. The categories Set, Set_*, Mon, Gp, Ab, RGraph, $\bigvee$-Lat and TorsFreeAb are regular. In these categories, regular epimorphisms are exactly the surjective homomorphisms. The categories Top of topological spaces and Cat are not regular.

The notion of a 'good functor' between regular categories is the following one.

Definition 1.8. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between the regular categories $\mathcal{C}$ and $\mathcal{D}$ is said to be *regular* if it preserves finite limits and regular epimorphisms.

Let us stress the fact that if moreover F reflects isomorphisms, 'it reflects what $\mathcal{C}$ has and F preserves'. This is made precise in Lemma 1.10.

Definition 1.9. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be *conservative* if it reflects isomorphisms, i.e., if, for any morphism f in $\mathcal{C}$, f is an isomorphism if and only if $F(f)$ is.

Lemma 1.10. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a regular conservative functor between regular categories. Then F is faithful and reflects finite limits and regular epimorphisms.

As announced above, there is a factorisation system in each regular category.

Definition 1.11. [43] In a category $\mathcal{C}$, the morphism $e\colon A \to B$ is said to be *orthogonal* to the morphism $m\colon C \to D$ (denoted $e \perp m$) if any commutative square $ge = mf$ admits a unique diagonal d such that $de = f$ and $md = g$.

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dotted, "d"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

Definition 1.12. [43] A *factorisation system* on a category $\mathcal{C}$ is a pair $(\mathcal{E}, \mathcal{M})$ where both $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms of $\mathcal{C}$ such that

1. every isomorphism belongs to both $\mathcal{E}$ and $\mathcal{M}$,
2. both $\mathcal{E}$ and $\mathcal{M}$ are closed under composition,
3. for each $e \in \mathcal{E}$ and each $m \in \mathcal{M}$, e is orthogonal to m,
4. every morphism $f \in \mathcal{C}$ can be factorised as $f = me$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

By a *(regular epi, mono)-factorisation system* in a category $\mathcal{C}$, we mean a factorisation system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ is the class of all regular epimorphisms and $\mathcal{M}$ the class of all monomorphisms in $\mathcal{C}$.

Proposition 1.13. Every regular category $\mathcal{C}$ admits a (regular epi, mono)-factorisation system.

In a regular category, regular epimorphisms coincide with other kinds of epimorphisms.

Definition 1.14. An epimorphism f in a category is said to be *extremal* if, given $f = ip$ with i a monomorphism, i is necessarily an isomorphism. The epimorphism f is said to be *strong* if it is orthogonal to any monomorphism.

Proposition 1.15. In any category $\mathcal{C}$,

1. every regular epimorphism is strong,
2. every strong epimorphism is extremal,
3. the composition of two strong epimorphisms is a strong epimorphism,
4. if a composite gf is a strong (resp. extremal) epimorphism, so is g,
5. a morphism which is both a monomorphism and an extremal epimorphism is an isomorphism.

If $\mathcal{C}$ has finite limits,

6. if a morphism f is such that $f = ip$ with i a monomorphism implies that i is an isomorphism, then f is an epimorphism and so an extremal epimorphism,
7. if f is orthogonal to any monomorphism, then f is an epimorphism and so a strong epimorphism,
8. strong epimorphisms coincide with extremal epimorphisms.

If moreover $\mathcal{C}$ is a regular category,

9. regular epimorphisms coincide with strong epimorphisms,

10. if $f\colon A \twoheadrightarrow B$ and $g\colon C \twoheadrightarrow D$ are regular epimorphisms, their product $f \times g\colon A \times C \twoheadrightarrow B \times D$ is also a regular epimorphism.

We can characterise regular categories without mentioning regular epimorphisms but only strong ones.

Proposition 1.16. A category $\mathcal{C}$ is regular if and only if it satisfies the following conditions:

1. finite limits exist,

2. every morphism f can be factorised as $f = ip$ with i a monomorphism and p a strong epimorphism,

3. the pullback of a strong epimorphism along any morphism is again a strong epimorphism.

Regular categories provide a good context for the calculus of relations. We recall here the definition of a relation and how to compose binary ones in a regular category.

Definition 1.17. A *subobject* of an object A in a category $\mathcal{C}$ is the isomorphism class $[m]$ of a monomorphism $m\colon I \rightarrowtail A$. We write $\mathrm{Sub}(A)$ for the class of subobjects of A. It is actually a preordered class if we consider that $[n] \leqslant [m]$ if n factors through m. If, for each object A in $\mathcal{C}$, $\mathrm{Sub}(A)$ is actually a set, we say that $\mathcal{C}$ is *well-powered.*

We will often refer to a subobject $[m]$ by one of its representative monomorphism m.

Example 1.18. As in a regular category, the factorisation of a morphism $f\colon A \to B$ as a regular epimorphism p followed by a monomorphism i is unique up to isomorphism, the subobject of B represented by i depends only on f. It is called the *image* of f and is denoted by $\mathrm{Im}(f)$.

Definition 1.19. Let $n \geqslant 1$ be a natural number and $\mathcal{C}$ a category with finite products. An *n-ary relation* in $\mathcal{C}$ is a subobject $r\colon R \rightarrowtail A_1 \times \cdots \times A_n$ of an n-ary product.

We will often denote this relation by R (if there is no ambiguity about the subobject $[r]$). If $\mathcal{C}$ is a regular category and $(r_1, r_2)\colon R \rightarrowtail A \times B$ and $(s_1, s_2)\colon S \rightarrowtail B \times C$ are two binary relations in $\mathcal{C}$, the *composite relation* $S \circ R = SR$ is given by the image of $(r_1p_1, s_2p_2)\colon P \to A \times C$, where (P, p_1, p_2) is the pullback of s_1 along r_2.

$$\begin{tikzcd}
 & & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
A & & B & & C
\end{tikzcd}$$

This composition is associative, i.e., $T \circ (S \circ R) = (T \circ S) \circ R$ if T is another binary relation $T \rightarrowtail C \times D$. Moreover, if Δ_A denotes the *diagonal relation* $(1_A, 1_A)\colon A \rightarrowtail A \times A$, we have $R \circ \Delta_A = R = \Delta_B \circ R$. We will also write R^{op} for the *dual relation* $(r_2, r_1)\colon R \rightarrowtail B \times A$ and if $(r_1', r_2')\colon R' \rightarrowtail A \times B$ is another binary relation, $R \cap R' \rightarrowtail A \times B$ denotes their intersection given by their pullback.

To conclude this section, let us recall the notion of an exact category from [9]. In order to do so, we first need the notion of an equivalence relation.

Definition 1.20. Let $(r_1, r_2)\colon R \rightarrowtail A \times A$ be a binary relation on A in the finitely complete category $\mathcal{C}$.

1. We say that R is a *reflexive relation* if there exists a morphism $\delta\colon A \to R$ satisfying the identities $r_1 \circ \delta = 1_A = r_2 \circ \delta$. This means $\Delta_A \leqslant R$.

2. The relation R is said to be *symmetric* if there exists a morphism $\sigma\colon R \to R$ such that $r_1 \circ \sigma = r_2$ and $r_2 \circ \sigma = r_1$. This means $R^{\mathrm{op}} \leqslant R$ which implies $R^{\mathrm{op}} = R$.

3. We say that R is *transitive* if there exists a morphism $\tau\colon P \to R$ such that $r_1 \circ \tau = r_1 \circ p_1$ and $r_2 \circ \tau = r_2 \circ p_2$ where (P, p_1, p_2) is

the pullback of r_1 along r_2.

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "r_1"] \\
R \arrow[r, "r_2"'] & A
\end{tikzcd}$$

If $\mathcal{C}$ is regular, this means $RR \leqslant R$.

4. An *equivalence relation* on A is a binary relation on A which is reflexive, symmetric and transitive.

Example 1.21. If $f\colon A \to B$ is a morphism in a finitely complete category, its kernel pair $(r_1, r_2)\colon R \rightarrowtail A \times A$ is an equivalence relation on A. Such equivalence relations are said to be *effective*.

Definition 1.22. [9] A category $\mathcal{C}$ is said to be *exact* if it is regular and such that every equivalence relation is effective.

Example 1.23. The categories Set, Set_*, Mon, Gp, Ab and RGraph are exact. The category TorsFreeAb is regular but not exact.

1.3 Birkhoff's approach

The study of universal algebra originates in the thirties when Birkhoff proposed in [13] an intuitive way to unify 'algebraic structures'. His idea was to fix a set of operation symbols of prescribed arity (the signature) together with a set of equations built up from these operation symbols. An algebra is then nothing but a set A equipped with an operation $\sigma^A\colon A^n \to A$ for each operation symbol σ of arity n satisfying the required equations. This presentation has the advantage to be very intuitive as it immediately describes what we have in mind when we think of an 'algebraic-like category'. However, since it appears before [37], it does not make an explicit use of category theory, which can simplify the presentation, as it was done by Lawvere in [76] and recalled in the next section.

Birkhoff considered only the one-sorted finitary case. Since further we will need the many-sorted infinitary case, we immediately describe this more general setting. As good references for the subject, one can cite [4] and [5].

Definition 1.24. Let S be a set. An *S-sorted signature of algebras* is a set Σ together with an arity function assigning to each $\sigma \in \Sigma$ a collection $(s_i \in S)_{i<n}$ for some cardinal n, and an element $s \in S$.

We then denote the arity of σ by $\prod_{i<n} s_i \to s$ (or $\sigma \colon 1 \to s$ if $n = 0$). The elements of S are called *sorts* and those of Σ *operation symbols*.

Definition 1.25. Let S be a set of sorts and Σ an S-sorted signature of algebras. A *Σ-algebra* is an S-sorted set A (i.e., an object of Set^S) equipped with an operation

$$\sigma^A \colon \prod_{i<n} A_{s_i} \to A_s$$

for each operation symbol $\sigma \in \Sigma$ of arity $\prod_{i<n} s_i \to s$. If B is also a Σ-algebra, a *Σ-homomorphism* $f \colon A \to B$ is an S-sorted function such that, for each $\sigma \colon \prod_{i<n} s_i \to s$ in Σ and each family $(a_i \in A_{s_i})_{i<n}$, the identity

$$f_s(\sigma^A(a_i)) = \sigma^B(f_{s_i}(a_i))$$

holds. We get in this way the category Σ-Alg and its forgetful functor $U_\Sigma \colon \Sigma\text{-Alg} \to \mathrm{Set}^S$.

For the sake of brevity, when there is no ambiguity, we will sometimes write f instead of f_s for the s-component of an S-sorted function f.

These Σ-algebras are S-sorted sets equipped with operations. To define monoids, groups and so forth, we need to require these operations to satisfy some equations. To define such equations, we now describe the left adjoint of U_Σ.

If X is an S-sorted set, we denote by $\mathrm{Fr}_\Sigma(X)$ the smallest S-sorted set satisfying the following conditions:

1. for each $s \in S$, $X_s \subseteq \mathrm{Fr}_\Sigma(X)_s$,
2. for each $\sigma \colon \prod_{i<n} s_i \to s$ in Σ and each family $(t_i \in \mathrm{Fr}_\Sigma(X)_{s_i})_{i<n}$, the formal expression $\sigma((t_i)_{i<n})$ belongs to $\mathrm{Fr}_\Sigma(X)_s$.

The fact that $\sigma((t_i)_{i<n})$ is a formal expression means that the two elements $\sigma((t_i)_{i<n})$ and $\sigma'((t'_i)_{i<n'})$ are equal if and only if $n = n'$, $s_i = s'_i$ and $t_i = t'_i$ for all $i < n$, and $\sigma = \sigma'$. Elements of $\mathrm{Fr}_\Sigma(X)_s$ are called *terms* of sort s in the variables from X. If I is the disjoint union $\bigsqcup_{s\in S} X_s$ and if, for each $i \in I$, s_i denotes the corresponding sort, we may write $t\colon \prod_{i\in I} s_i \to s$ for a term t of sort s. This S-sorted set $\mathrm{Fr}_\Sigma(X)$ is equipped with the obvious Σ-algebra structure and we consider the S-sorted inclusion $X \hookrightarrow U_\Sigma \mathrm{Fr}_\Sigma(X)$. This is the reflection of X along U_Σ so that we have an adjunction $\mathrm{Fr}_\Sigma \dashv U_\Sigma$.

Definition 1.26. Let S be a set of sorts and Σ an S-sorted signature of algebras. An *equation* in Σ is a pair (t_1, t_2) of terms of the same sort s in the variables from an S-sorted set X. We also write $t_1 = t_2$ for the equation (t_1, t_2). A Σ-algebra A is said to *satisfy* the equation (t_1, t_2) if for any S-sorted function $f\colon X \to U_\Sigma(A)$, the unique Σ-homomorphism $\overline{f}\colon \mathrm{Fr}_\Sigma(X) \to A$ extending f

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[dr, "f"'] & & U_\Sigma \mathrm{Fr}_\Sigma(X) \arrow[dl, dotted, "\overline{f}"] \\
& U_\Sigma(A) &
\end{tikzcd}$$

is such that $\overline{f}_s(t_1) = \overline{f}_s(t_2)$.

Definition 1.27. An *algebraic theory* is a triple (S, Σ, E) where S is a set of sorts, Σ an S-sorted signature of algebras and E a set of equations for Σ. It is *one-sorted* if S is a singleton and we often denote it by (Σ, E) in this case.

We can now define algebraic categories.

Definition 1.28. Let (S, Σ, E) be an algebraic theory. An (S, Σ, E)-*algebra* is a Σ-algebra which satisfies all equations of E. The full subcategory of Σ-Alg given by (S, Σ, E)-algebras is denoted by (S, Σ, E)-Alg with $U_{(S,\Sigma,E)}\colon (S, \Sigma, E)\text{-Alg} \to \mathrm{Set}^S$ the forgetful functor. A category which is equivalent to a category of the form (S, Σ, E)-Alg is called an *algebraic category*. We say it is *S-sorted* if S is specified. For example, it is a *one-sorted algebraic category* if (S, Σ, E) is one-sorted.

Before giving some examples of algebraic categories, let us define finitary ones (or more generally λ-ary ones).

Definition 1.29. An infinite cardinal λ is called *regular* if, for any set I with $\#I < \lambda$ and any family of sets $(X_i)_{i\in I}$ with $\#X_i < \lambda$ for each $i \in I$, we have $\#\bigcup_{i\in I} X_i < \lambda$.

Definition 1.30. An algebraic theory (S, Σ, E) is said to be λ-*ary* for a regular cardinal λ if

1. λ is larger than n for any operation symbol $\sigma\colon \prod_{i<n} s_i \to s$ of Σ,
2. $\#\coprod_{s\in S} X_s < \lambda$ for any equation (t_1, t_2) of E in the variables from the S-sorted set X.

In particular, it is called *finitary* if it is $\aleph_0$-ary for the regular cardinal $\aleph_0 = \#\mathbb{N}$. A category which is equivalent to a category of the form (S, Σ, E)-Alg for a λ-ary (resp. finitary) algebraic theory (S, Σ, E) is called a λ-*ary (resp. finitary) algebraic category.*

Note that due to the following classical theorem of set theory (see e.g. [54]), for any algebraic theory, one can find a regular cardinal λ for which it is λ-ary.

Theorem 1.31. Given a set $(\lambda_i)_{i\in I}$ of cardinals, there exists a regular cardinal λ such that $\lambda_i < \lambda$ for each $i \in I$.

Example 1.32. The category Mon is a one-sorted finitary algebraic category. Indeed, consider S to be the singleton $\{*\}$ and $\Sigma = \{0, +\}$ with $0\colon 1 \to *$ and $+\colon *^2 \to *$. The expression $(y + x) + (0 + y)$, technically written $+(+(y, x), +(0(), y))$, is an example of a term of Σ in the variables from $\{x, y\}$. The set of equations E is given by

$$E = \{x + 0 = x, 0 + x = x, (x + y) + z = x + (y + z)\}$$

where the first two equations are in the variables from $\{x\}$ and the last one from $\{x, y, z\}$. Similarly, Set, Set$_*$, Gp, Ab, Rng and Vect$_\mathbb{R}$ are finitary one-sorted algebraic categories. RGraph is a two-sorted finitary algebraic category, where *two-sorted* has the meaning $\#S = 2$. The category $\bigvee$-Lat might be thought of as algebraic, but it is a priori not because we need an operation for each cardinality (taking the join of a subset is an operation which depends on the cardinality of this subset). Therefore, in this case, Σ will not be a set.

Let us now give some basic properties of algebraic categories. For that we need to introduce the notion of a λ-filtered colimit, which appeared for the first time in [7] in the finitary case.

Definition 1.33. Let λ be a regular cardinal. A small category $\mathcal{C}$ is said to be λ-*filtered* if the following conditions hold:

1. $\mathcal{C}$ is not empty,
2. given a set I with $\#I < \lambda$ and a family of objects $(A_i \in \mathcal{C})_{i \in I}$, there exists an object A of $\mathcal{C}$ and morphisms $f_i \colon A_i \to A$ for each $i \in I$,
3. given a set I with $\#I < \lambda$ and a family of parallel morphisms $(f_i \colon A \to B)_{i \in I}$ in $\mathcal{C}$, there exists a morphism $g \colon B \to C$ such that $gf_i = gf_j$ for all $i, j \in I$.

If $\lambda = \aleph_0$, we say that $\mathcal{C}$ is *filtered*. The dual notions are respectively called λ-*cofiltered categories* and *cofiltered categories*.

In what follows, by a λ-*limit*, we mean a limit over a diagram of shape $\mathcal{D}$ with $\#\operatorname{ar}(\mathcal{D}) < \lambda$, where $\operatorname{ar}(\mathcal{D})$ denotes the set of arrows of the small category $\mathcal{D}$.

Theorem 1.34. Let (S, Σ, E) be a λ-ary algebraic theory for a regular cardinal λ. Then (S, Σ, E)-Alg is complete, cocomplete, exact and its λ-limits commute with its λ-filtered colimits. The forgetful functor $U_{(S,\Sigma,E)} \colon (S, \Sigma, E)\text{-Alg} \to \mathrm{Set}^S$ has a left adjoint, is regular conservative and preserves λ-filtered colimits.

In particular, regular epimorphisms in (S, Σ, E)-Alg coincide with the componentwise surjective Σ-homomorphisms. The left adjoint $\mathrm{Fr}_{(S,\Sigma,E)} :$ $\mathrm{Set}^S \to (S, \Sigma, E)$-Alg of $U_{(S,\Sigma,E)}$ can be described using the notion of a theorem.

Definition 1.35. Let (S,Σ,E) be an algebraic theory and (t_1,t_2) an equation in Σ. We say that $t_1 = t_2$ is a *theorem* of (S,Σ,E) if every (S,Σ,E)-algebra satisfies the equation (t_1,t_2).

To construct the free (S,Σ,E)-algebra $\mathrm{Fr}_{(S,\Sigma,E)}(X)$ on the S-sorted set X, we let $\mathrm{Fr}_{(S,\Sigma,E)}(X)_s$ be the set of equivalence classes of terms of sort s in the variables from X, where two terms t_1, t_2 are identified if and only if $t_1 = t_2$ is a theorem of (S,Σ,E). The structure of (S,Σ,E)-algebra on $\mathrm{Fr}_{(S,\Sigma,E)}(X)$ and the reflection $X \to U_{(S,\Sigma,E)}(\mathrm{Fr}_{(S,\Sigma,E)}(X))$ are constructed in the obvious way.

Let us conclude this section with some characterisations of algebraic categories. To be able to state them, we need to recall some classical definitions (see e.g. [14]).

Definition 1.36. An object P in a category $\mathcal{C}$ is said to be a *strong projective object* (or *projective object* in short) if, for any morphism $g\colon P \to B$ and strong epimorphism $f\colon A \twoheadrightarrow B$, there exists a morphism $h\colon P \to A$ such that $fh = g$.

$$\begin{tikzcd} & P \arrow[ld, dotted, "h"'] \arrow[d, "g"] \\ A \arrow[r, two heads, "f"'] & B \end{tikzcd}$$

Example 1.37. In an algebraic category (S,Σ,E)-Alg, the free algebras $\mathrm{Fr}_{(S,\Sigma,E)}(X)$ are projective objects.

Definition 1.38. Let λ be a regular cardinal. We say that an object P in a category $\mathcal{C}$ is λ*-presentable* if the representable functor $\mathcal{C}(P,-)\colon \mathcal{C} \to \mathrm{Set}$ preserves λ-filtered colimits. If $\lambda = \aleph_0$, P is said in that case to be *finitely presentable.*

As examples of λ-presentable objects, let us give their classical characterisation in λ-ary algebraic categories.

Proposition 1.39. Let (S,Σ,E) be a λ-ary algebraic theory for a regular cardinal λ. The λ-presentable objects in (S,Σ,E)-Alg are exactly the (S,Σ,E)-algebras P which can be expressed as a coequaliser

$$\begin{tikzcd} \mathrm{Fr}_{(S,\Sigma,E)}(X) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Fr}_{(S,\Sigma,E)}(Y) \arrow[r, two heads] & P \end{tikzcd}$$

where X and Y are S-sorted sets such that $\#\coprod_{s\in S} X_s < \lambda$ and $\#\coprod_{s\in S} Y_s < \lambda$.

Example 1.40. In view of the above proposition, for a regular cardinal λ, a set X is λ-presentable in Set if and only if $\#X < \lambda$.

Definition 1.41. A set $\mathcal{G}$ of objects in a category $\mathcal{C}$ is said to be a *set of generators* if for any pair of parallel morphisms $f, g\colon A \rightrightarrows B$ with $f \neq g$, there exists a $G \in \mathcal{G}$ and a morphism $h\colon G \to A$ such that $fh \neq gh$. If $\mathcal{G}$ is the singleton $\{G\}$, we then say that G is a *generator*.

Definition 1.42. A set $\mathcal{G}$ of generators in a category $\mathcal{C}$ is called *strong* if, given any morphism $g\colon A \to C$, any monomorphism $m\colon B \rightarrowtail C$ and any family of morphisms $(h_f\colon G \to B)_{G\in\mathcal{G},\, f\in\mathcal{C}(G,A)}$ satisfying $mh_f = gf$ for each $G \in \mathcal{G}$ and each $f\colon G \to A$, there exists a (unique) morphism $d\colon A \to B$ such that $md = g$. If $\mathcal{G}$ is the singleton $\{G\}$, we then say that G is a *strong generator*.

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "h_f"'] & A \arrow[d, "g"] \arrow[dl, dotted, "d"] \\
B \arrow[r, tail, "m"'] & C
\end{tikzcd}$$

Proposition 1.43. Let $\mathcal{G}$ be a set of objects in a category $\mathcal{C}$ with small coproducts. For an object $A \in \mathcal{C}$, we consider the unique morphism

$$\gamma_A\colon \coprod_{\substack{G\in\mathcal{G}\\ f\in\mathcal{C}(G,A)}} \operatorname{dom}(f) \longrightarrow A$$

satisfying $\gamma_A \circ \iota_f = f$ for any $G \in \mathcal{G}$ and any $f\colon G \to A$, where ι_f is the coproduct injection. Then, $\mathcal{G}$ is a set of generators if and only if γ_A is an epimorphism for each object $A \in \mathcal{C}$. It is a strong set of generators if and only if γ_A is a strong epimorphism for each object A.

Example 1.44. Let (S, Σ, E) be an algebraic theory. For a sort $s \in S$, we denote by G_s the free (S, Σ, E)-algebra on the S-sorted set X defined by $X_s = \{*\}$ and $X_{s'} = \varnothing$ for $s' \neq s$. The set $\mathcal{G} = \{G_s \mid s \in S\}$ is a strong set of generators in (S, Σ, E)-Alg.

We can now give the classical abstract characterisation of algebraic categories. The finitary one-sorted case was proved by Lawvere in [76].

Theorem 1.45. Let λ be a regular cardinal. The following conditions are equivalent for a category $\mathcal{C}$:

1. $\mathcal{C}$ is a λ-ary algebraic category,
2. (a) $\mathcal{C}$ is exact,
   (b) $\mathcal{C}$ has a strong set of generators $\mathcal{G}$ such that each object of $\mathcal{G}$ is projective and λ-presentable,
   (c) for each set X and function $f : X \to \mathcal{G}$, the coproduct $\coprod_{x \in X} f(x)$ exists in $\mathcal{C}$.

Let us write its one-sorted version.

Theorem 1.46. Let λ be a regular cardinal. The following conditions are equivalent for a category $\mathcal{C}$:

1. $\mathcal{C}$ is a one-sorted λ-ary algebraic category,
2. (a) $\mathcal{C}$ is exact,
   (b) $\mathcal{C}$ has a projective λ-presentable strong generator G,
   (c) for each set X, the copower $\coprod_X G$ exists in $\mathcal{C}$.

And to conclude, here is the famous Birkhoff Variety Theorem.

Theorem 1.47. [13, 6] Let S be a set of sorts and Σ an S-sorted signature of algebras. The following conditions on a full subcategory $\mathcal{C}$ of Σ-Alg are equivalent:

1. there exists a set E of equations of Σ such that $\mathcal{C}$ is the category (S, Σ, E)-Alg,
2. (a) $\mathcal{C}$ is closed under small products, i.e., if I is a set and $(A_i)_{i \in I}$ a family of objects in $\mathcal{C}$, then the product $\prod_{i \in I} A_i$ in Σ-Alg is actually in $\mathcal{C}$,
   (b) $\mathcal{C}$ is closed under subobjects, i.e., if $f \colon A \rightarrowtail B$ is a monomorphism in Σ-Alg with $B \in \mathcal{C}$, then $A \in \mathcal{C}$,
   (c) $\mathcal{C}$ is closed under quotients, i.e., if $f \colon A \twoheadrightarrow B$ is a regular epimorphism in Σ-Alg with $A \in \mathcal{C}$, then $B \in \mathcal{C}$.

1.4 Lawvere theories

Birkhoff's presentation of universal algebra does not use the categorical point of view. In his thesis [76], Lawvere proposed an equivalent, but much more elegant approach using category theory. In this presentation, a theory is now a small category with finite products and an algebra for this theory is a finite product preserving functor. A homomorphism is then nothing but a natural transformation. Lawvere considered in his PhD thesis only the finitary one-sorted case, but it is not hard to generalise it to the other cases (see e.g. [5]). We only briefly recall the original case here.

Definition 1.48. [76] A *Lawvere theory* $\mathcal{T}$ is a category whose set of objects is $\mathbb{N}$, n being the n-th power of 1 with fixed projections $p_1, \dots, p_n \colon n \to 1$ and such that $p_1 = 1_1$ if $n = 1$.

Definition 1.49. [76] Let $\mathcal{T}$ be a Lawvere theory. A *$\mathcal{T}$-algebra* is a finite product preserving functor $A \colon \mathcal{T} \to \mathrm{Set}$. A *homomorphism of $\mathcal{T}$-algebras* (or *$\mathcal{T}$-homomorphism*) $f \colon A \to B$ is a natural transformation $A \Rightarrow B$. This gives rise to the category $\mathcal{T}$-Alg.

This category of $\mathcal{T}$-algebras comes equipped with the faithful and conservative functor $U_{\mathcal{T}} \colon \mathcal{T}\text{-Alg} \to \mathrm{Set}$ of evaluation at 1. We thus consider that a $\mathcal{T}$-algebra A is a structure on its underlying set $A(1)$ (by abuse of notation, also sometimes denoted A) and that a $\mathcal{T}$-homomorphism $f \colon A \to B$ is a function $f \colon A(1) \to B(1)$ satisfying some properties. This forgetful functor has a left adjoint $\mathrm{Fr}_{\mathcal{T}} \colon \mathrm{Set} \to \mathcal{T}\text{-Alg}$. The reflection of the set $\{1, \dots, n\}$ for a natural number $n \in \mathbb{N}$ is given by the $\mathcal{T}$-algebra $\mathcal{T}(n, -) \colon \mathcal{T} \to \mathrm{Set}$.

Now, for each finitary one-sorted algebraic theory (Σ, E), one can construct a Lawvere theory $\mathcal{T}_{(\Sigma,E)}$ in the following way: For natural numbers $n, m \in \mathbb{N}$, $\mathcal{T}_{(\Sigma,E)}(n, m)$ is the set

$$(\Sigma, E)\text{-Alg}(\mathrm{Fr}_{(\Sigma,E)}(\{1, \dots, m\}), \mathrm{Fr}_{(\Sigma,E)}(\{1, \dots, n\})).$$

Composition and identities in $\mathcal{T}_{(\Sigma,E)}$ are computed as in the dual category $(\Sigma, E)\text{-Alg}^{\mathrm{op}}$, while the chosen projections are the images of the n maps

$\{1\} \to \{1, \ldots, n\}$ under $\mathrm{Fr}_{(\Sigma,E)}$. We can then construct an equivalence of categories between (Σ, E)-Alg and $\mathcal{T}_{(\Sigma,E)}$-Alg making the triangle

$$\begin{tikzcd}
(\Sigma, E)\text{-Alg} \arrow[rr, "\sim"] \arrow[dr, "U_{(\Sigma,E)}"'] & & \mathcal{T}_{(\Sigma,E)}\text{-Alg} \arrow[dl, "U_{\mathcal{T}_{(\Sigma,E)}}"] \\
& \mathrm{Set} &
\end{tikzcd}$$

commute.

Conversely, given a Lawvere theory $\mathcal{T}$, we can construct a finitary one-sorted algebraic theory $(\Sigma_{\mathcal{T}}, E_{\mathcal{T}})$ as follows: The set of operation symbols of arity n (i.e., $*^n \to *$) is given by $\mathcal{T}(n, 1)$. To each n-ary term t of $\Sigma_{\mathcal{T}}$ (let us say in the variables from $\{x_1, \ldots, x_n\}$), we associate an operation symbol $\overline{t}\colon n \to 1$ recursively:

- $\overline{x_i}$ is the chosen projection $p_i\colon n \to 1$ for each $1 \leqslant i \leqslant n$,
- if $\overline{t_1}, \ldots, \overline{t_m}$ have been defined and if $\sigma\colon m \to 1$ is an m-ary operation symbol, then $\overline{\sigma(t_1, \ldots, t_m)}$ is defined as the composite $\sigma \circ (\overline{t_1}, \ldots, \overline{t_m})$.

$$\begin{tikzcd}
n \arrow[r, "(\overline{t_1},\ldots,\overline{t_m})"] & m \arrow[r, "\sigma"] & 1
\end{tikzcd}$$

We then define $E_{\mathcal{T}}$ as the set of equations $t = \overline{t}(x_1, \ldots, x_n)$ for each $n \in \mathbb{N}$ and each n-ary term t of $\Sigma_{\mathcal{T}}$. One can easily prove that the Lawvere theory $\mathcal{T}_{(\Sigma_{\mathcal{T}},E_{\mathcal{T}})}$ is isomorphic to $\mathcal{T}$ (with the same chosen projections). We thus also have an equivalence of categories between $(\Sigma_{\mathcal{T}}, E_{\mathcal{T}})$-Alg and $\mathcal{T}$-Alg making the triangle

$$\begin{tikzcd}
(\Sigma_{\mathcal{T}}, E_{\mathcal{T}})\text{-Alg} \arrow[rr, "\sim"] \arrow[dr, "U_{(\Sigma_{\mathcal{T}},E_{\mathcal{T}})}"'] & & \mathcal{T}\text{-Alg} \arrow[dl, "U_{\mathcal{T}}"] \\
& \mathrm{Set} &
\end{tikzcd}$$

commutative. A finitary one-sorted algebraic category is thus a category which is equivalent to $\mathcal{T}$-Alg for some Lawvere theory $\mathcal{T}$. We can thus see a $\mathcal{T}$-algebra A as the structure of a $(\Sigma_{\mathcal{T}}, E_{\mathcal{T}})$-algebra over the set $A(1)$ and a $\mathcal{T}$-homomorphism $f\colon A \to B$ as a function $f\colon A(1) \to B(1)$ preserving these structures.

In view of this, we will call a term t of $\Sigma_{\mathcal{T}}$ a *term* of $\mathcal{T}$. It gives rise to a morphism $\bar{t}\colon n \to 1$ in $\mathcal{T}$ and so to the n-ary operation

$$t^A\colon\ A(1)^n \xrightarrow{\cong} A(n) \xrightarrow{A(\bar{t})} A(1)$$

for any $\mathcal{T}$-algebra A. If (Σ, E) is a finitary one-sorted algebraic theory such that $\mathcal{T}_{(\Sigma,E)}$ is isomorphic to $\mathcal{T}$ (with the same chosen projections), each such map $f\colon n \to 1$ in $\mathcal{T}$ is associated to a morphism $\mathrm{Fr}_{(\Sigma,E)}(\{1\}) \to \mathrm{Fr}_{(\Sigma,E)}(\{1,\ldots,n\})$, thus to (an equivalent class of) an n-ary term $\widetilde{f}$ of Σ. If (t_1, t_2) is an equation of $\Sigma_{\mathcal{T}}$, we say that $t_1 = t_2$ is a *theorem* of $\mathcal{T}$ if one (thus all) equivalent conditions bellow is satisfied:

1. $t_1 = t_2$ is a theorem of $(\Sigma_{\mathcal{T}}, E_{\mathcal{T}})$,
2. each $\mathcal{T}$-algebra $A\colon \mathcal{T} \to \mathrm{Set}$ satisfies $A(\overline{t_1}) = A(\overline{t_2})$,
3. for each $\mathcal{T}$-algebra A, $t_1^A = t_2^A$,
4. $\overline{t_1} = \overline{t_2}$,
5. $\widetilde{\overline{t_1}} = \widetilde{\overline{t_2}}$ is a theorem of (Σ, E) for a (Σ, E) as above.

We conclude this section with the notion of a morphism of Lawvere theories and its associated algebraic functor.

Definition 1.50. A morphism $\mathcal{T} \to \mathcal{R}$ between the Lawvere theories $\mathcal{T}$ and $\mathcal{R}$ is a finite product preserving functor $F\colon \mathcal{T} \to \mathcal{R}$ such that for each $n \in \mathbb{N}$, $F(n) = n$ and for each $1 \leqslant i \leqslant n$, $F(p_i^{\mathcal{T}}) = p_i^{\mathcal{R}} \in \mathcal{R}(n,1)$ for the chosen projections $p_i^{\mathcal{T}}$ and $p_i^{\mathcal{R}}$.

Such a morphism associates to each n-ary operation symbol σ of $\Sigma_{\mathcal{T}}$ (i.e., a map $n \to 1$ in $\mathcal{T}$) an n-ary operation symbol σ^ι of $\Sigma_{\mathcal{R}}$. Hence, each term t of $\mathcal{T}$ is interpreted as a term t^ι of $\mathcal{R}$.

Let us denote by Th[Set] (resp. Th[Set$_*$]) the Lawvere theory corresponding to Set (resp. Set$_*$), i.e., $\mathcal{T}_{(\Sigma,E)}$ for (Σ, E) given by $\Sigma = \varnothing$ (resp. $\Sigma = \{0\colon 1 \to *\}$) and $E = \varnothing$. Analogously, we write Th[Ab] (resp. Th[ComMon]) for the Lawvere theory corresponding to Ab (resp. ComMon). Note that these are unambiguously defined since if $\mathcal{T}$ and

$\mathcal{T}'$ are two Lawvere theories with an equivalence between the categories $\mathcal{T}$-Alg and $\mathcal{T}$'-Alg making the triangle

$$\begin{tikzcd}
\mathcal{T}\text{-Alg} \arrow[rr, "\sim"] \arrow[dr, "U_{\mathcal{T}}"'] & & \mathcal{T}'\text{-Alg} \arrow[dl, "U_{\mathcal{T}'}"] \\
& \mathrm{Set} &
\end{tikzcd}$$

commutative, there is an isomorphism of Lawvere theories $\mathcal{T} \cong \mathcal{T}'$.

Proposition 1.51. The theory Th[Set] is an initial object in the category of Lawvere theories and their morphisms.

Each morphism of Lawvere theories induces an algebraic functor.

Definition 1.52. Let $\mathcal{T}$ and $\mathcal{R}$ be two Lawvere theories. A functor $G\colon \mathcal{R}\text{-Alg} \to \mathcal{T}\text{-Alg}$ is said to be *algebraic* if there exists a morphism of Lawvere theories $F\colon \mathcal{T} \to \mathcal{R}$ such that G is naturally isomorphic to the functor F^* given by precomposition with F.

$$\begin{aligned} F^*\colon \mathcal{R}\text{-Alg} &\longrightarrow \mathcal{T}\text{-Alg} \\ A &\longmapsto AF \end{aligned}$$

They have an easy characterisation (see e.g. [14]).

Theorem 1.53. Let $G\colon \mathcal{R}\text{-Alg} \to \mathcal{T}\text{-Alg}$ be a functor where $\mathcal{T}$ and $\mathcal{R}$ are Lawvere theories. Then G is an algebraic functor if and only if $U_{\mathcal{R}}$ is naturally isomorphic to $U_{\mathcal{T}} \circ G$. In this case, G has a left adjoint.

$$\begin{tikzcd}
\mathcal{R}\text{-Alg} \arrow[rr, "G"] \arrow[dr, "U_{\mathcal{R}}"'] & \cong & \mathcal{T}\text{-Alg} \arrow[dl, "U_{\mathcal{T}}"] \\
& \mathrm{Set} &
\end{tikzcd}$$

For a morphism of Lawvere theories $F\colon \mathcal{T} \to \mathcal{R}$, the left adjoint to F^* is denoted by $\mathrm{Fr}_F\colon \mathcal{T}\text{-Alg} \to \mathcal{R}\text{-Alg}$.

1.5 Monadic categories

One slogan on monadic categories could be ‘monadic categories over Set^S are categories of algebras for which we allow an arbitrary large number of operations and equations, provided that free algebras exist'. We will then recover the previous examples of algebraic categories but also the category $\bigvee$-Lat of complete lattices which have a proper class of operations as we said in Example 1.32. Classical references for monads and monadic categories are [11] (where a monad is called a triple), [82] and the second volume of [14].

Definition 1.54. A *monad* $\mathbb{T}$ on a category $\mathcal{C}$ is a triple (T, η, μ) where $T\colon \mathcal{C} \to \mathcal{C}$ is an endofunctor and $\eta\colon 1_{\mathcal{C}} \Rightarrow T$, $\mu\colon TT \Rightarrow T$ are natural transformations such that the equalities

$$\mu \circ (\eta \star 1_T) = 1_T = \mu \circ (1_T \star \eta)$$

and

$$\mu \circ (\mu \star 1_T) = \mu \circ (1_T \star \mu)$$

hold.

$$\begin{tikzcd}
T \arrow[r, "\eta\star 1_T"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] & T \arrow[l, "1_T\star\eta"'] \arrow[dl, "1_T"] \\
& T &
\end{tikzcd}
\qquad
\begin{tikzcd}
TTT \arrow[r, "\mu\star 1_T"] \arrow[d, "1_T\star\mu"'] & TT \arrow[d, "\mu"] \\
TT \arrow[r, "\mu"'] & T
\end{tikzcd}$$

The main example of a monad is given by adjunctions.

Example 1.55. Let $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ be an adjunction with unit $\eta\colon 1_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon\colon FG \Rightarrow 1_{\mathcal{D}}$. This gives rise to a monad (T, η, μ) on $\mathcal{C}$ by setting $T = GF$ and $\mu = 1_G \star \varepsilon \star 1_F$.

Example 1.56. The powerset functor $\mathcal{P}\colon$ Set $\to$ Set together with the natural transformations $\eta\colon 1_{\text{Set}} \Rightarrow \mathcal{P}$ and $\mu\colon \mathcal{PP} \Rightarrow \mathcal{P}$ given by

$$\begin{aligned} \eta_X\colon X &\longrightarrow \mathcal{P}(X) \\ x &\longmapsto \{x\} \end{aligned} \quad \text{and} \quad \begin{aligned} \mu_X\colon \quad \mathcal{PP}(X) &\longrightarrow \mathcal{P}(X) \\ \{A_i | i \in I\} &\longmapsto \bigcup_{i\in I} A_i \end{aligned}$$

for each set X gives rise to the *powerset monad* $\mathbb{P} = (\mathcal{P}, \eta, \mu)$ on Set.

Definition 1.57. [38] Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on the category $\mathcal{C}$. A $\mathbb{T}$-*algebra* is pair (A, α) where A is an object of $\mathcal{C}$ and $\alpha\colon T(A) \to A$

a morphism satisfying the identities $\alpha \circ \eta_A = 1_A$ and $\alpha \circ T(\alpha) = \alpha \circ \mu_A$.

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & T(A) \arrow[d, "\alpha"] & TT(A) \arrow[r, "T(\alpha)"] \arrow[d, "\mu_A"'] & T(A) \arrow[d, "\alpha"] \\
 & A & T(A) \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

A *homomorphism* $f\colon (A, \alpha) \to (B, \beta)$ of $\mathbb{T}$-algebras (or $\mathbb{T}$-*homomorphism*) is a morphism $f\colon A \to B$ in $\mathcal{C}$ such that $\beta \circ T(f) = f \circ \alpha$.

$$\begin{tikzcd}
T(A) \arrow[r, "T(f)"] \arrow[d, "\alpha"'] & T(B) \arrow[d, "\beta"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

This forms the *Eilenberg-Moore category* $\mathcal{C}^{\mathbb{T}}$ for the monad $\mathbb{T}$.

Example 1.58. A $\mathbb{P}$-algebra for the powerset monad $\mathbb{P}$ can be seen as a complete lattice. The morphism $\alpha\colon \mathcal{P}(A) \to A$ sends a subset $S \subseteq A$ to its join $\bigvee S$. A homomorphism of $\mathbb{P}$-algebras is a map which preserves arbitrary joins. The Eilenberg-Moore category $\mathrm{Set}^{\mathbb{P}}$ is thus equivalent to the category $\bigvee$-Lat of complete lattices and sup-preserving maps.

Next proposition describes free $\mathbb{T}$-algebras and tells us that any monad is in fact induced by an adjunction as in Example 1.55.

Proposition 1.59. [38] Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on the category $\mathcal{C}$. The forgetful functor $U^{\mathbb{T}}\colon \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ and the free algebra functor

$$\begin{aligned}
F^{\mathbb{T}}\colon \mathcal{C} &\longrightarrow \mathcal{C}^{\mathbb{T}} \\
A &\longmapsto (T(A), \mu_A) \\
f &\longmapsto T(f)
\end{aligned}$$

form an adjunction $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ which induces, as in Example 1.55, the monad $\mathbb{T}$. The functor $U^{\mathbb{T}}$ is faithful, conservative, preserves, creates and reflects limits which exist in $\mathcal{C}$.

This Eilenberg-Moore adjunction can actually be seen as the terminal adjunction giving rise to the monad $\mathbb{T}$.

Proposition 1.60. [38] Let $\begin{tikzcd}\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]\end{tikzcd}$ be an adjunction inducing the monad $\mathbb{T}$ on $\mathcal{C}$. There exists a unique functor $K\colon \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ such that $U^{\mathbb{T}}K = G$ and $KF = F^{\mathbb{T}}$.

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "K"] \arrow[dr, shift left, "G"] & & \mathcal{C}^{\mathbb{T}} \arrow[dl, shift left, "U^{\mathbb{T}}"] \\
& \mathcal{C} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F^{\mathbb{T}}"] &
\end{tikzcd}$$

Definition 1.61. A functor $G\colon \mathcal{D} \to \mathcal{C}$ is said to be *monadic* if it has a left adjoint $F\colon \mathcal{C} \to \mathcal{D}$ and the functor $K\colon \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ from Proposition 1.60 is an equivalence of categories.

By abuse of notations, we will sometimes write that the category $\mathcal{D}$ is monadic over $\mathcal{C}$ if it is unambiguous which functor $\mathcal{D} \to \mathcal{C}$ we consider. As we said in the introduction of this section, monadic categories over Set^S can be thought of as 'algebraic-like categories' for which we allow an arbitrary large number of operations and equations, in such a way that free algebras exist. With Example 1.58 in mind, we already know that the category $\bigvee$-Lat (with its forgetful functor) is monadic over Set. We can also prove that algebraic categories are monadic over Set^S for some set S (see Theorem 1.66). Moreover, it is proved in [95] that the forgetful functor CompHaus → Set from the category of compact Hausdorff spaces is also monadic. On the other hand, the forgetful functor $U\colon (\bigvee\bigwedge)$-Lat → Set from the category $(\bigvee\bigwedge)$-Lat of complete lattices and maps preserving arbitrary suprema and infima is not monadic because it does not have a left adjoint (the 'free complete lattice' on $\{x, y, z\}$ is not a set but a proper class). For more details, we refer the reader to [88, 2]. Let us now give some properties of monadic categories over Set^S and their characterisation.

Theorem 1.62. Let S be a set and $U\colon \mathcal{C} \to \mathrm{Set}^S$ a monadic functor. Then $\mathcal{C}$ is exact, complete and cocomplete and U is regular and conservative.

Theorem 1.63. [35] For a category $\mathcal{C}$, the following conditions are equivalent:

1. there exists a monadic functor $U\colon \mathcal{C} \to \mathrm{Set}$,
2. (a) $\mathcal{C}$ is exact,
   (b) $\mathcal{C}$ has a projective strong generator P,
   (c) the copower $\coprod_X P$ exists for every set X.

And here is its many-sorted version.

Theorem 1.64. For a category $\mathcal{C}$, the following conditions are equivalent:

1. there exists a set S and a monadic functor $U\colon \mathcal{C} \to \mathrm{Set}^S$,
2. (a) $\mathcal{C}$ is exact,
   (b) $\mathcal{C}$ has a strong set of generators $\mathcal{G}$ such that each object of $\mathcal{G}$ is projective,
   (c) for each set X and function $f : X \to \mathcal{G}$, the coproduct $\coprod_{x\in X} f(x)$ exists in $\mathcal{C}$.

The analogy with Theorem 1.45 is striking. Here is the reason why.

Definition 1.65. Let λ be a regular cardinal and $\mathbb{T} = (T, \eta, \mu)$ a monad on a category $\mathcal{C}$. We say that $\mathbb{T}$ is a *λ-ary monad* if the functor $T\colon \mathcal{C} \to \mathcal{C}$ preserves λ-filtered limits. A monadic functor $G\colon \mathcal{D} \to \mathcal{C}$ is *λ-ary monadic* if the monad induced by the adjunction $F \dashv G$ is λ-ary.

Theorem 1.66. Let λ be a regular cardinal and S a set. The following conditions on a category $\mathcal{C}$ are equivalent:

1. $\mathcal{C}$ is an S-sorted λ-ary algebraic category,
2. there exists a λ-ary monadic functor $U\colon \mathcal{C} \to \mathrm{Set}^S$.

We conclude this section with a lemma about monadic categories over regular categories which will be useful in Chapter 5. It has been stated without proof in [101] and proved in [61].

Lemma 1.67. Let $\mathcal{C}$ be a regular category and $\mathbb{T} = (T, \eta, \mu)$ a monad on it. Then,

1. T preserves regular epimorphisms if and only if $U^{\mathbb{T}}\colon \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ does,
2. if T preserves regular epimorphisms, $U^{\mathbb{T}}$ reflects them. In this case, $\mathcal{C}^{\mathbb{T}}$ is regular.

Proof. 1. Firstly, suppose $U^{\mathbb{T}}$ preserves regular epimorphisms. Since $T = U^{\mathbb{T}}F^{\mathbb{T}}$, T preserves regular epimorphisms since $U^{\mathbb{T}}$ and $F^{\mathbb{T}}$ do. Conversely, suppose T preserves regular epimorphisms. Let $f\colon (A,\alpha) \to (B,\beta)$ be a morphism in $\mathcal{C}^{\mathbb{T}}$. Then, f factors as

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & B \\
& I \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

in $\mathcal{C}$, with p a regular epimorphism and m a monomorphism. So, $T(p)$ is a regular epimorphism and there exists a unique morphism ι making the following diagram commutative.

$$\begin{tikzcd}
T(A) \arrow[r, two heads, "T(p)"] \arrow[d, "p\alpha"'] & T(I) \arrow[d, "\beta T(m)"] \arrow[dl, "\iota"] \\
I \arrow[r, tail, "m"'] & B
\end{tikzcd}$$

Since m is a monomorphism and (B,β) a $\mathbb{T}$-algebra, (I,ι) is also a $\mathbb{T}$-algebra and p and m are $\mathbb{T}$-homomorphisms. Now, if we suppose f to be a regular epimorphism in $\mathcal{C}^{\mathbb{T}}$, m will also be. But since it is a monomorphism, it is an isomorphism in $\mathcal{C}^{\mathbb{T}}$ and so in $\mathcal{C}$. Thus, f is a regular epimorphism in $\mathcal{C}$.

2. Since $U^{\mathbb{T}}$ creates finite limits, $\mathcal{C}^{\mathbb{T}}$ has them. Now, let $f\colon (A,\alpha) \to (B,\beta)$ be a $\mathbb{T}$-homomorphism such that $f\colon A \twoheadrightarrow B$ is a regular epimorphism in $\mathcal{C}$. We denote by

$$\begin{tikzcd}
(R,\rho) \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & (A,\alpha)
\end{tikzcd}$$

the kernel pair of f in $\mathcal{C}^{\mathbb{T}}$. Since $U^{\mathbb{T}}$ preserves limits, it is also its kernel pair in $\mathcal{C}$. So, f is the coequaliser of the pair (r_1, r_2) in $\mathcal{C}$. Using this and the fact that $T(f)$ is an epimorphism in $\mathcal{C}$, one easily proves that

f is the coequaliser of (r_1, r_2) also in $\mathcal{C}^{\mathbb{T}}$. Therefore, $U^{\mathbb{T}}$ reflects regular epimorphisms. Moreover, since $U^{\mathbb{T}}$ preserves and reflects regular epimorphisms and since they are stable under pullbacks in $\mathcal{C}$, they are also in $\mathcal{C}^{\mathbb{T}}$. Finally, considering the construction done in Point 1, the kernel pair of f is also the kernel pair of p, which is a regular epimorphism in $\mathcal{C}^{\mathbb{T}}$. It is thus the coequaliser of its kernel pair, which proves that each kernel pair in $\mathcal{C}^{\mathbb{T}}$ has a coequaliser. $\square$

1.6 Essentially algebraic categories

Gabriel and Ulmer defined in [44] locally presentable categories as cocomplete categories which have a strong set of generators formed by λ-presentable objects for some regular cardinal λ. They characterised them as categories of λ-limit preserving functors from a small category with λ-limits to Set. Compared with Lawvere theories, the important difference is that products have been replaced by limits. The idea is thus that we are now allowed to consider operations defined not from A^n but from a regular subobject of it, i.e., a solution set of some given n-ary equations. This idea has been made precise in [4, 3]. These categories are not regular in general and we give here a syntactic characterisation when they are.

1.6.1 Definition, characterisations and free models

Definition 1.68. [44] Let λ be a regular cardinal. We say that the category $\mathcal{C}$ is *locally λ-presentable* if it is cocomplete and has a strong set of generators $\mathcal{G}$ such that each $G \in \mathcal{G}$ is λ-presentable. If $\lambda = \aleph_0$, we say in that case that $\mathcal{C}$ is *locally finitely presentable.* If $\mathcal{C}$ is locally λ-presentable for some regular cardinal λ, $\mathcal{C}$ is said to be *locally presentable.*

Before giving some examples, let us give two of their characterisations. In what follows, if λ is a regular cardinal and $\mathcal{D}$ a small category with λ-limits, we will denote by λ-Lex$(\mathcal{D}, \text{Set})$ the category of λ-limit preserving functors from $\mathcal{D}$ to Set and natural transformations. If $\lambda = \aleph_0$, we will denote it simply by Lex$(\mathcal{D}, \text{Set})$.

Theorem 1.69. [44] Let λ be a regular cardinal. For a category $\mathcal{C}$, the following conditions are equivalent:

1. $\mathcal{C}$ is locally λ-presentable,
2. $\mathcal{C}$ is equivalent to $\lambda\text{-Lex}(\mathcal{D}, \text{Set})$ for some small category $\mathcal{D}$ with λ-limits.

In that case, $\mathcal{D}$ can be chosen to be equivalent to the dual of the full subcategory of $\mathcal{C}$ of λ-presentable objects.

We now describe them in terms of algebras with partial operations.

Definition 1.70. [4] An *essentially algebraic theory* is a quintuple $\Gamma = (S, \Sigma, E, \Sigma_t, \text{Def})$ where

1. (S, Σ, E) is an algebraic theory,
2. Σ_t is a subset of Σ,
3. Def is a function assigning to each operation symbol $\sigma\colon \prod_{i<n} s_i \to s$ in $\Sigma \setminus \Sigma_t$ a set $\text{Def}(\sigma)$ of equations of Σ_t in the variables from X, the S-sorted set defined by $X_{s'} = \{x_i \,|\, i < n, s_i = s'\}$ for each $s' \in S$.

Definition 1.71. [4] Let λ be a regular cardinal. The essentially algebraic theory Γ is said to be λ-*ary* if

1. (S, Σ, E) is λ-ary,
2. for each $\sigma \in \Sigma \setminus \Sigma_t$, $\#\,\text{Def}(\sigma) < \lambda$.

If $\lambda = \aleph_0$, we say in this case that Γ is a *finitary essentially algebraic theory.*

Definition 1.72. [4] Let Γ be an essentially algebraic theory. A Γ-*model* is an S-sorted set A together with, for each operation symbol $\sigma\colon \prod_{i<n} s_i \to s$ in Σ, a partial function $\sigma^A\colon \prod_{i<n} A_{s_i} \to A_s$ such that:

1. for each $\sigma \in \Sigma_t$, σ^A is defined everywhere,

2. given $\sigma\colon \prod_{i<n} s_i \to s$ in $\Sigma \setminus \Sigma_t$ and a family $(a_i \in A_{s_i})_{i<n}$, $\sigma^A((a_i)_{i<n})$ is defined if and only if the elements a_i's satisfy all equations of $\mathrm{Def}(\sigma)$ in A,

3. A satisfies the equations of E wherever they are defined.

We can easily give a precise meaning to when some elements of A satisfy a given equation and when a term is defined at some elements using the recursive definition of a term (see page 28). This is left to the reader.

Definition 1.73. [4] Let Γ be an essentially algebraic theory and A, B two Γ-models. A Γ-*homomorphism* $f\colon A \to B$ is an S-sorted function such that, given $\sigma\colon \prod_{i<n} s_i \to s$ in Σ and a family $(a_i \in A_{s_i})_{i<n}$ such that $\sigma^A((a_i)_{i<n})$ is defined, the identity

$$f_s(\sigma^A(a_i)) = \sigma^B(f_{s_i}(a_i)) \tag{1}$$

holds.

Notice that if (1) holds for all $\sigma \in \Sigma_t$, then for each $\sigma' \in \Sigma \setminus \Sigma_t$, $\sigma'^B(f_{s_i}(a_i))$ is defined if $\sigma'^A(a_i)$ is, while the converse does not hold in general. The category of Γ-models and their homomorphisms is denoted by $\mathrm{Mod}(\Gamma)$. A category which is equivalent to some model category $\mathrm{Mod}(\Gamma)$ for an essentially algebraic theory Γ is called *essentially algebraic*. The notions of λ-*ary* and *finitary essentially algebraic categories* are defined in the obvious way.

Theorem 1.74. [44, 4, 3] A category $\mathcal{C}$ is locally presentable if and only if it is essentially algebraic. More precisely, if λ is a regular cardinal, $\mathcal{C}$ is locally λ-presentable if and only if it is λ-ary essentially algebraic.

Example 1.75. Of course, every algebraic category is essentially algebraic. As another example, one can say that Cat is locally finitely presentable. Indeed, the only partial operation defining small categories is the composition law. This is an operation $A^2 \to A$ (where A denotes the set of arrows) and it is defined on the set $\{(f,g) \in A^2 \mid c(f) = d(g)\}$ where d and c are the domain and codomain maps. Note that this solution set is nothing but the pullback of c along d.

If $\aleph_1$ denotes the successor cardinal of $\aleph_0$, the category Ban of (real) Banach spaces and linear mappings L of norm $||L|| \leqslant 1$ is locally $\aleph_1$-presentable [86, 4, 14].

To give another family of examples, we need a few definitions.

Definition 1.76. Let S be a set of sorts and Σ an S-sorted signature of algebras. An *implication* in Σ

$$\bigwedge_{i \in I} (t_i = t'_i) \Rightarrow (t = t')$$

is the collection of an S-sorted set X, an equation $(t = t')$ in Σ in the variables from X, a set I and for each $i \in I$, an equation $(t_i = t'_i)$ in Σ in the variables from X. A Σ-algebra A *satisfies* this implication when, for any S-sorted function $f \colon X \to U_\Sigma(A)$, if the unique Σ-homomorphism $\overline{f} \colon \mathrm{Fr}_\Sigma(X) \to A$ extending f satisfies $\overline{f}(t_i) = \overline{f}(t'_i)$ for each $i \in I$, then $\overline{f}(t) = \overline{f}(t')$.

Note that an equation can be seen as an implication with $I = \varnothing$.

Definition 1.77. A category $\mathcal{C}$ is said to be a *quasivariety* if there exists a set S of sorts, an S-sorted signature of algebras Σ and a set of implications in Σ such that $\mathcal{C}$ is equivalent to the full subcategory of Σ-Alg of Σ-algebras satisfying the given implications.

Example 1.78. The category TorsFreeAb is a quasivariety. Indeed, an object in it is an abelian group satisfying the implications

$$\begin{aligned} x + x = 0 &\Longrightarrow x = 0 \\ x + x + x = 0 &\Longrightarrow x = 0 \\ &\vdots \end{aligned}$$

for each element x.

Proposition 1.79. [4] Every quasivariety is essentially algebraic.

Let us go back to essentially algebraic categories.

Proposition 1.80. Let Γ be an essentially algebraic theory. The category $\operatorname{Mod}(\Gamma)$ is complete, cocomplete and the forgetful functor $U_\Gamma : \operatorname{Mod}(\Gamma) \to \mathrm{Set}^S$ is conservative and has a left adjoint.

The fact that U_Γ has a left adjoint implies it preserves limits so that small limits in $\operatorname{Mod}(\Gamma)$ are computed in each component as in Set. To describe this left adjoint $\mathrm{Fr}_\Gamma \colon \mathrm{Set}^S \to \operatorname{Mod}(\Gamma)$, we need to introduce the notion of an everywhere-defined term.

Definition 1.81. Let Γ be an essentially algebraic theory and $t_1, t_2 : \prod_{i \in I} s_i \to s$ two terms of Σ. We say that $t_1 = t_2$ is a *theorem* of Γ if, given a Γ-model A and a family $(a_i \in A_{s_i})_{i \in I}$ such that $t_1((a_i)_{i \in I})$ and $t_2((a_i)_{i \in I})$ are both defined, the equality $t_1((a_i)_{i \in I}) = t_2((a_i)_{i \in I})$ holds in A_s.

Definition 1.82. Let Γ be an essentially algebraic theory and X an S-sorted set. A term of Σ in the variables from X is said to be *everywhere-defined* if it belongs to the smallest S-subset Y of $\mathrm{Fr}_\Sigma(X)$ satisfying the following conditions:

1. for each $s \in S$, $X_s \subseteq Y_s$,
2. if $\sigma \colon \prod_{i<n} s_i \to s$ is in Σ and $(t_i \in Y_{s_i})_{i<n}$ is a family such that either $\sigma \in \Sigma_t$ or $\sigma \in \Sigma \backslash \Sigma_t$ and for each equation $(u_1, u_2) \in \operatorname{Def}(\sigma)$, $u_1((t_i)_{i<n}) = u_2((t_i)_{i<n})$ is a theorem of Γ, then the term $\sigma((t_i)_{i<n})$ belongs to Y_s.

Intuitively, everywhere-defined terms are terms which are everywhere defined in any Γ-model. We can now describe the left adjoint Fr_Γ to U_Γ as follows. If X is an S-sorted set, for each $s \in S$, $\mathrm{Fr}_\Gamma(X)_s$ is the set of equivalence classes of everywhere-defined terms of sort s in the variables from X, where we identify the two terms t_1 and t_2 if and only if $t_1 = t_2$ is a theorem of Γ. Notice that this is an equivalence relation since we only consider everywhere-defined terms. The Γ-model structure on $\mathrm{Fr}_\Gamma(X)$ and the S-sorted function $X \to U_\Gamma \mathrm{Fr}_\Gamma(X)$ are defined in the obvious way. This is the reflection of X along U_Γ and this describes the announced adjunction $\mathrm{Fr}_\Gamma \dashv U_\Gamma$.

1.6.2 Regular essentially algebraic categories

Essentially algebraic categories are in general not regular. For instance, Cat is locally finitely presentable (Example 1.75) but not regular (Example 1.7). However, we still have a *(strong epi, mono)-factorisation system* in those categories. By that we of course mean a factorisation system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ is the class of all strong epimorphisms and $\mathcal{M}$ the class of all monomorphisms.

Definition 1.83. Let A be a Γ-model for an essentially algebraic theory Γ. A *submodel* of A is a Γ-model B included in A for which the inclusion $B \hookrightarrow A$ is a Γ-homomorphism.

Proposition 1.84. Let Γ be an essentially algebraic theory. The category $\mathrm{Mod}(\Gamma)$ has a (strong epi, mono)-factorisation system.

Proof. Properties 1, 2 and 3 of Definition 1.12 are true in any category for the above cited classes $\mathcal{E}$ and $\mathcal{M}$. For the last one, if we have a homomorphism $f\colon A \to B$ of Γ-models, we can factorise it in the following way. We consider the submodel I of B defined by

$$I_s = \Big\{ t((f_{s_i}(a_i))_{i\in I}) \mid t\colon \prod_{i\in I} s_i \to s \text{ is a term of } \Sigma \text{ and } (a_i \in A_{s_i})_{i\in I} \text{ a} \\ \text{family such that } t((f_{s_i}(a_i))_{i\in I}) \text{ is defined in } B \Big\}$$

for each sort $s \in S$. I is the smallest submodel of B for which $f_s(a) \in I_s$ for all $s \in S$ and $a \in A_s$. This means that the corestriction $p\colon A \twoheadrightarrow I$ of f to I is an extremal epimorphism (and so a strong one since $\mathrm{Mod}(\Gamma)$ is complete) and f factors as $f = ip$ with i the inclusion $I \hookrightarrow B$. $\square$

As usual, we will refer to $I = \mathrm{Im}(f)$ as the *image* of f. Notice that in the case where Γ is finitary, we could have considered in the definition of $\mathrm{Im}(f)_s$ only finitary terms $t\colon \prod_{i=1}^{n} s_i \to s$ in the variables from a *finite S-sorted set* X (i.e., $\# \coprod_{s\in S} X_s < \aleph_0$). We also remark that $t((f_{s_i}(a_i))_{i\in I})$ could not have been written as $f_s(t((a_i)_{i\in I}))$ (as we do in the algebraic case) since $t((a_i)_{i\in I})$ might not be defined in A.

In [30], regular locally finitely presentable categories $\mathrm{Lex}(\mathcal{D}, \mathrm{Set})$ have been characterised in terms of the corresponding theory $\mathcal{D}$.

Definition 1.85. [30] A category $\mathcal{C}$ is said to be *weakly regular* if any commutative square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, two heads, "g"'] & D
\end{tikzcd}$$

with g a regular epimorphism can be factorised in a commutative diagram as

$$\begin{tikzcd}
A \arrow[dr, "l"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "h"'] & & \\
& A' \arrow[r, two heads, "g'"] \arrow[d, "k'"'] & B \arrow[d, "k"] \\
& C \arrow[r, two heads, "g"'] & D
\end{tikzcd}$$

where g' is a regular epimorphism.

Of course, any regular category is weakly regular.

Theorem 1.86. [30] Let $\mathcal{D}$ be a finitely complete small category. The following conditions are equivalent:

1. $\mathrm{Lex}(\mathcal{D}, \mathrm{Set})$ is regular,

2. $\mathcal{D}^{\mathrm{op}}$ is weakly regular.

Such characterisations in terms of the corresponding category $\mathcal{D}$ have been called ‘syntactic’ in [30]. However, it does not seem easy to derive from Theorem 1.86 a characterisation of those (finitary) Γ for which $\mathrm{Mod}(\Gamma)$ is regular. It is this second type of characterisations that we will call ‘syntactic’ in this thesis. We provide now such a syntactic characterisation of essentially algebraic theories for which the category of models is regular. The finitary case has been proved in [55].

Lemma 1.87. Let Γ be an essentially algebraic theory and $t\colon \prod_{i\in I} s_i \to s$ a term of Σ. If $(a_i \in A_{s_i})_{i\in I}$ are elements of a Γ-model A, we can find a strong epimorphism $q\colon A \twoheadrightarrow B$ in $\mathrm{Mod}(\Gamma)$ such that $t((q(a_i))_{i\in I})$ is defined in B and if $f\colon A \to C$ is a homomorphism such that $t((f(a_i))_{i\in I})$ is defined in C, then f factors uniquely through q.

Proof. We are going to prove this lemma by induction on the number of steps used in the construction of the term t. If t is a projection (or any everywhere-defined term), 1_A is the homomorphism we are looking for. Now, suppose t uses the operation symbols or projections

$$\sigma_j \in \Sigma \cup \{p_k \colon \prod_{i\in I} s_i \to s_k \mid k \in I\}$$

for each $j \in J$ as first step of its construction. Thus, t can be written as

$$t = t'((\sigma_j((x_i)_{i\in I}))_{j\in J})$$

where $t' \colon \prod_{j\in J} s'_j \to s$ uses less steps than t to be constructed. Let R be the smallest submodel of $A \times A$ which contains $(u((a_i)_{i\in I}), u'((a_i)_{i\in I}))$ for each $j \in J$ such that $\sigma_j \in \Sigma \setminus \Sigma_t$ and each equation $(u, u') \in \mathrm{Def}(\sigma_j)$. Let q_1 be the coequaliser of r_1 and r_2

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A \arrow[r, two heads, "q_1"] & B_1 \end{tikzcd}$$

where $r_i = \pi_i r$ with r the inclusion $R \hookrightarrow A \times A$ and π_1 and π_2 the projections. Thus, in B_1, all $\sigma_j((q_1(a_i))_{i\in I})$ are defined. Now, we use the induction hypothesis on t' to build a universal strong epimorphism $q_2 \colon B_1 \twoheadrightarrow B$ such that

$$\begin{aligned} t'((q_2(\sigma_j((q_1(a_i))_{i\in I})))_{j\in J}) &= t'((\sigma_j((q_2q_1(a_i))_{i\in I}))_{j\in J}) \\ &= t((q_2q_1(a_i))_{i\in I}) \end{aligned}$$

is defined. Let us prove that q_2q_1 is the strong epimorphism we are looking for. Let $f \colon A \to C$ be a homomorphism such that $t((f(a_i))_{i\in I})$ is defined. Since the kernel pair $R[f]$ of f contains $(u((a_i)_{i\in I}), u'((a_i)_{i\in I}))$ for all j such that $\sigma_j \in \Sigma \setminus \Sigma_t$ and all equations $(u, u') \in \mathrm{Def}(\sigma_j)$, we have $R \subseteq R[f]$ and $fr_1 = fr_2$. Therefore, f factors through q_1 as $f = gq_1$. Finally, g factors through q_2 since

$$\begin{aligned} t((f(a_i))_{i\in I}) &= t((gq_1(a_i))_{i\in I}) \\ &= t'((\sigma_j((gq_1(a_i))_{i\in I}))_{j\in J}) \\ &= t'((g(\sigma_j((q_1(a_i))_{i\in I})))_{j\in J}) \end{aligned}$$

is defined. □

Theorem 1.88. Let Γ be an essentially algebraic theory. Then $\mathrm{Mod}(\Gamma)$ is a regular category if and only if, for each term $t\colon \prod_{i\in I} s_i \to s$ of Σ, there exists in Γ

- a term $\pi\colon \prod_{j\in J} s'_j \to s$,
- for each $j \in J$, an everywhere-defined term $\alpha_j\colon s \to s'_j$,
- for each $j \in J$, an everywhere-defined term $\mu_j\colon \prod_{i\in I} s_i \to s'_j$

such that

1. $\pi((\alpha_j(x))_{j\in J})$ is an everywhere-defined term $s \to s$,
2. $\pi((\alpha_j(x))_{j\in J}) = x$ is a theorem of Γ,
3. $\alpha_j(t((x_i)_{i\in I})) = \mu_j((x_i)_{i\in I})$ is a theorem of Γ for each $j \in J$.

Proof. Since $\mathrm{Mod}(\Gamma)$ is complete and has a (strong epi, mono) factorisation system, by Proposition 1.16, it is regular if and only if strong epimorphisms are pullback stable. So, let us suppose the condition in the statement holds in Γ and consider a pullback square

$$\begin{tikzcd}
P \arrow[r, "p'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\
A \arrow[r, two heads, "p"'] & C
\end{tikzcd}$$

in $\mathrm{Mod}(\Gamma)$ with p a strong epimorphism. We have to prove that $\mathrm{Im}(p') = B$. So, let $b \in B_s$ for some $s \in S$. Since p is a strong epimorphism, there exists a term $t\colon \prod_{i\in I} s_i \to s$ of Σ and a family $(a_i \in A_{s_i})_{i\in I}$ such that $t((p(a_i))_{i\in I})$ is defined and equal to $f(b)$ (see the description of $\mathrm{Im}(p)$ from Proposition 1.84). Let π, α_j's and μ_j's be the terms given by the assumption for this t. For each $j \in J$,

$$\begin{aligned}
f(\alpha_j(b)) &= \alpha_j(f(b)) \\
&= \alpha_j(t((p(a_i))_{i\in I})) \\
&= \mu_j((p(a_i))_{i\in I}) \\
&= p(\mu_j((a_i)_{i\in I}))
\end{aligned}$$

since α_j and μ_j are everywhere-defined. But small limits in Mod(Γ) are computed in each sort as in Set. Hence, $d_j = (\mu_j((a_i)_{i\in I}), \alpha_j(b)) \in P_{s'_j}$ with

$$b = \pi((\alpha_j(b))_{j\in J}) = \pi((p'(d_j))_{j\in J}).$$

Therefore, $b \in \mathrm{Im}(p')_s$ and p' is a strong epimorphism.

Conversely, we assume that Mod(Γ) is regular and consider a term $t\colon \prod_{i\in I} s_i \to s$ of Σ. Let X and Y be the S-sorted sets defined by $X_{s'} = \{x_i \,|\, i \in I, s_i = s'\}$ for each $s' \in S$, $Y_s = \{y\}$ and $Y_{s'} = \varnothing$ for all $s' \neq s \in S$. We consider also the strong epimorphism $q\colon \mathrm{Fr}_\Gamma(X) \twoheadrightarrow B$ given by Lemma 1.87, for the term t and the elements $x_i \in \mathrm{Fr}_\Gamma(X)_{s_i}$. Thus $t((q(x_i))_{i\in I})$ is defined. Let $f\colon \mathrm{Fr}_\Gamma(Y) \to B$ be the unique map such that $f(y) = t((q(x_i))_{i\in I})$ and consider the pullback of q along f.

$$\begin{tikzcd}
P \arrow[r, two heads, "p"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fr}_\Gamma(Y) \arrow[d, "f"] \\
\mathrm{Fr}_\Gamma(X) \arrow[r, two heads, "q"'] & B
\end{tikzcd}$$

Since Mod(Γ) is regular, p is also a strong epimorphism. So, $y \in \mathrm{Im}(p)_s$ which means that there exists a term $\pi\colon \prod_{j\in J} s'_j \to s$ and elements $d_j \in P_{s'_j}$ for each $j \in J$ such that $\pi((p(d_j))_{j\in J})$ is defined and equal to y. Using the descriptions of P, $\mathrm{Fr}_\Gamma(X)$ and $\mathrm{Fr}_\Gamma(Y)$, it implies that, for each $j \in J$, there exist everywhere-defined terms $\alpha_j\colon s \to s'_j$ and $\mu_j\colon \prod_{i\in I} s_i \to s'_j$ such that $d_j = (\mu_j, \alpha_j)$. Thus, the equalities

$$\begin{aligned} y &= \pi((p(d_j))_{j\in J}) \\ &= \pi((\alpha_j(y))_{j\in J}) \end{aligned} \tag{2}$$

hold in $\mathrm{Fr}_\Gamma(Y)_s$ and for each $j \in J$, we have

$$\begin{aligned} \mu_j((q(x_i))_{i\in I}) &= q(\mu_j((x_i)_{i\in I})) \\ &= f(\alpha_j(y)) \\ &= \alpha_j(f(y)) \\ &= \alpha_j(t((q(x_i))_{i\in I})) \end{aligned} \tag{3}$$

in $B_{s'_j}$. Equalities (2) mean that $\pi((\alpha_j(x))_{j\in J})$ is everywhere-defined and $\pi((\alpha_j(x))_{j\in J}) = x$ is a theorem of Γ. With the universal properties of $\mathrm{Fr}_\Gamma(X)$ and q, equalities (3) mean that $\alpha_j(t((a_i)_{i\in I})) = \mu_j((a_i)_{i\in I})$ holds in any Γ-model A as soon as $t((a_i)_{i\in I})$ is defined. Indeed, if it is so, let $g\colon \mathrm{Fr}_\Gamma(X) \to A$ be the unique Γ-homomorphism such that $g(x_i) = a_i$ for all $i \in I$. Since $t((g(x_i))_{i\in I})$ is defined, g factors through q as $g = hq$ and

$$\begin{aligned}
\alpha_j(t((a_i)_{i\in I})) &= \alpha_j(t((hq(x_i))_{i\in I})) \\
&= h(\alpha_j(t((q(x_i))_{i\in I}))) \\
&= h(\mu_j((q(x_i))_{i\in I})) \\
&= \mu_j((hq(x_i))_{i\in I}) \\
&= \mu_j((a_i)_{i\in I}).
\end{aligned}$$ ☐

Using the remark after Proposition 1.84, if Γ is finitary, we are allowed to only consider finitary terms $t\colon \prod_{i=1}^n s_i \to s$.

Theorem 1.89. [55] Let Γ be a finitary essentially algebraic theory. Then $\mathrm{Mod}(\Gamma)$ is a regular category if and only if, for each finitary term $t\colon \prod_{i=1}^n s_i \to s$ of Σ, there exists in Γ:

- a finitary term $\pi\colon \prod_{j=1}^m s'_j \to s$,
- for each $1 \leqslant j \leqslant m$, an everywhere-defined term $\alpha_j\colon s \to s'_j$,
- for each $1 \leqslant j \leqslant m$, an everywhere-defined term $\mu_j\colon \prod_{i=1}^n s_i \to s'_j$

such that

1. $\pi(\alpha_1(x), \dots, \alpha_m(x))$ is an everywhere-defined term $s \to s$,
2. $\pi(\alpha_1(x), \dots, \alpha_m(x)) = x$ is a theorem of Γ,
3. $\alpha_j(t(x_1, \dots, x_n)) = \mu_j(x_1, \dots, x_n)$ is a theorem of Γ for each $1 \leqslant j \leqslant m$.

1.7 Sketches

In this section we deal with the concept of a sketch, introduced by Ehresmann in a slightly different form (see e.g. [36] or the second volume of [14]). We can get in this way a common framework for the notions of algebraic and essentially algebraic categories.

Definition 1.90. A *sketch* $\mathbb{S}$ is a triple $(\mathcal{S}, \mathcal{P}, \mathcal{I})$ where:

1. $\mathcal{S}$ is a small category,
2. $\mathcal{P}$ is a set of cones on functors $F\colon \mathcal{D} \to \mathcal{S}$, defined on small categories $\mathcal{D}$,
3. $\mathcal{I}$ is a set of cocones on functors $F\colon \mathcal{D} \to \mathcal{S}$, defined on small categories $\mathcal{D}$,

and we consider the two cones $\lambda\colon \Delta_X \Rightarrow F$ and $\lambda'\colon \Delta_X \Rightarrow F'$ equal if there exists an isomorphism $i\colon \mathcal{D} \to \mathcal{D}'$ of categories such that $F'i = F$ and $\lambda_D = \lambda'_{i(D)}$ for each $D \in \mathcal{D}$ (and similarly for cocones). A *morphism of sketches* $F\colon \mathbb{S} \to \mathbb{S}'$ is a functor $F\colon \mathcal{S} \to \mathcal{S}'$ which maps each cone in $\mathcal{P}$ on a cone in $\mathcal{P}'$ and each cocone in $\mathcal{I}$ on a cocone in $\mathcal{I}'$. This gives the category Sk of sketches.

Remark 1.91. Each small category $\mathcal{C}$ can be viewed as a sketch in several ways. Firstly, $\mathrm{I}(\mathcal{C}) = (\mathcal{C}, \varnothing, \varnothing)$ is the *indiscrete sketch* on $\mathcal{C}$. This gives rise to the adjunction

$$\begin{tikzcd} \mathrm{Sk} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend right, "\mathrm{I}"'] \end{tikzcd}$$

where U is the forgetful functor. Besides, $\mathrm{D}(\mathcal{C}) = (\mathcal{C}, \mathcal{P}, \mathcal{I})$ is the *discrete $\mathcal{V}$-sketch* on $\mathcal{C}$ where $\mathcal{P}$ (resp. $\mathcal{I}$) is the class of all small limit cones (resp. all small colimit cocones) in $\mathcal{C}$ (for a bigger universe $\mathcal{V} \ni \mathcal{U}$). As an intermediate step, we also have $\mathrm{FD}(\mathcal{C}) = (\mathcal{C}, \mathcal{P}, \mathcal{I})$, the *finitely discrete sketch* on $\mathcal{C}$ where $\mathcal{P}$ (resp. $\mathcal{I}$) is the set of all finite limit cones (resp. finite colimit cocones) in $\mathcal{C}$.

Definition 1.92. [36] Let $\mathbb{S} = (\mathcal{S}, \mathcal{P}, \mathcal{I})$ be a sketch. An $\mathbb{S}$-*model* is a functor $A\colon \mathcal{S} \to \mathrm{Set}$ which sends each cone in $\mathcal{P}$ (resp. each cocone in $\mathcal{I}$) to a limit (resp. a colimit) in Set. A *homomorphism* $f\colon A \to B$ of $\mathbb{S}$-models is a natural transformation $A \Rightarrow B$. The corresponding category is denoted by $\mathrm{Mod}(\mathbb{S})$.

The following characterisation is the idea behind Lawvere's work [76].

Theorem 1.93. For a category $\mathcal{C}$, the following conditions are equivalent:

1. $\mathcal{C}$ is an algebraic category,
2. $\mathcal{C}$ is equivalent to a category of the form $\mathrm{Mod}(\mathbb{S})$ for a sketch $\mathbb{S} = (\mathcal{S}, \mathcal{P}, \varnothing)$ where cones in $\mathcal{P}$ are cones over discrete diagrams.

Its essentially algebraic version is the idea behind Gabriel-Ulmer's work [44].

Theorem 1.94. For a category $\mathcal{C}$, the following conditions are equivalent:

1. $\mathcal{C}$ is locally presentable,
2. $\mathcal{C}$ is equivalent to a category of the form $\mathrm{Mod}(\mathbb{S})$ for a sketch $\mathbb{S} = (\mathcal{S}, \mathcal{P}, \varnothing)$.

1.8 Internal algebras and $\mathcal{T}$-enrichments

Replacing the category Set by an arbitrary category $\mathcal{C}$ (with enough products), we get the notion of an internal algebra.

Definition 1.95. Let λ be a regular cardinal, (S, Σ, E) a λ-ary algebraic theory and $\mathcal{C}$ a category with λ-products. An *internal* (S, Σ, E)*-algebra* A in $\mathcal{C}$ is a collection of objects $(A_s \in \mathcal{C})_{s \in S}$ together with, for each $\sigma \colon \prod_{i<n} s_i \to s$ in Σ, a morphism $\sigma^A \colon \prod_{i<n} A_{s_i} \to A_s$ such that they satisfy the equations of E. A *homomorphism* $f \colon A \to B$ of internal (S, Σ, E)-algebras is a collection of morphisms $(f_s \colon A_s \to B_s)_{s \in S}$ such that, for each $\sigma \colon \prod_{i<n} s_i \to s$ in Σ, the square

$$\begin{tikzcd}
\prod_{i<n} A_{s_i} \arrow[r, "\sigma^A"] \arrow[d, "\prod_{i<n} f_{s_i}"'] & A_s \arrow[d, "f_s"] \\
\prod_{i<n} B_{s_i} \arrow[r, "\sigma^B"'] & B_s
\end{tikzcd}$$

commutes. The corresponding category is denoted by (S, Σ, E)-Alg$(\mathcal{C})$ and the forgetful functor by $U_{(S,\Sigma,E)}\colon (S, \Sigma, E)\text{-Alg}(\mathcal{C}) \to \mathcal{C}^S$.

The sentence 'they satisfy the equations of E' has to be understood in the obvious sense: To each term $t\colon \prod_{i\in I} s_i \to s$ in Σ in less than λ variables, we associate a morphism $t^A\colon \prod_{i\in I} A_{s_i} \to A_s$ recursively by:

1. to the variable x_j of sort s_j for $j \in I$, we associate the j-th projection $\pi_j = x_j^A\colon \prod_{i\in I} A_{s_i} \to A_{s_j}$,
2. if for the terms $t_j\colon \prod_{i\in I} s_i \to s'_j$ $(j < n)$, the morphisms t_j^A have been defined and if $\sigma\colon \prod_{j<n} s'_j \to s$ is in Σ, we associate the morphism

$$\begin{tikzcd}\prod_{i\in I} A_{s_i} \arrow[r, "(t_j^A)_{j<n}"] & \prod_{j<n} A_{s'_j} \arrow[r, "\sigma^A"] & A_s\end{tikzcd}$$

   to the term $\sigma((t_j)_{j\in J})$.

Then, A satisfies the equation (t_1, t_2) if $t_1^A = t_2^A$.

The situation is easier for Lawvere theories.

Definition 1.96. Let $\mathcal{T}$ be a Lawvere theory and $\mathcal{C}$ a category with finite products. An *internal $\mathcal{T}$-algebra* in $\mathcal{C}$ is a finite product preserving functor $A\colon \mathcal{T} \to \mathcal{C}$. A *homomorphism* $f\colon A \to B$ of internal $\mathcal{T}$-algebras is a natural transformation $A \Rightarrow B$. This gives rise to the category $\mathcal{T}$-Alg$(\mathcal{C})$ and the functor $U_{\mathcal{T}}\colon \mathcal{T}\text{-Alg}(\mathcal{C}) \to \mathcal{C}$ of evaluation at 1.

As in the classical case, we have the following equivalences.

Proposition 1.97. Let $\mathcal{C}$ be a category with finite products.

1. If (Σ, E) is a one-sorted finitary algebraic theory, there is an equivalence between the categories (Σ, E)-Alg$(\mathcal{C})$ and $\mathcal{T}_{(\Sigma,E)}$-Alg$(\mathcal{C})$ making the triangle

$$\begin{tikzcd}(\Sigma, E)\text{-Alg}(\mathcal{C}) \arrow[rr, "\sim"] \arrow[dr, "U_{(\Sigma,E)}"'] & & \mathcal{T}_{(\Sigma,E)}\text{-Alg}(\mathcal{C}) \arrow[dl, "U_{\mathcal{T}_{(\Sigma,E)}}"] \\ & \mathcal{C} & \end{tikzcd}$$

   commute.

2. If $\mathcal{T}$ is a Lawvere theory, there is an equivalence between the categories $(\Sigma_{\mathcal{T}}, E_{\mathcal{T}})$-$\mathrm{Alg}(\mathcal{C})$ and $\mathcal{T}$-$\mathrm{Alg}(\mathcal{C})$ making the triangle

$$\begin{tikzcd}
(\Sigma_{\mathcal{T}}, E_{\mathcal{T}})\text{-}\mathrm{Alg}(\mathcal{C}) \arrow[rr, "\sim"] \arrow[dr, "U_{(\Sigma_{\mathcal{T}}, E_{\mathcal{T}})}"'] & & \mathcal{T}\text{-}\mathrm{Alg}(\mathcal{C}) \arrow[dl, "U_{\mathcal{T}}"] \\
& \mathcal{C} &
\end{tikzcd}$$

commute.

Example 1.98. In a category $\mathcal{C}$ with finite products, an *internal group* is an object G together with morphisms $0\colon 1 \to G$, $+\colon G^2 \to G$ and $-\colon G \to G$ satisfying the usual axioms for groups.

Without going into the details here, it is possible to define internal Γ-models in a complete category for an essentially algebraic theory Γ. For example, this leads to the notion of an *internal category* in a category $\mathcal{C}$ with finite limits. This is the collection of two objects A_0 and A_1 together with morphisms

$$\begin{tikzcd}
A_1 \times_{c,d} A_1 \arrow[r, "m"] & A_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & A_0 \arrow[l, bend left=50, "e"]
\end{tikzcd}$$

satisfying the well-known axioms for categories, where $A_1 \times_{c,d} A_1$ is the pullback of c along d.

$$\begin{tikzcd}
A_1 \times_{c,d} A_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_1 \arrow[d, "d"] \\
A_1 \arrow[r, "c"'] & A_0
\end{tikzcd}$$

Similarly, an *internal groupoid* is an internal category $\mathbb{A}$ together with an additional morphism $i\colon A_1 \to A_1$ which satisfies the axioms for inverses.

Let us now get back to the case of a Lawvere theory $\mathcal{T}$.

Definition 1.99. [72] Let $\mathcal{T}$ be a Lawvere theory. A *$\mathcal{T}$-enriched category* (or *$\mathcal{T}$-category* in short) is a category $\mathcal{C}$ equipped with a factorisation

Hom of the hom functor through $U_{\mathcal{T}}$.

$$\begin{tikzcd} & \mathcal{T}\text{-Alg} \arrow[d, "U_{\mathcal{T}}"] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[ur, "\mathrm{Hom}"] \arrow[r, "\mathrm{hom}"'] & \mathrm{Set} \end{tikzcd}$$

Such a factorisation is called a *$\mathcal{T}$-enrichment* of $\mathcal{C}$. A *$\mathcal{T}$-enriched functor* (or *$\mathcal{T}$-functor* in short) between the $\mathcal{T}$-enriched categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that for all objects $A, B \in \mathcal{C}$,

$$F\colon\ \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(A),F(B))$$

is a homomorphism of $\mathcal{T}$-algebras. Small $\mathcal{T}$-categories and $\mathcal{T}$-functors form the category $\mathcal{T}$-Cat.

If $\mathcal{C}$ has finite products, a $\mathcal{T}$-enrichment of $\mathcal{C}$ can be defined in an equivalent way as a section for the forgetful functor $U_{\mathcal{T}}\colon\ \mathcal{T}\text{-Alg}(\mathcal{C}) \to \mathcal{C}$. In order words, it is the assignment of an internal $\mathcal{T}$-algebra structure on each object of $\mathcal{C}$ in such a way that every morphism is a homomorphism of internal $\mathcal{T}$-algebras. Given a factorisation Hom as in Definition 1.99, one gets an internal $\mathcal{T}$-algebra structure on $A \in \mathcal{C}$ with

$$t^A = t(\pi_1, \dots, \pi_n)\colon A^n \to A$$

for each n-ary term t, where the π_i's are the product projections. Conversely, given a section of $U_{\mathcal{T}}$, one gets the Hom factorisation letting $t(f_1, \dots, f_n)$ be the map

$$\begin{tikzcd} A \arrow[r, "(f_1,\dots,f_n)"] & B^n \arrow[r, "t^B"] & B \end{tikzcd}$$

for each n-ary term t and morphisms $f_1, \dots, f_n\colon A \to B$.

It is easy to see that a $\mathcal{T}$-enrichment of a category $\mathcal{C}$ induces a $\mathcal{T}$-enrichment on every functor category $\mathcal{C}^{\mathcal{P}}$ for a small category $\mathcal{P}$, looking componentwise (and so on every power $\mathcal{C}^S$ for a set S). We can also notice that $\mathcal{T}$-enrichments on a category $\mathcal{C}$ are in one-to-one correspondence with $\mathcal{T}$-enrichments on its dual $\mathcal{C}^{\mathrm{op}}$. Hence, if $\mathcal{C}$ has finite coproducts, it gives rise to an internal $\mathcal{T}$-algebra structure on every object of $\mathcal{C}^{\mathrm{op}}$, which is called an *internal $\mathcal{T}$-co-algebra* in $\mathcal{C}$. We thus have a morphism

$$t^{A,\mathrm{op}} = t(\iota_1, \dots, \iota_n) \colon A \to nA$$

for each n-ary term t of $\mathcal{T}$, where $\iota_1, \dots, \iota_n$ are the coproduct injections.

Example 1.100. A Th[Set]-category is nothing but a usual category. A Th[Set$_*$]-category is also called a *pointed category.* If $\mathcal{C}$ has a terminal object 1, there is at most one Th[Set$_*$]-enrichment of $\mathcal{C}$. It exists if and only if 1 is also a initial object, called in that case a *zero object.* Then, for $A, B \in \mathcal{C}$, the morphism $0 \colon A \to B$ is the unique morphism $A \to 0 \to B$ which factors through the zero object $1 = 0$.

A Th[Ab]-category is also called a *preadditive category.* If $\mathcal{C}$ has finite products, there is at most one Th[Ab]-enrichment of $\mathcal{C}$. If it exists, we say that $\mathcal{C}$ is an *additive category.*

We would like now to describe $\mathcal{T}$-enrichments of algebraic categories.

Definition 1.101. Let s (resp. t) be an m-ary (resp. n-ary) term of a Lawvere theory $\mathcal{T}$. We say that s and t *commute* if

$$\begin{aligned} &t(s(x_{11}, \dots, x_{m1}), \dots, s(x_{1n}, \dots, x_{mn})) \\ &= s(t(x_{11}, \dots, x_{1n}), \dots, t(x_{m1}, \dots, x_{mn})) \end{aligned}$$

is a theorem of $\mathcal{T}$. If $\bar{s} \colon m \to 1$ and $\bar{t} \colon n \to 1$ are the corresponding morphisms in $\mathcal{T}$, this means that the square

$$\begin{tikzcd} m \cdot n \arrow[r, "\bar{s}^n"] \arrow[d, "\bar{t}^m"'] & n \arrow[d, "\bar{t}"] \\ m \arrow[r, "\bar{s}"'] & 1 \end{tikzcd}$$

commutes.

Definition 1.102. A morphism $F \colon \mathcal{T} \to \mathcal{R}$ of Lawvere theories is *central* if for each term t of $\mathcal{T}$, its interpretation as a term t^{ι} of $\mathcal{R}$ induced by F commutes with each term of $\mathcal{R}$.

Proposition 1.103. [40] Let $\mathcal{T}$ and $\mathcal{R}$ be two Lawvere theories. $\mathcal{T}$-enrichments of $\mathcal{R}$-Alg are in one-to-one correspondence with central morphisms $\mathcal{T} \to \mathcal{R}$.

As a consequence, we have a characterisation of commutative theories.

Definition 1.104. [79] A Lawvere theory $\mathcal{T}$ is *commutative* if the identity morphism $\mathcal{T} \to \mathcal{T}$ is central, i.e., if any two terms of $\mathcal{T}$ commute.

Example 1.105. The Lawvere theories Th[Set], Th[Set$_*$], Th[ComMon] and Th[Ab] are commutative.

Proposition 1.106. [79] The following conditions on a Lawvere theory $\mathcal{T}$ are equivalent:

1. $\mathcal{T}$ is commutative,
2. $\mathcal{T}$-Alg has a $\mathcal{T}$-enrichment defined as follows: given two $\mathcal{T}$-algebras A and B, an n-ary term t of $\mathcal{T}$ and n $\mathcal{T}$-homomorphisms $f_1, \ldots, f_n : A \to B$, the function $t(f_1, \ldots, f_n) \colon A \to B$ defined by

$$t(f_1, \ldots, f_n)(a) = t^B(f_1(a), \ldots, f_n(a))$$

for every $a \in A$ extends to a $\mathcal{T}$-homomorphism which is the value of the action of t on $(f_1, \ldots, f_n)$.

Let us now focus on $\mathcal{T}$-enrichments of $\mathrm{Mod}(\Gamma)$ for an essentially algebraic theory $\Gamma = (S, \Sigma, E, \Sigma_t, \mathrm{Def})$. We view $\mathcal{T}$ as $\mathcal{T}_{(\Sigma', E')}$ for a finitary one-sorted algebraic theory (Σ', E'). By an operation symbol (resp. an axiom) of $\mathcal{T}$ we thus mean an element of Σ' (resp. E'). Suppose we have a $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma)$ and let τ be an n-ary operation symbol of $\mathcal{T}$. For a given $s \in S$, we consider the S-sorted set X such that $X_s = \{x_1, \ldots, x_n\}$ and $X_{s'} = \varnothing$ for $s' \neq s$. The $\mathcal{T}$-algebra structure on $\mathrm{Fr}_\Gamma(X)$ gives rise to an interpretation of τ into an everywhere-defined term $\tau^s \colon s^n \to s$ of Γ ($\tau^s = (\tau^{\mathrm{Fr}_\Gamma(X)})_s(x_1, \ldots, x_n) \in \mathrm{Fr}_\Gamma(X)_s$). Given any Γ-model A and $a_1, \ldots, a_n \in A_s$, let $f \colon \mathrm{Fr}_\Gamma(X) \to A$ be the unique Γ-homomorphism such that $f_s(x_i) = a_i$ for each $1 \leqslant i \leqslant n$. Then, since f is also a $\mathcal{T}$-homomorphism, the square

$$\begin{tikzcd}
\mathrm{Fr}_\Gamma(X)^n \arrow[r, "\tau^{\mathrm{Fr}_\Gamma(X)}"] \arrow[d, "f^n"'] & \mathrm{Fr}_\Gamma(X) \arrow[d, "f"] \\
A^n \arrow[r, "\tau^A"'] & A
\end{tikzcd}$$

commutes. This implies that $(\tau^A)_s(a_1, \dots, a_n) = \tau^s(a_1, \dots, a_n)$. Hence, these interpretations turn axioms of $\mathcal{T}$ into theorems of Γ, in the sense that $t_1^s = t_2^s$ is a theorem of Γ for each axiom $t_1 = t_2$ of $\mathcal{T}$ where t_1^s and t_2^s are defined in the obvious way. Moreover, since $\tau^A \colon A^n \to A$ is a Γ-homomorphism for any Γ-model A,

$$\tau^s(\sigma((x_{1i})_{i<m}), \dots, \sigma((x_{ni})_{i<m})) = \sigma((\tau^{s_i}(x_{1i}, \dots, x_{ni}))_{i<m}) \tag{4}$$

is a theorem of Γ for each operation symbol $\sigma \colon \prod_{i<m} s_i \to s$ of Σ. These observations lead to the following proposition.

Proposition 1.107. Let Γ be an essentially algebraic theory and $\mathcal{T}$ a Lawvere theory. $\mathcal{T}$-enrichments of $\mathrm{Mod}(\Gamma)$ are in one-to-one correspondence with assignments, for each n-ary operation symbol τ of $\mathcal{T}$ and sort $s \in S$, of an everywhere-defined term $\tau^s \colon s^n \to s$ of Γ such that

1. these interpretations turn axioms of $\mathcal{T}$ into theorems of Γ at each sort $s \in S$,
2. for any operation symbols σ of Σ and τ of $\mathcal{T}$, σ commutes with the interpretations of τ, i.e., (4) is a theorem of Γ.

Proof. We proved above that each $\mathcal{T}$-enrichment gives rise to such an assignment. On the other hand, given such an assignment, we define an internal $\mathcal{T}$-algebra on every Γ-model A by letting $(\tau^A)_s = \tau^s \colon (A^n)_s = (A_s)^n \to A_s$ for all operation symbols τ of $\mathcal{T}$ and $s \in S$. It is routine verifications to check that this yields a Γ-homomorphism $\tau^A \colon A^n \to A$, that this gives rise to a $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma)$ and that these two applications are reciprocal inverses. ☐

If $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$, this reduces to the following corollary.

Corollary 1.108. Let Γ be an essentially algebraic theory. The category $\mathrm{Mod}(\Gamma)$ is pointed if and only if, for each $s \in S$, there exists a unique (up to theorems) everywhere-defined constant term 0^s of sort s such that, for each operation symbol $\sigma \colon \prod_{i<n} s_i \to s$ in Σ, $\sigma((0^{s_i})_{i<n}) = 0^s$ is a theorem of Γ.

Notice that the above conditions imply in particular that $\sigma((0^{s_i})_{i<n})$ is always defined since, if $\sigma \in \Sigma \setminus \Sigma_t$, for each equation $(t_1, t_2) \in \mathrm{Def}(\sigma)$, $t_1((0^{s_i})_{i<n}) = 0^{s'} = t_2((0^{s_i})_{i<n})$.

We would like now to describe the link between $\mathcal{T}$-categories and $\mathcal{R}$-categories for a morphism $\mathcal{T} \to \mathcal{R}$ of Lawvere theories.

Proposition 1.109. Each morphism of commutative Lawvere theories $F\colon \mathcal{T} \to \mathcal{R}$ gives rise to an adjunction between $\mathcal{R}$-Cat and $\mathcal{T}$-Cat.

$$\begin{tikzcd}
\mathcal{R}\text{-Cat} \arrow[r, bend right, "F^*"'] \arrow[r, phantom, "\perp"] & \mathcal{T}\text{-Cat} \arrow[l, bend right, "\mathrm{Fr}_F"']
\end{tikzcd}$$

Proof. Let $\mathcal{D}$ be a small $\mathcal{R}$-category. We define $F^*(\mathcal{D})$ by $\mathrm{ob}(F^*(\mathcal{D})) = \mathrm{ob}(\mathcal{D})$ and $F^*(\mathcal{D})(A, B) = F^*(\mathcal{D}(A, B))$ for all $A, B \in \mathcal{D}$. Composition and identities are defined as in $\mathcal{D}$ since F^* commutes with the forgetful functors $\mathcal{R}$-Alg $\to$ Set and $\mathcal{T}$-Alg $\to$ Set. We extend F^* to a functor in the obvious way.

Let now $\mathcal{C}$ be a small $\mathcal{T}$-category. We are going to construct a reflection of $\mathcal{C}$ along F^*. Let $\mathrm{ob}(\mathrm{Fr}_F(\mathcal{C})) = \mathrm{ob}(\mathcal{C})$ and $\mathrm{Fr}_F(\mathcal{C})(A, B) = \mathrm{Fr}_F(\mathcal{C}(A, B))$ for all $A, B \in \mathcal{C}$. We denote by $\widetilde{f}$ the image of $f \in \mathcal{C}(A, B)$ under the reflection $\mathcal{C}(A, B) \to F^*(\mathrm{Fr}_F(\mathcal{C}(A, B)))$. For such an f, we define $- \circ \widetilde{f}\colon \mathrm{Fr}_F(\mathcal{C}(B, C)) \to \mathrm{Fr}_F(\mathcal{C}(A, C))$ as the unique $\mathcal{R}$-homomorphism such that $\widetilde{g} \circ \widetilde{f} = \widetilde{gf}$ for each $g \in \mathcal{C}(B, C)$. Now, we define

$$- \circ -\colon \mathrm{Fr}_F(\mathcal{C}(A, B)) \longrightarrow \mathcal{R}\text{-Alg}(\mathrm{Fr}_F(\mathcal{C}(B, C)), \mathrm{Fr}_F(\mathcal{C}(A, C)))$$

as the unique $\mathcal{R}$-homomorphism such that $- \circ \widetilde{f}$ is the one defined above for each $f \in \mathcal{C}(A, B)$. Note that this can be done since $\mathcal{R}$ is commutative. Together with the assignment $1_A = \widetilde{1_A}$, this makes $\mathrm{Fr}_F(\mathcal{C})$ an $\mathcal{R}$-category and the obvious map $\mathcal{C} \to F^* \mathrm{Fr}_F(\mathcal{C})$ a $\mathcal{T}$-functor. The fact that this is the reflection of $\mathcal{C}$ along F^* also follows from the universal property of $\mathrm{Fr}_F\colon \mathcal{T}$-Alg $\to \mathcal{R}$-Alg. □

Corollary 1.110. Let $\mathcal{T}$ be a commutative Lawvere theory. Then, the forgetful functor $U_{\mathcal{T}}\colon \mathcal{T}\text{-Cat} \to \text{Cat}$ has a left adjoint.

$$\begin{tikzcd}
\mathcal{T}\text{-Cat} \arrow[r, bend right, "U_{\mathcal{T}}"'] \arrow[r, phantom, "\perp"] & \text{Cat} \arrow[l, bend right, "\mathrm{Fr}_{\mathcal{T}}"']
\end{tikzcd}$$

Proof. Let $\mathrm{Fr}_{\mathcal{T}}$ be Fr_F for F the unique morphism of Lawvere theories $\mathrm{Th}[\mathrm{Set}] \to \mathcal{T}$. ☐

The same occurs with sketches.

Definition 1.111. Let $\mathcal{T}$ be a Lawvere theory. A *$\mathcal{T}$-enriched sketch* (or *$\mathcal{T}$-sketch* in short) is a sketch $\mathbb{S}$ together with a $\mathcal{T}$-enrichment on the underlying category. A *morphism of $\mathcal{T}$-sketches* is a morphism of sketches which is a $\mathcal{T}$-functor. The corresponding category is denoted by $\mathcal{T}$-Sk.

Proposition 1.112. Let $F\colon \mathcal{T} \to \mathcal{R}$ be a morphism of commutative Lawvere theories. Then, the adjunction

$$\begin{tikzcd}
\mathcal{R}\text{-Cat} \arrow[r, bend right, "F^*"'] \arrow[r, phantom, "\perp"] & \mathcal{T}\text{-Cat} \arrow[l, bend right, "\mathrm{Fr}_F"']
\end{tikzcd}$$

of Proposition 1.109 gives rise to an adjunction between $\mathcal{R}$-Sk and $\mathcal{T}$-Sk.

$$\begin{tikzcd}
\mathcal{R}\text{-Sk} \arrow[r, bend right, "F^*"'] \arrow[r, phantom, "\perp"] & \mathcal{T}\text{-Sk}\,. \arrow[l, bend right, "\mathrm{Fr}_F"']
\end{tikzcd}$$

Proof. If $\mathbb{S} = (\mathcal{S}, \mathcal{P}, \mathcal{I})$ is an $\mathcal{R}$-sketch, let $F^*(\mathbb{S}) = (F^*(\mathcal{S}), \mathcal{P}, \mathcal{I})$ and if $\mathbb{S}' = (\mathcal{S}', \mathcal{P}', \mathcal{I}')$ is a $\mathcal{T}$-sketch, we define

$$\mathrm{Fr}_F(\mathbb{S}') = (\mathrm{Fr}_F(\mathcal{S}'), \eta_{\mathcal{S}'}(\mathcal{P}'), \eta_{\mathcal{S}'}(\mathcal{I}'))$$

where $\eta_{\mathcal{S}'}\colon \mathcal{S}' \to F^* \mathrm{Fr}_F(\mathcal{S}')$ is the unit and $\eta_{\mathcal{S}'}(\mathcal{P}')$ is the set of cones

$$1_{\eta_{\mathcal{S}'}} \star \lambda\colon \Delta_{\eta_{\mathcal{S}'}(X)} \Rightarrow \eta_{\mathcal{S}'} G$$

where $\lambda\colon \Delta_X \Rightarrow G$ runs through $\mathcal{P}'$ (and similarly for $\eta_{\mathcal{S}'}(\mathcal{I}')$). The rest of the proof follows easily from Proposition 1.109. ☐

The above results have their counter-part in 2-dimensional category theory. Without going into too much detail here, let us just recall the definition of a 2-category and cite $\mathcal{T}$-Cat and $\mathcal{T}$-Sk as examples.

Definition 1.113. A *2-category* consists of

- a category $\mathcal{C}$,
- for each $A, B \in \mathcal{C}$, a small category whose set of objects is $\mathcal{C}(A, B)$,
- for all morphisms $\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & C \end{tikzcd}$, an application

$$\begin{aligned} \mathcal{C}(A,B)(f,f') \times \mathcal{C}(B,C)(g,g') &\longrightarrow \mathcal{C}(A,C)(gf, g'f') \\ (\alpha, \beta) &\longmapsto \beta \star \alpha \end{aligned}$$

satisfying the following axioms

1. $1_g \star 1_f = 1_{gf}$ for any diagram $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f"'{name=D}] \arrow[Rightarrow, from=U, to=D, "1_f"] & B \arrow[r, bend left, "g"{name=U2}] \arrow[r, bend right, "g"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "1_g"] & C \end{tikzcd}$,

2. $\alpha \star 1_{1_A} = \alpha = 1_{1_B} \star \alpha$ for any diagram $\begin{tikzcd} A \arrow[r, bend left, "1_A"{name=U}] \arrow[r, bend right, "1_A"'{name=D}] \arrow[Rightarrow, from=U, to=D, "1_{1_A}"] & A \arrow[r, bend left, "f"{name=U2}] \arrow[r, bend right, "f'"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\alpha"] & B \arrow[r, bend left, "1_B"{name=U3}] \arrow[r, bend right, "1_B"'{name=D3}] \arrow[Rightarrow, from=U3, to=D3, "1_{1_B}"] & B \end{tikzcd}$,

3. $(\gamma \star \beta) \star \alpha = \gamma \star (\beta \star \alpha)$ for any diagram $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"] & B \arrow[r, bend left, "g"{name=U2}] \arrow[r, bend right, "g'"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\beta"] & C \arrow[r, bend left, "h"{name=U3}] \arrow[r, bend right, "h'"'{name=D3}] \arrow[Rightarrow, from=U3, to=D3, "\gamma"] & D \end{tikzcd}$,

4. $(\delta \star \beta) \circ (\gamma \star \alpha) = (\delta \circ \gamma) \star (\beta \circ \alpha)$ for any diagram $\begin{tikzcd}[column sep=large] A \arrow[r, bend left=70, "f"{name=U}] \arrow[r, "f'"{name=M, description}] \arrow[r, bend right=70, "f''"'{name=D}] \arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M, to=D, "\beta"] & B \arrow[r, bend left=70, "g"{name=U2}] \arrow[r, "g'"{name=M2, description}] \arrow[r, bend right=70, "g''"'{name=D2}] \arrow[Rightarrow, from=U2, to=M2, "\gamma"] \arrow[Rightarrow, from=M2, to=D2, "\delta"] & C \end{tikzcd}$.

We call a morphism in the underlying category $\mathcal{C}$ a *1-cell* and a morphism in any category $\mathcal{C}(A, B)$ a *2-cell*. An invertible 2-cell is called a *2-isomorphism*. If $f\colon A \to B$ is a 1-cell and $C \in \mathcal{C}$ an object, we write

$\mathcal{C}(C,f)$ for the composition functor

$$\begin{aligned}\mathcal{C}(C,f)\colon \mathcal{C}(C,A) &\longrightarrow \mathcal{C}(C,B)\\ g &\longmapsto fg\\ \alpha &\longmapsto 1_f \star \alpha\end{aligned}$$

and $\mathcal{C}(f,C)\colon \mathcal{C}(B,C) \to \mathcal{C}(A,C)$ for the analogous functor.

Example 1.114. Each category $\mathcal{C}$ can be turned into a 2-category, thinking of each hom-set $\mathcal{C}(A,B)$ as a discrete category. Small categories, functors and natural transformations organise themselves in a 2-category, also denoted by Cat. More generally, if $\mathcal{T}$ is a Lawvere theory, small $\mathcal{T}$-categories, $\mathcal{T}$-functors and natural transformations form the 2-category $\mathcal{T}$-Cat, while $\mathcal{T}$-sketches, morphisms of $\mathcal{T}$-sketches and natural transformations form the 2-category $\mathcal{T}$-Sk.

Chapter 2

Matrix conditions

Now that we gave some definitions of categories of algebras and their general characterisations, we would like to study categorical properties that distinguish one from the others. Which properties does Gp have and not Set? What are the differences and common properties of Gp and Ab? The notion of an abelian category [81, 25, 52] gives one of the first answers: it contains Ab, Mod_R (the category of modules over the ring R) and the categories of sheaves of abelian groups on a site, but these are essentially the only examples. We thus want to study less restrictive algebraic categorical conditions, in order to encompass categories such as Gp, Mon and LieAlg_k (the category of Lie algebras over a field k). This is what we call 'non-abelian categorical algebra'.

One of the first major steps in that direction has been realised using Mal'tsev's result which characterises finitary one-sorted algebraic categories in which the equality $RS = SR$ holds for any congruences (i.e., equivalence relations in the sense of Definition 1.20) R and S on a same object as the ones whose theory has a ternary term $p(x, y, z)$ satisfying the axioms $p(x, y, y) = x$ and $p(x, x, y) = y$ [87]. This property of commutativity of equivalence relations can be stated in any regular category and is what defines Mal'tsev categories among them [28]. It is equivalent to the condition that each binary relation is difunctional, a property that can be stated in any category with finite limits and that distinguishes Mal'tsev categories among them [29].

Another property studied in non-abelian categorical algebra is unitality [18]. A pointed finitely complete category is said to be unital if, for any objects X and Y, the morphisms

$$X \xrightarrow{(1_X,0)} X \times Y \xleftarrow{(0,1_Y)} Y$$

are jointly strongly epimorphic. A Lawvere theory has a unital category of algebras if and only if it has a unique nullary term 0 and a binary term $u(x,y)$ such that $u(x,0)=x$ and $u(0,x)=x$ are theorems.

Many other such properties have been defined in categorical algebra and often have a characterisation of the finitary algebraic categories which satisfy it. Among them, one can cite: strongly unital categories [18], subtractive categories [63], n-permutable categories [27] and protomodular categories [17].

In [64], Z. Janelidze proposed a way to unify many of these categorical properties. Given a matrix of terms in a Lawvere theory $\mathcal{T}$ (often Th[Set] or Th[Set$_*$]), we can consider those $\mathcal{T}$-categories in which relations satisfy a property described by the columns of the given matrix. In this way, we recover the examples of Mal'tsev, (strongly) unital and subtractive categories. Finitary one-sorted algebraic categories which satisfy such an exactness property are characterised by the existence of a term satisfying some axioms given by the lines of the matrix. We provide in this chapter a generalisation of this result characterising such essentially algebraic categories.

An extension of these matrix conditions introduced in [67] encompasses in the regular context some more examples such as n-permutability. These matrices are also recalled in Section 2.2.

2.1 Categories with M-closed relations

2.1.1 Definitions and algebraic characterisations

Let us start by recalling simple matrix conditions in general; examples will be provided in the next subsection.

Definition 2.1. [64] Let $\mathcal{T}$ be a Lawvere theory. A *simple extended* *matrix of terms in* $\mathcal{T}$ is a matrix

$$M = \left(\begin{array}{ccc|c} t_{11} & \cdots & t_{1b} & u_1 \\ \vdots & & \vdots & \vdots \\ t_{a1} & \cdots & t_{ab} & u_a \end{array} \right) \tag{5}$$

where the t_{ij}'s and the u_i's are k-ary terms of $\mathcal{T}$ (in the variables from a finite set X such that $\#X = k$) with $a \geqslant 1$, $b \geqslant 0$ and $k \geqslant 0$.

Definition 2.2. [64] Let M be a simple extended matrix of terms in the Lawvere theory $\mathcal{T}$ as in (5) and $\mathcal{C}$ a $\mathcal{T}$-category with finite products. We say that the a-ary relation $r\colon R \rightarrowtail A^a$ in $\mathcal{C}$ is *M-closed* when, for each object Y in $\mathcal{C}$ and all morphisms $y_1, \ldots, y_k\colon Y \to A$, if, for each $j \in \{1, \ldots, b\}$, the morphism

$$(t_{1j}(y_1, \ldots, y_k), \ldots, t_{aj}(y_1, \ldots, y_k))\colon Y \to A^a$$

factors through r, then so does the morphism

$$(u_1(y_1, \ldots, y_k), \ldots, u_a(y_1, \ldots, y_k))\colon Y \to A^a.$$

Definition 2.3. [64] Let M be a simple extended matrix of terms in the Lawvere theory $\mathcal{T}$ as in (5) and $\mathcal{C}$ a $\mathcal{T}$-category with finite products. We say that the a-ary relation $r\colon R \rightarrowtail A_1 \times \cdots \times A_a$ in $\mathcal{C}$ is *strictly M-closed* when, for any object Y in $\mathcal{C}$ and any family of morphisms

$$(y_{ii'}\colon Y \to A_i)_{i \in \{1,\ldots,a\}, i' \in \{1,\ldots,k\}},$$

if, for each $j \in \{1, \ldots, b\}$, the morphism

$$(t_{1j}(y_{11}, \ldots, y_{1k}), \ldots, t_{aj}(y_{a1}, \ldots, y_{ak}))\colon Y \to A_1 \times \cdots \times A_a$$

factors through r, then so does the morphism

$$(u_1(y_{11}, \ldots, y_{1k}), \ldots, u_a(y_{a1}, \ldots, y_{ak}))\colon Y \to A_1 \times \cdots \times A_a.$$

Here is the link between M-closedness and strict M-closedness.

Theorem 2.4. [64] Let $\mathcal{T}$ be a Lawvere theory, M a simple extended matrix of terms in $\mathcal{T}$ as in (5) and $\mathcal{C}$ a finitely complete $\mathcal{T}$-category. Then, the following conditions are equivalent:

1. every relation $r\colon R \rightarrowtail A^a$ in $\mathcal{C}$ is M-closed,
2. every relation $r\colon R \rightarrowtail A_1 \times \cdots \times A_a$ in $\mathcal{C}$ is strictly M-closed.

If the above conditions are satisfied, we say that $\mathcal{C}$ *has M-closed relations*. With this matrix notation, we have an easy characterisation of algebraic categories with M-closed relations.

Theorem 2.5. [64] Let $F\colon \mathcal{T} \to \mathcal{R}$ be a central morphism of Lawvere theories. Let also M be a simple extended matrix of terms in $\mathcal{T}$ as in (5). Then, the $\mathcal{T}$-enriched category $\mathcal{R}$-Alg (induced by F) has M-closed relations if and only if there exists a b-ary term p in $\mathcal{R}$ such that

$$p(t^\iota_{i1}(x_1,\dots,x_k),\dots,t^\iota_{ib}(x_1,\dots,x_k)) = u^\iota_i(x_1,\dots,x_k)$$

is a theorem of $\mathcal{R}$ for each $i \in \{1,\dots,a\}$, where t^ι denotes the interpretation in $\mathcal{R}$ induced by F of the term t in $\mathcal{T}$.

We now generalise this result to essentially algebraic categories.

Theorem 2.6. Let Γ be an essentially algebraic theory, $\mathcal{T}$ a Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$ as in (5). Given a $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma)$ (as in Proposition 1.107), $\mathrm{Mod}(\Gamma)$ has M-closed relations if and only if, for each $s \in S$, there exists a term $p^s\colon s^b \to s$ of Γ such that

1. for each $i \in \{1,\dots,a\}$, $p^s(t^s_{i1}(x_1,\dots,x_k),\dots,t^s_{ib}(x_1,\dots,x_k))$ is an everywhere-defined term $s^k \to s$,
2. for each $i \in \{1,\dots,a\}$,

   $$p^s(t^s_{i1}(x_1,\dots,x_k),\dots,t^s_{ib}(x_1,\dots,x_k)) = u^s_i(x_1,\dots,x_k)$$

   is a theorem of Γ.

Proof. Notice firstly that a relation $r\colon R \rightarrowtail A_1 \times \cdots \times A_a$ in $\mathrm{Mod}(\Gamma)$ can be seen as a submodel of $A_1 \times \cdots \times A_a$. We suppose that terms as in the statement exist in Γ and we are going to prove $\mathrm{Mod}(\Gamma)$ has M-closed relations. So let R be a submodel of A^a and $y_1, \dots, y_k \colon Y \to A$ homomorphisms such that

$$(t_{1j}(y_1, \dots, y_k), \dots, t_{aj}(y_1, \dots, y_k)) \colon Y \to A^a$$

factors through R for every $j \in \{1, \dots, b\}$. We have to prove that

$$(u_1(y_1, \dots, y_k), \dots, u_a(y_1, \dots, y_k)) \colon Y \to A^a$$

also factors through R. In other words, we must show that, for each $s \in S$ and each $x \in Y_s$,

$$(u_1^s(y_1(x), \dots, y_k(x)), \dots, u_a^s(y_1(x), \dots, y_k(x)))$$

belongs to R_s. But we know that

$$(t_{1j}^s(y_1(x), \dots, y_k(x)), \dots, t_{aj}^s(y_1(x), \dots, y_k(x))) \in R_s$$

for every $j \in \{1, \dots, b\}$. Thus

$$\begin{aligned}
&(u_1^s(y_1(x), \dots, y_k(x)), \dots, u_a^s(y_1(x), \dots, y_k(x))) \\
&= (p^s(t_{11}^s(y_1(x), \dots, y_k(x)), \dots, t_{1b}^s(y_1(x), \dots, y_k(x))), \dots \\
&\qquad \dots, p^s(t_{a1}^s(y_1(x), \dots, y_k(x)), \dots, t_{ab}^s(y_1(x), \dots, y_k(x)))) \\
&= p^s((t_{11}^s(y_1(x), \dots, y_k(x)), \dots, t_{a1}^s(y_1(x), \dots, y_k(x))), \dots \\
&\qquad \dots, (t_{1b}^s(y_1(x), \dots, y_k(x)), \dots, t_{ab}^s(y_1(x), \dots, y_k(x)))) \\
&\in R_s
\end{aligned}$$

since R is closed under the operation p^s.

Conversely, suppose $\mathrm{Mod}(\Gamma)$ has M-closed relations and let $s \in S$ be a sort. We denote by X the S-sorted set defined by $X_s = \{x_1, \dots, x_k\}$ and $X_{s'} = \varnothing$ if $s' \neq s$. Let R be the smallest submodel of $\mathrm{Fr}_\Gamma(X)^a$ such that

$$(t_{1j}^s(x_1, \dots, x_k), \dots, t_{aj}^s(x_1, \dots, x_k)) \in R_s$$

for each $j \in \{1, \dots, b\}$. It is not hard to prove that R is actually given by

$$\begin{aligned}R_{s'} = \Big\{ &(q(t^s_{11}(x_1,\dots,x_k),\dots,t^s_{1b}(x_1,\dots,x_k)),\dots\\ &\dots,q(t^s_{a1}(x_1,\dots,x_k),\dots,t^s_{ab}(x_1,\dots,x_k)))\mid\\ &q\colon s^b \to s' \text{ is a term of } \Gamma \text{ such that}\\ &q(t^s_{i1}(x_1,\dots,x_k),\dots,t^s_{ib}(x_1,\dots,x_k))\colon s^k \to s'\\ &\text{is everywhere-defined for all } i \in \{1,\dots,a\}\Big\}\end{aligned}$$

for each $s' \in S$. Let now Y be the S-sorted set defined by $Y_s = \{y\}$ and $Y_{s'} = \varnothing$ if $s' \neq s$. For $l \in \{1,\dots,k\}$, we consider the unique Γ-homomorphism $y_l\colon \mathrm{Fr}_\Gamma(Y) \to \mathrm{Fr}_\Gamma(X)$ which sends y on x_l. The morphism

$$(t_{1j}(y_1,\dots,y_k),\dots,t_{aj}(y_1,\dots,y_k))\colon \mathrm{Fr}_\Gamma(Y) \to \mathrm{Fr}_\Gamma(X)^a$$

factors through R for each $j \in \{1,\dots,b\}$ since its image at y belongs to R_s. But $\mathrm{Mod}(\Gamma)$ has M-closed relations, so we can say that the morphism

$$(u_1(y_1,\dots,y_k),\dots,u_a(y_1,\dots,y_k))\colon \mathrm{Fr}_\Gamma(Y) \to \mathrm{Fr}_\Gamma(X)^a$$

factors through R. Evaluated at y, this means

$$(u^s_1(x_1,\dots,x_k),\dots,u^s_a(x_1,\dots,x_k)) \in R_s.$$

In view of the description of R_s, this gives the expected term p^s. $\square$

2.1.2 Examples

We now give some examples of categorical properties induced by simple extended matrices.

Example 2.7. We consider $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$, $M = \left(\begin{array}{ccc|c} x & y & y & x \\ x & x & y & y \end{array}\right)$ and $\mathcal{C}$ a finitely complete category. A binary relation $R \rightarrowtail A \times B$ is said to be *difunctional* when it is strictly M-closed. Explicitly, this means that, given morphisms $y_{11}, y_{12}\colon Y \rightrightarrows A$ and $y_{21}, y_{22}\colon Y \rightrightarrows B$ in $\mathcal{C}$ such that (y_{11}, y_{21}), (y_{12}, y_{21}) and $(y_{12}, y_{22})\colon Y \to A \times B$ factor through R, then (y_{11}, y_{22}) also factors through R. This is of course equivalent to R being strictly $\left(\begin{array}{ccc|c} x & y & y & x \\ u & u & v & v \end{array}\right)$-closed. If $\mathcal{C} = \mathrm{Set}$, we can view a relation $R \rightarrowtail A \times B$ as a classical relation $R \subseteq A \times B$. It is difunctional when the implication

$$a_1 R b_1 \wedge a_2 R b_1 \wedge a_2 R b_2 \Rightarrow a_1 R b_2$$

holds for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. A finitely complete category $\mathcal{C}$ is then called a *Mal'tsev category* when it has M-closed relations [29].

As a particular case of Theorem 2.5, we have the following Mal'tsev's result.

Corollary 2.8. [87, 75] Let $\mathcal{R}$ be a Lawvere theory. $\mathcal{R}$-Alg is a Mal'tsev category if and only if $\mathcal{R}$ contains a ternary term $p(x, y, z)$ such that $p(x, y, y) = x$ and $p(x, x, y) = y$ are theorems of $\mathcal{R}$.

Such a ternary term $p(x, y, z)$ is called a *Mal'tsev operation.*

Example 2.9. Gp is a Mal'tsev category. A Mal'tsev operation p is obtained as $p(x, y, z) = xy^{-1}z$. The categories Heyt of Heyting algebras, TopGp of topological groups, Ab and $\mathrm{Set}^{\mathrm{op}}$ are also a Mal'tsev categories but Set, Mon and Top are not.

Here are some characterisations of Mal'tsev categories.

Theorem 2.10. [29] Let $\mathcal{C}$ be a finitely complete category. The following statements are equivalent.

1. $\mathcal{C}$ is a Mal'tsev category,
2. any relation $r\colon R \rightarrowtail A \times B$ in $\mathcal{C}$ is difunctional,
3. any relation $r\colon R \rightarrowtail A \times A$ in $\mathcal{C}$ is difunctional,
4. any reflexive relation in $\mathcal{C}$ is an equivalence relation,
5. any reflexive relation in $\mathcal{C}$ is symmetric,
6. any reflexive relation in $\mathcal{C}$ is transitive.

In a regular context, we have even more characterisations.

Theorem 2.11. [28] Let $\mathcal{C}$ be a regular category. The following statements are equivalent.

1. $\mathcal{C}$ is a Mal'tsev category,
2. the composite of two equivalence relations on the same object is an equivalence relation,
3. if R and S are equivalence relations on the same object, then the equality $RS = SR$ holds.

We now focus on unital categories.

Example 2.12. [18, 64] Let $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$, $M = \left(\begin{array}{cc|c} x & 0 & x \\ 0 & x & x \end{array}\right)$ and $\mathcal{C}$ be a finitely complete pointed category. When $\mathcal{C}$ has M-closed relations, we say it is *unital*.

Let us again give the corresponding particular case of Theorem 2.5.

Corollary 2.13. Let $\mathcal{R}$ be a Lawvere theory. $\mathcal{R}$-Alg is unital if and only if $\mathcal{R}$ has a unique (up to theorems) constant term 0 and a binary term $u(x, y)$ such that $u(x, 0) = x$ and $u(0, x) = x$ are theorems of $\mathcal{R}$.

A direct proof of this fact can be found in [15]. A binary term $u(x, y)$ satisfying such axioms is called a *Jónsson-Tarski operation* [71].

Example 2.14. The categories Mon, Gp, Ab and $\mathrm{Set}_*^{\mathrm{op}}$ are unital but Set_* is not.

In order to recover the usual definition of unital categories, let us generalise the notion of strong and extremal epimorphisms.

Definition 2.15. Let I be a set and $(f_i \colon A_i \to B)_{i \in I}$ a family of morphisms with a common codomain in a category $\mathcal{C}$. We say that these morphisms are *jointly epimorphic* if, for any pair of parallel arrows $g, h \colon B \rightrightarrows C$, $gf_i = hf_i$ for each $i \in I$ implies $g = h$. Such jointly epimorphic morphisms are said to be *jointly extremaly epimorphic* if any monomorphism $m \colon C \rightarrowtail B$ through which each f_i factors is necessarily an isomorphism. Jointly epimorphic morphisms $(f_i)_{i \in I}$ are said to be *jointly strongly epimorphic* if, given any monomorphism $m \colon C \rightarrowtail D$, any family $(g_i \colon A_i \to C)_{i \in I}$ and any morphism $h \colon B \to D$ such that $hf_i = mg_i$ for each $i \in I$, there exists a morphism $d \colon B \to C$ such that $md = h$ (and $df_i = g_i$ for each $i \in I$).

$$\begin{tikzcd}
A_i \arrow[r, "f_i"] \arrow[d, "g_i"'] & B \arrow[d, "h"] \arrow[dl, dotted, "d"] \\
C \arrow[r, tail, "m"'] & D
\end{tikzcd}$$

This terminology is consistent with the previously defined one in the following sense.

Proposition 2.16. Let I be a set and $(f_i \colon A_i \to B)_{i \in I}$ a family of morphisms in a category $\mathcal{C}$ in which the coproduct $\coprod_{i \in I} A_i$ exists. Let also $f \colon \coprod_{i \in I} A_i \to B$ be the morphism induced by the f_i's. Then, these morphisms $(f_i)_{i \in I}$ are jointly epimorphic (resp. jointly extremaly epimorphic, resp. jointly strongly epimorphic) if and only if f is an epimorphism (resp. an extremal epimorphism, resp. a strong epimorphism).

As in the classical case, it gets simpler with finite limits.

Proposition 2.17. In a category $\mathcal{C}$ with finite limits, jointly extremaly epimorphic families of morphisms coincide with jointly strongly epimorphic ones. Moreover, if I is a set and $(f_i \colon A_i \to B)_{i \in I}$ a family of morphisms in $\mathcal{C}$ such that, if they factor through a common monomorphism m, this m is an isomorphism, then this family $(f_i)_{i \in I}$ is jointly epimorphic and so jointly strongly epimorphic.

We can now give the classical characterisation of unital categories.

Proposition 2.18. [18] The following conditions on a finitely complete pointed category $\mathcal{C}$ are equivalent:

1. $\mathcal{C}$ is unital,

2. for each objects X, Y in $\mathcal{C}$, the morphisms $(1_X, 0)$ and $(0, 1_Y)$ are jointly strongly epimorphic,

$$\begin{tikzcd}
X \arrow[r, "{(1_X,0)}"] & X \times Y & Y \arrow[l, "{(0,1_Y)}"']
\end{tikzcd}$$

3. for each object X in $\mathcal{C}$, the morphisms $(1_X, 0)$ and $(0, 1_X)$ are jointly strongly epimorphic.

$$\begin{tikzcd} X \arrow[r, "(1_X{,}0)"] & X \times X & X \arrow[l, "(0{,}1_X)"'] \end{tikzcd}$$

Let us now say a word on strongly unital categories.

Example 2.19. [18, 64] Let $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$ and $M = \left(\begin{array}{ccc|c} x & 0 & 0 & x \\ x & x & y & y \end{array}\right)$. A finitely complete pointed category $\mathcal{C}$ with M-closed relations is called a *strongly unital category.*

Example 2.20. The category Gp is strongly unital. Although Mon is unital, it is not strongly unital.

Proposition 2.21. [18, 15] The following conditions on a finitely complete pointed category $\mathcal{C}$ are equivalent:

1. $\mathcal{C}$ is strongly unital,

2. in $\mathcal{C}$, for every diagram

$$\begin{tikzcd} A \arrow[r, shift left, "s"] & C \arrow[l, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, shift right, "t"'] \end{tikzcd}$$

where $fs = 1_A$, $gt = 1_B$ and $ft = 0$, the factorisation

$$(f, g) \colon C \to A \times B$$

is a strong epimorphism.

Our last example is about subtractive categories.

Example 2.22. [63] We consider $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$ and $M = \left(\begin{array}{cc|c} x & 0 & x \\ x & x & 0 \end{array}\right)$. A finitely complete pointed category with M-closed relations is called a *subtractive category.*

Let us particularise Theorem 2.5 one more time.

Corollary 2.23. [100, 63] Let $\mathcal{R}$ be a Lawvere theory. $\mathcal{R}$-Alg is subtractive if and only if $\mathcal{R}$ has a unique (up to theorems) constant term

0 and a binary term $u(x,y)$ such that $u(x,0)=x$ and $u(x,x)=0$ are theorems of $\mathcal{R}$.

Such a binary term u is called a *subtraction*.

Here is how these properties are linked to each other.

Proposition 2.24. [18, 15, 63] Let $\mathcal{C}$ be a finitely complete pointed category. The following implications hold:

1. if $\mathcal{C}$ is a Mal'tsev category, it is strongly unital,
2. $\mathcal{C}$ is strongly unital if and only if it is unital and subtractive.

In order to stress another link between those properties, we recall the construction of the category of points.

Definition 2.25. [17] A *point* in a category $\mathcal{C}$ is a pair of morphisms $(p\colon A \twoheadrightarrow I, s\colon I \rightarrowtail A)$ such that $ps = 1_I$. A morphism of points $(p,s) \to (p',s')$ is a pair of morphisms $(u\colon A \to A', v\colon I \to I')$ such that $vp = p'u$ and $us = s'v$.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, two heads, shift right, "p"'] & A' \arrow[d, two heads, shift right, "p'"'] \\
I \arrow[r, "v"'] \arrow[u, tail, shift right, "s"'] & I' \arrow[u, tail, shift right, "s'"']
\end{tikzcd}$$

This forms the category $\mathrm{Pt}(\mathcal{C})$ of points of $\mathcal{C}$.

Together with this category, we have a forgetful functor

$$\begin{array}{rrcl}
\pi\colon & \mathrm{Pt}(\mathcal{C}) & \longrightarrow & \mathcal{C} \\
& A \underset{p}{\overset{s}{\leftrightarrows}} I & \longmapsto & I \\
& (u,v) & \longmapsto & v.
\end{array}$$

Proposition 2.26. [17] Let $\mathcal{C}$ be a category with pullbacks of split epimorphisms along arbitrary morphisms. The functor $\pi\colon \mathrm{Pt}(\mathcal{C}) \to \mathcal{C}$ is a fibration.

This fibration is called *the fibration of points of* $\mathcal{C}$. The fibre over $I \in \mathcal{C}$ of this fibration is denoted by $\mathrm{Pt}_I(\mathcal{C})$ and is a pointed category (the zero object is $(1_I, 1_I)$). If $v\colon J \to I$ is a morphism of $\mathcal{C}$, the change of base functor $v^*\colon \operatorname{Pt}_I(\mathcal{C}) \to \operatorname{Pt}_J(\mathcal{C})$ sends (p,s) to (q,t), where (B,q,u) is the pullback of p along v

$$\begin{tikzcd}
B \arrow[r, "u"] \arrow[d, shift right, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, shift right, "p"'] \\
J \arrow[r, "v"'] \arrow[u, shift right, "t"'] & I \arrow[u, shift right, "s"']
\end{tikzcd}$$

and t is the unique morphism such that $qt = 1_J$ and $ut = sv$. If $f\colon (p,s) \to (p',s')$ is a morphism in the fibre $\operatorname{Pt}_I(\mathcal{C})$,

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, shift right, "p"'] & & A' \arrow[dl, shift right, "p'"'] \\
& I \arrow[ul, shift right, "s"'] \arrow[ur, shift right, "s'"'] &
\end{tikzcd}$$

the corresponding morphism $v^*(f)\colon (q,s) \to (q',s')$ is the unique morphism satisfying $q'v^*(f) = q$ and $u'v^*(f) = fu$.

Looking at these fibres leads to the following characterisations of Mal'tsev categories.

Theorem 2.27. [18, 63] Let $\mathcal{C}$ be a finitely complete category. The following statements are equivalent:

1. $\mathcal{C}$ is a Mal'tsev category,
2. for each $I \in \mathcal{C}$, $\operatorname{Pt}_I(\mathcal{C})$ is a Mal'tsev category,
3. for each $I \in \mathcal{C}$, $\operatorname{Pt}_I(\mathcal{C})$ is unital,
4. for each $I \in \mathcal{C}$, $\operatorname{Pt}_I(\mathcal{C})$ is strongly unital,
5. for each $I \in \mathcal{C}$, $\operatorname{Pt}_I(\mathcal{C})$ is subtractive.

A generalisation of the previous theorem using the matrix presentation can be found in [65]. The equivalence between points 1 and 3 can be stated in the following way (see also Proposition 2.18).

Theorem 2.28. [18] A finitely complete category $\mathcal{C}$ is a Mal'tsev category if and only if, for any pullback of split epimorphisms,

$$\begin{tikzcd}
P \arrow[r, shift right, two heads, "f'"'] \arrow[d, shift left, "g'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, shift right, "r_Y"'] \arrow[d, shift right, two heads, "g"'] \\
X \arrow[u, shift left, "l_X"] \arrow[r, shift left, two heads, "f"] & Z \arrow[l, shift left, "s"] \arrow[u, shift right, "t"']
\end{tikzcd}$$

the induced morphisms l_X and r_Y (defined by $g'l_X = 1_X$, $f'l_X = tf$, $g'r_Y = sg$ and $f'r_Y = 1_Y$) are jointly strongly epimorphic.

2.1.3 Approximate co-operations

We have seen that Mal'tsev finitary one-sorted algebraic categories are characterised by the presence of a Mal'tsev operation $p(x, y, z)$ (see Corollary 2.8). Of course, in a general category, such a characterisation does not exist any more. D. Bourn and Z. Janelidze showed however that in the regular context (with binary coproducts), an approximate Mal'tsev co-operation still exists [21]. Approximate stands here for the fact that it is not a map $Y \to 3Y$ but instead $p\colon Z \to 3Y$ with an approximation $d\colon Z \to Y$. This co-operation p satisfies the Mal'tsev axioms, up to the approximation d. Mal'tsev categories are then characterised by the condition that such a couple (p, d) can be found with d being a regular epimorphism. Z. Janelidze generalised this with his matrix presentation in [68]. Let us first recall the Mal'tsev case and then the more general matrix case.

Definition 2.29. [21] Let $\mathcal{C}$ be a finitely complete category with binary coproducts and Y an object of $\mathcal{C}$. An *approximate Mal'tsev co-operation* on Y is a morphism $p\colon Z \to 3Y$ together with an *approximation* $d\colon Z \to Y$ such that the square

$$\begin{tikzcd}
Z \arrow[r, "p"] \arrow[d, "d"'] & 3Y \arrow[d, "\left(\begin{smallmatrix} \iota_1 & \iota_1 \\ \iota_2 & \iota_1 \\ \iota_2 & \iota_2 \end{smallmatrix}\right)"] \\
Y \arrow[r, "(\iota_1, \iota_2)"'] & (2Y)^2
\end{tikzcd}$$

commutes, where ι_1 and ι_2 are the coproduct injections.

For each object Y, one can build the *universal approximate Mal'tsev* *co-operation* on Y, denoted (p^Y, d^Y), by considering the following pullback.

$$\begin{tikzcd}
W(Y) \arrow[r, "p^Y"] \arrow[d, "d^Y"'] \arrow[dr, phantom, very near start, "\lrcorner"] & 3Y \arrow[d, "\left(\begin{smallmatrix} \iota_1 & \iota_1 \\ \iota_2 & \iota_1 \\ \iota_2 & \iota_2 \end{smallmatrix}\right)"] \\
Y \arrow[r, "(\iota_1, \iota_2)"'] & (2Y)^2
\end{tikzcd}$$

Theorem 2.30. [21] Let $\mathcal{C}$ be a regular category with binary coproducts. The following statements are equivalent:

1. $\mathcal{C}$ is a Mal'tsev category,
2. for each $Y \in \mathcal{C}$, there is an approximate Mal'tsev co-operation on Y for which the approximation d is a regular epimorphism,
3. for each $Y \in \mathcal{C}$, the universal approximate Mal'tsev co-operation on Y is such that d^Y is a regular epimorphism.

Let us now discuss its generalisation appearing in [68].

Definition 2.31. [68] Let $\mathcal{T}$ be a Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$ as in (5). Let also $\mathcal{C}$ be a finitely complete $\mathcal{T}$-category with finite coproducts and Y one of its objects. An *approximate co-solution for M on Y* is a morphism $p\colon Z \to bY$ together with an *approximation* $d\colon Z \to Y$ such that the square

$$\begin{tikzcd}
Z \arrow[r, "p"] \arrow[d, "d"'] & bY \arrow[d, "\left(\begin{smallmatrix} t_{11}^{Y,\mathrm{op}} & \cdots & t_{a1}^{Y,\mathrm{op}} \\ \vdots & & \vdots \\ t_{1b}^{Y,\mathrm{op}} & \cdots & t_{ab}^{Y,\mathrm{op}} \end{smallmatrix}\right)"] \\
Y \arrow[r, "(u_1^{Y,\mathrm{op}}, \dots, u_a^{Y,\mathrm{op}})"'] & (kY)^a
\end{tikzcd}$$

commutes.

For each object Y, one can build the *universal approximate co-solution for M on Y*, denoted (p^Y, d^Y), by considering the following pullback square.

$$\begin{tikzcd}
W(Y) \arrow[r, "p^Y"] \arrow[d, "d^Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & bY \arrow[d, "\begin{pmatrix} t_{11}^{Y,\mathrm{op}} & \cdots & t_{a1}^{Y,\mathrm{op}} \\ \vdots & & \vdots \\ t_{1b}^{Y,\mathrm{op}} & \cdots & t_{ab}^{Y,\mathrm{op}} \end{pmatrix}"] \\
Y \arrow[r, "(u_1^{Y,\mathrm{op}},\dots,u_a^{Y,\mathrm{op}})"'] & (kY)^a
\end{tikzcd}$$

And here is the generalisation of Theorem 2.30.

Theorem 2.32. [68] Let $\mathcal{T}$ be a Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$. The following conditions on a regular $\mathcal{T}$-category $\mathcal{C}$ with finite coproducts are equivalent:

1. $\mathcal{C}$ has M-closed relations,
2. for each $Y \in \mathcal{C}$, there is an approximate co-solution for M on Y for which the approximation d is a regular epimorphism,
3. for each $Y \in \mathcal{C}$, the universal approximate co-solution for M on Y is such that d^Y is a regular epimorphism.

2.1.4 Categories with M-closed strong relations

Considering only strong relations instead of all relations, one gets the notion of a category with M-closed strong relations. If M is the Mal'tsev matrix of Example 2.7, we recover weakly Mal'tsev categories introduced in [91], while if M is the matrix defining unital categories (see Example 2.12), we get the notion of a weakly unital category [90]. In both cases, the idea is to replace 'jointly strongly epimorphic' by 'jointly epimorphic' in their respective characterisations.

Strong monomorphisms are dual to strong epimorphisms, i.e., these are the monomorphisms m for which $e \perp m$ for every epimorphism e. In addition to the dual properties of those in Proposition 1.15, one also has the following.

Proposition 2.33. In a category with pullbacks, a morphism m such that any epimorphism is orthogonal to it is necessarily a monomorphism, and thus a strong monomorphism. Moreover, the pullback of a strong monomorphism along an arbitrary morphism is again a strong monomorphism.

Definition 2.34. Let $\mathcal{C}$ be a category with finite limits and $a \geqslant 1$ a natural number. An *a-ary strong relation* in $\mathcal{C}$ is a relation $r\colon R \rightarrowtail A_1 \times \cdots \times A_a$ which is also a strong monomorphism.

Theorem 2.35. Let $\mathcal{T}$ be a Lawvere theory, M a simple extended matrix of terms in $\mathcal{T}$ as in (5) and $\mathcal{C}$ a finitely complete $\mathcal{T}$-category. Then, the following conditions are equivalent:

1. every strong relation $r\colon R \rightarrowtail A^a$ in $\mathcal{C}$ is M-closed,
2. every strong relation $r\colon R \rightarrowtail A_1 \times \cdots \times A_a$ in $\mathcal{C}$ is strictly M-closed.

Proof. 2 ⇒ 1 being trivial, let us prove 1 ⇒ 2. So, let us consider a strong relation $r\colon R \rightarrowtail A_1 \times \cdots \times A_a$ in $\mathcal{C}$. Since r is strong, its pullback s along $\pi_1 \times \cdots \times \pi_a$ is also strong, where $\pi_i\colon A_1 \times \cdots \times A_a \to A_i$ is the i-th projection.

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "r"] \\
(A_1 \times \cdots \times A_a)^a \arrow[r, "\pi_1 \times \cdots \times \pi_a"'] & A_1 \times \cdots \times A_a
\end{tikzcd}$$

We conclude the proof by Proposition 1.9 in [64] which says that r is strictly M-closed if and only if s is M-closed. □

If the above conditions are satisfied, we say that $\mathcal{C}$ *has M-closed strong relations.* In view of the following examples, we could also have written that $\mathcal{C}$ is ‘weakly with M-closed relations’.

Example 2.36. [91, 69] Let $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$, $M = \left(\begin{array}{ccc|c} x & y & y & x \\ x & x & y & y \end{array}\right)$ and $\mathcal{C}$ be a finitely complete category. When $\mathcal{C}$ has M-closed strong relations, we say it is *weakly Mal'tsev.*

The reader should compare the above example and following proposition with Example 2.7 and Theorem 2.28.

Proposition 2.37. [91, 69] A finitely complete category $\mathcal{C}$ is weakly Mal'tsev if and only if, for any pullback of split epimorphisms,

$$\begin{tikzcd}
P \arrow[r, two heads, shift right, "f'"'] \arrow[d, shift left, "g'"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[l, shift right, "r_Y"'] \arrow[d, shift right, "g"'] \\
X \arrow[u, shift left, "l_X"] \arrow[r, two heads, shift left, "f"] & Z \arrow[l, shift left, "s"] \arrow[u, shift right, "t"']
\end{tikzcd}$$

the induced morphisms l_X and r_Y (defined by $g'l_X = 1_X$, $f'l_X = tf$, $g'r_Y = sg$ and $f'r_Y = 1_Y$) are jointly epimorphic.

Example 2.38. Of course, every Mal'tsev category is weakly Mal'tsev. The category of commutative monoids with cancellation is a weakly Mal'tsev category but not a Mal'tsev one (this is due to G. Janelidze, see [91]). $\mathrm{Top}^{\mathrm{op}}$ is also a weakly Mal'tsev category (see [92]). This example can be generalised to other topological contexts using $(\mathbb{T}, V)$-categories (see [92, 32, 84] for more details).

Example 2.39. Let $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$, $M = \left(\begin{array}{cc|c} x & 0 & x \\ 0 & x & x \end{array}\right)$ and $\mathcal{C}$ be a finitely complete pointed category. If $\mathcal{C}$ has M-closed strong relations, it is called *weakly unital* [90].

Proposition 2.40. The following conditions on a pointed category $\mathcal{C}$ with finite limits are equivalent:

1. $\mathcal{C}$ is weakly unital,
2. for all objects X and Y in $\mathcal{C}$, the product injections
$$X \xrightarrow{(1_X,0)} X \times Y \xleftarrow{(0,1_Y)} Y$$
are jointly epimorphic,
3. for any object X in $\mathcal{C}$, the product injections
$$X \xrightarrow{(1_X,0)} X \times X \xleftarrow{(0,1_X)} X$$
are jointly epimorphic.

Proof. We first prove 1 ⇒ 2. Let $f, g\colon X \times Y \rightrightarrows Z$ be morphisms such that $f(1_X, 0) = g(1_X, 0)$ and $f(0, 1_Y) = g(0, 1_Y)$. Their equaliser $e\colon E \rightarrowtail X \times Y$ is a strong relation through which $(p_X, 0)$ and $(0, p_Y)\colon X \times Y \to X \times Y$ factor. Thus, by assumption, $1_{X\times Y} = (p_X, p_Y)\colon X \times Y \to X \times Y$ also factors through it, so that e is an isomorphism and $f = g$.

2 ⇒ 3 being trivial, it remains to prove 3 ⇒ 1. So, given a strong relation $r\colon R \rightarrowtail A^2$ and a morphism $x\colon X \to A$ such that $(x, 0)$ and $(0, x)\colon X \to A^2$ factor through r, we have to prove that (x, x) also factors through r. Let us consider the pullback s of r along x^2.

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "r"] \\
X^2 \arrow[r, "x^2"'] & A^2
\end{tikzcd}$$

The relation s is strong, $(1_X, 0)$ and $(0, 1_X)\colon X \to X^2$ factor through it and we only have to prove that $(1_X, 1_X)$ also factors through s. But since $(1_X, 0)$ and $(0, 1_X)$ are supposed to be jointly epimorphic, s is an epimorphism. Together with the fact that it is also a strong monomorphism, s is an isomorphism and so $(1_X, 1_X)$ factors through it. □

Again, compare this with Example 2.12 and Proposition 2.18.

2.2 Categories with (M, X)-closed relations

In order to encompass also the example of n-permutable categories [27], we present in this section a more general type of matrices [67] which, in a regular context, also give rise to conditions on categories.

Definition 2.41. [67] Let $\mathcal{T}$ be a Lawvere theory. An *extended matrix* (M, X) *of terms in* $\mathcal{T}$ is given by a matrix

$$M = \left(\begin{array}{ccc|ccc} t_{11} & \cdots & t_{1b} & u_{11} & \cdots & u_{1b'} \\ \vdots & & \vdots & \vdots & & \vdots \\ t_{a1} & \cdots & t_{ab} & u_{a1} & \cdots & u_{ab'} \end{array}\right) \tag{6}$$

with $a \geqslant 1$, $b \geqslant 0$, $b' \geqslant 0$ and where the u_{ij}'s are k-ary terms of $\mathcal{T}$ in the variables from a finite set X' with $\#X' = k$ and the t_{ij}'s are l-ary terms of $\mathcal{T}$ in the variables from $X'' \subseteq X'$ with $\#X'' = l$ and $X = X' \setminus X''$.

For simplicity, we will often think of X'' as $\{x_1, \dots, x_l\}$ and X as $\{x_{l+1}, \dots, x_k\}$.

Definition 2.42. [67] Let $\mathcal{T}$ be a Lawvere theory, (M, X) an extended matrix of terms in $\mathcal{T}$ as in (6) and $\mathcal{C}$ a regular $\mathcal{T}$-category. An a-ary relation $r\colon R \rightarrowtail A^a$ in $\mathcal{C}$ is said to be (M, X)-*closed* if, when we consider the pullbacks

$$\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d, tail, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & R^b \arrow[d, tail, "r^b"] \\
A^l \arrow[r, "\begin{pmatrix} t_{11}^A & \cdots & t_{1b}^A \\ \vdots & & \vdots \\ t_{a1}^A & \cdots & t_{ab}^A \end{pmatrix}"'] & (A^a)^b
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "g'"] \arrow[d, tail, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & R^{b'} \arrow[d, tail, "r^{b'}"] \\
A^k \arrow[r, "\begin{pmatrix} u_{11}^A & \cdots & u_{1b'}^A \\ \vdots & & \vdots \\ u_{a1}^A & \cdots & u_{ab'}^A \end{pmatrix}"'] & (A^a)^{b'}
\end{tikzcd}$$

and

$$\begin{tikzcd}
T \arrow[r, tail, "t'"] \arrow[dd, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, tail, "g"] \\
& A^k \cong A^l \times A^{k-l} \arrow[d, "\pi_1 = (p_1, \dots, p_l)"] \\
P \arrow[r, tail, "f"'] & A^l
\end{tikzcd}$$

then t is a regular epimorphism (or, in other words, f factors through the image of $\pi_1 g$). Here, $p_j\colon A^k \to A$ is the j-th projection for $1 \leqslant j \leqslant k$.

We also have a description of (M, X)-closedness in terms of generalised elements as in the following proposition.

Proposition 2.43. Let (M, X) be an extended matrix of terms in the Lawvere theory $\mathcal{T}$ as in (6). Let also $r\colon R \rightarrowtail A^a$ be an a-ary relation in the regular $\mathcal{T}$-category $\mathcal{C}$. Then, R is (M, X)-closed if and only if, for each morphism $y = (y_1, \dots, y_l)\colon Y \to A^l$ in $\mathcal{C}$ such that

$$(t_{1j}(y_1, \dots, y_l), \dots, t_{aj}(y_1, \dots, y_l))\colon Y \to A^a$$

factors through r for each $j \in \{1, \dots, b\}$, there exists a regular epimorphism $p\colon Z \twoheadrightarrow Y$ and morphisms $z_{l+1}, \dots, z_k\colon Z \to A$ such that the map $Z \to A^a$

$$(u_{1j}(y_1p, \dots, y_lp, z_{l+1}, \dots, z_k), \dots, u_{aj}(y_1p, \dots, y_lp, z_{l+1}, \dots, z_k))$$

factors through r for each $j \in \{1, \dots, b'\}$.

Proof. For the 'if part', it suffices to consider, using the notations of Definition 2.42, $y = f$. We then get a regular epimorphism $p\colon Z \twoheadrightarrow P$ and morphisms $z_{l+1}, \dots, z_k\colon Z \to A$ such that, for each $j \in \{1, \dots, b'\}$, the morphism $Z \to A^a$

$$(u_{1j}(p_1fp, \dots, p_lfp, z_{l+1}, \dots, z_k), \dots, u_{aj}(p_1fp, \dots, p_lfp, z_{l+1}, \dots, z_k))$$

factors as rw_j. Considering the morphisms

$$(w_1, \dots, w_{b'})\colon Z \to R^{b'}$$

and

$$(p_1fp, \dots, p_lfp, z_{l+1}, \dots, z_k)\colon Z \to A^k,$$

we get a morphism $z\colon Z \to Q$ such that

$$gz = (p_1fp, \dots, p_lfp, z_{l+1}, \dots, z_k)$$

which implies $\pi_1 gz = fp$. Therefore, p factors through t and so t is a regular epimorphism.

For the 'only if part', let $v_j\colon Y \to R$ be the unique morphism such that

$$rv_j = (t_{1j}(y_1, \dots, y_l), \dots, t_{aj}(y_1, \dots, y_l))$$

for each $j \in \{1, \dots, b\}$. Then, let $h\colon Y \to P$ be the unique morphism such that $fh = (y_1, \dots, y_l)$ and $f'h = (v_1, \dots, v_b)$. Eventually, we construct p as the pullback of t along h

$$\begin{tikzcd} Z \arrow[r, "p'"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "t"] \\ Y \arrow[r, "h"'] & P \end{tikzcd}$$

and $z_j = p_j g t' p'$ for each $j \in \{l+1, \dots, k\}$. It remains to see that

$$\begin{aligned} & u_{ij}(y_1 p, \dots, y_l p, z_{l+1}, \dots, z_k) \\ & = u_{ij}(p_1 f h p, \dots, p_l f h p, p_{l+1} g t' p', \dots, p_k g t' p') \\ & = u_{ij}(p_1 f t p', \dots, p_l f t p', p_{l+1} g t' p', \dots, p_k g t' p') \\ & = u_{ij}(p_1 g t' p', \dots, p_k g t' p') \\ & = u_{ij}(p_1, \dots, p_k) g t' p' \\ & = u_{ij}^A g t' p' \end{aligned}$$

for all $1 \leqslant i \leqslant a$ and $1 \leqslant j \leqslant b'$, and so

$$\begin{aligned} & (u_{1j}(y_1 p, \dots, y_l p, z_{l+1}, \dots, z_k), \dots, u_{aj}(y_1 p, \dots, y_l p, z_{l+1}, \dots, z_k)) \\ & = (u_{1j}^A, \dots, u_{aj}^A) g t' p' \\ & = r p_j g' t' p' \end{aligned}$$

factors through r for each $j \in \{1, \dots, b'\}$. □

Definition 2.44. Let $\mathcal{T}$ be a Lawvere theory and (M, X) an extended matrix of terms in $\mathcal{T}$ as in (6). We say that the regular $\mathcal{T}$-category $\mathcal{C}$ *has* (M, X)*-closed relations* if every a-ary relation $R \rightarrowtail A^a$ in $\mathcal{C}$ is (M, X)-closed.

Matrices from Section 2.1 are particular examples of extended matrices.

Example 2.45. If M is a simple extended matrix of terms in the Lawvere theory $\mathcal{T}$, it gives rise to the extended matrix $(M, \varnothing)$ of terms in $\mathcal{T}$. In this case, $b' = 1$ and $l = k$.

Proposition 2.46. [67] Let $\mathcal{T}$ be a Lawvere theory, M a simple extended matrix of terms in $\mathcal{T}$ as in (5) and $\mathcal{C}$ a regular $\mathcal{T}$-category. An a-ary relation $R \rightarrowtail A^a$ in $\mathcal{C}$ is M-closed if and only if it is $(M, \varnothing)$-closed. Thus $\mathcal{C}$ has M-closed relations if and only if it has $(M, \varnothing)$-closed relations.

Let us now discuss the case of n-permutable categories. If $R \rightarrowtail X \times Y$ and $S \rightarrowtail Y \times X$ are two binary relations in a regular category, and $n \geqslant 1$ a natural number, we denote by $(R, S)_n$ the relation given by the composite

$$(R, S)_n = RSR \cdots$$

which contains exactly n occurrences of R or S. If $R = S$ and $X = Y$, we write as usual $(R, R)_n = R^n$.

Definition 2.47. [27] Let $n \geqslant 2$ be a natural number. A regular category $\mathcal{C}$ is said to be *n-permutable* if, given two equivalence relations R and S on a same object, the identity

$$(R, S)_n = (S, R)_n$$

holds.

Example 2.48. In view of Theorem 2.11, a 2-permutable category is nothing else than a regular Mal'tsev category. A 3-permutable category is also called a *Goursat category* [27].

Theorem 2.49. [27, 70] Let $n \geqslant 2$ be a natural number and $\mathcal{C}$ a regular category. The following conditions are equivalent:

1. $\mathcal{C}$ is n-permutable,
2. $(R, S)_n = (S, R)_n$ for any two effective equivalence relations R and S on the same object,
3. $(P, P^{\mathrm{op}})_{n+1} \leqslant (P, P^{\mathrm{op}})_{n-1}$ for any binary relation P,
4. $(E, E^{\mathrm{op}})_{n-1}$ is an equivalence relation for any reflexive relation E,
5. $E^{\mathrm{op}} \leqslant E^{n-1}$ for any reflexive relation E.

Example 2.50. We consider here the case $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$ and

$$(M, X) = \left(\left(\begin{array}{ccc|ccccc} x & y & y & x & z_1 & z_2 & \cdots & z_{n-2} \\ x & x & y & z_1 & z_2 & \cdots & z_{n-2} & y \end{array} \right), \{z_1, \ldots, z_{n-2}\} \right)$$

for a natural number $n \geqslant 2$. We can prove that a regular category has (M, X)-closed relations if and only if it is n-permutable, but it would be very long to do it directly. We will instead prove it as an application of Barr's Embedding Theorem 4.2 (see Proposition 4.6).

As for simple extended matrices, we can nicely characterise finitary one-sorted algebraic categories with (M, X)-closed relations.

Theorem 2.51. [67] Let $F\colon \mathcal{T} \to \mathcal{R}$ be a central morphism of Lawvere theories. Let also (M, X) be an extended matrix of terms in $\mathcal{T}$ as in (6). Then, the regular $\mathcal{T}$-enriched category $\mathcal{R}$-Alg (induced by F) has (M, X)-closed relations if and only if there exist b-ary terms $p_1, \dots, p_{b'}$ and l-ary terms $q_1, \dots, q_{k-l}$ in $\mathcal{R}$ such that

$$\begin{aligned}&p_j(t^\iota_{i1}(x_1, \dots, x_l), \dots, t^\iota_{ib}(x_1, \dots, x_l))\\&\quad = u^\iota_{ij}(x_1, \dots, x_l, q_1(x_1, \dots, x_l), \dots, q_{k-l}(x_1, \dots, x_l))\end{aligned}$$

is a theorem of $\mathcal{R}$ for each $i \in \{1, \dots, a\}$ and each $j \in \{1, \dots, b'\}$, where t^ι denotes the interpretation in $\mathcal{R}$ induced by F of the term t in $\mathcal{T}$.

We now prove a similar characterisation of essentially algebraic categories with (M, X)-closed relations. Since images in those categories are a bit harder to describe than in the algebraic case, we need more terms here.

Theorem 2.52. Let Γ be an essentially algebraic theory such that $\mathrm{Mod}(\Gamma)$ is regular, $\mathcal{T}$ a Lawvere theory and (M, X) an extended matrix of terms in $\mathcal{T}$ as in (6). Given a $\mathcal{T}$-enrichment of $\mathrm{Mod}(\Gamma)$ (as in Proposition 1.107), $\mathrm{Mod}(\Gamma)$ has (M, X)-closed relations if and only if, for each $s \in S$, there exists in Γ

- a term $\pi^s\colon \prod_{u \in U} s_u \to s$,
- for every $v \in \{1, \dots, k\}$ and $u \in U$, an everywhere-defined term $q^u_v\colon s^l \to s_u$,
- for every $j \in \{1, \dots, b'\}$ and $u \in U$, a term $p^u_j\colon s^b \to s_u$

such that

1. the term
$$p^u_j(t^s_{i1}(x_1,\dots,x_l),\dots,t^s_{ib}(x_1,\dots,x_l))\colon s^l\to s_u$$
is everywhere-defined for all $i\in\{1,\dots,a\}$, $j\in\{1,\dots,b'\}$ and $u\in U$,

2. the theorem
$$\begin{aligned}&p^u_j(t^s_{i1}(x_1,\dots,x_l),\dots,t^s_{ib}(x_1,\dots,x_l))\\&\quad=u^{s_u}_{ij}(q^u_1(x_1,\dots,x_l),\dots,q^u_k(x_1,\dots,x_l))\end{aligned}$$
holds in Γ for all $i\in\{1,\dots,a\}$, $j\in\{1,\dots,b'\}$ and $u\in U$,

3. the term
$$\pi^s((q^u_v(x_1,\dots,x_l))_{u\in U})\colon s^l\to s$$
is everywhere-defined for each $v\in\{1,\dots,l\}$,

4. the theorem
$$\pi^s((q^u_v(x_1,\dots,x_l))_{u\in U})=x_v$$
holds in Γ for each $v\in\{1,\dots,l\}$.

Proof. Suppose firstly that such terms are given. Let $R\subseteq A^a$ be an a-ary relation on A in $\mathrm{Mod}(\Gamma)$. Let also P and Q be as in the definition of (M,X)-closedness. We have to prove that $f\colon P\rightarrowtail A^l$ factors through the image of $\pi_1 g\colon Q\to A^l$. Let $s\in S$ and $(a_1,\dots,a_l)\in P_s\subseteq A^l_s$. We know that
$$(t^s_{1j}(a_1,\dots,a_l),\dots,t^s_{aj}(a_1,\dots,a_l))\in R_s$$
for each $j\in\{1,\dots,b\}$. So, for each $j\in\{1,\dots,b'\}$ and $u\in U$,
$$\begin{aligned}&\Big(u^{s_u}_{1j}(q^u_1(a_1,\dots,a_l),\dots,q^u_k(a_1,\dots,a_l)),\dots\\&\qquad\qquad\dots,u^{s_u}_{aj}(q^u_1(a_1,\dots,a_l),\dots,q^u_k(a_1,\dots,a_l))\Big)\\&=\big(p^u_j(t^s_{11}(a_1,\dots,a_l),\dots,t^s_{1b}(a_1,\dots,a_l)),\dots\\&\qquad\qquad\dots,p^u_j(t^s_{a1}(a_1,\dots,a_l),\dots,t^s_{ab}(a_1,\dots,a_l))\big)\end{aligned}$$

$$\begin{aligned}&= p^u_j\left((t^s_{11}(a_1,\dots,a_l),\dots,t^s_{a1}(a_1,\dots,a_l)),\dots\right.\\&\qquad\qquad\qquad\qquad\left.\dots,(t^s_{1b}(a_1,\dots,a_l),\dots,t^s_{ab}(a_1,\dots,a_l))\right)\\&\in R_{s_u}\end{aligned}$$

since R is a submodel of A^a. This means that

$$b_u = (q^u_1(a_1,\dots,a_l),\dots,q^u_k(a_1,\dots,a_l)) \in Q_{s_u} \subseteq A^k_{s_u}$$

for all $u \in U$. Therefore,

$$\begin{aligned}(a_1,\dots,a_l) &= (\pi^s((q^u_1(a_1,\dots,a_l))_{u\in U}),\dots,\pi^s((q^u_l(a_1,\dots,a_l))_{u\in U}))\\&= \pi^s(((q^u_1(a_1,\dots,a_l),\dots,q^u_l(a_1,\dots,a_l)))_{u\in U})\\&= \pi^s((\pi_1 g(b_u))_{u\in U})\\&\in \mathrm{Im}(\pi_1 g)_s\end{aligned}$$

and R is (M, X)-closed.

Conversely, let us suppose that $\mathrm{Mod}(\Gamma)$ has (M, X)-closed relations. Let $s \in S$ and Y be the S-sorted set such that $Y_s = \{y_1,\dots,y_l\}$ and $Y_{s'} = \varnothing$ for $s' \neq s$. We denote by R the smallest a-ary homomorphic relation on $\mathrm{Fr}_\Gamma(Y)$ such that

$$(t^s_{1j}(y_1,\dots,y_l),\dots,t^s_{aj}(y_1,\dots,y_l)) \in R_s$$

for each $j \in \{1,\dots,b\}$. It is easy to prove that for each $s' \in S$,

$$\begin{aligned}R_{s'} = \Big\{&a\text{-tuple of everywhere-defined terms}\\&(t(t^s_{11}(y_1,\dots,y_l),\dots,t^s_{1b}(y_1,\dots,y_l)),\dots\\&\dots,t(t^s_{a1}(y_1,\dots,y_l),\dots,t^s_{ab}(y_1,\dots,y_l)))\mid\\&t\colon s^b \to s' \text{ is a term of } \Gamma\Big\}\\\subseteq{}& \mathrm{Fr}_\Gamma(Y)^a_{s'}.\end{aligned}$$

Let P and Q be as in the definition of (M, X)-closedness for R. Since $(y_1,\dots,y_l) \in P_s$ and R is (M, X)-closed, $(y_1,\dots,y_l) \in \mathrm{Im}(\pi_1 g)_s$. Therefore, there exists a term $\pi^s\colon \prod_{u\in U} s_u \to s$ and an element $q^u \in Q_{s_u}$ for each $u \in U$ such that

$$\pi^s((\pi_1 g(q^u))_{u \in U}) = (y_1, \dots, y_l).$$

So, for each $u \in U$, using the description of Q_{s_u}, there exist everywhere-defined terms $q_1^u, \dots, q_k^u \colon s^l \to s_u$ such that

$$(u_{1j}^{s_u}(q_1^u, \dots, q_k^u), \dots, u_{aj}^{s_u}(q_1^u, \dots, q_k^u)) \in R_{s_u}$$

for any $j \in \{1, \dots, b'\}$ and the term

$$\pi^s((q_v^u(y_1, \dots, y_l))_{u \in U})$$

is everywhere-defined and equal to y_v for each $v \in \{1, \dots, l\}$. The above description of R_{s_u} gives the terms $p_j^u \colon s^b \to s_u$ for every $u \in U$ and $j \in \{1, \dots, b'\}$ with the required properties. ☐

Of course, using the remark after Proposition 1.84, if Γ is finitary, we can consider that π^s is a finitary term $\prod_{u=1}^{m} s_u \to s$. This case has been proved in [59]. If we consider the matrix from Example 2.50, we get in this way a syntactic characterisation of n-permutable locally (finitely) presentable categories. Another characterisation of regular n-permutable categories of the form $\mathrm{Lex}(\mathcal{D}, \mathrm{Set})$ for a small finitely complete category $\mathcal{D}$ is proved in [47] in terms of $\mathcal{D}$. Since finitely presentable objects in the category $\mathrm{Mod}(\Gamma)$ are hard to understand, it does not seem obvious how to lift the characterisation from [47] to the above syntactic one. Moreover, with our Theorem 2.6, we described Mal'tsev locally presentable categories with no assumption of regularity.

Let us conclude this chapter with a generalisation of approximate co-solutions for M (Definition 2.31 and Theorem 2.32) from the previous section to the extended matrices (M, X). The n-permutable case appears in [97] while the general case is from [59]. In view of Theorem 2.51, we now need several approximate co-operations.

Definition 2.53. [59] Let $\mathcal{T}$ be a Lawvere theory and (M, X) an extended matrix of terms in $\mathcal{T}$ as in (6). Let also $\mathcal{C}$ be a regular $\mathcal{T}$-category with finite coproducts and Y an object of $\mathcal{C}$. An *approximate co-solution* *for* (M, X) on Y is a span

$$\begin{tikzcd}
 & Z \arrow[dl, "(p_1,\ldots,p_{b'})"'] \arrow[d, "d"] \arrow[dr, "(q_1,\ldots,q_{k-l})"] & \\
(bY)^{b'} & Y & (lY)^{k-l}
\end{tikzcd}$$

satisfying

$$\begin{pmatrix} t_{i1}^{Y,\mathrm{op}} \\ \vdots \\ t_{ib}^{Y,\mathrm{op}} \end{pmatrix} p_j = u_{ij}(\iota_1 d, \ldots, \iota_l d, q_1, \ldots, q_{k-l}) \colon Z \to lY$$

for all $i \in \{1, \ldots, a\}$ and $j \in \{1, \ldots, b'\}$, where $\iota_1, \ldots, \iota_l \colon Y \to lY$ are the coproduct injections.

For any object Y in $\mathcal{C}$, there is a universal such. Let L be the following product,

$$\begin{tikzcd}
 & L = (bY)^{b'} \times Y \times (lY)^{k-l} \arrow[dl, "(p'_1,\ldots,p'_{b'})"'] \arrow[d, "d'"] \arrow[dr, "(q'_1,\ldots,q'_{k-l})"] & \\
(bY)^{b'} & Y & (lY)^{k-l}
\end{tikzcd}$$

consider the equaliser

$$\begin{tikzcd}[column sep=huge]
W(Y) \arrow[r, tail, "e"] & L \arrow[r, shift left, "\left(\begin{pmatrix} t_{i1}^{Y,\mathrm{op}} \\ \vdots \\ t_{ib}^{Y\,\mathrm{op}} \end{pmatrix} p'_j\right)_{\substack{i\in\{1,\ldots,a\}\\ j\in\{1,\ldots,b'\}}}"] \arrow[r, shift right, "\left(u_{ij}(\iota_1 d',\ldots,\iota_l d',q'_1,\ldots,q'_{k-l})\right)_{\substack{i\in\{1,\ldots,a\}\\ j\in\{1,\ldots,b'\}}}"'] & (lY)^{a\times b'}
\end{tikzcd}$$

and let $d^Y = d'e$, $(q_1^Y, \ldots, q_{k-l}^Y) = (q'_1 e, \ldots, q'_{k-l} e)$ and $(p_1^Y, \ldots, p_{b'}^Y) = (p'_1 e, \ldots, p'_{b'} e)$. Then

$$\begin{tikzcd}
 & W(Y) \arrow[dl, "(p_1^Y,\ldots,p_{b'}^Y)"'] \arrow[d, "d^Y"] \arrow[dr, "(q_1^Y,\ldots,q_{k-l}^Y)"] & \\
(bY)^{b'} & Y & (lY)^{k-l}
\end{tikzcd}$$

is the *universal approximate co-solution for* (M, X) on Y in the sense that any other factorises uniquely through it.

Theorem 2.54. [59] Let $\mathcal{T}$ be a Lawvere theory and (M, X) an extended matrix of terms in $\mathcal{T}$. The following conditions on a regular $\mathcal{T}$-category $\mathcal{C}$ with finite coproducts are equivalent:

1. $\mathcal{C}$ has (M, X)-closed relations,
2. for each $Y \in \mathcal{C}$, there is an approximate co-solution for (M, X) on Y for which d is a regular epimorphism,
3. for each $Y \in \mathcal{C}$, the universal approximate co-solution for (M, X) on Y is such that d^Y is a regular epimorphism.

Proof. $3 \Rightarrow 2$ is obvious and $2 \Rightarrow 3$ follows from the universality of $W(Y)$. Let us prove $1 \Rightarrow 2$. So, let $Y \in \mathcal{C}$ and consider the pullbacks

$$\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & (bY)^b \arrow[d, "\begin{pmatrix} t_{11}^{Y,\mathrm{op}} & \cdots & t_{a1}^{Y,\mathrm{op}} \\ \vdots & & \vdots \\ t_{1b}^{Y,\mathrm{op}} & \cdots & t_{ab}^{Y,\mathrm{op}} \end{pmatrix}^b"] \\
(lY)^l \arrow[r, "\begin{pmatrix} t_{11}^{lY} & \cdots & t_{1b}^{lY} \\ \vdots & & \vdots \\ t_{a1}^{lY} & \cdots & t_{ab}^{lY} \end{pmatrix}"'] & ((lY)^a)^b
\end{tikzcd}$$

$$\begin{tikzcd}
Q \arrow[r, "g'"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & (bY)^{b'} \arrow[d, "\begin{pmatrix} t_{11}^{Y,\mathrm{op}} & \cdots & t_{a1}^{Y,\mathrm{op}} \\ \vdots & & \vdots \\ t_{1b}^{Y,\mathrm{op}} & \cdots & t_{ab}^{Y,\mathrm{op}} \end{pmatrix}^{b'}"] \\
(lY)^k \arrow[r, "\begin{pmatrix} u_{11}^{lY} & \cdots & u_{1b'}^{lY} \\ \vdots & & \vdots \\ u_{a1}^{lY} & \cdots & u_{ab'}^{lY} \end{pmatrix}"'] & ((lY)^a)^{b'}
\end{tikzcd}$$

and

$$\begin{tikzcd}
T \arrow[r, "t'"] \arrow[dd, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, "g"] \\
 & (lY)^k \cong (lY)^l \times (lY)^{k-l} \arrow[d, "\pi_1"] \\
P \arrow[r, "f"'] & (lY)^l.
\end{tikzcd}$$

Since the image of

$$\begin{tikzcd}
bY \arrow[rr, "{\begin{pmatrix} t_{11}^{Y,\mathrm{op}} & \cdots & t_{a1}^{Y,\mathrm{op}} \\ \vdots & & \vdots \\ t_{1b}^{Y,\mathrm{op}} & \cdots & t_{ab}^{Y,\mathrm{op}} \end{pmatrix}}"] & & (lY)^a
\end{tikzcd}$$

is an (M, X)-closed relation, t is a regular epimorphism. In addition, the diagram

$$\begin{tikzcd}
Y \arrow[r, "{(\iota_1,\ldots,\iota_b)}"] \arrow[d, "{(\iota_1,\ldots,\iota_l)}"'] & (bY)^b \arrow[d, "{\begin{pmatrix} t_{11}^{Y,\mathrm{op}} & \cdots & t_{a1}^{Y,\mathrm{op}} \\ \vdots & & \vdots \\ t_{1b}^{Y,\mathrm{op}} & \cdots & t_{ab}^{Y,\mathrm{op}} \end{pmatrix}^b}"] \\
(lY)^l \arrow[r, "{\begin{pmatrix} t_{11}^{lY} & \cdots & t_{1b}^{lY} \\ \vdots & & \vdots \\ t_{a1}^{lY} & \cdots & t_{ab}^{lY} \end{pmatrix}}"'] & ((lY)^a)^b
\end{tikzcd}$$

commutes and so $(\iota_1, \ldots, \iota_l)$ factors through f. Hence, if we consider the pullback

$$\begin{tikzcd}
Z \arrow[r, "d'"] \arrow[dd, two heads, "d"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, "g"] \\
 & (lY)^k \cong (lY)^l \times (lY)^{k-l} \arrow[d, "\pi_1"] \\
Y \arrow[r, "{(\iota_1,\ldots,\iota_l)}"'] & (lY)^l
\end{tikzcd}$$

the morphism $d : Z \twoheadrightarrow Y$, being a pullback of t, is a regular epimorphism. Therefore, we have the expected approximate co-solution for (M, X)

on Y:

$$\begin{tikzcd}
 & Z \arrow[ld, "g'd'"'] \arrow[d, two heads, "d"] \arrow[rd, "\pi_2 gd'"] & \\
(bY)^{b'} & Y & (lY)^{k-l}
\end{tikzcd}$$

It is actually an approximate co-solution for (M, X) on Y since, for each $i \in \{1, \dots, a\}$ and each $j \in \{1, \dots, b'\}$, by definition of the pullback Q, we know that

$$\begin{aligned}
\begin{pmatrix} t_{i1}^{Y,\mathrm{op}} \\ \vdots \\ t_{ib}^{Y,\mathrm{op}} \end{pmatrix} p'_j g' d' &= u_{ij}^{lY} gd' \\
&= u_{ij}(p''_1, \dots, p''_k) gd' \\
&= u_{ij}(p''_1 gd', \dots, p''_k gd') \\
&= u_{ij}(\iota_1 d, \dots, \iota_l d, p''_{l+1} gd', \dots, p''_k gd')
\end{aligned}$$

where $p'_1, \dots, p'_{b'} \colon (bY)^{b'} \to bY$ and $p''_1, \dots, p''_k \colon (lY)^k \to lY$ are the product projections.

It remains thus to prove $3 \Rightarrow 1$. Let $r \colon R \rightarrowtail A^a$ be an a-ary relation in $\mathcal{C}$. We are going to use Proposition 2.43 to prove that R is (M, X)-closed. So, we consider a morphism $y = (y_1, \dots, y_l) \colon Y \to A^l$ such that

$$(t_{1j}(y_1, \dots, y_l), \dots, t_{aj}(y_1, \dots, y_l)) \colon Y \to A^a$$

factors through r for each $j \in \{1, \dots, b\}$. By assumption, we have a regular epimorphism $d^Y \colon W(Y) \twoheadrightarrow Y$ and a morphism

$$z_v = \begin{pmatrix} y_1 \\ \vdots \\ y_l \end{pmatrix} q_{v-l}^Y \colon W(Y) \to A$$

for each $v \in \{l+1, \dots, k\}$. Now, for each $j \in \{1, \dots, b'\}$,

$$\begin{aligned}
&(u_{1j}(y_1 d^Y, \dots, y_l d^Y, z_{l+1}, \dots, z_k), \dots, u_{aj}(y_1 d^Y, \dots, y_l d^Y, z_{l+1}, \dots, z_k)) \\
&= \Bigg(\begin{pmatrix} y_1 \\ \vdots \\ y_l \end{pmatrix} u_{1j}(\iota_1 d^Y, \dots, \iota_l d^Y, q_1^Y, \dots, q_{k-l}^Y), \dots \\
&\qquad\qquad \dots, \begin{pmatrix} y_1 \\ \vdots \\ y_l \end{pmatrix} u_{aj}(\iota_1 d^Y, \dots, \iota_l d^Y, q_1^Y, \dots, q_{k-l}^Y) \Bigg)
\end{aligned}$$

$$= \left(\begin{pmatrix} y_1 \\ \vdots \\ y_l \end{pmatrix} \begin{pmatrix} t_{11}^{Y,\mathrm{op}} \\ \vdots \\ t_{1b}^{Y,\mathrm{op}} \end{pmatrix} p_j, \ldots, \begin{pmatrix} y_1 \\ \vdots \\ y_l \end{pmatrix} \begin{pmatrix} t_{a1}^{Y,\mathrm{op}} \\ \vdots \\ t_{ab}^{Y,\mathrm{op}} \end{pmatrix} p_j \right)$$

$$= \begin{pmatrix} t_{11}(y_1,\ldots,y_l) & \cdots & t_{a1}(y_1,\ldots,y_l) \\ \vdots & & \vdots \\ t_{1b}(y_1,\ldots,y_l) & \cdots & t_{ab}(y_1,\ldots,y_l) \end{pmatrix} p_j$$

which factors through r by assumption on $y_1, \ldots, y_l$. This proves that R is (M, X)-closed. $\square$

Chapter 3

Unconditional exactness properties

The free cofiltered limit completion of a small finitely complete category $\mathcal{C}$ is obtained as the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Lex}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ [7, 44]. One of the crucial part of Barr's proof for his embedding theorem [10] is the fact that, if $\mathcal{C}$ is regular, then so is $\mathrm{Lex}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$. Since we will need the same 'preservation' if $\mathcal{C}$ has (M, X)-closed relations for our embedding theorems, we are going to study in this chapter such properties which are 'preserved' under this completion.

We thus devote the first section of this chapter to precisely define what we call an 'unconditional exactness property'. Roughly speaking, this is a property which can be stated as 'if we start with a finite diagram of a given shape in $\mathcal{C}$ and add to it some finite (co)limits, then some finite (co)cones are also (co)limits'. This is called unconditional to contrast with properties of the kind: 'if some (co)cones are (co)limits, then some others are as well'. Many properties can be expressed as unconditional exactness properties: regularity, having (M, X)-closed relations, being Mal'tsev exact, abelian, and so forth (see Examples 3.15, 3.16 and 3.17).

We then prove in Section 3.2 that these unconditional exactness properties are 'preserved' under the free cofiltered limit completion. This generalises many already existing theorems (but not all), see Remark 3.22. As we said above, this will be used as a crucial tool in our embedding theorems. In addition, it will be shown in Subsection 3.2.2 how this preservation property can be used to remove an assumption on the existence of some colimits in a theorem.

3.1 Definition and examples

Since we would like to have as example of unconditional exactness properties the one of having (M, X)-closed relations for an extended matrix (M, X) of terms in a commutative Lawvere theory $\mathcal{T}$, we will actually define $\mathcal{T}$-unconditional exactness properties. Roughly speaking, it is the collection of

- a shape of diagram $\mathcal{E}_0$,
- an algorithm of construction of some finite (co)limits from this diagram (and some factorisations induced by them),
- some specified finite (co)cones in the resulting diagram.

In this language, a $\mathcal{T}$-category $\mathcal{C}$ satisfies it if, when we start with a diagram of shape $\mathcal{E}_0$ in $\mathcal{C}$ and when we add to it the finite (co)limits specified by the algorithm, then the specified finite (co)cones are (co)limits. This is equivalent to ask that some specified morphisms are isomorphisms.

In order to define them properly, we need to recall the notion of a conditional graph (see for instance the first volume of [14]). Graph theory was invented by Euler [39] in 1736, see e.g. [99] for recent developments.

Definition 3.1. The category Graph of *graphs* is the two-sorted algebraic category Σ-Alg where Σ is the S-sorted signature of algebras with $S = \{*_A, *_O\}$ defined by $\Sigma = \{d, c \colon *_A \to *_O\}$. As usual, if G is the graph $d, c \colon A \rightrightarrows O$, $A = \mathrm{ar}(G)$ is the set of *arrows* of G, $O = \mathrm{ob}(G)$ is the set of *objects* of G and an arrow $f \in A$ is represented as $f \colon d(f) \to c(f)$.

Definition 3.2. Let G be a graph. A *path* in G is a non-empty finite sequence $p = (A_1, f_1, A_2, f_2, \dots, A_n)$ alternating objects A_i and arrows f_i, the first and the last item being objects and each f_i is an arrow $A_i \to A_{i+1}$. In this case we write $p \colon A_1 \to A_n$.

Definition 3.3. A *commutativity condition* on a graph G is a pair of paths $((A_1, f_1, \dots, A_n), (B_1, g_1, \dots, B_m))$ with $A_1 = B_1$ and $A_n = B_m$.

Definition 3.4. A *conditional graph* is a graph G together with a set of commutativity conditions. A *morphism* $F\colon G \to G'$ *of conditional graphs* is a morphism of graphs such that, for each commutativity condition $((A_1, f_1, \ldots, A_n), (B_1, g_1, \ldots, B_m))$ of G,

$$((F(A_1), F(f_1), \ldots, F(A_n)), (F(B_1), F(g_1), \ldots, F(B_m)))$$

is a commutativity condition of G'. This gives rise to the category CondGraph.

Each small category $\mathcal{C}$ has of course an underlying graph. But it has also an underlying conditional graph, considering the set of all commutativity conditions $((A_1, f_1, \ldots, A_n), (B_1, g_1, \ldots, B_m))$ such that

$$f_{n-1} \circ \cdots \circ f_1 = g_{m-1} \circ \cdots \circ g_1$$

(where the left hand side is considered as 1_{A_1} if $n = 1$ and similarly for the right hand side). We thus have a forgetful functor

$$U\colon \text{Cat} \to \text{CondGraph}\,.$$

Proposition 3.5. The forgetful functor $U\colon$ Cat $\to$ CondGraph has a left adjoint Path: CondGraph $\to$ Cat.

The construction of Path can be found in [14]. It goes as follows: For a conditional graph G, let $\mathcal{D}$ be the category defined by $\text{ob}(\mathcal{D}) = \text{ob}(G)$ and for $A, B \in \text{ob}(G)$, $\mathcal{D}(A, B)$ is the set of paths $A \to B$. Composition is the concatenation of paths and 1_A is the path (A). Then, let $\mathcal{R}$ be the smallest subcategory of $\mathcal{D} \times \mathcal{D}$ which satisfies the conditions

1. $\text{ob}(\mathcal{R}) = \{(A, A) \mid A \in \mathcal{D}\}$,
2. for any $A, B \in \mathcal{D}$, $\mathcal{R}(A, B)$ contains the commutativity conditions of G made of parallel paths $A \to B$,
3. $\mathcal{R}(A, B) \subseteq \mathcal{D}(A, B) \times \mathcal{D}(A, B)$ is an equivalence relation for any $A, B \in \mathcal{D}$

where, for the sake of brevity, we denoted $\mathcal{R}((A, A), (B, B))$ by $\mathcal{R}(A, B)$. We then set $\text{ob}(\text{Path}(G)) = \text{ob}(G)$ and for any objects $A, B \in G$,

$\mathrm{Path}(A,B)$ is the quotient of $\mathcal{D}(A,B)$ by $\mathcal{R}(A,B)$. Composition and identities are computed as in $\mathcal{D}$. The unit $G \to U(\mathrm{Path}(G))$ is the identity on objects and sends $f\colon A \to B$ to $[(A,f,B)]$.

We now need to prove a few lemmas which deal with other universal constructions.

Lemma 3.6. [58] Let $\mathcal{T}$ be a Lawvere theory and $\mathcal{C}$ a $\mathcal{T}$-category. Suppose we have, for all $A,B \in \mathcal{C}$, a subset $S(A,B) \subseteq \mathcal{C}(A,B) \times \mathcal{C}(A,B)$. Let $S = \bigcup_{A,B\in\mathcal{C}} S(A,B)$. Then, there exists a unique (up to isomorphism) $\mathcal{T}$-functor $Q\colon \mathcal{C} \to \mathcal{D}$ satisfying $Q(f) = Q(g)$ for all $(f,g) \in S$ and such that, for any $\mathcal{T}$-functor $F\colon \mathcal{C} \to \mathcal{D}'$ satisfying $F(f) = F(g)$ for each $(f,g) \in S$, there exists a unique $\mathcal{T}$-functor $\overline{F}\colon \mathcal{D} \to \mathcal{D}'$ such that $\overline{F}Q = F$.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "Q"] \arrow[dr, "\forall F"'] & & \mathcal{D} \arrow[dl, dotted, "\exists!\,\overline{F}"] \\
& \mathcal{D}' &
\end{tikzcd}$$

Proof. By a $\mathcal{T}$-subcategory of a $\mathcal{T}$-category $\mathcal{E}$, we mean a subcategory $\mathcal{E}' \subseteq \mathcal{E}$ such that $t(f_1,\dots,f_n) \in \mathcal{E}'(A,B)$ for all $f_1,\dots,f_n \in \mathcal{E}'(A,B)$ and n-ary term t of $\mathcal{T}$. Let now $\mathcal{R}$ be the smallest $\mathcal{T}$-subcategory of $\mathcal{C} \times \mathcal{C}$ such that

1. $\mathrm{ob}(\mathcal{R}) = \{(A,A) \mid A \in \mathcal{C}\}$,
2. $S(A,B) \subseteq \mathcal{R}(A,B)$,
3. $\mathcal{R}(A,B) \subseteq \mathcal{C}(A,B) \times \mathcal{C}(A,B)$ is an equivalence relation for any $A,B \in \mathcal{C}$

where, for the sake of brevity, we denoted $\mathcal{R}((A,A),(B,B))$ by $\mathcal{R}(A,B)$. Now, let $\mathrm{ob}(\mathcal{D}) = \mathrm{ob}(\mathcal{C})$ and $\mathcal{D}(A,B) = \mathcal{C}(A,B)/\mathcal{R}(A,B)$ for all objects $A,B \in \mathcal{C}$. $\mathcal{D}$ is a $\mathcal{T}$-category where the identities, the composition and the $\mathcal{T}$-enrichment are induced by the ones in $\mathcal{C}$. Moreover, the quotient map $Q\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$-functor such that $Q(f) = Q(g)$ for each $(f,g) \in S$.

If we have a $\mathcal{T}$-functor $F\colon \mathcal{C} \to \mathcal{D}'$ satisfying $F(f) = F(g)$ for every $(f,g) \in S$, to prove the existence of $\overline{F}$, it suffices to consider the $\mathcal{T}$-subcategory $\mathcal{R}'$ of $\mathcal{C} \times \mathcal{C}$ such that $\mathrm{ob}(\mathcal{R}') = \{(A,A) \mid A \in \mathcal{C}\}$ and $\mathcal{R}'(A,B) = \{(f,g) \in \mathcal{C}(A,B) \times \mathcal{C}(A,B) \mid F(f) = F(g)\}$. By definition of $\mathcal{R}$, $\mathcal{R} \subseteq \mathcal{R}'$, and we can construct $\overline{F}$ with the universal property of the quotient. Uniqueness follows easily. □

We will denote this $\mathcal{T}$-category $\mathcal{D}$ by $\mathcal{C}/S$. The notations $U_{\mathcal{T}}$, $\mathrm{Fr}_{\mathcal{T}}$, U and Path in the next lemma come from Corollary 1.110 and Proposition 3.5.

Lemma 3.7. [58] Let $\mathcal{T}$ be a commutative Lawvere theory, $\mathcal{C}$ a small $\mathcal{T}$-category and $F\colon UU_{\mathcal{T}}(\mathcal{C}) \to G$ a morphism of conditional graphs. Then, there exists a unique (up to isomorphism) morphism of conditional graphs $\widehat{F}\colon G \to UU_{\mathcal{T}}(\mathcal{D})$ with $\mathcal{D}$ a small $\mathcal{T}$-category such that $\widehat{F}F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$-functor and satisfying the following universal property: if $H\colon G \to UU_{\mathcal{T}}(\mathcal{D}')$ is a morphism of conditional graphs with $\mathcal{D}'$ a small $\mathcal{T}$-category such that $HF\colon \mathcal{C} \to \mathcal{D}'$ is a $\mathcal{T}$-functor, there exists a unique $\mathcal{T}$-functor $\overline{H}\colon \mathcal{D} \to \mathcal{D}'$ such that $UU_{\mathcal{T}}(\overline{H})\widehat{F} = H$.

$$\begin{tikzcd}
UU_{\mathcal{T}}(\mathcal{C}) \arrow[r, "F"] & G \arrow[rr, "\widehat{F}"] \arrow[dr, "\forall H"'] & & UU_{\mathcal{T}}(\mathcal{D}) \arrow[dl, dotted, "\exists !\, UU_{\mathcal{T}}(\overline{H})"] \\
& & UU_{\mathcal{T}}(\mathcal{D}') &
\end{tikzcd}$$

Moreover, the analogous property also holds if $\mathcal{D}'$ is a (not necessarily small) $\mathcal{T}$-category.

Proof. Let $\widehat{F}$ be the composite

$$\begin{tikzcd}
& G \arrow[rr, "F_1"] & & U(\mathrm{Path}(G)) \arrow[dll, "U(F_2)"'] \\
U(U_{\mathcal{T}}(\mathrm{Fr}_{\mathcal{T}}(\mathrm{Path}(G)))) \arrow[rrr, "UU_{\mathcal{T}}(F_3)"'] & & & U(U_{\mathcal{T}}(\mathrm{Fr}_{\mathcal{T}}(\mathrm{Path}(G))/S))
\end{tikzcd}$$

where

$$\begin{aligned} S(A,B) = \{ & (t(F_2F_1F(h_1), \dots, F_2F_1F(h_n)), F_2F_1F(t(h_1, \dots, h_n))) \mid \\ & t \text{ is an } n\text{-ary term of } \mathcal{T} \text{ and } h_1, \dots, h_n \in \mathcal{C}(A,B)\}. \end{aligned}$$

The result then follows from the universal properties involved in the construction. Since the construction of $\widehat{F}$ does not change if we consider a bigger universe $\mathcal{V} \ni \mathcal{U}$, the universal property also holds if $\mathcal{D}'$ is not small. □

This small $\mathcal{T}$-category $\mathcal{D}$ will be denoted by $\mathcal{T}$-Path(F). Next definition describes how to formally add a (co)cone in a $\mathcal{T}$-sketch (or an induced morphism). In order to shorten notations, if there is no ambiguity, we will not always write the functors U and $U_{\mathcal{T}}$ when we apply them.

Definition 3.8. [58] Let $\mathcal{T}$ be a commutative Lawvere theory and $\mathbb{S} = (\mathcal{S}, \mathcal{P}, \mathcal{I})$ a $\mathcal{T}$-sketch. A *universal augmentation* of $\mathbb{S}$ is a morphism of $\mathcal{T}$-sketches $F \colon \mathbb{S} \to \mathbb{S}' = (\mathcal{S}', \mathcal{P}', \mathcal{I}')$ which can be built up from $\mathbb{S}$ by one of these four operations:

a. Choose a diagram $G \colon \mathcal{D} \to \mathcal{S}$ with $\mathcal{D}$ a finite category. Then, define the conditional graph $\overline{\mathcal{S}}$ as follows:

- the set of objects of $\overline{\mathcal{S}}$ is

$$\mathrm{ob}(\overline{\mathcal{S}}) = \mathrm{ob}(\mathcal{S}) \sqcup \{L\},$$

- the set of arrows of $\overline{\mathcal{S}}$ is

$$\mathrm{ar}(\overline{\mathcal{S}}) = \mathrm{ar}(\mathcal{S}) \sqcup \{\, L \xrightarrow{\lambda_D} G(D) \ \mid D \in \mathcal{D}\},$$

- the commutativity conditions on $\overline{\mathcal{S}}$ are the ones from $UU_{\mathcal{T}}(\mathcal{S})$ and

$$((L, \lambda_{D_1}, G(D_1), G(d), G(D_2)), (L, \lambda_{D_2}, G(D_2)))$$

for any map $d \colon D_1 \to D_2$ in $\mathcal{D}$.

Then, define $F \colon \mathcal{S} \to \mathcal{S}'$ to be the composite

$$\mathcal{S} \hookrightarrow \overline{\mathcal{S}} \to \mathcal{T}\text{-Path}(\mathcal{S} \hookrightarrow \overline{\mathcal{S}}),$$

$\mathcal{P}'$ to be induced by $\mathcal{P}$ and $\lambda \colon \Delta_L \Rightarrow G$ and $\mathcal{I}'$ to be induced by $\mathcal{I}$.

b. Choose a diagram $G \colon \mathcal{D} \to \mathcal{S}$ with $\mathcal{D}$ a finite category. Then, define the conditional graph $\overline{\mathcal{S}}$ as follows:

- the set of objects of $\overline{\mathcal{S}}$ is

$$\mathrm{ob}(\overline{\mathcal{S}}) = \mathrm{ob}(\mathcal{S}) \sqcup \{C\},$$

- the set of arrows of $\overline{\mathcal{S}}$ is

$$\mathrm{ar}(\overline{\mathcal{S}}) = \mathrm{ar}(\mathcal{S}) \sqcup \{\, G(D) \xrightarrow{\lambda_D} C \mid D \in \mathcal{D}\},$$

- the commutativity conditions on $\overline{\mathcal{S}}$ are the ones from $UU_{\mathcal{T}}(\mathcal{S})$ and

$$((G(D_1), \lambda_{D_1}, C), (G(D_1), G(d), G(D_2), \lambda_{D_2}, C))$$

  for any map $d\colon D_1 \to D_2$ in $\mathcal{D}$.

Then, define $F\colon \mathcal{S} \to \mathcal{S}'$ to be the composite

$$\mathcal{S} \hookrightarrow \overline{\mathcal{S}} \to \mathcal{T}\text{-Path}(\mathcal{S} \hookrightarrow \overline{\mathcal{S}}),$$

$\mathcal{P}'$ to be induced by $\mathcal{P}$ and $\mathcal{I}'$ to be induced by $\mathcal{I}$ and $\lambda\colon G \Rightarrow \Delta_C$.

c. Choose a cone $\lambda\colon \Delta_L \Rightarrow G$ in $\mathcal{P}$ and any cone $\mu\colon \Delta_X \Rightarrow G$ over $G\colon \mathcal{D} \to \mathcal{S}$. Define the conditional graph $\overline{\mathcal{S}}$ as follows:

- the objects of $\overline{\mathcal{S}}$ are the same as the ones of $\mathcal{S}$,
- the set of arrows of $\overline{\mathcal{S}}$ is

$$\mathrm{ar}(\overline{\mathcal{S}}) = \mathrm{ar}(\mathcal{S}) \sqcup \{\, X \xrightarrow{m} L \,\},$$

- the commutativity conditions on $\overline{\mathcal{S}}$ are the ones from $UU_{\mathcal{T}}(\mathcal{S})$ and

$$((X, \mu_D, G(D)), (X, m, L, \lambda_D, G(D)))$$

  for any object $D \in \mathcal{D}$.

Then, define $F\colon \mathcal{S} \to \mathcal{S}'$ to be the composite

$$\mathcal{S} \hookrightarrow \overline{\mathcal{S}} \to \mathcal{T}\text{-Path}(\mathcal{S} \hookrightarrow \overline{\mathcal{S}}),$$

$\mathcal{P}'$ to be induced by $\mathcal{P}$ and $\mathcal{I}'$ to be induced by $\mathcal{I}$.

d. Choose a cocone $\lambda\colon G \Rightarrow \Delta_C$ in $\mathcal{I}$ and any cocone $\mu\colon G \Rightarrow \Delta_X$ over $G\colon \mathcal{D} \to \mathcal{S}$. Define the conditional graph $\overline{\mathcal{S}}$ as follows:

- the objects of $\overline{\mathcal{S}}$ are the same as the ones of $\mathcal{S}$,
- the set of arrows of $\overline{\mathcal{S}}$ is

$$\mathrm{ar}(\overline{\mathcal{S}}) = \mathrm{ar}(\mathcal{S}) \sqcup \{\, C \xrightarrow{m} X \,\},$$

- the commutativity conditions on $\overline{\mathcal{S}}$ are the ones from $UU_{\mathcal{T}}(\mathcal{S})$ and

$$((G(D), \lambda_D, C, m, X), (G(D), \mu_D, X))$$

for any object $D \in \mathcal{D}$.

Then, define $F\colon \mathcal{S} \to \mathcal{S}'$ to be the composite

$$\mathcal{S} \hookrightarrow \overline{\mathcal{S}} \to \mathcal{T}\text{-Path}(\mathcal{S} \hookrightarrow \overline{\mathcal{S}}),$$

$\mathcal{P}'$ to be induced by $\mathcal{P}$ and $\mathcal{I}'$ to be induced by $\mathcal{I}$.

Definition 3.9. Let $\mathcal{T}$ be a Lawvere theory and $\mathcal{E}$ a $\mathcal{T}$-category. We say that $\mathcal{E}$ *has finitely presentable hom-algebras* if, for all $A, B \in \mathcal{E}$, $\mathcal{E}(A, B)$ is a finitely presentable object in $\mathcal{T}$-Alg.

Note that a better name would have been 'locally finitely presentable', but this terminology is already used (Definition 1.68). In view of Example 1.40, if $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$ or $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$, a category $\mathcal{E}$ has finitely presentable hom-algebras if and only if $\mathcal{E}(A, B)$ is a finite set for each pair of objects $A, B \in \mathcal{E}$. We are now able to define $\mathcal{T}$-unconditional exactness properties.

Definition 3.10. [58] Let $\mathcal{T}$ be a commutative Lawvere theory. A *basic $\mathcal{T}$-unconditional exactness property* is a morphism of $\mathcal{T}$-sketches $e\colon \mathrm{I}(\mathcal{E}_0) \to \mathbb{E}_1$ where $\mathcal{E}_0$ is a small $\mathcal{T}$-category with finitely presentable hom-algebras and e is the composite of

$$\mathrm{I}(\mathcal{E}_0) = \mathbb{S}_0 \longrightarrow \mathbb{S}_1 \longrightarrow \dots \longrightarrow \mathbb{S}_n \xrightarrow{1_{\mathcal{S}_n}} \mathbb{E}_1$$

where each $\mathbb{S}_i \to \mathbb{S}_{i+1}$ is a universal augmentation of $\mathbb{S}_i = (\mathcal{S}_i, \mathcal{P}_i, \mathcal{I}_i)$ and $\mathbb{E}_1$ is $(\mathcal{S}_n, \mathcal{P}, \mathcal{I})$ with $\mathcal{P} \supseteq \mathcal{P}_n$ and $\mathcal{I} \supseteq \mathcal{I}_n$ containing only (co)cones over finite diagrams. A $\mathcal{T}$*-unconditional exactness property* is a class of basic $\mathcal{T}$-unconditional exactness properties.

Definition 3.11. [58] Let $\mathcal{T}$ be a commutative Lawvere theory. We say that a $\mathcal{T}$-category $\mathcal{C}$ *satisfies* the basic $\mathcal{T}$-unconditional exactness property $e\colon \mathrm{I}(\mathcal{E}_0) \to \mathbb{E}_1$ if, for any $\mathcal{T}$-functor $H\colon \mathcal{E}_0 \to \mathcal{C}$, there exists a morphism of $\mathcal{V}$-$\mathcal{T}$-sketches (for some bigger universe $\mathcal{V} \ni \mathcal{U}$) $\overline{H}\colon \mathbb{E}_1 \to \mathrm{FD}(\mathcal{C})$ such that $\overline{H}e = H$.

$$\begin{tikzcd}
\mathrm{I}(\mathcal{E}_0) \arrow[rr, "e"] \arrow[dr, "\forall H"'] & & \mathbb{E}_1 \arrow[dl, dotted, "\exists\,\overline{H}"] \\
& \mathrm{FD}(\mathcal{C}) &
\end{tikzcd}$$

We say that a $\mathcal{T}$-category $\mathcal{C}$ *satisfies* a $\mathcal{T}$-unconditional exactness property P if it satisfies all basic ones in P.

Note that this definition does not depend on the bigger universe $\mathcal{V}$ we choose. In the next lemma, we also choose any bigger universe $\mathcal{V} \ni \mathcal{U}$ and, for the sake of brevity, we denote the $\mathcal{V}$-2-category $\mathcal{V}$-$\mathcal{T}$-Sk of $\mathcal{V}$-$\mathcal{T}$-sketches simply by SK.

Lemma 3.12. [58] Let $\mathcal{T}$ be a commutative Lawvere theory and e : $\mathrm{I}(\mathcal{E}_0) \to \mathbb{E}_1$ a basic $\mathcal{T}$-unconditional exactness property. Then, the following statements hold:

1. If $\mathcal{C}$ is a $\mathcal{T}$-category, the composition functor

$$\mathrm{SK}(e, \mathrm{FD}(\mathcal{C}))\colon\ \mathrm{SK}(\mathbb{E}_1, \mathrm{FD}(\mathcal{C})) \to \mathrm{SK}(\mathrm{I}(\mathcal{E}_0), \mathrm{FD}(\mathcal{C}))$$

is full and faithful. Thus, each natural transformation $\alpha\colon H \Rightarrow H'$ between $\mathcal{T}$-functors $\mathcal{E}_0 \rightrightarrows \mathcal{C}$ with extensions $\overline{H}, \overline{H'}\colon \mathbb{E}_1 \rightrightarrows \mathrm{FD}(\mathcal{C})$ extends uniquely as $\beta\colon \overline{H} \Rightarrow \overline{H'}$ with $\beta \star 1_e = \alpha$.

$$\begin{tikzcd}
\mathrm{I}(\mathcal{E}_0) \arrow[rr, "e"] \arrow[dr, bend left, "H", ""{name=A, below}] \arrow[dr, bend right, "H'"', ""{name=B, above}] & & \mathbb{E}_1 \arrow[dl, bend right, "\overline{H}"', ""{name=C, below}] \arrow[dl, bend left, "\overline{H'}", ""{name=D, above}] \\
& \mathrm{FD}(\mathcal{C}) &
\arrow[Rightarrow, from=A, to=B, "\alpha"] \arrow[Rightarrow, dotted, from=C, to=D, "\beta"']
\end{tikzcd}$$

2. A $\mathcal{T}$-category $\mathcal{C}$ satisfies e if and only if

$$\mathrm{SK}(e, \mathrm{FD}(\mathcal{C}))\colon \mathrm{SK}(\mathbb{E}_1, \mathrm{FD}(\mathcal{C})) \to \mathrm{SK}(\mathrm{I}(\mathcal{E}_0), \mathrm{FD}(\mathcal{C}))$$

   is essentially surjective on objects. This means that for each $\mathcal{T}$-functor $H\colon \mathcal{E}_0 \to \mathcal{C}$, there exists a morphism of $\mathcal{V}$-$\mathcal{T}$-sketches $\overline{H} : \mathbb{E}_1 \to \mathrm{FD}(\mathcal{C})$ such that $\overline{H}e$ is isomorphic to H.

$$\begin{tikzcd}
\mathrm{I}(\mathcal{E}_0) \arrow[rr, "e"] \arrow[dr, "\forall H"'] & \cong & \mathbb{E}_1 \arrow[dl, dotted, "\exists\, \overline{H}"] \\
& \mathrm{FD}(\mathcal{C}) &
\end{tikzcd}$$

3. If a $\mathcal{T}$-category $\mathcal{C}$ satisfies e, the extension $\overline{H}$ of H is unique up to isomorphism.

4. If $H : \mathcal{C} \rightleftarrows \mathcal{D} : K$ is an equivalence of $\mathcal{T}$-categories (with H and K $\mathcal{T}$-functors), $\mathcal{C}$ satisfies e if and only if $\mathcal{D}$ does.

Proof. 1. $\mathrm{SK}(1_{\mathcal{S}_n}, \mathrm{FD}(\mathcal{C}))\colon \mathrm{SK}(\mathbb{E}_1, \mathrm{FD}(\mathcal{C})) \to \mathrm{SK}(\mathbb{S}_n, \mathrm{FD}(\mathcal{C}))$ is obviously full and faithful. So, it remains to prove the same for $\mathrm{SK}(F, \mathrm{FD}(\mathcal{C}))$ for a universal augmentation

$$F\colon \mathbb{S} \hookrightarrow \overline{\mathbb{S}} \to \mathcal{T}\text{-Path}(\mathbb{S} \hookrightarrow \overline{\mathbb{S}}) = \mathbb{S}'.$$

   Faithfulness follows from the fact that each object of $\mathbb{S}'$ which does not come from $\mathbb{S}$ is sent to a (co)limit by $\overline{H'}$. Besides, fullness follows from the universal property of those (co)limits and the fact that each morphism of $\mathbb{S}'$ is on the form

$$t(f_{11} \cdots f_{1m_1}, \ldots, f_{n1} \cdots f_{nm_n})$$

   where t is an n-ary term of $\mathcal{T}$ and the f_{ij}'s come from $\overline{\mathbb{S}}$.

2. The 'only if part' is trivial. Let us prove the 'if part'. A morphism of $\mathcal{V}$-$\mathcal{T}$-sketches $\mathbb{S}_n \to \mathrm{FD}(\mathcal{C})$ extends to $\mathbb{E}_1$ if it does only up to isomorphism. Indeed, both statements mean that some diagrams in $\mathcal{C}$ are (co)limits. Similarly, a morphism of $\mathcal{V}$-$\mathcal{T}$-sketches $\mathbb{S}_i \to \mathrm{FD}(\mathcal{C})$ extends to $\mathbb{S}_{i+1}$ if it does only up to isomorphism. Indeed, both

statements mean that some diagram in $\mathcal{C}$ has a limit (Operation **a** in Definition 3.8), or some diagram in $\mathcal{C}$ has a colimit (Operation **b**) or are trivial (Operations **c** and **d**). The result follows then from Point 1 in which we proved that $\mathrm{SK}(F, \mathrm{FD}(\mathcal{C}))$ is conservative for each universal augmentation F.

3. It follows directly from Point 1.
4. Direct consequence of Point 2. □

The notion of a $\mathcal{T}$-unconditional exactness property is self-dual.

Proposition 3.13. [58] Let $\mathcal{T}$ be a commutative Lawvere theory and $e\colon \mathrm{I}(\mathcal{E}_0) \to \mathbb{E}_1$ a basic $\mathcal{T}$-unconditional exactness property. Then, the dual morphism

$$e^{\mathrm{op}}\colon \mathrm{I}(\mathcal{E}_0^{\mathrm{op}}) \longrightarrow \mathbb{E}_1^{\mathrm{op}} = (\mathcal{S}_n^{\mathrm{op}}, \mathcal{I}^{\mathrm{op}}, \mathcal{P}^{\mathrm{op}})$$

is also a basic $\mathcal{T}$-unconditional exactness property. Moreover, a $\mathcal{T}$-category $\mathcal{C}$ satisfies e if and only if $\mathcal{C}^{\mathrm{op}}$ satisfies e^{op}.

Proof. The conditions on $\mathcal{E}_0^{\mathrm{op}}$ and $\mathbb{E}_1^{\mathrm{op}}$ hold since they do for $\mathcal{E}_0$ and $\mathbb{E}_1$. Then, it suffices to notice that $F\colon \mathbb{S}_i \to \mathbb{S}_{i+1}$ is a universal augmentation if and only if $F^{\mathrm{op}}\colon \mathbb{S}_i^{\mathrm{op}} \to \mathbb{S}_{i+1}^{\mathrm{op}}$ is. Indeed, $\mathrm{Path}(G^{\mathrm{op}}) \cong (\mathrm{Path}(G))^{\mathrm{op}}$ for a conditional graph G, $\mathrm{Fr}_{\mathcal{T}}(\mathcal{D}^{\mathrm{op}}) \cong (\mathrm{Fr}_{\mathcal{T}}(\mathcal{D}))^{\mathrm{op}}$ for a small category $\mathcal{D}$ and $\mathcal{C}^{\mathrm{op}}/S^{\mathrm{op}} \cong (\mathcal{C}/S)^{\mathrm{op}}$ for a small $\mathcal{T}$-category $\mathcal{C}$ and sets $S(A,B) \subseteq \mathcal{C}(A,B) \times \mathcal{C}(A,B)$. Thus, Operations **a** and **b** are dual to each other as well as Operations **c** and **d**. The second part of the statement is obvious. □

Before giving some examples of $\mathcal{T}$-unconditional exactness properties, let us prove they interact well with morphisms of Lawvere theories.

Proposition 3.14. [58] Let $F\colon \mathcal{T} \to \mathcal{R}$ be a morphism of commutative Lawvere theories and $e\colon \mathrm{I}(\mathcal{E}_0) \longrightarrow \mathbb{E}_1$ a basic $\mathcal{T}$-unconditional exactness property. Then,

$$\mathrm{Fr}_F(e)\colon \mathrm{Fr}_F(\mathrm{I}(\mathcal{E}_0)) \longrightarrow \mathrm{Fr}_F(\mathbb{E}_1)$$

is a basic $\mathcal{R}$-unconditional exactness property. Moreover, an $\mathcal{R}$-category $\mathcal{C}$ satisfies $\mathrm{Fr}_F(e)$ if and only if the $\mathcal{T}$-category $F^*(\mathcal{C})$ satisfies e.

Proof. By construction, $\mathrm{Fr}_F(\mathrm{I}(\mathcal{E}_0)) = \mathrm{I}(\mathrm{Fr}_F(\mathcal{E}_0))$. Since $\mathrm{Fr}_F\colon \mathcal{T}\text{-Alg} \to \mathcal{R}\text{-Alg}$ preserves coequalisers, in view of Proposition 1.39, it sends finitely presentable $\mathcal{T}$-algebras to finitely presentable $\mathcal{R}$-algebras. So, $\mathrm{Fr}_F(\mathcal{E}_0)$ is a small $\mathcal{R}$-category with finitely presentable hom-algebras. Also by construction, $\mathrm{Fr}_F(\mathbb{E}_1)$ has only finite specified (co)cones. To prove that $\mathrm{Fr}_F(e)$ is a basic $\mathcal{R}$-unconditional exactness property, it remains to show that, if $G\colon \mathbb{S} \to \mathbb{S}'$ is a universal augmentation of the $\mathcal{T}$-sketch $\mathbb{S}$, then $\mathrm{Fr}_F(G)\colon\ \mathrm{Fr}_F(\mathbb{S}) \to \mathrm{Fr}_F(\mathbb{S}')$ is a universal augmentation of the $\mathcal{R}$-sketch $\mathrm{Fr}_F(\mathbb{S})$. In order to do so, we first remark that a (co)cone over $\mathcal{D} \to \mathcal{S}$ gives rise to a (co)cone over $\mathcal{D} \to \mathcal{S} \to \mathrm{Fr}_F(\mathcal{S})$ as in Proposition 1.112. With this in mind, we can construct, for each operation of Definition 3.8, two morphisms of conditional graphs H and K such that the following diagram in CondGraph (where we omit to write U, $U_{\mathcal{T}}$ and F^*)

$$\begin{tikzcd}
\mathcal{S} \arrow[rr, "G"] \arrow[dr] \arrow[dd, "\eta_{\mathcal{S}}"'] & & \mathcal{S}' \arrow[dd, "\eta_{\mathcal{S}'}"] \\
& \overline{\mathcal{S}} \arrow[ur] \arrow[dd, "H"] & \\
\mathrm{Fr}_F(\mathcal{S}) \arrow[rr, crossing over, "\mathrm{Fr}_F(G)"'] \arrow[dr] & & \mathrm{Fr}_F(\mathcal{S}') \\
& \overline{\mathrm{Fr}_F(\mathcal{S})} \arrow[ur, "K"'] &
\end{tikzcd}$$

commutes and H and $\mathrm{Fr}_F(\mathcal{S}) \to \overline{\mathrm{Fr}_F(\mathcal{S})}$ are jointly epimorphic (using notations of Definition 3.8). The universal properties involved in this construction imply that $\mathrm{Fr}_F(\mathcal{S}') = \mathcal{R}\text{-Path}(\mathrm{Fr}_F(\mathcal{S}) \to \overline{\mathrm{Fr}_F(\mathcal{S})})$, which proves that $\mathrm{Fr}_F(G)$ is a universal augmentation of $\mathrm{Fr}_F(\mathbb{S})$.

For the second part of the statement, we remark that $F^*(\mathrm{FD}(\mathcal{C})) = \mathrm{FD}(F^*(\mathcal{C}))$. The result is then a straightforward consequence of the universal property of Fr_F. ☐

In view of Proposition 1.51, the above result says in particular that each Th[Set]-unconditional exactness property can be considered as a

$\mathcal{T}$-unconditional exactness property for each commutative Lawvere theory $\mathcal{T}$. Let us now conclude this section with some examples of unconditional exactness properties.

Example 3.15. One can express regularity as a Th[Set]-unconditional exactness property. Indeed, having equalisers can be expressed via the morphism $\mathrm{I}(\mathcal{E}_0) \to \mathbb{S}_1$ where $\mathcal{E}_0$ is the category $\bullet_A \rightrightarrows \bullet_B$ and $\mathbb{S}_1$ is obtained via Operation **a** representing the equaliser. One can do analogously with the properties of having a terminal object and binary products. For the coequalisers of kernel pairs, we can proceed as follows: $\mathcal{E}_0$ is the arrow category $\bullet_A \to \bullet_B$. We then construct $\mathbb{S}_1$ using Operation **a** for the kernel pair of the arrow and $\mathbb{S}_2$ is constructed using Operation **b** for the corresponding coequaliser. We still need to express the condition that regular epimorphisms are stable under pullbacks. This property can be rephrased as follows: for each pair of arrows f, g with the same codomain, the map p' (constructed as in the diagram bellow) is the coequaliser of s_1 and s_2.

$$\begin{tikzcd}
R[p'] \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & A \times_I (I \times_B C) \arrow[dd, "g''"'] \arrow[dr, "p'"] \arrow[dr, phantom, "\lrcorner", very near start] & & C \arrow[dd, "g"] \\
& & I \times_B C \arrow[ur, "i'"] \arrow[dd, "g'"] \arrow[dr, phantom, "\lrcorner", very near start] & \\
R[f] \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A \arrow[rr, "f"'] \arrow[dr, "\mathrm{coeq}(r_1,r_2)=p"'] & & B \\
& & I \arrow[ur, "\exists!\, i"'] &
\end{tikzcd}$$

Therefore, to express this property as a Th[Set]-unconditional exactness property, we can start with $\mathcal{E}_0$ given by

$$\begin{tikzcd}
& \bullet_C \arrow[d, "g"] \\
\bullet_A \arrow[r, "f"'] & \bullet_B .
\end{tikzcd}$$

Then, we use an Operation **a** to create $\begin{tikzcd} \bullet_R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & \bullet_A \end{tikzcd}$, an Operation **b** to have $p\colon \bullet_A \to \bullet_I$, an Operation **d** to get $i\colon \bullet_I \to \bullet_B$, and so forth. Hence, we construct $\mathbb{S}_6$ in this way. Eventually, to construct $\mathbb{E}_1$, we add a cocone which encodes the information that p' is the coequaliser of s_1 and s_2.

Example 3.16. Let now $\mathcal{T}$ be any commutative Lawvere theory and (M, X) an extended matrix of terms in $\mathcal{T}$. Using Proposition 3.14, we know that being a regular $\mathcal{T}$-category is an $\mathcal{T}$-unconditional exactness property. On the other hand, for a regular $\mathcal{T}$-category, to have (M, X)-closed relations it is equivalent to consider any morphism $r = (r_1, \dots, r_a)\colon R \to A^a$ and require that t constructed as in Definition 2.42 is a regular epimorphism. This can be expressed as a $\mathcal{T}$-unconditional exactness property in a analogous way than above (but much longer), starting with $\mathcal{E}_0 = \mathrm{Fr}_{\mathcal{T}}(\mathcal{E}_0')$ where $\mathcal{E}_0'$ is the finite category below.

$$\begin{tikzcd} \bullet_R \arrow[r, shift left=2, "r_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "r_a"'] & \bullet_A \end{tikzcd}$$

In view of Proposition 1.39, $\mathcal{E}_0$ has finitely presentable hom-algebras.

Example 3.17. In addition to the above examples, we can also say that

1. having limits of shape $\mathcal{D}$ for a finite category $\mathcal{D}$,
2. having colimits of shape $\mathcal{D}$ for a finite category $\mathcal{D}$,
3. being a groupoid,
4. being a preorder,
5. being pointed,
6. being exact Mal'tsev

are Th[Set]-unconditional exactness properties and

7. being additive,
8. being abelian

are Th[Set$_*$]-unconditional exactness properties.

In view of Proposition 3.13, their dual properties are also Th[Set] (resp. Th[Set$_*$])-unconditional exactness properties.

3.2 Preservation under $\mathcal{C} \hookrightarrow \mathrm{Lex}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$

3.2.1 The theorem

The aim of this subsection is to prove that if a small finitely complete $\mathcal{T}$-category $\mathcal{C}$ satisfies some $\mathcal{T}$-unconditional exactness property, then so does its free cofiltered limit completion $\mathrm{Lex}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$. For the sake of brevity, let us denote $\mathrm{Lex}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ by $\widetilde{\mathcal{C}}$ and the full Yoneda embedding $\mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$ by i. We will often identify C and $i(C) = \mathcal{C}(C, -)$ for an object $C \in \mathcal{C}$, and thus consider $\mathcal{C}$ as a full subcategory of $\widetilde{\mathcal{C}}$. To fix notations, let us recall the definition of comma categories.

Definition 3.18. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor and D an object in $\mathcal{D}$. The *comma category* $(D \downarrow F)$ has pairs (C, d) as objects where C is an object of $\mathcal{C}$ and $d \colon D \to F(C)$ a morphism in $\mathcal{D}$. A morphism $(C, d) \to (C', d')$ in $(D \downarrow F)$ is a map $c \colon C \to C'$ in $\mathcal{C}$ such that $F(c)d = d'$. We write $D^{\downarrow}$ for the forgetful functor $(D \downarrow F) \to \mathcal{C}$.

Next theorem sums up some important properties of $\widetilde{\mathcal{C}}$. Point 1 has already been stated (Proposition 1.80) since the dual of $\widetilde{\mathcal{C}}$ is locally finitely presentable (Theorem 1.69).

Theorem 3.19. [7, 44] The following statements hold for a small finitely complete category $\mathcal{C}$:

1. $\widetilde{\mathcal{C}}$ is complete and cocomplete,
2. in $\widetilde{\mathcal{C}}$, cofiltered limits commute with limits and finite colimits,
3. the embedding $i \colon \mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$ preserves all colimits and finite limits,
4. for all $A \in \widetilde{\mathcal{C}}$, $(A, (c)_{(C,c) \in (A \downarrow i)})$ is the cofiltered limit of the functor

$$\begin{tikzcd}(A \downarrow i) \arrow[r, "A^{\downarrow}"] & \mathcal{C} \arrow[r, hook, "i"] & \widetilde{\mathcal{C}}.\end{tikzcd}$$

5. $i\colon \mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$ is the free cofiltered limit completion of $\mathcal{C}$, in the sense that, for any functor $F\colon \mathcal{C} \to \mathcal{D}$ to a category $\mathcal{D}$ with cofiltered limits, there exists a unique (up to isomorphism) functor $\overline{F}\colon \widetilde{\mathcal{C}} \to \mathcal{D}$ preserving cofiltered limits and such that $\overline{F}i$ is isomorphic to F.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, hook, "i"] \arrow[dr, "\forall F"'] & \arrow[d, phantom, "\cong"] & \widetilde{\mathcal{C}} \arrow[dl, dotted, "\exists!\, \overline{F}"] \\
& \mathcal{D} &
\end{tikzcd}$$

Now, if $\mathcal{C}$ is a small finitely complete $\mathcal{T}$-category for a Lawvere theory $\mathcal{T}$, we want to construct a $\mathcal{T}$-enrichment on $\widetilde{\mathcal{C}}$ in order to make i a $\mathcal{T}$-functor. Since $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ have finite products, this can be done in the following way. For each object $A \in \widetilde{\mathcal{C}}$ and n-ary term t of $\mathcal{T}$, we set t^A to be the unique morphism making the square

$$\begin{tikzcd}
A^n \arrow[r, "t^A"] \arrow[d, "c^n"'] & A \arrow[d, "c"] \\
C^n \arrow[r, "t^C"'] & C
\end{tikzcd}$$

commute for each $(C, c) \in (A \downarrow i)$. There is a unique such morphism according to Point 4 of the above theorem. This makes $\widetilde{\mathcal{C}}$ a $\mathcal{T}$-category and $i\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ a $\mathcal{T}$-functor. Moreover, this extends Gabriel-Ulmer's result.

Proposition 3.20. Let $\mathcal{T}$ be a Lawvere theory and $\mathcal{C}$ a small finitely complete $\mathcal{T}$-category. The embedding $i\colon \mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$ is the free cofiltered limit $\mathcal{T}$-completion of $\mathcal{C}$, meaning the following properties hold:

1. i is a $\mathcal{T}$-functor and $\widetilde{\mathcal{C}}$ has cofiltered limits,

2. for any $\mathcal{T}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is a $\mathcal{T}$-category with cofiltered limits, there exists a unique (up to isomorphism) $\mathcal{T}$-functor $\overline{F}\colon \widetilde{\mathcal{C}} \to \mathcal{D}$ preserving cofiltered limits and such that $\overline{F}i$ is isomorphic to F.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, hook, "i"] \arrow[dr, "\forall F"'] & \arrow[d, phantom, "\cong"] & \widetilde{\mathcal{C}} \arrow[dl, dotted, "\exists!\, \overline{F}"] \\
& \mathcal{D} &
\end{tikzcd}$$

Proof. Point 1 follows from the construction of the $\mathcal{T}$-enrichment of $\widetilde{\mathcal{C}}$. For Point 2, by Theorem 3.19.5, it suffices to prove that $\overline{F}$ is a $\mathcal{T}$-functor knowing that F is. So let t be an n-ary term of $\mathcal{T}$ and $f_1, \dots, f_n \colon A \to B$ morphisms in $\widetilde{\mathcal{C}}$. We denote by $\alpha \colon \overline{F}i \Rightarrow F$ the given natural isomorphism. We have to prove the equality

$$\overline{F}(t(f_1, \dots, f_n)) = t(\overline{F}(f_1), \dots, \overline{F}(f_n))$$

holds. Since $\overline{F}$ preserves cofiltered limits and in view of Theorem 3.19.4, we only have to show that

$$\overline{F}(c)\overline{F}(t(f_1, \dots, f_n)) = \overline{F}(c)t(\overline{F}(f_1), \dots, \overline{F}(f_n))$$

for any morphism $c \colon B \to C$ with $C \in \mathcal{C}$. It then suffices to compute:

$$\begin{aligned}
&\overline{F}(c)\overline{F}(t(f_1, \dots, f_n)) \\
&= \overline{F}(t(cf_1, \dots, cf_n)) \\
&= \overline{F}(t(\pi_1, \dots, \pi_n))\overline{F}((cf_1, \dots, cf_n)) \\
&= \alpha_C^{-1} F(t(\pi_1, \dots, \pi_n))\alpha_{C^n}\overline{F}((cf_1, \dots, cf_n)) \\
&= \alpha_C^{-1} t(F(\pi_1), \dots, F(\pi_n))\alpha_{C^n}\overline{F}((cf_1, \dots, cf_n)) \\
&= t(\alpha_C^{-1} F(\pi_1)\alpha_{C^n}, \dots, \alpha_C^{-1} F(\pi_n)\alpha_{C^n})\overline{F}((cf_1, \dots, cf_n)) \\
&= t(\overline{F}(\pi_1), \dots, \overline{F}(\pi_n))\overline{F}((cf_1, \dots, cf_n)) \\
&= t(\overline{F}(cf_1), \dots, \overline{F}(cf_n)) \\
&= \overline{F}(c)t(\overline{F}(f_1), \dots, \overline{F}(f_n))
\end{aligned}$$

where $\pi_1, \dots, \pi_n \colon C^n \to C$ are the product projections. ☐

We can now prove our main theorem about $\mathcal{T}$-unconditional exactness properties.

Theorem 3.21. [58] Let $\mathcal{T}$ be a commutative Lawvere theory and P a $\mathcal{T}$-unconditional exactness property. If a small finitely complete $\mathcal{T}$-category $\mathcal{C}$ satisfies P, then $\widetilde{\mathcal{C}} = \mathrm{Lex}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ also satisfies P.

Proof. We can of course suppose without loss of generality that P is a basic $\mathcal{T}$-unconditional exactness property $e \colon \mathrm{I}(\mathcal{E}_0) \to \mathbb{E}_1$. Let $\mathcal{V}$ be a bigger universe such that $\mathcal{U} \in \mathcal{V}$. For the sake of brevity, let us write SK for the $\mathcal{V}$-2-category $\mathcal{V}$-$\mathcal{T}$-Sk, E_0 for $\mathrm{I}(\mathcal{E}_0)$, E_1 for $\mathbb{E}_1$, C for $\mathrm{FD}(\mathcal{C})$ and $\widetilde{C}$ for $\mathrm{FD}(\widetilde{\mathcal{C}})$ in SK.

By Lemma 3.12, we know that $\mathrm{SK}(e, C)\colon \mathrm{SK}(E_1, C) \to \mathrm{SK}(E_0, C)$ is an equivalence of categories and we have to prove that $\mathrm{SK}(e, \widetilde{C}) : \mathrm{SK}(E_1, \widetilde{C}) \to \mathrm{SK}(E_0, \widetilde{C})$ is essentially surjective on objects. Moreover, Theorem 3.19.4 tells us that for each $A \in \widetilde{\mathcal{C}}$, $(A, (b)_{(B,b) \in (A \downarrow i)})$ is the cofiltered limit of $i \circ A^{\downarrow}\colon (A \downarrow i) \to \widetilde{\mathcal{C}}$.

If $E \in \mathrm{SK}$ has only finite specified (co)cones, $\mathrm{SK}(E, \widetilde{C})$ has cofiltered limits and they are computed componentwise, meaning they are preserved and jointly reflected by the evaluation functors

$$\mathrm{ev}_K\colon \mathrm{SK}(E, \widetilde{C}) \to \widetilde{\mathcal{C}}$$

for $K \in E$. Indeed, if $G\colon \mathcal{D} \to \mathrm{SK}(E, \widetilde{C})$ is a cofiltered diagram, let L be its componentwise limit in the $\mathcal{V}$-category of $\mathcal{V}$-functors from $U(E)$ to $\widetilde{\mathcal{C}}$. Then L is actually a morphism of $\mathcal{V}$-$\mathcal{T}$-sketches since all $G(D)$ are, L is computed componentwise and by Theorem 3.19.2. Therefore it is the limit of G.

In addition, if $g\colon E_0 \to \widetilde{C}$ is a morphism in SK, the comma category $(g \downarrow \mathrm{SK}(E_0, i))$ is cofiltered. Indeed, it is small since $\mathcal{E}_0$ and $\mathcal{C}$ are and it has finite limits since $\mathcal{C}$ (and so the category of $\mathcal{T}$-functors from $\mathcal{E}_0$ to $\mathcal{C}$) has and i preserves them by Theorem 3.19.3.

Finally, let us prove that for each $K \in E_0$, the evaluation functor $\mathrm{ev}_K\colon \mathrm{SK}(E_0, C) \to \mathcal{C}$ has a right adjoint. This proof follows the ideas of [72]. Since $E_0 = \mathrm{I}(\mathcal{E}_0)$, $\mathrm{SK}(E_0, C)$ is just $\mathcal{T}\text{-}\mathrm{Cat}(\mathcal{E}_0, \mathcal{C})$. So, let $B \in \mathcal{C}$ and let us construct its coreflection $R\colon \mathcal{E}_0 \to \mathcal{C}$ along ev_K. Let $K' \in \mathcal{E}_0$. Since $\mathcal{E}_0$ has finitely presentable hom-algebras, $\mathcal{E}_0(K', K)$ can be seen as the coequaliser in $\mathcal{T}$-Alg

$$\begin{tikzcd} \mathrm{Fr}_{\mathcal{T}}(\{1, \dots, m_{K'}\}) \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & \mathrm{Fr}_{\mathcal{T}}(\{g_1^{K'}, \dots, g_{n_{K'}}^{K'}\}) \arrow[r, two heads] & \mathcal{E}_0(K', K) \end{tikzcd}$$

where $h(j)$ (resp. $k(j)$) is the $n_{K'}$-ary term $h_j^{K'}$ (resp. $k_j^{K'}$) of $\mathcal{T}$ for each

$j \in \{1, \ldots, m_{K'}\}$. Let $\mathcal{D}_{K'}$ be the finite category defined by

$$\mathrm{ob}(\mathcal{D}_{K'}) = \{Y, Z_1, \ldots, Z_{m_{K'}}\}$$

and

$$\mathrm{ar}(\mathcal{D}_{K'}) = \{1_W \mid W \in \mathrm{ob}(\mathcal{D}_{K'})\} \cup \{\begin{tikzcd} Y \arrow[r, shift left, "y_j"] \arrow[r, shift right, "z_j"'] & Z_j \end{tikzcd} \mid 1 \leqslant j \leqslant m_{K'}\}.$$

Then, we consider the functor $F\colon \mathcal{D}_{K'} \to \mathcal{C}$ defined by

$$F(\begin{tikzcd} Y \arrow[r, shift left, "y_j"] \arrow[r, shift right, "z_j"'] & Z_j \end{tikzcd}) = \begin{tikzcd} B^{n_{K'}} \arrow[r, shift left, "(h_j^{K'})^B"] \arrow[r, shift right, "(k_j^{K'})^B"'] & B \end{tikzcd}$$

for each $j \in \{1, \ldots, m_{K'}\}$. Since $\mathcal{C}$ is finitely complete, we can consider its limit $(R(K'), (s_W^{K'})_{W \in \mathcal{D}_{K'}})$. This construction gives us, for all $B' \in \mathcal{C}$, an isomorphism of $\mathcal{T}$-algebras

$$\begin{aligned} &\mathcal{C}(B', R(K')) \cong \mathcal{T}\text{-}\mathrm{Alg}(\mathcal{E}_0(K', K), \mathcal{C}(B', B)) \\ &\qquad f \mapsto (\widehat{f}\colon \mathcal{E}_0(K', K) \to \mathcal{C}(B', B)\colon [g_i^{K'}] \mapsto \pi_i \circ s_Y^{K'} \circ f) \end{aligned}$$

where $\pi_i\colon B^{n_{K'}} \to B$ is the i-th projection. Let us denote by $s^{K'}$: $\mathcal{E}_0(K', K) \to \mathcal{C}(R(K'), B)$ the morphism $\widehat{1_{R(K')}}$. This is the unique morphism such that $s^{K'}([g_i^{K'}]) = \pi_i \circ s_Y^{K'}$ for each $1 \leqslant i \leqslant n_{K'}$. Now, if $k\colon K_1 \to K_2$ is an arrow in $\mathcal{E}_0$, $R(k)$ is the morphism $R(K_1) \to R(K_2)$ corresponding to the $\mathcal{T}$-homomorphism

$$\mathcal{E}_0(K_2, K) \to \mathcal{C}(R(K_1), B)\colon g \mapsto s^{K_1}(gk).$$

In other words, it is the unique morphism $R(K_1) \to R(K_2)$ such that $s^{K_2}(g)R(k) = s^{K_1}(gk)$ for each $g\colon K_2 \to K$. This makes R into a $\mathcal{T}$-functor $\mathcal{E}_0 \to \mathcal{C}$, and together with the map $s^K(1_K)\colon R(K) \to B$, the expected coreflection of B. Indeed, if R' is another $\mathcal{T}$-functor $\mathcal{E}_0 \to \mathcal{C}$ and $x\colon R'(K) \to B$ a morphism in $\mathcal{C}$, we have to prove there is a unique natural transformation $\alpha\colon R' \Rightarrow R$ such that $s^K(1_K)\alpha_K = x$. Naturality of α imposes that, for each $K' \in \mathcal{E}_0$ and each $1 \leqslant i \leqslant n_{K'}$,

$$\pi_i s_Y^{K'} \alpha_{K'} = s^{K'}([g_i^{K'}])\alpha_{K'}$$

$$
\begin{aligned}
&= s^K(1_K)R([g_i^{K'}])\alpha_{K'}\\
&= s^K(1_K)\alpha_K R'([g_i^{K'}])\\
&= xR'([g_i^{K'}]).
\end{aligned}
$$

This implies that such an α is unique. To prove the existence, the equality above defines $\alpha_{K'}$ in view of the definition of $R(K')$. Then, the equality $s^{K'}(g)\alpha_{K'} = xR'(g)$ holds for any $g\colon K' \to K$ since it does for the generators $[g_i^{K'}]$. This implies $s^K(1_K)\alpha_K = x$. Moreover, if $k\colon K_1 \to K_2$ is a morphism in $\mathcal{E}_0$, one has

$$
\begin{aligned}
\pi_i s_Y^{K_2}\alpha_{K_2}R'(k) &= xR'([g_i^{K_2}]k)\\
&= s^{K_1}([g_i^{K_2}]k)\alpha_{K_1}\\
&= s^{K_2}([g_i^{K_2}])R(k)\alpha_{K_1}\\
&= \pi_i s_Y^{K_2}R(k)\alpha_{K_1}
\end{aligned}
$$

for each $1 \leqslant i \leqslant n_{K_2}$, which prove the naturality of α. Therefore, we have constructed a coreflection of B along $\mathrm{ev}_K\colon \mathrm{SK}(E_0, C) \to \mathcal{C}$ which proves this evaluation functor has a right adjoint.

Since we can apply the same construction to $\widetilde{\mathcal{C}}$ and since $i\colon \mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$ preserves finite limits, the evaluation functors $\mathrm{ev}_K\colon \mathrm{SK}(E_0, \widetilde{C}) \to \widetilde{\mathcal{C}}$ also have right adjoints and the square

$$
\begin{tikzcd}
\mathrm{SK}(E_0, C) \arrow[r, "\mathrm{ev}_K"'] \arrow[d, "\mathrm{SK}(E_0,i)"'] & \mathcal{C} \arrow[l, bend right, "\top"] \arrow[d, "i"] \\
\mathrm{SK}(E_0, \widetilde{C}) \arrow[r, "\mathrm{ev}_K"'] & \widetilde{\mathcal{C}} \arrow[l, bend right, "\top"]
\end{tikzcd}
$$

satisfies the Beck-Chevalley condition, meaning that the rightward and leftward squares commute and i (resp. $\mathrm{SK}(E_0, i)$) maps the counit (resp. the unit) of the first adjunction to the counit (resp. the unit) of the second one.

We have now all the ingredients to prove that

$$\mathrm{SK}(e, \widetilde{C})\colon \mathrm{SK}(E_1, \widetilde{C}) \to \mathrm{SK}(E_0, \widetilde{C})$$

is essentially surjective on objects and conclude the proof. So let $g\colon$

$E_0 \to \widetilde{C}$ be a morphism in SK and consider, for each $K \in E_0$, the comma categories $(g \downarrow \mathrm{SK}(E_0, i))$ and $(g(K) \downarrow i)$. Since the construction of comma categories is functorial, the Beck-Chevalley condition above induces an adjunction such that the rightward and leftward squares below commute.

$$\begin{tikzcd}
(g \downarrow \mathrm{SK}(E_0, i)) \arrow[r] \arrow[d, "g^{\downarrow}"'] & (g(K) \downarrow i) \arrow[l, bend right, "\top"'] \arrow[d, "g(K)^{\downarrow}"] \\
\mathrm{SK}(E_0, C) \arrow[r, "\mathrm{ev}_K"'] \arrow[d, "\mathrm{SK}(E_0,i)"'] & \mathcal{C} \arrow[l, bend right, "\top"'] \arrow[d, "i"] \\
\mathrm{SK}(E_0, \widetilde{C}) \arrow[r, "\mathrm{ev}_K"'] & \widetilde{\mathcal{C}} \arrow[l, bend right, "\top"']
\end{tikzcd}$$

Since $g(K)$ is the limit of $i \circ g(K)^{\downarrow}$, the top adjunction implies that $g(K)$ is also the limit of $\mathrm{ev}_K \circ \mathrm{SK}(E_0, i) \circ g^{\downarrow}$. In addition, we know that $(g \downarrow \mathrm{SK}(E_0, i))$ is cofiltered and such limits are jointly reflected by the evaluation functors $\mathrm{ev}_K \colon \mathrm{SK}(E_0, \widetilde{C}) \to \widetilde{\mathcal{C}}$. This means that g is the limit of $\mathrm{SK}(E_0, i) \circ g^{\downarrow}$.

By assumption, we know that $\mathrm{SK}(e, C)$ is an equivalence of categories. Let us write

$$T \colon \mathrm{SK}(E_0, C) \to \mathrm{SK}(E_1, C)$$

for its pseudo-inverse. We thus have the following diagram, in which the rightward square commutes.

$$\begin{tikzcd}
& (g \downarrow \mathrm{SK}(E_0, i)) \arrow[d, "g^{\downarrow}"] \\
\mathrm{SK}(E_1, C) \arrow[r, "\mathrm{SK}(e,C)"'] \arrow[d, "\mathrm{SK}(E_1,i)"'] & \mathrm{SK}(E_0, C) \arrow[l, bend right, "T"'] \arrow[d, "\mathrm{SK}(E_0,i)"] \\
\mathrm{SK}(E_1, \widetilde{C}) \arrow[r, "\mathrm{SK}(e,\widetilde{C})"'] & \mathrm{SK}(E_0, \widetilde{C})
\end{tikzcd}$$

Since it exists, we denote the cofiltered limit of $\mathrm{SK}(E_1, i) \circ T \circ g^{\downarrow}$ by $l \in \mathrm{SK}(E_1, \widetilde{C})$. We would like to prove $\mathrm{SK}(e, \widetilde{C})(l) = le \cong g$. To see this, we are actually going to prove that le is also the limit of $\mathrm{SK}(E_0, i) \circ g^{\downarrow}$. Since the evaluation functors $\mathrm{ev}_K\colon \mathrm{SK}(E_0,\widetilde{C}) \to \widetilde{\mathcal{C}}$ jointly reflect cofiltered limits, it is enough to show that $le(K)$ is the limit of $\mathrm{ev}_K \circ \mathrm{SK}(E_0,i) \circ g^{\downarrow}$. But l is the limit of $\mathrm{SK}(E_1,i) \circ T \circ g^{\downarrow}$ and $\mathrm{ev}_{e(K)}\colon \mathrm{SK}(E_1,\widetilde{C}) \to \widetilde{\mathcal{C}}$ preserves it. Therefore, $le(K)$ is the limit of

$$\begin{aligned} \mathrm{ev}_{e(K)} \circ \mathrm{SK}(E_1,i) \circ T \circ g^{\downarrow} &= \mathrm{ev}_K \circ \mathrm{SK}(e,\widetilde{C}) \circ \mathrm{SK}(E_1,i) \circ T \circ g^{\downarrow} \\ &= \mathrm{ev}_K \circ \mathrm{SK}(E_0,i) \circ \mathrm{SK}(e,C) \circ T \circ g^{\downarrow} \\ &\cong \mathrm{ev}_K \circ \mathrm{SK}(E_0,i) \circ g^{\downarrow} \end{aligned}$$

which concludes the proof. □

Remark 3.22. In the literature, particular cases of Theorem 3.21 have already been proved, for the following $\mathcal{T}$-unconditional exactness properties:

- regularity (for $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$) (Theorem 2.2 in [10]),
- coregularity (for $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$) (Section 1 in [33]),
- additivity (for $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$) (Proposition 2.5 in [34]),
- abelianness (for $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$) (Proposition 2.12 in [34]),
- being exact Mal'tsev with pushouts (for $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$) (Proposition 3.2 in [16]),
- being coregular co-Mal'tsev (for $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$) (Corollary 2.6 in [47]).

Analogous results have also been proved for properties which might not be some $\mathcal{T}$-unconditional exactness properties. Without giving definitions here, we can cite:

- being a pretopos (Theorem 2.4 in [10]),
- being co-extensive (Theorem 5 in [30]),
- being extensive (Theorem 8 in [30]),
- being co-(cartesian closed) (Proposition 2.8 in [34]).

In [16] and [30], it is also shown that one needs more conditions on a small co-exact category $\mathcal{C}$ in order to have $\widetilde{\mathcal{C}}$ co-exact (without going into details, $\mathcal{C}$ needs to be co-pro-exact). Exactness is thus not a Th[Set]-unconditional exactness property.

Let us finally say that, in the particular case where $\mathcal{E}_0$ is of the form $\mathrm{Fr}_{\mathcal{T}}(\mathcal{E}'_0)$ for some finite category $\mathcal{E}'_0$, one could also have used Lemma 5.1 in [85] to prove Theorem 3.21.

3.2.2 An application

Theorem 3.21 will be used as a crucial tool in our embedding theorems. We explain in this subsection another of its applications. The idea is quite simple: suppose P and Q are 'exactness properties' involving finite limits with P being an unconditional one. If we can prove that, for each finitely complete category with finite colimits, the implication $P \Rightarrow Q$ holds, then we can often (depending on the nature of Q) prove this implication also holds for any finitely complete category (without the assumption on colimits).

Indeed, let $\mathcal{C}$ be such a category satisfying P. Up to a change of universe, we can suppose it is small. By Theorem 3.21, $\widetilde{\mathcal{C}}$ also satisfies P. It thus satisfies Q since it is cocomplete. In view of the embedding $i\colon \mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$, it often implies that $\mathcal{C}$ satisfies Q too. We provide such an example here.

Moreover, in order to apply Theorem 3.21 in the case $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$, we need to check that some category $\mathcal{E}_0$ has its hom-sets $\mathcal{E}_0(A, B)$ finite. This is not always obvious and we provide an algorithm which makes it easier.

As explained above, we are going to use Theorem 3.21 in order to get a characterisation of n-permutable categories. We are actually going to prove only one direction here, the complete theorem will be proved in Chapter 4 (see Theorem 4.11). So, for a natural number $n \geqslant 3$, we consider in a regular category the diagram

$$\begin{tikzcd}
 & & & & L \arrow[rrrr, "\lambda"] \arrow[dllll, "p_1"'] \arrow[ddll, "p_2"] \arrow[dddl, "p_3"] \arrow[ddddd, "p_{n-2}"] \arrow[ddddddr, "p_{n-1}"] & & & & U \times_W V \arrow[dl, dotted, "q_1"'] \arrow[ddddddr, shift left, two heads, "q_2"] \\
X \arrow[rrrrrrr, dotted, "\gamma"] \arrow[ddr, shift right, "f"'] & & & & & & & U \arrow[ur, dotted] \arrow[ddddddr, dotted, shift right, two heads, "h"'] & & \\
 & & X \arrow[dl, shift right, "g"'] \arrow[dd, shift right, "f"'] & & & & & & & \\
 & Y \arrow[uul, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[ddddrrrrrrr, dotted, "\beta"] & & X \arrow[dl, shift right, "g"'] & & & & & & \\
 & & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[dddrrrrrr, dotted, "\cdots\beta"] & \ddots & & & & & & \\
 & & & \ddots & X \arrow[dd, shift right, "f"'] & & & & & \\
 & & & & & X \arrow[rrrr, "\delta"] \arrow[dl, shift right, "g"'] & & & & V \arrow[dl, shift right, "k"'] \arrow[uuuuuul, shift left] \\
 & & & & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "t"'] \arrow[rrrr, two heads, "\beta"] & & & & W \arrow[ur, shift right, "v"'] \arrow[uuuuuul, dotted, shift right, "u"'] \arrow[llll, bend left, "i"] \arrow[llll, bend left=20, "i'"'] &
\end{tikzcd}$$

(7)

in which the equalities

$$fs = gt = 1_Y,\ \beta i = \beta i' = kv = hu = 1_W,\ \beta g = \beta f = h\gamma = k\delta,$$
$$\gamma s = u\beta,\ \delta t = v\beta,\ ft = i\beta,\ gs = i'\beta,\ q_1\lambda = \gamma p_1 \text{ and } q_2\lambda = \delta p_{n-1}$$

hold, the square

$$\begin{tikzcd}
U \times_W V \arrow[r, two heads, "q_2"] \arrow[d, two heads, "q_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, two heads, "k"] \\
U \arrow[r, two heads, "h"'] & W
\end{tikzcd}$$

is a pullback and $(L, p_1, \ldots, p_{n-1})$ is the limit of the zig-zag formed by the alternating split epimorphisms f and g. The following proposition is due to D. Rodelo in [60].

Proposition 3.23. [60] Let $n \geqslant 3$ be a natural number and $\mathcal{C}$ a regular category with finite coproducts. If, for each diagram (7) in $\mathcal{C}$ for which γ and δ are regular epimorphisms, the morphism λ turns out to be a regular epimorphism as well, then $\mathcal{C}$ is n-permutable.

Proof. The particular case of Theorem 2.54 for the extended matrix of Example 2.50 already appears in [97]. It tells us it is enough to construct, for each object Y in $\mathcal{C}$, a commutative diagram

$$\begin{tikzcd}
 & & & Y \arrow[llld, "\iota_1"'] \arrow[rrrd, "\iota_2"] & & & \\
2Y & & & & & & 2Y \\
 & & & Z \arrow[uu, two heads, "d"] \arrow[llld, "p'_1"'] \arrow[lldd, "p'_2"] \arrow[lddd, "p'_3"] \arrow[rrdd, "p'_{n-2}"] \arrow[rrrd, "p'_{n-1}"] & & & \\
3Y \arrow[uu, "1_Y+\nabla_Y"] \arrow[d, "\nabla_Y+1_Y"'] & & & & & & 3Y \arrow[uu, "\nabla_Y+1_Y"'] \arrow[d, "1_Y+\nabla_Y"] \\
2Y & 3Y \arrow[l, "1_Y+\nabla_Y"] \arrow[d] & & & & 3Y \arrow[r, "\nabla_Y+1_Y"'] & 2Y \\
 & 2Y & 3Y \arrow[l] & & \cdots & &
\end{tikzcd}$$

where d is a regular epimorphism, $\iota_1, \iota_2 \colon Y \to 2Y$ the coproduct injections and $\nabla_Y \colon 2Y \to Y$ the codiagonal $\left(\begin{smallmatrix} 1_Y \\ 1_Y \end{smallmatrix}\right)$. In order to do so, we consider diagram (7), with $f = \delta = \nabla_Y + 1_Y$, $s = \left(\begin{smallmatrix} \iota_2 \\ \iota_3 \end{smallmatrix}\right)$, $g = \gamma = 1_Y + \nabla_Y$, $t = \left(\begin{smallmatrix} \iota_1 \\ \iota_2 \end{smallmatrix}\right)$, $\beta = k = h = \nabla_Y$, $v = i = \iota_1$ and $u = i' = \iota_2$. Since γ and δ are split epimorphisms, by assumption, the unique λ satisfying the conditions is a regular epimorphism. We then consider the kernel pair of ∇_Y, the induced morphism j

$$\begin{tikzcd}
Y \arrow[dr, dotted, "j"] \arrow[drr, bend left, "\iota_2"] \arrow[ddr, bend right, "\iota_1"'] & & \\
 & R[\nabla_Y] \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & 2Y \arrow[d, "\nabla_Y"] \\
 & 2Y \arrow[r, "\nabla_Y"'] & Y
\end{tikzcd}$$

and its pullback j' along the regular epimorphism λ.

$$\begin{tikzcd}
Z \arrow[r, two heads, "\lambda'"] \arrow[d, "j'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "j"] \\
L \arrow[r, two heads, "\lambda"'] & R[\nabla_Y]
\end{tikzcd}$$

We get in this way the expected commutative diagram

$$\begin{tikzcd}
 & & & Y \arrow[dlll, "\iota_1"'] \arrow[drrr, "\iota_2"] & & & \\
2Y & & & & & & 2Y \\
 & & & Z \arrow[uu, two heads, "\lambda'"'] \arrow[dlll, "p_1j'"] \arrow[ddll, "p_2j'"] \arrow[dddl, "p_3j'"] \arrow[ddrr, "p_{n-2}j'"'] \arrow[drrr, "p_{n-1}j'"'] & & & \\
3Y \arrow[uu, "1_Y+\nabla_Y"] \arrow[d, "\nabla_Y+1_Y"'] & & & & & & 3Y \arrow[uu, "\nabla_Y+1_Y"'] \arrow[d, "1_Y+\nabla_Y"] \\
2Y & 3Y \arrow[l, "1_Y+\nabla_Y"] \arrow[d] & & & & 3Y \arrow[r, "\nabla_Y+1_Y"'] & 2Y \\
 & 2Y & 3Y \arrow[l] & & \cdots & &
\end{tikzcd}$$

where λ' is a regular epimorphism. $\square$

We would like now to remove the coproduct assumption in the previous proposition. For that purpose, we first need to transform the property of being regular and satisfying

P: for any diagram (7), if γ and δ are regular epimorphisms, then so is λ

into a Th[Set]-unconditional exactness property. In a regular context, this property is equivalent to the following one:

P': For any diagram (7), let $\gamma = mp$ and $\delta = m'p'$ be the (regular epi, mono)-factorisations of γ and δ (which are constructed via the coequaliser of their kernel pairs). Then, considering the pullback of hm along km',

$$\begin{tikzcd}
I \times_W J \arrow[r, "q_4"] \arrow[d, "q_3"'] \arrow[dr, phantom, "\lrcorner", very near start] & J \arrow[d, "km'"] \\
I \arrow[r, "hm"'] & W
\end{tikzcd}$$

the unique morphism $\overline{\lambda}\colon L \to I \times_W J$ such that $q_3\overline{\lambda} = pp_1$ and $q_4\overline{\lambda} = p'p_{n-1}$ is a regular epimorphism.

$$\begin{tikzcd}
& L \arrow[rrr, "\overline{\lambda}"] \arrow[dl, "p_1"'] \arrow[ddr, "p_{n-1}"] & & & I \times_W J \arrow[rrr] \arrow[dl, "q_3"'] \arrow[ddr, "q_4"] & & & U \times_W V \arrow[dl, "q_1"'] \arrow[ddr, "q_2"] & \\
X \arrow[rrr, two heads, "p"'] & & & I \arrow[rrr, tail, "m"'] & & & U & & \\
& & X \arrow[rrr, two heads, "p'"'] & & & J \arrow[rrr, tail, "m'"'] & & & V
\end{tikzcd}$$

This Property P' is implied by P since we get a new diagram (7) replacing U by I and V by J (u factors through m as $u = mpsi$ and v through m' as $v = m'p'ti$). The converse implication is obvious.

To prove the above property is a Th[Set]-unconditional exactness property, it remains to show the diagram we start with to build (7) is of the form $D\colon \mathcal{E}_0 \to \mathcal{C}$ for some finite category $\mathcal{E}_0$. In order to do that, we are going to prove a lemma about the finiteness of $\mathrm{Path}(G)$ for a finite conditional graph G (i.e., with $\mathrm{ob}(G)$ and $\mathrm{ar}(G)$ finite). If X is an object of G, we denote by $G \setminus X$ the conditional graph constructed from G by removing X and all arrows from or to X. We consider the biggest set of commutativity conditions on $G \setminus X$ such that the composite

$$G \setminus X \hookrightarrow G \to U(\mathrm{Path}(G))$$

is a morphism of conditional graphs. Intuitively, we consider as commutativity conditions any pair of parallel paths in $G \setminus X$ inducing the same composite in the category generated by G.

Lemma 3.24. [60] Let G be a finite conditional graph and X one of its objects. Suppose that for any pair of arrows $f\colon Y \to X$ and $g\colon X \to Z$ in G, there exists a path $(A_1, h_1, \dots, h_{n-1}, A_n)\colon Y \to Z$ in $G \setminus X$ such that the composites $g \circ f$ and $h_{n-1} \circ \cdots \circ h_1$ are equal in $\mathrm{Path}(G)$. Then, $\mathrm{Path}(G)$ is finite if and only if $\mathrm{Path}(G \setminus X)$ is finite.

Proof. Firstly, we can suppose without loss of generality that all commutativity conditions of $G \setminus X$ are also in G. Indeed, we can add them to the ones of G keeping $\mathrm{Path}(G)$ the same. Thus $G \setminus X \hookrightarrow G$ is a morphism of conditional graphs.

This inclusion morphism $G \setminus X \hookrightarrow G$ gives rise to a faithful functor

$$\mathrm{Path}(G \setminus X) \to \mathrm{Path}(G).$$

Indeed, suppose that two parallel paths of $G \setminus X$ are equalised by the composite

$$G \setminus X \to U(\mathrm{Path}(G \setminus X)) \to U(\mathrm{Path}(G)).$$

Since the natural square

$$\begin{tikzcd} G \setminus X \arrow[r, hook] \arrow[d] & G \arrow[d] \\ U(\mathrm{Path}(G \setminus X)) \arrow[r] & U(\mathrm{Path}(G)) \end{tikzcd}$$

commutes, they are also equalised by $G \setminus X \hookrightarrow G \to U(\mathrm{Path}(G))$. This means it is a commutativity condition on $G \setminus X$ and so they correspond to the same arrow in $\mathrm{Path}(G \setminus X)$. This shows the 'only if' part.

Let us now prove the 'if part'. If $\mathrm{Path}(G \setminus X)$ is finite, there exists a $N \in \mathbb{N}$ such that any morphism of $\mathrm{Path}(G \setminus X)$ can be represented by a path of at most N arrows of $G \setminus X$. Consider a path $(A_1, f_1, \dots, A_n)$ in G. By the assumption on X, we know that this path is equal in $\mathrm{Path}(G)$ to a path $(B_1, h_1, \dots, B_m)$ where $h_2, \dots, h_{m-2}$ are in $G \setminus X$. We can thus suppose $m - 3 \leqslant N$, which proves that $\mathrm{Path}(G)$ is finite since $\mathrm{ob}(\mathrm{Path}(G)) = \mathrm{ob}(G) = \mathrm{ob}(G \setminus X) \cup \{X\}$ is and

$$\#\,\mathrm{ar}(\mathrm{Path}(G)) \leqslant \#\,\mathrm{ob}(G) + \sum_{i=1}^{N+2} \#\,\mathrm{ar}(G)^i < \aleph_0. \qquad \square$$

This lemma gives us an easy way to prove that some category generated by a finite conditional graph is finite. In what follows, for the sake of brevity, we write $f_{n-1} \cdots f_1 = g_{m-1} \cdots g_1$ for the commutativity condition $((A_1, f_1, \dots, A_n), (B_1, g_1, \dots, B_m))$ and $G \setminus \{X_1, \dots, X_n\}$ is defined recursively as $(G \setminus \{X_1, \dots, X_{n-1}\}) \setminus X_n$. Remark that

$$G \setminus \{X_1, \dots, X_n\} = G \setminus \{X_{\sigma(1)}, \dots, X_{\sigma(n)}\}$$

for any permutation σ of $\{1, \dots, n\}$. Let us apply Lemma 3.24 here with the conditional graph G given by

$$\begin{tikzcd}
 & X \arrow[r, bend left, "f"] \arrow[r, "g"] \arrow[d, "\delta"'] \arrow[ddl, bend right, "\gamma"'] & Y \arrow[l, "s"] \arrow[l, bend left, "t"] \arrow[d, bend right, "\beta"'] \\
 & V \arrow[r, "k"] & W \arrow[l, "v"] \arrow[u, "i"'] \arrow[u, bend right, "i'"'] \arrow[dll, "u"] \\
U \arrow[urr, "h"] & &
\end{tikzcd}$$

where the commutativity conditions are

$$fs = gt = 1_Y,\ \beta i = \beta i' = kv = hu = 1_W,\ \beta g = \beta f = h\gamma = k\delta,$$
$$\gamma s = u\beta,\ \delta t = v\beta,\ ft = i\beta \text{ and } gs = i'\beta.$$

Notice that Property P' is built up from a diagram of shape $\mathrm{Path}(G)$ by adding some finite (co)limits to it. Due to the equalities $h\gamma = \beta f$ and $hu = 1_W$, $\mathrm{Path}(G)$ is finite if and only if $\mathrm{Path}(G \setminus \{U\})$ is. Since $k\delta = \beta g$ and $kv = 1_W$, we only have to prove that the category $\mathrm{Path}(G \setminus \{U, V\})$ is finite. Then, with the equalities $ft = i\beta$ and $gs = i'\beta$, it is enough to show that $\mathrm{Path}(G \setminus \{U, V, W\})$ is finite. Since $fs = gt = 1_Y$, if we add some formal arrows $y, y' \colon Y \rightrightarrows Y$ and the conditions $ft = y$ and $gs = y'$, we only have to show that the category generated by the graph

$$\begin{tikzcd}
Y \arrow[loop left, "y"] \arrow[loop right, "y'"]
\end{tikzcd}$$

and the commutativity conditions coming from $\mathrm{Path}(G)$ is finite. But this is obvious since $yy = ftft = i\beta i\beta = i\beta = y$, $yy' = y$, $y'y = y'$ and $y'y' = y'$.

We are now able to remove the coproduct assumption from Proposition 3.23.

Proposition 3.25. [60] Let $n \geqslant 3$ be a natural number and $\mathcal{C}$ a regular category. If, for each diagram (7) in $\mathcal{C}$ for which γ and δ are regular epimorphisms, the morphism λ turns out to be a regular epimorphism as well, then $\mathcal{C}$ is n-permutable.

Proof. Up to a change of universe, we can suppose $\mathcal{C}$ to be small. Since the property of being regular and satisfying P is equivalent to the one of being regular and satisfying P', which is a Th[Set]-unconditional exactness property by the discussion above, we know from Theorem 3.21 that $\widetilde{\mathcal{C}}$ is also a regular category which satisfies P. Since it has small colimits, we deduce from Proposition 3.23 that $\widetilde{\mathcal{C}}$ is n-permutable. Finally, since the embedding $\mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$ is full, faithful and preserves finite limits and colimits, this implies that $\mathcal{C}$ is also n-permutable. □

The converse of this proposition will be proved in the next chapter using our embedding theorem for n-permutable categories (see Theorem 4.11).

Chapter 4

Embedding theorems

The idea behind embedding theorems is to provide a representative element among a collection of categories, such that each category in that collection has a 'nice' embedding in the representative category (or one of its powers). Probably the most famous one is the Yoneda embedding which embeds any small category $\mathcal{C}$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ with a full and faithful functor which preserves and reflects small limits. Another such embedding theorem has been proved by Barr [10]: for any small regular category, one can build a small category $\mathcal{P}$ and a full and faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Set}^{\mathcal{P}}$ which preserves and reflects finite limits and regular epimorphisms. Such theorems have a very practical consequence: if one has to prove a statement about small limits in any small category (or about finite limits and regular epimorphisms in any small regular category), it is enough to prove this statement in the powers of Set. If the statement is 'componentwise', it can be further restricted to show it only in Set. Up to a change of universe, one can also remove the smallness assumption in this technique.

For such applications, notice that we actually do not need the embedding to be full but only faithful and conservative. Indeed, looking at the components, fullness only tells us that it reflects isomorphisms. In other words, the fullness of an embedding $\mathcal{C} \hookrightarrow \mathrm{Set}^{\mathcal{P}}$ can not be easily stated in terms of the functors $\mathcal{C} \hookrightarrow \mathrm{Set}^{\mathcal{P}} \to \mathrm{Set}$, while the reflection of isomorphisms simply means that these functors jointly reflect isomorphisms. With this idea in mind, Z. Janelidze proposed another version of Barr's embedding theorem: any small regular category has a faithful conservative embedding in $\mathrm{Set}^{\mathrm{Sub}(1)}$ which preserves and reflects finite limits and regular epimorphisms (here 1 is the terminal object of the category being embedded). This phenomenon already occurred with abelian categories: Lubkin proved in [80] that each small abelian category admits a faithful conservative functor $\mathcal{C} \hookrightarrow \mathrm{Ab}$ which preserves finite limits and finite colimits, while Mitchell showed that any small abelian category has a full and faithful embedding which also preserves finite limits and finite colimits in a category of modules Mod_R for a ring R (see [93, 42]).

After recalling Yoneda and Barr's embedding theorems in Sections 4.1 and 4.2 respectively, we turn our attention in Section 4.3 to the case of categories with (M, X)-closed relations for an extended matrix (M, X) of terms in a commutative Lawvere theory $\mathcal{T}$. There, using characterisations 1.88 and 2.51, we construct a regular essentially algebraic category $\mathrm{Mod}(\Gamma)$ with (M, X)-closed relations and prove that any small regular $\mathcal{T}$-category with (M, X)-closed relations admits a faithful conservative $\mathcal{T}$-enriched embedding in $\mathrm{Mod}(\Gamma)^{\mathrm{Sub}(1)}$ which preserves and reflects finite limits and regular epimorphisms. A similar result is proved for protomodular categories in Section 4.4, but we need there an assumption on the existence of some colimits.

The last section of this chapter is devoted to an embedding theorem for categories with M-closed strong relations for a simple extended matrix M. In that case, the embedding is full, faithful and preserves finite limits. It is actually a factorisation of the Yoneda embedding through $\mathrm{Part}_M^{\mathcal{C}^{\mathrm{op}}}$ where objects in Part_M are also defined using partial operations. One of the major differences with essentially algebraic categories is that, for monomorphisms $f\colon A \rightarrowtail B$ in Part_M (which are not strong), the fact that $p(f(a_1), \dots, f(a_m))$ is defined does not imply that $p(a_1, \dots, a_m)$ is.

4.1 Yoneda embedding

As detailed in [83], Mac Lane learned about the Yoneda lemma in 1954 in a café at the Gare du Nord at Paris from Yoneda himself. We explain in this section how the embedding derived from it can be used to reduce proofs about limits to the particular category Set, or $\mathcal{T}$-Alg in the $\mathcal{T}$-

enriched case.

If $\mathcal{C}$ is a small category, the Yoneda embedding is given by

$$\begin{aligned} Y\colon \mathcal{C} &\longrightarrow \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \\ C &\longmapsto \mathcal{C}(-,C). \end{aligned}$$

This functor Y is full and faithful (and so conservative), preserves limits and reflects small limits.

Now, consider a statement of the form $P \Rightarrow Q$ where P and Q are conjunctions of properties which can be expressed as

1. some finite diagram is commutative,
2. some finite diagram is a limit diagram,
3. some morphism is a monomorphism,
4. some morphism is an isomorphism,
5. some morphism factors through a given monomorphism.

Then, this statement $P \Rightarrow Q$ is valid in all $\mathcal{V}$-categories (for all universes $\mathcal{V}$) if and only if it is valid in $\mathcal{V}$-Set (for all universes $\mathcal{V}$). The 'only if part' being obvious, let us prove the 'if part'. Let $\mathcal{C}$ be a $\mathcal{V}$-category. Up to a change of universe, it is a $\mathcal{W}$-small category. Then, every part of the statement $P \Rightarrow Q$ is preserved and reflected by the Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{W}\text{-Set}^{\mathcal{C}^{\mathrm{op}}}$. Indeed, f being a monomorphism can be expressed as the fact that its kernel pair (r_1, r_2) is such that $r_1 = r_2$, and the fact that f factors through the monomorphism m is equivalent to say that the pullback of m along f is an isomorphism. It is thus enough to prove $P \Rightarrow Q$ in $\mathcal{W}\text{-Set}^{\mathcal{C}^{\mathrm{op}}}$. But each part of this statement can be reduced 'componentwise' to $\mathcal{W}$-Set, which concludes the proof.

Of course, since the choice of our base universe $\mathcal{U}$ is arbitrary, this roughly means that the statement $P \Rightarrow Q$ is valid in all categories if and only if it is valid in Set. Let us give an example.

Proposition 4.1. Every reflexive difunctional binary relation in a finitely complete category is an equivalence relation.

Proof. Let M be the Mal'tsev matrix of Example 2.7. A relation is M-closed if and only if it $(M, \varnothing)$-closed (Proposition 2.46). Since $k = l$ in this case, in view of Definition 2.42, this only means that a morphism factors through another monomorphism. Being an equivalence relation can also be expressed in such a way. In view of the discussion above, it suffices thus to prove the statement in Set. In this category, for a relation $R \subseteq A \times A$, being reflexive, symmetric or transitive has the classical meaning. We recall that being difunctional means

$$a_1 R a'_1 \wedge a_2 R a'_1 \wedge a_2 R a'_2 \Rightarrow a_1 R a'_2$$

for all $a_1, a'_1, a_2, a'_2 \in A$. A reflexive difunctional relation is thus symmetric since

$$aRa' \Rightarrow a'Ra' \wedge aRa' \wedge aRa \Rightarrow a'Ra$$

for all $a, a' \in A$ and it is transitive since

$$aRa' \wedge a'Ra'' \Rightarrow aRa' \wedge a'Ra' \wedge a'Ra'' \Rightarrow aRa''$$

for all $a, a', a'' \in A$. ☐

We conclude this section with the $\mathcal{T}$-enriched version of the Yoneda embedding. If $\mathcal{T}$ is a commutative Lawvere theory and $\mathcal{C}$ a small $\mathcal{T}$-category, the Yoneda embedding Y factors as

$$\begin{tikzcd} & \mathcal{T}\text{-Alg}^{\mathcal{C}^{\mathrm{op}}} \arrow[d, "U_{\mathcal{T}}^{\mathcal{C}^{\mathrm{op}}}"] \\ \mathcal{C} \arrow[ur, "Y_{\mathcal{T}}"] \arrow[r, "Y"'] & \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \end{tikzcd}$$

where $U_{\mathcal{T}}^{\mathcal{C}^{\mathrm{op}}}$ is the functor which acts by composition with $U_{\mathcal{T}}$. This $\mathcal{T}$-enriched Yoneda embedding $Y_{\mathcal{T}}$ is a full and faithful $\mathcal{T}$-functor which also preserves and reflects small limits. In the same way as above, with this embedding, one can reduce proofs about finite limits from $\mathcal{T}$-categories to $\mathcal{T}$-Alg:

Let $\mathcal{T}$ be a commutative Lawvere theory. Suppose we are given a statement of the form $P \Rightarrow Q$ where P and Q are conjunctions of properties which can be expressed as

1. some finite diagram is commutative,
2. some finite diagram is a limit diagram,
3. the equality $t(f_1, \dots, f_n) = g$ holds for an n-ary term t of $\mathcal{T}$ and parallel morphisms $f_1, \dots, f_n, g$,
4. some morphism is a monomorphism,
5. some morphism is an isomorphism,
6. some morphism factors through a given monomorphism.

Then, this statement $P \Rightarrow Q$ is valid in all $\mathcal{V}$-$\mathcal{T}$-categories (for all universes $\mathcal{V}$) if and only if it is valid in $\mathcal{V}$-$\mathcal{T}$-Alg (for all universes $\mathcal{V}$).

4.2 Barr's embedding

In this section, we state Barr's embedding theorem, prove Z. Janelidze's variant in which the power is replaced by the set Sub(1) of subobjects of the terminal object 1 (but loosing fullness) and give some applications of this embedding.

Theorem 4.2. [10] Let $\mathcal{C}$ be a small regular category. There exists a small category $\mathcal{P}$ and a full and faithful regular embedding $\mathcal{C} \hookrightarrow \mathrm{Set}^{\mathcal{P}}$.

An enriched version of this theorem can be found in [31]. A crucial tool in Barr's proof is the free cofiltered limit completion $\mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$. He first showed that if $\mathcal{C}$ is regular, then so is $\widetilde{\mathcal{C}}$ (which is also a consequence of Theorem 3.21 and Example 3.15). The next step was to construct a $\mathcal{C}$-projective covering in $\widetilde{\mathcal{C}}$.

Definition 4.3. Let $\mathcal{C}$ be a small regular category. An object $P \in \widetilde{\mathcal{C}}$ is said to be $\mathcal{C}$-*projective* if, for any regular epimorphism $f \colon A \twoheadrightarrow B$ in $\mathcal{C}$ and morphism $g \colon P \to B$ in $\widetilde{\mathcal{C}}$, there exists a morphism $h \colon P \to A$ such that $fh = g$.

$$\begin{tikzcd}
& P \arrow[d, "g"] \arrow[dl, dotted, "h"'] \\
A \arrow[r, two heads, "f\in\mathcal{C}"'] & B
\end{tikzcd}$$

Using the Yoneda embedding, this exactly means that the functor $P\colon \mathcal{C} \to \mathsf{Set}$ preserves regular epimorphisms (compare with Definition 1.36). Barr constructed a regular epimorphism $\widehat{X} \twoheadrightarrow X$ for each $X \in \widetilde{\mathcal{C}}$ such that $\widehat{X}$ is a $\mathcal{C}$-projective object. This argument goes actually back to [52] in the case of an abelian category. In this context, Grothendieck constructed such a $\mathcal{C}$-projective covering in a functorial way. This functorial part was omitted by Barr since he did not need it. In order to complete this result, we provide here the functorial construction in the regular context.

Theorem 4.4. Let $\mathcal{C}$ be a small regular category. Then $\widetilde{\mathcal{C}}$ is regular and there exists a functor $\widehat{()}\colon \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$ and a natural transformation $e\colon \widehat{()} \Rightarrow 1_{\widetilde{\mathcal{C}}}$ such that, for every object $X \in \widetilde{\mathcal{C}}$, $\widehat{X}$ is $\mathcal{C}$-projective and $e_X\colon \widehat{X} \twoheadrightarrow X$ is a regular epimorphism.

Proof. As mentioned above, $\widetilde{\mathcal{C}}$ is regular in view of Theorem 3.21 and Example 3.15. Now, for $X \in \widetilde{\mathcal{C}}$, let us describe Barr's construction of the regular epimorphism $e_X\colon \widehat{X} \twoheadrightarrow X$. We are going to recursively construct some regular epimorphisms

$$\begin{tikzcd}\cdots \arrow[r, two heads] & X^{[n]} \arrow[r, two heads, "e^n"] & X^{[n-1]} \arrow[r, two heads] & \cdots \arrow[r, two heads] & X^{[1]} \arrow[r, two heads, "e^1"] & X^{[0]} = X\end{tikzcd}$$

such that, if we have morphisms

$$\begin{tikzcd}X^{[i+1]} \arrow[r, two heads, "e^{i+1}"] & X^{[i]} \arrow[d, "g"] \\ C \arrow[r, two heads, "f"'] & C'\end{tikzcd}$$

with $C, C' \in \mathcal{C}$, f a regular epimorphism and $i \in \mathbb{N}$, there exists a morphism $h\colon X^{[i+1]} \to C$ such that $fh = ge^{i+1}$. We suppose

$$\begin{tikzcd}X^{[n]} \arrow[r, two heads, "e^n"] & X^{[n-1]}\end{tikzcd}$$

has been defined and we are going to construct

$$\begin{tikzcd}X^{[n+1]} \arrow[r, two heads, "e^{n+1}"] & X^{[n]}\end{tikzcd}$$

with the required properties. We well-order (using the Axiom of Choice) the set of diagrams

$$\begin{tikzcd}
 & X^{[n]} \arrow[d, "g"] \\
C \arrow[r, two heads, "f"'] & C'
\end{tikzcd} \tag{8}$$

where $C, C' \in \mathcal{C}$, f is a regular epimorphism and g does not factor through e^n (if $n \geqslant 1$). Then, we construct with a transfinite recursion a sequence (indexed by ordinals) of regular epimorphisms

$$\begin{tikzcd}
\cdots \arrow[r, two heads] & X^{[n]}_{\omega+1} \arrow[r, two heads, "e^n_{\omega+1}"] & X^{[n]}_{\omega} \arrow[r, two heads] & \cdots \arrow[r, two heads] & X^{[n]}_1 \arrow[r, two heads, "e^n_1"] & X^{[n]}_0 = X^{[n]}
\end{tikzcd} \tag{9}$$

in the following way:

Firstly, $X^{[n]}_0 = X^{[n]}$. Then, if α is a limit ordinal, $X^{[n]}_\alpha$ is the limit of (9) for the $X^{[n]}_\beta$ with $\beta < \alpha$. For such an ordinal β, the leg $e^n_{\alpha,\beta} \colon X^{[n]}_\alpha \twoheadrightarrow X^{[n]}_\beta$ of this limit is a regular epimorphism. Indeed, if we consider the following commutative diagram on the right,

$$\begin{tikzcd}
R[e^n_{\alpha,\beta}] \arrow[d, shift left] \arrow[d, shift right] & \cdots & R[e^n_{\beta+1}e^n_{\beta+2}] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & R[e^n_{\beta+1}] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & X^{[n]}_\beta \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \cdots \arrow[r] & X^{[n]}_1 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & X^{[n]}_0 \arrow[d, shift left] \arrow[d, shift right] \\
X^{[n]}_\alpha \arrow[d, "e^n_{\alpha,\beta}"'] & \cdots & X^{[n]}_{\beta+2} \arrow[r, two heads, "e^n_{\beta+2}"] \arrow[d, "e^n_{\beta+1}e^n_{\beta+2}"'] & X^{[n]}_{\beta+1} \arrow[r, two heads, "e^n_{\beta+1}"] \arrow[d, "e^n_{\beta+1}"] & X^{[n]}_\beta \arrow[r, two heads] \arrow[d, "1"] & \cdots \arrow[r, two heads] & X^{[n]}_1 \arrow[r, two heads, "e^n_1"] \arrow[d, "1"] & X^{[n]}_0 \arrow[d, "1"] \\
X^{[n]}_\beta & \cdots & X^{[n]}_\beta \arrow[r, two heads, "1"'] & X^{[n]}_\beta \arrow[r, two heads, "1"'] & X^{[n]}_\beta \arrow[r, two heads] & \cdots \arrow[r, two heads] & X^{[n]}_1 \arrow[r, two heads, "e^n_1"'] & X^{[n]}_0
\end{tikzcd}$$

$X^{[n]}_\alpha$ is by definition the limit of the middle row and $X^{[n]}_\beta$ the limit of the stationary third row. The top row is formed by the kernel pair of the downward morphisms. Since limits commute with limits, its limit is the kernel pair of $e^n_{\alpha,\beta}$. The downward morphisms being regular epimorphisms, they are the coequaliser of they kernel pairs. Since cofiltered limits commute in $\widetilde{\mathcal{C}}$ with finite colimits, $e^n_{\alpha,\beta}$ is the coequaliser of its kernel pair, and so a regular epimorphism.

Finally, suppose we have defined $X^{[n]}_\alpha$ and let us define $X^{[n]}_{\alpha+1}$. Consider the α-th diagram of shape (8) and define $e^n_{\alpha+1}$ as in the following pullback.

$$\begin{tikzcd}
X^{[n]}_{\alpha+1} \arrow[r, two heads, "e^{n}_{\alpha+1}"] \arrow[dd, "k_{X,f,g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X^{[n]}_{\alpha} \arrow[d, two heads] \\
 & X^{[n]} \arrow[d, "g"] \\
C \arrow[r, two heads, "f"'] & C'
\end{tikzcd}$$

We stop this process when we run through all diagrams of shape (8). We then take $X^{[n+1]}$ as the limit of (9) with $e^{n+1}\colon X^{[n+1]} \twoheadrightarrow X^{[n]}$ the corresponding leg. By construction, it satisfies the required properties.

We have thus constructed a sequence (indexed by $\mathbb{N}$) of regular epimorphisms

$$\begin{tikzcd}
\cdots \arrow[r, two heads] & X^{[2]} \arrow[r, two heads, "e^{2}"] & X^{[1]} \arrow[r, two heads, "e^{1}"] & X^{[0]} = X
\end{tikzcd}$$

with the stated properties. We finally write $\widehat{X}$ for its limit with $e_{X,n} : \widehat{X} \twoheadrightarrow X^{[n]}$ the corresponding leg and $e_X = e_{X,0}$.

Now, let us construct, for each pair of morphisms (f, g) as in the diagram

$$\begin{tikzcd}
 & \widehat{X} \arrow[d, "g"] \\
C \arrow[r, two heads, "f"'] & C'
\end{tikzcd}$$

where $C, C' \in \mathcal{C}$ and f is a regular epimorphism, a canonical morphism $c(f, g)\colon \widehat{X} \to C$ such that $f \circ c(f, g) = g$. By the Yoneda lemma, g is represented by an element of $\widehat{X}(C')$. But, by duality, $\widehat{X}$ is the filtered colimit of

$$\begin{tikzcd}
X = X^{[0]} \arrow[r, tail] & X^{[1]} \arrow[r, tail] & X^{[2]} \arrow[r, tail] & \cdots
\end{tikzcd}$$

in $\mathrm{Lex}(\mathcal{C}, \mathrm{Set})$. Moreover, filtered colimit in $\mathrm{Lex}(\mathcal{C}, \mathrm{Set})$ are computed as in $\mathrm{Set}^{\mathcal{C}}$, so componentwise. Hence, there exists an $n \in \mathbb{N}$ such that g factors through $e_{X,n}$. Consider the least such n and

$$X^{[n]} \xrightarrow{h(g)} C'$$

the unique map such that $h(g)e_{X,n} = g$. Suppose now that

$$\begin{tikzcd}
 & X^{[n]} \arrow[d, "h(g)"] \\
C \arrow[r, two heads, "f"'] & C'
\end{tikzcd}$$

is the α-th diagram of shape (8). Then, we construct $c(f, g)$ canonically as $k_{X,f,h(g)}e_{X,n,\alpha+1}$, where $e_{X,n,\alpha}$ and $d_{X,n,\alpha}$ are the canonical projections as pictured below.

$$\begin{tikzcd}
\widehat{X} \arrow[d, "e_{X,n,\alpha+1}"] \arrow[rd, bend left, "e_{X,n,\alpha}"] \arrow[ddd, bend right=60, "c(f,g)"'] & \\
X^{[n]}_{\alpha+1} \arrow[r, two heads, "e^n_{\alpha+1}"] \arrow[rd, phantom, "\lrcorner", very near start] \arrow[dd, "k_{X,f,h(g)}"] & X^{[n]}_{\alpha} \arrow[d, "d_{X,n,\alpha}"] \\
 & X^{[n]} \arrow[d, "h(g)"] \\
C \arrow[r, two heads, "f"'] & C'
\end{tikzcd}$$

Thus,

$$f \circ c(f, g) = h(g) \circ e_{X,n} = g$$

and $\widehat{X}$ is $\mathcal{C}$-projective.

We extend now this construction functorialy on arrows. Let $f \colon X \to Y$ be a morphism in $\widetilde{\mathcal{C}}$. Let us suppose by recursion that we have written fe_X as $d_{Y,n}f_n$, where, again, $d_{Y,n}$ is the projection as below.

$$\begin{tikzcd}
\widehat{X} \arrow[rrr, two heads, "e_X"] \arrow[rrd, "f_n"] & & & X \arrow[d, "f"] \\
\widehat{Y} \arrow[r, two heads] & Y^{[n+1]} \arrow[r, two heads] & Y^{[n]} \arrow[r, two heads, "d_{Y,n}"'] & Y
\end{tikzcd}$$

Let us extend it to $Y^{[n+1]}$. We are going to extend it to $Y^{[n]}_{\alpha}$, for every α, by a transfinite recursion. The 0-th step is already done and we do it for limit ordinals in a unique way by definition of a limit. Now, suppose we have written f_n as $d_{Y,n,\alpha}f_{n,\alpha}$. So, we can draw the diagram

$$\begin{tikzcd}
\widehat{X} \arrow[d, dotted] \arrow[dr, bend left, "f_{n,\alpha}"] \arrow[ddd, bend right=60, "c(f',gf_n)"'] & \\
Y^{[n]}_{\alpha+1} \arrow[r, two heads, "e^n_{\alpha+1}"] \arrow[dd, "k_{Y,f',g}"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y^{[n]}_{\alpha} \arrow[d, "d_{Y,n,\alpha}"] \\
 & Y^{[n]} \arrow[d, "g"] \\
C \arrow[r, two heads, "f'"'] & C'
\end{tikzcd}$$

where f' and g forms the α-th diagram of shape (8). We then define $f_{n,\alpha+1}$ as the unique arrow $\widehat{X} \to Y^{[n]}_{\alpha+1}$ which makes commute this diagram. We can thus extend those $f_{n,\alpha}$'s uniquely to $Y^{[n+1]}$, defining f_{n+1}. Eventually, we extend those f_n's uniquely to $\widehat{Y}$ and define $\widehat{f}$.

It remains to prove that this construction is functorial. If we started with $f = 1_X$, we have to prove by induction that $(1_X)_n$ is the canonical projection $e_{X,n}\colon \widehat{X} \twoheadrightarrow X^{[n]}$. It is obviously true if $n = 0$. Let us suppose it is true for n. In order to prove it for $n+1$, let us show by transfinite induction that $(1_X)_{n,\alpha}$ is the canonical projection $e_{X,n,\alpha}\colon \widehat{X} \twoheadrightarrow X^{[n]}_{\alpha}$. It is obviously true for 0 and the limit ordinals. If we suppose it for α, we have to prove it for $\alpha+1$. We denote by (f',g) the α-th diagram of shape (8). We know that $e^n_{\alpha+1}(1_X)_{n,\alpha+1} = (1_X)_{n,\alpha} = e_{X,n,\alpha} = e^n_{\alpha+1}e_{X,n,\alpha+1}$ and $k_{X,f',g}(1_X)_{n,\alpha+1} = c(f', ge_{X,n})$. But $g = h(ge_{X,n})$ since g does not factor through e^n. Thus, $k_{X,f',g}(1_X)_{n,\alpha+1} = c(f', ge_{X,n}) = k_{X,f',g}e_{X,n,\alpha+1}$ and $(1_X)_{n,\alpha+1} = e_{X,n,\alpha+1}$ by definition of a pullback.

Finally, let us consider $f\colon X \to Y$ and $g\colon Y \to Z$ in $\widetilde{\mathcal{C}}$. We want to show that $\widehat{gf} = \widehat{g}\widehat{f}$. With the same inductions as above, we only have to prove that if $e_{Z,n}\widehat{gf} = e_{Z,n}\widehat{g}\widehat{f}$ and $e_{Z,n,\alpha}\widehat{gf} = e_{Z,n,\alpha}\widehat{g}\widehat{f}$, then $e_{Z,n,\alpha+1}\widehat{gf} = e_{Z,n,\alpha+1}\widehat{g}\widehat{f}$. In order words, we know that $(gf)_n = g_n\widehat{f}$ and $(gf)_{n,\alpha} = g_{n,\alpha}\widehat{f}$ and we want to show $(gf)_{n,\alpha+1} = g_{n,\alpha+1}\widehat{f}$. We already know it is true if we compose it by $e^n_{\alpha+1}$. So, it remains to prove that $k_{Z,f',g'}(gf)_{n,\alpha+1} = k_{Z,f',g'}g_{n,\alpha+1}\widehat{f}$ where (f',g') is the α-th diagram of shape (8). Since $k_{Z,f',g'g_n,\alpha+1}\widehat{f} = c(f', g'g_n)\widehat{f}$, let us suppose that

$$\begin{tikzcd} & Y^{[n']} \arrow[d, "h(g'g_n)"] \\ C \arrow[r, two heads, "f'"'] & C' \end{tikzcd}$$

is the β-th diagram of shape (8). Thus,

$$\begin{aligned} k_{Z,f',g'g_n,\alpha+1}\widehat{f} &= c(f', g'g_n)\widehat{f} \\ &= k_{Y,f',h(g'g_n)}e_{Y,n',\beta+1}\widehat{f} \\ &= k_{Y,f',h(g'g_n)}f_{n',\beta+1} \\ &= c(f', h(g'g_n)f_{n'}). \end{aligned}$$

But we know that

$$h(g'g_n)f_{n'} = h(g'g_n)e_{Y,n'}\widehat{f} = g'g_n\widehat{f} = g'(gf)_n$$

by assumption. Therefore,

$$k_{Z,f',g'g_n,\alpha+1}\widehat{f} = c(f', h(g'g_n)f_{n'}) = c(f', g'(gf)_n) = k_{Z,f',g'(gf)_{n,\alpha+1}}$$

which concludes the proof. ☐

We are able to prove a variant of Barr's embedding theorem, for which the power $\mathcal{P}$ is now known to be the set (or discrete category) Sub(1). However, this embedding is not full but only conservative. These changes are due to Z. Janelidze. We recall from Lemma 1.10 that a regular conservative functor between regular categories is faithful, preserves and reflects finite limits and regular epimorphisms.

Theorem 4.5. Let $\mathcal{C}$ be a small regular category with 1 as terminal object. Then, there exists a regular conservative embedding $\mathcal{C} \hookrightarrow \mathrm{Set}^{\mathrm{Sub}(1)}$.

Proof. Let $e\colon \widehat{()} \Rightarrow 1_{\widetilde{\mathcal{C}}}$ be the natural transformation given by Theorem 4.4. If I is a subobject of 1, let $I_\star$ be the coproduct in $\widetilde{\mathcal{C}}$ of the $\widehat{C}$'s for all $C \in \mathcal{C}$ such that the image of the unique morphism $C \to 1$ is I. Then, if $I \in \mathrm{Sub}(1)$ and $C \in \mathcal{C}$, we define $\phi(C)_I$ as the set

$\widetilde{\mathcal{C}}(I_\star, C)$. If $f\colon C \to C'$ is an arrow in $\mathcal{C}$, we set $\phi(f)_I$ to be the map $f \circ -\colon \phi(C)_I \to \phi(C')_I$ acting by composition with f. This defines a functor $\phi\colon \mathcal{C} \to \mathrm{Set}^{\mathrm{Sub}(1)}$. Let us check it satisfies the required properties.

Firstly, to prove ϕ preserves finite limits, it is enough to show that $\phi(-)_I\colon \mathcal{C} \to \mathrm{Set}$ preserves them for each $I \in \mathrm{Sub}(1)$. By the Yoneda lemma, $\phi(-)_I$ is isomorphic to $I_\star\colon \mathcal{C} \to \mathrm{Set}$ which preserves finite limits by definition.

We see it preserves regular epimorphisms in a similar way. Indeed, we must show $\phi(-)_I$ (or equivalently $I_\star$) preserves them for any $I \in \mathrm{Sub}(1)$. In order words, we have to prove $I_\star \in \widetilde{\mathcal{C}}$ is a $\mathcal{C}$-projective object, which follows directly from the fact it is the coproduct of $\mathcal{C}$-projective objects.

It remains to show ϕ is conservative. For each object $C' \in \mathcal{C}$, we consider the image factorisation of the unique morphism $C' \to 1$.

$$\begin{tikzcd} C' \arrow[r, two heads, "p"] & I \arrow[r, tail] & 1 \end{tikzcd}$$

For each $C'' \in \mathcal{C}$ such that I is also the image of $C'' \to 1$, since $\widehat{C''}$ is $\mathcal{C}$-projective, there exists a morphism $g_{C''}\colon \widehat{C''} \to C'$ making the square

$$\begin{tikzcd} \widehat{C''} \arrow[r, two heads, "e_{C''}"] \arrow[d, "g_{C''}"'] & C'' \arrow[d, two heads] & \\ C' \arrow[r, two heads, "p"'] & I \arrow[r, tail] & 1 \end{tikzcd}$$

commute. In particular, we choose $g_{C'} = e_{C'}$. This gives an induced morphism $g\colon I_\star \twoheadrightarrow C'$ which is a regular epimorphism since $e_{C'} = g\iota_{C'}$ is. Now, if $f\colon C \to C'$ is a morphism in $\mathcal{C}$ such that $\phi(f)_I$ is surjective, there exists a morphism $h\colon I_\star \to C$ in $\widetilde{\mathcal{C}}$ satisfying $fh = g$, which implies that f is a regular epimorphism. Therefore, ϕ reflects regular epimorphisms and it remains to prove it reflects monomorphisms.

So let $f\colon C \to D$ be a morphism in $\mathcal{C}$ such that $\phi(f)_I$ is injective for any $I \in \mathrm{Sub}(1)$. If we are given two morphisms $h, k\colon C' \rightrightarrows C$ in $\mathcal{C}$ such that $fh = fk$, we consider I and $g\colon I_\star \twoheadrightarrow C'$ defined as above. We thus know that $fhg = fkg$ which implies $hg = kg$ by assumption on f. Since g is a regular epimorphism, this means $h = k$ and f is a monomorphism.

Hence, ϕ reflects monomorphisms and isomorphisms. $\square$

In the same way we explained in Section 4.1, Theorems 4.2 and 4.5 allow us to reduce the proof of statements about finite limits and regular epimorphisms in regular categories to Set: Suppose we are given a statement of the form $P \Rightarrow Q$ where P and Q are conjunctions of properties which can be expressed as

1. some finite diagram is commutative,
2. some finite diagram is a limit diagram,
3. some morphism is a monomorphism,
4. some morphism is a regular epimorphism,
5. some morphism is an isomorphism,
6. some morphism factors through a given monomorphism.

Then, this statement $P \Rightarrow Q$ is valid in all regular $\mathcal{V}$-categories (for all universes $\mathcal{V}$) if and only if it is valid in $\mathcal{V}$-Set (for all universes $\mathcal{V}$).

As an application of this theorem, we can now prove in a quicker way the characterisation of n-permutable categories in terms of a matrix condition we left unproven in Example 2.50.

Proposition 4.6. [60] Let $n \geqslant 2$ be a natural number, (M, X) the extended matrix

$$(M,X) = \left(\left(\begin{array}{ccc|cccccc} x & y & y & x & z_1 & z_2 & \cdots & z_{n-2} \\ x & x & y & z_1 & z_2 & \cdots & z_{n-2} & y \end{array} \right), \{z_1, \dots, z_{n-2}\} \right)$$

of terms in Th[Set] and $\mathcal{C}$ a regular category. Then the following conditions are equivalent:

1. $\mathcal{C}$ is n-permutable,
2. $(\Delta_X \cap R)R^{\mathrm{op}}(\Delta_X \cap R) \leqslant R^{n-1}$ for any binary relation $R \rightarrowtail X \times X$ in $\mathcal{C}$,
3. $\mathcal{C}$ has (M, X)-closed relations.

Proof. Let us first prove that a binary relation $R \rightarrowtail X \times X$ is (M,X)-closed if and only if

$$(\Delta_X \cap R)R^{\mathrm{op}}(\Delta_X \cap R) \leqslant R^{n-1}.$$

In view of Definition 2.42 and the construction of the composition, dual and intersection of relations, it is enough to prove these implications only in Set. So let $R \subseteq X^2$ be a relation in Set. With the notations of Definition 2.42,

$$P = \{(x,x') \in X^2 \,|\, xRx,\, x'Rx,\, x'Rx'\}$$

and

$$Q = \{(x,x',x_1,\dots,x_{n-2}) \in X^n \,|\, xRx_1,\, x_1Rx_2,\dots,x_{n-2}Rx'\}.$$

Thus, R is (M,X)-closed if and only if, for each $(x,x') \in P$, there exist $x_1,\dots,x_{n-2} \in X$ such that $(x,x',x_1,\dots,x_{n-2}) \in Q$. On the other hand, $x(\Delta_X \cap R)R^{\mathrm{op}}(\Delta_X \cap R)x'$ exactly means that xRx, $x'Rx$ and $x'Rx'$, while $xR^{n-1}x'$ holds if and only if there exist $x_1,\dots,x_{n-2} \in X$ such that xRx_1, x_1Rx_2, $\dots$, $x_{n-2}Rx'$. This already proves $2 \Leftrightarrow 3$.

The implication $2 \Rightarrow 1$ follows directly from Point 5 of Theorem 2.49 since for any reflexive relation E on X, the equalities

$$E^{\mathrm{op}} = \Delta_X E^{\mathrm{op}} \Delta_X = (\Delta_X \cap E)E^{\mathrm{op}}(\Delta_X \cap E)$$

hold.

It remains to prove the implication $1 \Rightarrow 2$, which is due to D. Rodelo in [60]. Let $r\colon R \rightarrowtail X \times X$ be a binary relation in $\mathcal{C}$. We treat the case $n = 2k-1$ is odd first. In that case, we define, for each $1 \leqslant i \leqslant k$, the relation s_i via the pullback

$$\begin{tikzcd}
S_i \arrow[r] \arrow[d, tail, "s_i"'] \arrow[dr, phantom, very near start, "\lrcorner"] & R \arrow[d, tail, "r"] \\
X^k \times X^k \arrow[r, "\pi_i \times \pi_i"'] & X \times X
\end{tikzcd}$$

where $\pi_1,\dots,\pi_k\colon X^k \to X$ are the product projections. We also define for each $1 \leqslant j \leqslant k-1$ the relation t_j via the pullback

$$\begin{tikzcd}
T_j \arrow[rr] \arrow[d, tail, "t_j"'] \arrow[dr, phantom, "\lrcorner", very near start] & & R \arrow[d, tail, "r"] \\
X^k \times X^k \arrow[r, "\pi_{j+1}\times\pi_j"'] & X \times X \arrow[r, "\mathrm{tw}"'] & X \times X
\end{tikzcd}$$

where tw is the twisting isomorphism. We then write $P \rightarrowtail X^k \times X^k$ for the intersection of all these relations $s_1, \dots, s_k, t_1, \dots, t_{k-1}$ (formed via a finite limit). By Theorem 2.49, we know that

$$(P, P^{\mathrm{op}})_{n+1} \leqslant (P, P^{\mathrm{op}})_{n-1}.$$

Now that all the constructions have been done, it remains to prove that this implies the required inequality. By Barr's embedding theorem, it suffices to do it in Set. There, the relation $P \subseteq X^k \times X^k$ is defined by $(a_1, \dots, a_k)P(b_1, \dots, b_k)$ if and only if

$$\begin{cases} a_i R b_i, & \text{for each } 1 \leqslant i \leqslant k \\ b_j R a_{j+1}, & \text{for each } 1 \leqslant j \leqslant k-1. \end{cases}$$

Let $x, y \in X$ be such that xRx, yRx and yRy. We have to show $xR^{n-1}y$. From the relations

$$\begin{array}{rcl}
(x, \dots, x) & P & (x, \dots, x) \\
(y, x, \dots, x) & P & (x, \dots, x) \\
(y, x, \dots, x) & P & (y, x, \dots, x) \\
(y, y, x \dots, x) & P & (y, x, \dots, x) \\
 & \vdots & \\
(y, \dots, y, x) & P & (y, \dots, y, x) \\
(y, \dots, y) & P & (y, \dots, y, x),
\end{array}$$

we get $(x, \dots, x)(P, P^{\mathrm{op}})_{2k=n+1}(y, \dots, y)$. By assumption, we conclude that $(x, \dots, x)(P, P^{\mathrm{op}})_{n-1}(y, \dots, y)$, i.e.,

$$(x, \dots, x) \quad P \quad (z_{11}, \dots, z_{1k})$$

$$\begin{array}{rcl}
(z_{21},\dots,z_{2k}) & P & (z_{11},\dots,z_{1k}) \\
(z_{21},\dots,z_{2k}) & P & (z_{31},\dots,z_{3k}) \\
(z_{41},\dots,z_{4k}) & P & (z_{31},\dots,z_{3k}) \\
 & \vdots & \\
(z_{n-3,1},\dots,z_{n-3,k}) & P & (z_{n-2,1},\dots,z_{n-2,k}) \\
(y,\dots,y) & P & (z_{n-2,1},\dots,z_{n-2,k}),
\end{array}$$

for some $(z_{i1},\dots,z_{ik}) \in X^k$, $1 \leqslant i \leqslant n-2$. From these relations, we get xRz_{11}, $z_{11}Rz_{22}$, $z_{22}Rz_{32}$, $z_{32}Rz_{43}$,..., $z_{n-3,k-1}Rz_{n-2,k-1}$ and $z_{n-2,k-1}Ry$, which implies $xR^{n-1}y$.

It remains to treat the case $n = 2k$ is even. In a similar way, we define the relation P, suppose we have $(P, P^{\mathrm{op}})_{n+1} \leqslant (P, P^{\mathrm{op}})_{n-1}$ and prove the required inequality in the particular case $\mathcal{C} = \mathrm{Set}$. So let $x, y \in X$ be such that xRx, yRx and yRy and we want to show that $xR^{n-1}y$. Again, the relations

$$\begin{array}{rcl}
(x,\dots,x) & P & (x,\dots,x) \\
(y,x,\dots,x) & P & (x,\dots,x) \\
(y,x,\dots,x) & P & (y,x,\dots,x) \\
(y,y,x\dots,x) & P & (y,x,\dots,x) \\
 & \vdots & \\
(y,\dots,y,x) & P & (y,\dots,y,x) \\
(y,\dots,y) & P & (y,\dots,y,x) \\
(y,\dots,y) & P & (y,\dots,y),
\end{array}$$

tell us that $(x,\dots,x)(P,P^{\mathrm{op}})_{2k+1=n+1}(y,\dots,y)$. From this, it follows that $(x,\dots,x)(P,P^{\mathrm{op}})_{n-1}(y,\dots,y)$, which means

$$\begin{array}{rcl}
(x,\dots,x) & P & (z_{11},\dots,z_{1k}) \\
(z_{21},\dots,z_{2k}) & P & (z_{11},\dots,z_{1k}) \\
(z_{21},\dots,z_{2k}) & P & (z_{31},\dots,z_{3k})
\end{array}$$

$$\begin{array}{ccc}
(z_{41}, \dots, z_{4k}) & P & (z_{31}, \dots, z_{3k}) \\
 & \vdots & \\
(z_{n-2,1}, \dots, z_{n-2,k}) & P & (z_{n-3,1}, \dots, z_{n-3,k}) \\
(z_{n-2,1}, \dots, z_{n-2,k}) & P & (y, \dots, y),
\end{array}$$

for some $(z_{i1}, \dots, z_{ik}) \in X^k$, $1 \leqslant i \leqslant n-2$. This implies in particular that xRz_{11}, $z_{11}Rz_{22}$, $z_{22}Rz_{32}$, $z_{32}Rz_{43}, \dots$, $z_{n-3,k-1}Rz_{n-2,k}$ and $z_{n-2,k}Ry$. Therefore, $xR^{n-1}y$ which concludes the proof. □

Note that some equivalences similar to 1 ⇔ 2 were already mentioned in [70].

4.3 Embedding for categories with (M, X)-closed relations

Putting together Theorems 2.54, 3.21 and 4.4, we are now going to prove an embedding theorem for regular $\mathcal{T}$-categories with (M, X)-closed relations. As a first step, we construct a finitary essentially algebraic theory $\Gamma_{(M,X)}$ for which the category of models $\mathrm{Mod}(\Gamma_{(M,X)})$ will be our 'representative category'. By that we mean $\mathrm{Mod}(\Gamma_{(M,X)})$ is a regular $\mathcal{T}$-category with (M, X)-closed relations and every small regular $\mathcal{T}$-category with (M, X)-closed relations admits a regular conservative $\mathcal{T}$-enriched embedding in $\mathrm{Mod}(\Gamma_{(M,X)})^{\mathrm{Sub}(1)}$. We will conclude this section with a similar embedding theorem for exact $\mathcal{T}$-categories with (M, X)-closed relations, using the exact completion (introduced in [77], see also [98, 73]) of the regular category $\mathrm{Mod}(\Gamma_{(M,X)})$.

4.3.1 Construction of $\Gamma_{(M,X)}$

Firstly, if Γ and Γ' are two essentially algebraic theories, we will write $\Gamma \subseteq \Gamma'$ to mean $S \subseteq S'$, $\Sigma \subseteq \Sigma'$, $E \subseteq E'$, $\Sigma_t \subseteq \Sigma'_t$, $\Sigma \setminus \Sigma_t \subseteq \Sigma' \setminus \Sigma'_t$ and $\mathrm{Def}(\sigma) = \mathrm{Def}'(\sigma)$ for all $\sigma \in \Sigma \setminus \Sigma_t$. In this case, we have a forgetful functor $U\colon \mathrm{Mod}(\Gamma') \to \mathrm{Mod}(\Gamma)$.

Let now $\mathcal{T}$ be a commutative Lawvere theory. We will see it as $\mathcal{T}_{(\Sigma,E)}$ for a finitary one-sorted algebraic theory (Σ, E). By an operation symbol (resp. an axiom) of $\mathcal{T}$, we thus mean an element of Σ (resp. E). For the sake of brevity, for each natural number r, we denote by $\Sigma_r^{\mathcal{T}}$ the set of r-ary operation symbols of $\mathcal{T}$. Let also

$$(M,X) = \left(\left(\begin{array}{ccc|ccc} t_{11} & \cdots & t_{1b} & u_{11} & \cdots & u_{1b'} \\ \vdots & & \vdots & \vdots & & \vdots \\ t_{a1} & \cdots & t_{ab} & u_{a1} & \cdots & u_{ab'} \end{array} \right), X \right)$$

be an extended matrix of terms in $\mathcal{T}$. We are going to construct recursively a series of finitary essentially algebraic theories

$$\Gamma^0 \subseteq \Delta^1 \subseteq \cdots \subseteq \Gamma^n \subseteq \Delta^{n+1} \subseteq \cdots$$

and a $\mathcal{T}$-enrichment on the $\mathrm{Mod}(\Gamma^n)$'s and $\mathrm{Mod}(\Delta^n)$'s. Let us first define $\Gamma^0 = (S^0, \Sigma^0, E^0, \Sigma_t^0, \mathrm{Def}^0)$:

- $S^0 = \{\star\}$,
- $\Sigma^0 = \Sigma_t^0 = \{\tau^\star \colon \star^r \to \star \,|\, r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}\}$,
- $E^0 = \{\text{all axioms from } \mathcal{T} \text{ for the } \tau^\star\text{'s}\}$.

We consider the obvious $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma^0) \cong \mathcal{T}\text{-Alg}$ (see Proposition 1.107). Now, let us suppose we have defined

$$\Gamma^0 \subseteq \Delta^1 \subseteq \cdots \subseteq \Delta^n \subseteq \Gamma^n$$

and the $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma^n)$ (with $\Gamma^n = (S^n, \Sigma^n, E^n, \Sigma_t^n, \mathrm{Def}^n)$). We are going to construct

$$\Delta^{n+1} = (S'^{n+1}, \Sigma'^{n+1}, E'^{n+1}, \Sigma_t'^{n+1}, \mathrm{Def}'^{n+1})$$

first (below $\overline{S}^0 = S^0$ and $\overline{S}^n = S^n \setminus S^{n-1}$ if $n \geqslant 1$):

$$S'^{n+1} = S^n \cup \{(s,0),(s,1) \,|\, s \in \overline{S}^n\} \cong S^n \sqcup \overline{S}^n \sqcup \overline{S}^n,$$

$$\begin{aligned} \Sigma_t'^{n+1} = \Sigma_t^n &\cup \{\tau^{(s,0)} \colon (s,0)^r \to (s,0) \,|\, r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \\ &\cup \{\tau^{(s,1)} \colon (s,1)^r \to (s,1) \,|\, r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \end{aligned}$$

$$\begin{aligned}
&\cup \{\alpha^s \colon s \to (s,0) \mid s \in \overline{S}^n\}\\
&\cup \{\rho_1^s, \dots, \rho_{b'}^s \colon s^b \to (s,0) \mid s \in \overline{S}^n\}\\
&\cup \{\kappa_1^s, \dots, \kappa_{k-l}^s \colon s^l \to (s,0) \mid s \in \overline{S}^n\}\\
&\cup \{\eta^s, \varepsilon^s \colon (s,0) \to (s,1) \mid s \in \overline{S}^n\},
\end{aligned}$$

$$\Sigma'^{n+1} = \Sigma^n \cup \Sigma_t'^{n+1} \cup \{\pi^s \colon (s,0) \to s \mid s \in \overline{S}^n\},$$

$$\begin{aligned}
&E'^{n+1} =\\
&E^n \cup \{u_{ij}^{(s,0)}(\alpha^s(x_1), \dots, \alpha^s(x_l), \kappa_1^s(x_1, \dots, x_l), \dots, \kappa_{k-l}^s(x_1, \dots, x_l))\\
&\qquad = \rho_j^s(t_{i1}^s(x_1, \dots, x_l), \dots, t_{ib}^s(x_1, \dots, x_l)) \mid\\
&\qquad\qquad\qquad\qquad 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b', s \in \overline{S}^n\}\\
&\quad \cup \{\eta^s(\alpha^s(x)) = \varepsilon^s(\alpha^s(x)) \mid s \in \overline{S}^n\}\\
&\quad \cup \{\pi^s(\alpha^s(x)) = x \mid s \in \overline{S}^n\}\\
&\quad \cup \{\alpha^s(\pi^s(x)) = x \mid s \in \overline{S}^n\}\\
&\quad \cup \{\text{all axioms from } \mathcal{T} \text{ for the } \tau^{(s,0)}\text{'s and the } \tau^{(s,1)}\text{'s} \mid s \in \overline{S}^n\}\\
&\quad \cup \{\tau^{(s,0)}(\alpha^s(x_1), \dots, \alpha^s(x_r)) = \alpha^s(\tau^s(x_1, \dots, x_r)) \mid\\
&\qquad\qquad\qquad\qquad r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\}\\
&\quad \cup \{\tau^{(s,0)}(\rho_j^s(x_{11}, \dots, x_{1b}), \dots, \rho_j^s(x_{r1}, \dots, x_{rb}))\\
&\qquad = \rho_j^s(\tau^s(x_{11}, \dots, x_{r1}), \dots, \tau^s(x_{1b}, \dots, x_{rb})) \mid\\
&\qquad\qquad\qquad\qquad 1 \leqslant j \leqslant b', r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\}\\
&\quad \cup \{\tau^{(s,0)}(\kappa_v^s(x_{11}, \dots, x_{1l}), \dots, \kappa_v^s(x_{r1}, \dots, x_{rl}))\\
&\qquad = \kappa_v^s(\tau^s(x_{11}, \dots, x_{r1}), \dots, \tau^s(x_{1l}, \dots, x_{rl})) \mid\\
&\qquad\qquad\qquad\qquad 1 \leqslant v \leqslant k-l, r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\}\\
&\quad \cup \{\tau^{(s,1)}(\eta^s(x_1), \dots, \eta^s(x_r)) = \eta^s(\tau^{(s,0)}(x_1, \dots, x_r)) \mid\\
&\qquad\qquad\qquad\qquad r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\}\\
&\quad \cup \{\tau^{(s,1)}(\varepsilon^s(x_1), \dots, \varepsilon^s(x_r)) = \varepsilon^s(\tau^{(s,0)}(x_1, \dots, x_r)) \mid\\
&\qquad\qquad\qquad\qquad r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\}\\
&\quad \cup \{\tau^s(\pi^s(x_1), \dots, \pi^s(x_r)) = \pi^s(\tau^{(s,0)}(x_1, \dots, x_r)) \mid\\
&\qquad\qquad\qquad\qquad r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\}
\end{aligned}$$

and

$$\begin{cases} \mathrm{Def}'^{n+1}(\sigma) = \mathrm{Def}^n(\sigma) \ \text{ if } \ \sigma \in \Sigma^n \setminus \Sigma^n_t \\ \mathrm{Def}'^{n+1}(\pi^s) = \{\eta^s(x) = \varepsilon^s(x)\} \ \text{ for } \ s \in \overline{S}^n. \end{cases}$$

Hence, we have $\Gamma^n \subseteq \Delta^{n+1}$ and we consider the obvious $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Delta^{n+1})$. Let now T^{n+1} be the set of finitary terms $\theta\colon \prod_{i=1}^m s_i \to s$ of Σ'^{n+1} which are not terms of Σ'^n (where we consider $\Sigma'^0 = \varnothing$). We then define Γ^{n+1} as:

$$S^{n+1} = S'^{n+1} \cup \{s_\theta, s'_\theta \,|\, \theta \in T^{n+1}\} \cong S'^{n+1} \sqcup T^{n+1} \sqcup T^{n+1},$$

$$\begin{aligned} \Sigma^{n+1}_t = \Sigma'^{n+1}_t &\cup \{\tau^{s_\theta}\colon s^r_\theta \to s_\theta \,|\, r \in \mathbb{N}, \tau \in \Sigma^{\mathcal{T}}_r, \theta \in T^{n+1}\} \\ &\cup \{\tau^{s'_\theta}\colon (s'_\theta)^r \to s'_\theta \,|\, r \in \mathbb{N}, \tau \in \Sigma^{\mathcal{T}}_r, \theta \in T^{n+1}\} \\ &\cup \{\alpha_\theta\colon s \to s_\theta \,|\, \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\ &\cup \{\mu_\theta\colon \prod_{i=1}^m s_i \to s_\theta \,|\, \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\ &\cup \{\eta_\theta, \varepsilon_\theta\colon s_\theta \to s'_\theta \,|\, \theta \in T^{n+1}\}, \end{aligned}$$

$$\Sigma^{n+1} = \Sigma'^{n+1} \cup \Sigma^{n+1}_t \cup \{\pi_\theta\colon s_\theta \to s \,|\, \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\},$$

$$\begin{aligned} &E^{n+1} = \\ &E'^{n+1} \cup \{\eta_\theta(\alpha_\theta(x)) = \varepsilon_\theta(\alpha_\theta(x)) \,|\, \theta \in T^{n+1}\} \\ &\qquad \cup \{\pi_\theta(\alpha_\theta(x)) = x \,|\, \theta \in T^{n+1}\} \\ &\qquad \cup \{\alpha_\theta(\pi_\theta(x)) = x \,|\, \theta \in T^{n+1}\} \\ &\qquad \cup \{\alpha_\theta(\theta(x_1, \dots, x_m)) = \mu_\theta(x_1, \dots, x_m) \,|\, \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\ &\qquad \cup \{\text{all axioms from } \mathcal{T} \text{ for the } \tau^{s_\theta}\text{'s and the } \tau^{s'_\theta}\text{'s} \,|\, \theta \in T^{n+1}\} \\ &\qquad \cup \{\tau^{s_\theta}(\alpha_\theta(x_1), \dots, \alpha_\theta(x_r)) = \alpha_\theta(\tau^s(x_1, \dots, x_r)) \,| \\ &\qquad\qquad\qquad\qquad r \in \mathbb{N}, \tau \in \Sigma^{\mathcal{T}}_r, \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \end{aligned}$$

$$\begin{aligned}
&\cup \{\tau^{s_\theta}(\mu_\theta(x_{11},\dots,x_{1m}),\dots,\mu_\theta(x_{r1},\dots,x_{rm}))\\
&\qquad = \mu_\theta(\tau^{s_1}(x_{11},\dots,x_{r1}),\dots,\tau^{s_m}(x_{1m},\dots,x_{rm})) \mid\\
&\qquad\qquad\qquad r\in\mathbb{N},\tau\in\Sigma_r^{\mathcal{T}},\theta\colon \prod_{i=1}^m s_i\to s\in T^{n+1}\}\\
&\cup \{\tau^{s'_\theta}(\eta_\theta(x_1),\dots,\eta_\theta(x_r)) = \eta_\theta(\tau^{s_\theta}(x_1,\dots,x_r)) \mid\\
&\qquad\qquad\qquad r\in\mathbb{N},\tau\in\Sigma_r^{\mathcal{T}},\theta\in T^{n+1}\}\\
&\cup \{\tau^{s'_\theta}(\varepsilon_\theta(x_1),\dots,\varepsilon_\theta(x_r)) = \varepsilon_\theta(\tau^{s_\theta}(x_1,\dots,x_r)) \mid\\
&\qquad\qquad\qquad r\in\mathbb{N},\tau\in\Sigma_r^{\mathcal{T}},\theta\in T^{n+1}\}\\
&\cup \{\tau^{s}(\pi_\theta(x_1),\dots,\pi_\theta(x_r)) = \pi_\theta(\tau^{s_\theta}(x_1,\dots,x_r)) \mid\\
&\qquad\qquad\qquad r\in\mathbb{N},\tau\in\Sigma_r^{\mathcal{T}},\theta\colon \prod_{i=1}^m s_i\to s\in T^{n+1}\}
\end{aligned}$$

and

$$\begin{cases} \mathrm{Def}^{n+1}(\sigma) = \mathrm{Def}'^{n+1}(\sigma) \text{ if } \sigma\in\Sigma'^{n+1}\setminus\Sigma_t'^{n+1}\\ \mathrm{Def}^{n+1}(\pi_\theta) = \{\eta_\theta(x)=\varepsilon_\theta(x)\} \text{ for } \theta\in T^{n+1}.\end{cases}$$

Thus, we have $\Delta^{n+1}\subseteq\Gamma^{n+1}$ and we consider the obvious $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma^{n+1})$. In this way, the expected series

$$\Gamma^0\subseteq\Delta^1\subseteq\Gamma^1\subseteq\cdots$$

has been constructed. We then set $\Gamma_{(M,X)}$ to be the union of these finitary essentially algebraic theories. By that we obviously mean $S_{(M,X)} = \bigcup_{n\in\mathbb{N}} S^n$, $\Sigma_{(M,X)} = \bigcup_{n\in\mathbb{N}}\Sigma^n$, $E_{(M,X)} = \bigcup_{n\in\mathbb{N}} E^n$, $\Sigma_{t,(M,X)} = \bigcup_{n\in\mathbb{N}}\Sigma_t^n$ and $\mathrm{Def}_{(M,X)}(\sigma) = \mathrm{Def}^n(\sigma)$ for all $n\in\mathbb{N}$ and $\sigma\in\Sigma^n\setminus\Sigma_t^n$. We provide $\mathrm{Mod}(\Gamma_{(M,X)})$ with the $\mathcal{T}$-enrichment coming from the $\mathcal{T}$-enrichments on the $\mathrm{Mod}(\Gamma^n)$'s. Remark that for each $\pi\colon s'\to s\in\Sigma_{(M,X)}\setminus\Sigma_{t,(M,X)}$, there are three corresponding operation symbols in $\Sigma_{t,(M,X)}$, these are $\alpha\colon s\to s'$ and $\eta,\varepsilon\colon s'\rightrightarrows s''$.

Proposition 4.7. [59] Let $\mathcal{T}$ be a commutative Lawvere theory and (M,X) an extended matrix of terms in $\mathcal{T}$. Then the $\mathcal{T}$-enriched category $\mathrm{Mod}(\Gamma_{(M,X)})$ is regular with (M,X)-closed relations.

Proof. It is the 'Γ ingredient' of the construction which makes the category $\mathrm{Mod}(\Gamma_{(M,X)})$ regular. Indeed, each finitary term θ of $\Sigma_{(M,X)}$ is in

T^{n+1} for some $n \in \mathbb{N}$, which makes the conditions of Theorem 1.89 hold.

On the other hand, the 'Δ part' of the construction ensures that $\mathrm{Mod}(\Gamma_{(M,X)})$ has (M,X)-closed relations. To see that, it suffices to use Theorem 2.52 with the terms $\pi^s\colon (s,0) \to s$,

$$\alpha^s \circ p_1, \dots, \alpha^s \circ p_l, \kappa_1^s, \dots, \kappa_{k-l}^s \colon s^l \to (s,0)$$

(where $p_1, \dots, p_l\colon s^l \to s$ are the projections), and

$$\rho_1^s, \dots, \rho_{b'}^s \colon s^b \to (s,0). \qquad \square$$

4.3.2 Proof of the embedding theorem

Let us now prove our embedding theorem. The Mal'tsev case already appears in [55].

Theorem 4.8. [59] Let $\mathcal{T}$ be a commutative Lawvere theory, (M,X) an extended matrix of terms in $\mathcal{T}$ and $\mathcal{C}$ a small regular $\mathcal{T}$-category with (M,X)-closed relations. Let 1 be the terminal object in $\mathcal{C}$. Then, there exists a regular conservative $\mathcal{T}$-enriched embedding

$$\phi\colon \mathcal{C} \hookrightarrow \mathrm{Mod}(\Gamma_{(M,X)})^{\mathrm{Sub}(1)}.$$

Moreover, for each morphism $f\colon C \to C'$ in $\mathcal{C}$, each $I \in \mathrm{Sub}(1)$ and each $s \in S_{(M,X)}$,

$$(\mathrm{Im}\,\phi(f)_I)_s = \{(\phi(f)_I)_s(x) \,|\, x \in (\phi(C)_I)_s\}.$$

Proof. By Theorem 3.21 and Examples 3.15 and 3.16, we know that $\widetilde{\mathcal{C}}$ is a regular $\mathcal{T}$-category with (M,X)-closed relations. In what follows, we denote by $e\colon \widehat{()} \Rightarrow 1_{\widetilde{\mathcal{C}}}$ the natural transformation given by Theorem 4.4. If $C \in \mathcal{C}$ and $P \in \mathrm{Sub}(1)$, we are going to construct $\phi(C)_P \in \mathrm{Mod}(\Gamma_{(M,X)})$. More precisely, we are going to construct a $\Gamma_{(M,X)}$-model $\phi(C)_P$ satisfying the following conditions:

1. For each $s \in S_{(M,X)}$, $(\phi(C)_P)_s = \widetilde{\mathcal{C}}(P_s, C)$ for some $\mathcal{C}$-projective object $P_s \in \widetilde{\mathcal{C}}$.

2. For each $s \in S_{(M,X)}$ and r-ary operation symbol τ of $\mathcal{T}$,

$$\tau^s : \widetilde{\mathcal{C}}(P_s, C)^r \longrightarrow \widetilde{\mathcal{C}}(P_s, C)$$

is the operation τ coming from the $\mathcal{T}$-enrichment of $\widetilde{\mathcal{C}}$.

3. For each $\pi \colon s' \to s \in \Sigma_{(M,X)} \setminus \Sigma_{t,(M,X)}$ and its corresponding $\alpha \colon s \to s'$, there is a given regular epimorphism

$$\begin{tikzcd} P_{s'} \arrow[r, two heads, "l_\alpha"] & P_s \end{tikzcd}$$

in $\widetilde{\mathcal{C}}$ such that

$$\begin{aligned} \alpha \colon \widetilde{\mathcal{C}}(P_s, C) &\longrightarrow \widetilde{\mathcal{C}}(P_{s'}, C) \\ f &\longmapsto f l_\alpha \end{aligned}$$

and

$$\begin{aligned} \pi \colon \widetilde{\mathcal{C}}(P_{s'}, C) &\longrightarrow \widetilde{\mathcal{C}}(P_s, C) \\ g &\longmapsto \text{the unique } f \text{ such that } f l_\alpha = g \end{aligned}$$

where π is defined if and only if such an f exists. For the corresponding operation symbols $\eta, \varepsilon \colon s' \rightrightarrows s''$, we consider the kernel pair (v, w) of l_α.

$$\begin{tikzcd} \widehat{R} \arrow[r, two heads, "e_R"] & R \arrow[r, shift left, "v"] \arrow[r, shift right, "w"'] & P_{s'} \arrow[r, two heads, "l_\alpha"] & P_s \end{tikzcd}$$

We require then $P_{s''} = \widehat{R}$,

$$\begin{aligned} \eta \colon \widetilde{\mathcal{C}}(P_{s'}, C) &\longrightarrow \widetilde{\mathcal{C}}(P_{s''}, C) \\ g &\longmapsto g v e_R \end{aligned}$$

and

$$\begin{aligned} \varepsilon \colon \widetilde{\mathcal{C}}(P_{s'}, C) &\longrightarrow \widetilde{\mathcal{C}}(P_{s''}, C) \\ g &\longmapsto g w e_R. \end{aligned}$$

4. For each sort $s \in S_{(M,X)}$, we consider the universal approximate co-solution for (M, X) on P_s

$$\begin{tikzcd}
 & \widehat{W(P_s)} \arrow[d, two heads, "e_{W(P_s)}"] & \\
 & W(P_s) \arrow[dl, "(p_1^{P_s},\dots,p_{b'}^{P_s})"'] \arrow[d, two heads, "d^{P_s}"] \arrow[dr, "(q_1^{P_s},\dots,q_{k-l}^{P_s})"] & \\
(bP_s)^{b'} & P_s & (lP_s)^{k-l}
\end{tikzcd}$$

where d^{P_s} is a regular epimorphism by Theorem 2.54. We require then $P_{(s,0)} = \widehat{W(P_s)}$,

$$\begin{aligned} \rho_j^s \colon \widetilde{\mathcal{C}}(P_s, C)^b &\longrightarrow \widetilde{\mathcal{C}}(P_{(s,0)}, C) \\ (f_1, \dots, f_b) &\longmapsto \begin{pmatrix} f_1 \\ \vdots \\ f_b \end{pmatrix} p_j^{P_s} e_{W(P_s)} \end{aligned}$$

for each $j \in \{1, \dots, b'\}$ and

$$\begin{aligned} \kappa_v^s \colon \widetilde{\mathcal{C}}(P_s, C)^l &\longrightarrow \widetilde{\mathcal{C}}(P_{(s,0)}, C) \\ (f_1, \dots, f_l) &\longmapsto \begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix} q_v^{P_s} e_{W(P_s)} \end{aligned}$$

for each $v \in \{1, \dots, k-l\}$.

5. For each finitary term $\theta \colon \prod_{i=1}^m s_i \to s$ of $\Sigma_{(M,X)}$, there is a given morphism $l_{\mu_\theta} \colon P_{s_\theta} \to P_{s_1} + \cdots + P_{s_m}$ such that

$$\begin{aligned} \mu_\theta \colon \widetilde{\mathcal{C}}(P_{s_1}, C) \times \cdots \times \widetilde{\mathcal{C}}(P_{s_m}, C) &\longrightarrow \widetilde{\mathcal{C}}(P_{s_\theta}, C) \\ (f_1, \dots, f_m) &\longmapsto \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} l_{\mu_\theta}. \end{aligned}$$

Since $\Gamma_{(M,X)}$ is the union of the series

$$\Gamma^0 \subseteq \Delta^1 \subseteq \Gamma^1 \subseteq \cdots$$

of essentially algebraic theories, to construct a $\Gamma_{(M,X)}$-model $\phi(C)_P$, it is enough to construct recursively a Γ^n-model for each $n \in \mathbb{N}$ such that they agree on the common sorts and operations.

Firstly, to define a Γ^0-model, we set $P_\star$ to be the coproduct in $\widetilde{\mathcal{C}}$ of the $\widehat{C'}$'s for all $C' \in \mathcal{C}$ such that the image of the unique morphism $C' \to 1$ is $P \in \mathrm{Sub}(1)$. $P_\star$ is $\mathcal{C}$-projective since it is the coproduct of $\mathcal{C}$-projective objects. The Γ^0-structure is then imposed by conditions 1 and 2.

Now, we suppose we have defined a Γ^n-model satisfying the above conditions. We are going to extend it to a Γ^{n+1}-model with the same properties. Firstly, we extend it to a Δ^{n+1}-model. Let $s \in \overline{S}^n$. Condition 4 above imposes the constructions of $P_{(s,0)}$, the ρ^s_j's and the κ^s_v's. Moreover, condition 3 with $l_{\alpha^s} = d^{P_s} e_{W(P_s)}$ from condition 4 defines α^s, π^s, $P_{(s,1)}$, η^s and ε^s and condition 2 forces the construction of the $\tau^{(s,0)}$'s and the $\tau^{(s,1)}$'s. It follows then from the definitions that this gives a Δ^{n+1}-model which satisfies conditions 1–5. Indeed, to see that the operations τ^s for $\tau \in \Sigma^{\mathcal{T}}_r$ commute with the other ones, it suffices to use the fact that, in a $\mathcal{T}$-category with finite coproducts, if we are given morphisms $(x_{ij} \colon X \to Y)_{i \in \{1,\dots,r\}, j \in \{1,\dots,r'\}}$, then the equality

$$\tau\left(\begin{pmatrix} x_{11} \\ \vdots \\ x_{1r'} \end{pmatrix}, \dots, \begin{pmatrix} x_{r1} \\ \vdots \\ x_{rr'} \end{pmatrix}\right) = \begin{pmatrix} \tau(x_{11},\dots,x_{r1}) \\ \vdots \\ \tau(x_{1r'},\dots,x_{rr'}) \end{pmatrix}$$

holds, which can be seen by composing with the coproduct injections. We can also compute for $f_1, \dots, f_l \colon P_s \to C$, $1 \leqslant i \leqslant a$ and $1 \leqslant j \leqslant b'$,

$$\begin{aligned}
&\rho^s_j(t^s_{i1}(f_1,\dots,f_l),\dots,t^s_{ib}(f_1,\dots,f_l)) \\
&= \begin{pmatrix} t_{i1}(f_1,\dots,f_l) \\ \vdots \\ t_{ib}(f_1,\dots,f_l) \end{pmatrix} p^{P_s}_j e_{W(P_s)} \\
&= \begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix} \begin{pmatrix} t_{i1}(\iota_1,\dots,\iota_l) \\ \vdots \\ t_{ib}(\iota_1,\dots,\iota_l) \end{pmatrix} p^{P_s}_j e_{W(P_s)} \\
&= \begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix} u_{ij}(\iota_1 d^{P_s},\dots,\iota_l d^{P_s}, q^{P_s}_1,\dots,q^{P_s}_{k-l}) e_{W(P_s)}
\end{aligned}$$

$$
\begin{aligned}
&= u_{ij}(f_1 d^{P_s} e_{W(P_s)}, \dots, f_l d^{P_s} e_{W(P_s)},\\
&\qquad\qquad \begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix} q_1^{P_s} e_{W(P_s)}, \dots, \begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix} q_{k-l}^{P_s} e_{W(P_s)})\\
&= u_{ij}^{(s,0)}(\alpha^s(f_1), \dots, \alpha^s(f_l), \kappa_1^s(f_1, \dots, f_l), \dots, \kappa_{k-l}^s(f_1, \dots, f_l)).
\end{aligned}
$$

It remains to extend it to a Γ^{n+1}-model. In order to simplify the proof, we are going to construct P_{s_θ}, l_{μ_θ} and l_{α_θ} for each finitary term $\theta\colon \prod_{i=1}^m s_i \to s$ of Σ'^{n+1} such that it matches the previous construction if θ is actually a term of Σ'^n. Then, condition 3 will force the construction of α_θ, π_θ, $P_{s'_\theta}$, η_θ and ε_θ, condition 2 will define the τ^{s_θ} and $\tau^{s'_\theta}$'s, and condition 5 will impose the definition of μ_θ. We are going to do it recursively in such a way that the equality

$$\alpha_\theta(\theta(f_1, \dots, f_m)) = \mu_\theta(f_1, \dots, f_m)$$

holds for any cospan $(f_i\colon P_{s_i} \to C)_{i \in \{1,\dots,m\}}$ such that $\theta(f_1, \dots, f_m)$ is defined.

Firstly, let $\theta = p_j\colon \prod_{i=1}^m s_i \to s_j$ be a projection ($1 \leqslant j \leqslant m$). In this case, we define $P_{s_\theta} = P_{s_j}$, $l_{\mu_\theta} = \iota_j\colon P_{s_j} \to P_{s_1} + \cdots + P_{s_m}$ and $l_{\alpha_\theta} = 1_{P_{s_j}}$. Obviously, one has

$$\alpha_\theta(\theta(f_1, \dots, f_m)) = f_j = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \iota_j = \mu_\theta(f_1, \dots, f_m)$$

for any cospan $(f_i\colon P_{s_i} \to C)_{i \in \{1,\dots,m\}}$.

Secondly, let $\theta\colon \prod_{i=1}^m s_i \to s'$ be a finitary term of Σ'^{n+1} for which l_{μ_θ} and l_{α_θ} have been constructed. If $\pi\colon s' \to s \in \Sigma'^{n+1} \setminus \Sigma_t'^{n+1}$ has corresponding $\alpha\colon s \to s'$, we define $P_{s_{\pi(\theta)}} = P_{s_\theta}$, $l_{\alpha_{\pi(\theta)}} = l_\alpha l_{\alpha_\theta}$ and $l_{\mu_{\pi(\theta)}} = l_{\mu_\theta}$.

$$\begin{tikzcd}
P_{s_{\pi(\theta)}} = P_{s_\theta} \arrow[rr, "l_{\mu_{\pi(\theta)}} = l_{\mu_\theta}"] \arrow[d, two heads, "l_{\alpha_\theta}"'] \arrow[dr, two heads, "l_{\alpha_{\pi(\theta)}}"] & & P_{s_1} + \cdots + P_{s_m} \\
P_{s'} \arrow[r, two heads, "l_\alpha"'] & P_s &
\end{tikzcd}$$

If the cospan $(f_i\colon P_{s_i} \to C)_{i \in \{1,\dots,m\}}$ is such that $\theta(f_1, \dots, f_m)\colon P_{s'} \to C$

is defined, we know from the previous step in the recursion that

$$\theta(f_1, \ldots, f_m) l_{\alpha_\theta} = \alpha_\theta(\theta(f_1, \ldots, f_m)) = \mu_\theta(f_1, \ldots, f_m).$$

If moreover $\pi(\theta(f_1, \ldots, f_m))\colon P_s \to C$ is defined, we have

$$\pi(\theta(f_1, \ldots, f_m)) l_\alpha = \theta(f_1, \ldots, f_m).$$

In this case,

$$\begin{aligned} \alpha_{\pi(\theta)}(\pi(\theta(f_1, \ldots, f_m))) &= \pi(\theta(f_1, \ldots, f_m)) l_{\alpha_{\pi(\theta)}} \\ &= \pi(\theta(f_1, \ldots, f_m)) l_\alpha l_{\alpha_\theta} \\ &= \theta(f_1, \ldots, f_m) l_{\alpha_\theta} \\ &= \mu_\theta(f_1, \ldots, f_m) \\ &= \mu_{\pi(\theta)}(f_1, \ldots, f_m). \end{aligned}$$

Eventually, let us suppose $\sigma\colon \prod_{i=1}^r s'_i \to s \in \Sigma_t'^{n+1}$ is an operation symbol and for each $1 \leqslant j \leqslant r$, $\theta_j\colon \prod_{i=1}^m s_i \to s'_j$ is a finitary term of Σ'^{n+1} for which $l_{\mu_{\theta_j}}$ and $l_{\alpha_{\theta_j}}$ have been defined. We already have a corresponding morphism $l_\sigma\colon P_s \to P_{s'_1} + \cdots + P_{s'_r}$ such that

$$\begin{aligned} \sigma\colon \widetilde{\mathcal{C}}(P_{s'_1}, C) \times \cdots \times \widetilde{\mathcal{C}}(P_{s'_r}, C) &\to \widetilde{\mathcal{C}}(P_s, C) \\ (f_1, \ldots, f_r) &\mapsto \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} l_\sigma \end{aligned}$$

(if $\sigma = \tau^s$ for some $\tau \in \Sigma_r^{\mathcal{T}}$, we have $l_{\tau^s} = \tau(\iota_1, \ldots, \iota_r)\colon P_s \to rP_s$). Let us consider the following diagram where the square is a pullback.

$$\begin{tikzcd} P_{s_\theta} = \widehat{U} \arrow[r, two heads, "e_U"] & U \arrow[r, "u_1"] \arrow[d, two heads, "u_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_{s_{\theta_1}} + \cdots + P_{s_{\theta_r}} \arrow[r, "\begin{pmatrix} l_{\mu_{\theta_1}} \\ \vdots \\ l_{\mu_{\theta_r}} \end{pmatrix}"] \arrow[d, two heads, "l_{\alpha_{\theta_1}} + \cdots + l_{\alpha_{\theta_r}}"] & P_{s_1} + \cdots + P_{s_m} \\ & P_s \arrow[r, "l_\sigma"'] & P_{s'_1} + \cdots + P_{s'_r} & \end{tikzcd}$$

Denoting the term $\sigma(\theta_1, \ldots, \theta_r)\colon \prod_{i=1}^m s_i \to s$ by θ, we define $P_{s_\theta} = \widehat{U}$,

$l_{\alpha_\theta} = u_2 e_U$ and

$$l_{\mu_\theta} = \begin{pmatrix} l_{\mu_{\theta_1}} \\ \vdots \\ l_{\mu_{\theta_r}} \end{pmatrix} u_1 e_U.$$

Then, if the cospan $(f_i \colon P_{s_i} \to C)_{i \in \{1,\dots,m\}}$ is such that $\theta_j(f_1,\dots,f_m) : P_{s'_j} \to C$ is defined for each $1 \leqslant j \leqslant r$,

$$\begin{aligned}
\alpha_\theta(\theta(f_1,\dots,f_m)) &= \sigma(\theta_1(f_1,\dots,f_m),\dots,\theta_r(f_1,\dots,f_m))l_{\alpha_\theta} \\
&= \begin{pmatrix} \theta_1(f_1,\dots,f_m) \\ \vdots \\ \theta_r(f_1,\dots,f_m) \end{pmatrix} l_\sigma u_2 e_U \\
&= \begin{pmatrix} \theta_1(f_1,\dots,f_m) \\ \vdots \\ \theta_r(f_1,\dots,f_m) \end{pmatrix} (l_{\alpha_{\theta_1}} + \cdots + l_{\alpha_{\theta_r}}) u_1 e_U \\
&= \begin{pmatrix} \alpha_{\theta_1}(\theta_1(f_1,\dots,f_m)) \\ \vdots \\ \alpha_{\theta_r}(\theta_r(f_1,\dots,f_m)) \end{pmatrix} u_1 e_U \\
&= \begin{pmatrix} \mu_{\theta_1}(f_1,\dots,f_m) \\ \vdots \\ \mu_{\theta_r}(f_1,\dots,f_m) \end{pmatrix} u_1 e_U \\
&= \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \begin{pmatrix} l_{\mu_{\theta_1}} \\ \vdots \\ l_{\mu_{\theta_r}} \end{pmatrix} u_1 e_U \\
&= \mu_\theta(f_1,\dots,f_m)
\end{aligned}$$

using the previous steps in the recursion.

This ends the construction of P_{s_θ}, l_{α_θ} and l_{μ_θ} for each finitary term θ of Σ'^{n+1}. Similarly as above, this defines a Γ^{n+1}-model which satisfies conditions 1–5. This concludes the recursive construction of our Γ^n-model for each $n \in \mathbb{N}$. Considering them all together, we get a $\Gamma_{(M,X)}$-model $\phi(C)_P$.

Now, if $f \colon C \to C' \in \mathcal{C}$ and $P \in \mathrm{Sub}(1)$, we define a morphism $\phi(f)_P \colon \phi(C)_P \to \phi(C')_P$ by

$$\begin{aligned}
(\phi(f)_P)_s \colon \widetilde{\mathcal{C}}(P_s, C) &\longrightarrow \widetilde{\mathcal{C}}(P_s, C') \\
g &\longmapsto fg
\end{aligned}$$

for all $s \in S_{(M,X)}$. By conditions 2–5, $\phi(f)_P$ is a $\Gamma_{(M,X)}$-homomorphism. This defines the expected functor $\phi \colon \mathcal{C} \to \mathrm{Mod}(\Gamma_{(M,X)})^{\mathrm{Sub}(1)}$.

To prove ϕ is a $\mathcal{T}$-enriched functor, we only need to show that $\phi(-)_P$ is $\mathcal{T}$-enriched for all $P \in \mathrm{Sub}(1)$ since the $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma_{(M,X)})^{\mathrm{Sub}(1)}$ is computed componentwise. We thus have to prove that given $\tau \in \Sigma_r^{\mathcal{T}}$, parallel morphisms $f_1, \ldots, f_r \colon C \to C'$ in $\mathcal{C}$, $P \in \mathrm{Sub}(1)$ and $s \in S_{(M,X)}$, $(\phi(\tau(f_1, \ldots, f_r))_P)_s = \tau(\phi(f_1)_P, \ldots, \phi(f_r)_P)_s$. This holds due to condition 2, since given $g \in \widetilde{\mathcal{C}}(P_s, C)$,

$$\begin{aligned}(\phi(\tau(f_1, \ldots, f_r))_P)_s(g) &= \tau(f_1, \ldots, f_r)g \\ &= \tau(f_1 g, \ldots, f_r g) \\ &= \tau^s(f_1 g, \ldots, f_r g) \\ &= \tau^s((\phi(f_1)_P)_s(g), \ldots, (\phi(f_r)_P)_s(g)) \\ &= \tau(\phi(f_1)_P, \ldots, \phi(f_r)_P)_s(g).\end{aligned}$$

Similarly as we described for Barr's Embedding Theorem 4.5, ϕ preserves finite limits and regular epimorphisms since for each $P \in \mathrm{Sub}(1)$ and each $s \in S_{(M,X)}$, $P_s \colon \mathcal{C} \to \mathrm{Set}$ preserves them. Note that if a homomorphism f of $\Gamma_{(M,X)}$-models is such that f_s is surjective for each $s \in S_{(M,X)}$, it is a strong epimorphism and so a regular epimorphism.

Again, in the same way we did for Theorem 4.5, if $f \colon C \to C' \in \mathcal{C}$ is such that $(\phi(f)_P)_s$ is surjective (resp. injective) for each $P \in \mathrm{Sub}(1)$ and each $s \in S_{(M,X)}$ (or even just for $s = \star$), then f is a regular epimorphism (resp. a monomorphism). This implies that ϕ is conservative.

It remains to check that, for $f \colon C \to C' \in \mathcal{C}$, $P \in \mathrm{Sub}(1)$ and $s \in S_{(M,X)}$,

$$(\mathrm{Im}\, \phi(f)_P)_s = \{(\phi(f)_P)_s(x) \mid x \in (\phi(C)_P)_s\}.$$

Consider $\pi \colon s' \to s \in \Sigma_{(M,X)} \setminus \Sigma_{t,(M,X)}$ and $x \in \widetilde{\mathcal{C}}(P_{s'}, C)$ such that $\pi((\phi(f)_P)_{s'}(x))$ is defined. So, there exists $g \colon P_s \to C'$ making the square

$$\begin{tikzcd} P_{s'} \arrow[r, "x"] \arrow[d, "l_\alpha"'] & C \arrow[d, "f"] \\ P_s \arrow[r, "g"'] & C' \end{tikzcd}$$

commute (with $\alpha \colon s \to s'$ corresponding to π). Let $f = iq$ be the image factorisation of f. Since l_α is a strong epimorphism, there exists a $g' \colon P_s \to \mathrm{Im}(f)$ such that $ig' = g$. Since P_s is $\mathcal{C}$-projective, there exists a morphism $y \colon P_s \to C$ such that $qy = g'$. Thus, $fy = g$ and $(\phi(f)_P)_s(y) = g = \pi((\phi(f)_P)_{s'}(x))$. Therefore, in view of the description of images in categories of Γ-models given in Proposition 1.84 for any essentially algebraic theory Γ, this concludes the proof. $\square$

The reader may have noticed we actually defined in this proof, for each $P \in \mathrm{Sub}(1)$, an internal $\Gamma_{(M,X)}$-co-model in $\widetilde{\mathcal{C}}$.

4.3.3 Applications

As we previously explained for Yoneda and Barr's embedding theorems, Theorem 4.8 gives a way to reduce the proof of statements about finite limits and regular epimorphisms in regular $\mathcal{T}$-categories with (M, X)-closed relations to the particular case of $\mathrm{Mod}(\Gamma_{(M,X)})$. With more details, let (M, X) be an extended matrix of terms in the commutative Lawvere theory $\mathcal{T}$ and suppose we are given a statement of the form $P \Rightarrow Q$ where P and Q are conjunctions of properties which can be expressed as

1. some finite diagram is commutative,
2. some finite diagram is a limit diagram,
3. the equality $t(f_1, \dots, f_n) = g$ holds for an n-ary term t of $\mathcal{T}$ and parallel morphisms $f_1, \dots, f_n, g$,
4. some morphism is a monomorphism,
5. some morphism is a regular epimorphism,
6. some morphism is an isomorphism,
7. some morphism factors through a given monomorphism.

Then, this statement $P \Rightarrow Q$ is valid in all regular $\mathcal{V}$-$\mathcal{T}$-categories with (M, X)-closed relations (for all universes $\mathcal{V}$) if and only if it is valid in $\mathcal{V}$-$\mathrm{Mod}(\Gamma_{(M,X)})$ (for all universes $\mathcal{V}$).

Remark 4.9. At a first glance, one could think this technique will be hard to use in practice, in view of the difficult definition of $\mathrm{Mod}(\Gamma_{(M,X)})$. However, due to the additional property in Theorem 4.8, we can suppose that the homomorphisms $f\colon A \to B$ considered in the given statement have an easy description of their images in $\mathrm{Mod}(\Gamma_{(M,X)})$, i.e.,

$$(\mathrm{Im}\, f)_s = \{f_s(a) \,|\, a \in A_s\}$$

for each $s \in S_{(M,X)}$ (compare with the one given in Proposition 1.84). In particular, if f is a regular epimorphism, f_s will be a surjective function for each $s \in S_{(M,X)}$. Therefore, in practice, it seems we will never have to use the operations α_θ, μ_θ, η_θ, ε_θ and π_θ. They were built only to make $\mathrm{Mod}(\Gamma_{(M,X)})$ a regular category.

We now show on concrete examples how to use this embedding theorem to prove some results using elements and operations. We recall that, for the sake of brevity, we sometimes write f instead of f_s for the s-th component of an S-sorted function f. Firstly, we give an example in the regular subtractive context.

Lemma 4.10. [24] Let $\mathcal{C}$ be a regular subtractive category and d an approximate subtraction (i.e., a morphism $d\colon A \times A \to B$ such that $d(1_A, 1_A) = 0$).

$$\begin{tikzcd} A \arrow[d, "{(1_A,1_A)}"'] \arrow[dr, "0"] & \\ A \times A \arrow[r, "d"'] & B \end{tikzcd}$$

Let also $x, y, z, w\colon C \to A$ be four morphisms in $\mathcal{C}$ such that $d(x, y) = d(z, t)$. Then $d(x, z) = d(y, t)$.

Proof. By our Embedding Theorem 4.8, it is enough to prove this lemma in $\mathrm{Mod}(\Gamma_{(M,X)})$ with $(M, X) = \left(\left(\begin{array}{cc|c} x & 0 & x \\ x & x & 0 \end{array} \right), \varnothing \right)$. So, let $s \in S_{(M,X)}$ and $c \in C_s$. We can compute:

$$\begin{aligned} \alpha^s(d(x(c), z(c))) &= \rho_1^s(d(x(c), z(c)), 0^s) \\ &= \rho_1^s(d(x(c), z(c)), d(z(c), z(c))) \end{aligned}$$

$$
\begin{aligned}
&= d(\rho_1^s((x(c), z(c)), (z(c), z(c))))\\
&= d(\rho_1^s(x(c), z(c)), \rho_1^s(z(c), z(c)))\\
&= d(\rho_1^s(x(c), z(c)), 0^{(s,0)})\\
&= d(\rho_1^s(x(c), z(c)), \rho_1^s(y(c), y(c)))\\
&= d(\rho_1^s((x(c), y(c)), (z(c), y(c))))\\
&= \rho_1^s(d(x(c), y(c)), d(z(c), y(c)))\\
&= \rho_1^s(d(z(c), t(c)), d(z(c), y(c)))\\
&= d(\rho_1^s(z(c), z(c)), \rho_1^s(t(c), y(c)))\\
&= d(\rho_1^s(y(c), y(c)), \rho_1^s(t(c), y(c)))\\
&= \rho_1^s(d(y(c), t(c)), d(y(c), y(c)))\\
&= \rho_1^s(d(y(c), t(c)), 0^s)\\
&= \alpha^s(d(y(c), t(c))).
\end{aligned}
$$

Since $\pi^s(\alpha^s(x))$ is everywhere-defined and $\pi^s(\alpha^s(x)) = x$ is a theorem of $\Gamma_{(M,X)}$, $\alpha^s \colon B_s \to B_{(s,0)}$ is injective. We can thus deduce from the above calculation that $d(x(c), z(c)) = d(y(c), t(c))$. $\square$

As announced above, we now prove the converse implication of Proposition 3.25. For a natural number $n \geqslant 3$, we are going to use the diagram below in a regular category $\mathcal{C}$, of which (7) is a particular case:

$$\begin{tikzcd}
 & & & & L \arrow[rrrrr, "\lambda"] \arrow[dllll, swap, "p_1"] \arrow[ddll, "p_2"] \arrow[dddl, "p_3"] \arrow[dddddr, "p_{n-2}"] \arrow[ddddddrrr, "p_{n-1}"] & & & & & U \times_W V \arrow[dl, dotted, shift left, "q_1"] \arrow[ddddddr, shift left, "q_2"] & \\
X \arrow[rrrrrrrr, dotted, "\gamma"] \arrow[ddr, shift right, swap, "f"] & & & & & & & & U \arrow[ur, dotted, shift left] \arrow[ddddddr, dotted, shift right, swap, "h"] & & \\
 & & X \arrow[dl, shift right, swap, "g"] \arrow[dd, shift right, swap, "f"] & & & & & & & & \\
 & Y \arrow[uul, shift right, swap, "s"] \arrow[ur, shift right, swap, "t"] \arrow[ddddrrrrrrrr, dotted, "\beta_1"] & & X \arrow[dl, shift right, swap, "g"] & & & & & & & \\
 & & Y \arrow[uu, shift right, swap, "s"] \arrow[ur, shift right, swap, "t"] \arrow[dddrrrrrrr, dotted, "\beta_2"] & & \ddots & & & & & & \\
 & & & \ddots & & X \arrow[ddr, shift right, swap, "f"] & & & & & \\
 & & & & & & & X \arrow[rrr, "\delta"] \arrow[dl, shift right, swap, "g"] & & & V \arrow[uuuuuul, shift left] \arrow[dl, shift right, swap, "k"] \\
 & & & & & & Y \arrow[uul, shift right, swap, "s"] \arrow[ur, shift right, swap, "t"] \arrow[rrr, "\beta_{n-2}"] & & & W \arrow[uuuuuul, dotted, shift right, swap, "u"] \arrow[ur, shift right, swap, "v"] \arrow[lll, bend left, "i_2"] \arrow[lll, bend left=40, swap, "i_{n-2}"] &
\end{tikzcd}$$

(10)

in which the equalities

$$\begin{aligned}
&fs = gt = 1_Y,\ kv = hu = 1_W,\ h\gamma = \beta_1 f,\ \gamma s = u\beta_1,\ k\delta = \beta_{n-2}g,\\
&\delta t = v\beta_{n-2},\ q_1\lambda = \gamma p_1,\ q_2\lambda = \delta p_{n-1} \text{ and } \beta_j i_j = 1_W,\\
&\beta_j f = \beta_{j-1}g,\ ft = i_j\beta_{j-1} \text{ for each } 2 \leqslant j \leqslant n-2
\end{aligned}$$

hold, $(U \times_W V, q_1, q_2)$ is the pullback of k along h and $(L, p_1, \dots, p_{n-1})$ is the limit of the zig-zag formed by the alternating split epimorphisms f and g.

Theorem 4.11. [60] Let $n \geqslant 3$ be a natural number and $\mathcal{C}$ a regular category. The following statements are equivalent:

1. $\mathcal{C}$ is n-permutable,
2. for each diagram (7) in $\mathcal{C}$, if γ and δ are regular epimorphisms, then λ is also a regular epimorphism,
3. for each diagram (10) in $\mathcal{C}$, if γ and δ are regular epimorphisms, then λ is also a regular epimorphism.

Proof. 2 ⇒ 1 being the content of Proposition 3.25 and 3 ⇒ 2 being trivial, it remains to prove 1 ⇒ 3. Due to our Embedding Theorem 4.8, it is enough to prove it in $\mathrm{Mod}(\Gamma_{(M,X)})$ for

$$(M, X) = \left(\left(\begin{array}{ccc|ccccc} x & y & y & x & z_1 & z_2 & \cdots & z_{n-2} \\ x & x & y & z_1 & z_2 & \cdots & z_{n-2} & y \end{array} \right), \{z_1, \dots, z_{n-2}\} \right).$$

Moreover, using Remark 4.9, we can suppose without loss of generality that γ and δ are surjective in each sort. So, let $s' \in S_{(M,X)}$ (to avoid clashes of notations with the section s of f), $a \in U_{s'}$ and $b \in V_{s'}$ be such that $h(a) = k(b)$. We must prove that (a, b) is in the image of λ. Since $\gamma_{s'}$ and $\delta_{s'}$ are surjective, there exist $x, x' \in X_{s'}$ such that $\gamma(x) = a$ and $\delta(x') = b$. This implies the equalities

$$\beta_1 f(x) = h\gamma(x) = h(a) = k(b) = k\delta(x') = \beta_{n-2}g(x') \tag{11}$$

hold. Let us also compute the following identities:

$$\begin{aligned}
\gamma(sg)^{n-2}(x') &= \gamma sg(sg)^{n-3}(x') \\
&= u\beta_1 g(sg)^{n-3}(x') \\
&= u\beta_2 f(sg)^{n-3}(x') \\
&= u\beta_2 fsg(sg)^{n-4}(x') \\
&= u\beta_2 g(sg)^{n-4}(x') \\
&= \cdots \qquad\qquad (12) \\
&= u\beta_{n-3} g(sg)^1(x') \\
&= u\beta_{n-2} fsg(x') \\
&= u\beta_{n-2} g(x') \\
&\overset{(11)}{=} u\beta_1 f(x) \\
&= \gamma sf(x)
\end{aligned}$$

and

$$\begin{aligned}
\delta(tf)^{n-2}(x) &= \delta tf(tf)^{n-3}(x) \\
&= v\beta_{n-2} f(tf)^{n-3}(x) \\
&= v\beta_{n-3} g(tf)^{n-3}(x) \\
&= v\beta_{n-3} gtf(tf)^{n-4}(x) \\
&= v\beta_{n-3} f(tf)^{n-4}(x) \\
&= \cdots \qquad\qquad (13) \\
&= v\beta_2 f(tf)^1(x) \\
&= v\beta_1 gtf(x) \\
&= v\beta_1 f(x) \\
&\overset{(11)}{=} v\beta_{n-2} g(x') \\
&= \delta tg(x').
\end{aligned}$$

For $2 \leqslant j \leqslant n-2$, we also find

$$\begin{aligned}
ftg(sg)^{n-j-1}(x') &= i_j\beta_{j-1} g(sg)^{n-j-1}(x') \\
&= i_j\beta_j f(sg)^{n-j-1}(x')
\end{aligned}$$

$$\begin{aligned}
&= i_j\beta_j fsg(sg)^{n-j-2}(x') \\
&= i_j\beta_j g(sg)^{n-j-2}(x') \\
&= \cdots \\
&= i_j\beta_{n-3}g(sg)^1(x') \\
&= i_j\beta_{n-2}fsg(x') \qquad (14) \\
&= i_j\beta_{n-2}g(x') \\
&\overset{(11)}{=} i_j\beta_1 f(x) \\
&= i_j(\beta_{j-1}i_{j-1})(\beta_{j-2}i_{j-2})\cdots(\beta_2 i_2)\beta_1 f(x) \\
&= (i_j\beta_{j-1})(i_{j-1}\beta_{j-2})(i_{j-2}\cdots\beta_2)(i_2\beta_1)f(x) \\
&= (ft)^{j-1}f(x) \\
&= f(tf)^{j-1}(x).
\end{aligned}$$

Now, we define

$$x_1 = \rho_1^{s'}(x, sf(x), (sg)^{n-2}(x')) \in X_{(s',0)}$$

and for $2 \leqslant j \leqslant n-1$,

$$x_j = \rho_j^{s'}((tf)^{j-1}(x), tg(sg)^{n-j-1}(x'), (sg)^{n-j-1}(x')) \in X_{(s',0)}.$$

We can compute the following identities:

$$\begin{aligned}
f(x_1) &= \rho_1^{s'}(f(x), fsf(x), f(sg)^{n-2}(x')) \\
&= \rho_1^{s'}(f(x), f(x), g(sg)^{n-3}(x')) \\
&= \kappa_1^{s'}(f(x), g(sg)^{n-3}(x')) \\
&= \rho_2^{s'}(f(x), g(sg)^{n-3}(x'), g(sg)^{n-3}(x')) \\
&= \rho_2^{s'}(gtf(x), gtg(sg)^{n-3}(x'), g(sg)^{n-3}(x')) \\
&= g(x_2)
\end{aligned}$$

and for each $2 \leqslant j \leqslant n-2$,

$$\begin{aligned}
f(x_j) &= \rho_j^{s'}(f(tf)^{j-1}(x), ftg(sg)^{n-j-1}(x'), f(sg)^{n-j-1}(x')) \\
&\overset{(14)}{=} \rho_j^{s'}(f(tf)^{j-1}(x), f(tf)^{j-1}(x), f(sg)^{n-j-1}(x'))
\end{aligned}$$

$$\begin{aligned}
&= \kappa_j^{s'}(f(tf)^{j-1}(x), f(sg)^{n-j-1}(x'))\\
&= \rho_{j+1}^{s'}(f(tf)^{j-1}(x), f(sg)^{n-j-1}(x'), f(sg)^{n-j-1}(x'))\\
&= \rho_{j+1}^{s'}(g(tf)^{j}(x), g(sg)^{n-j-2}(x'), g(sg)^{n-j-2}(x'))\\
&= g(x_{j+1}).
\end{aligned}$$

This exactly means that $(x_1, \ldots, x_{n-1}) \in L_{(s',0)}$. Moreover, the equalities

$$\begin{aligned}
\gamma(x_1) &= \rho_1^{s'}(\gamma(x), \gamma sf(x), \gamma(sg)^{n-2}(x'))\\
&\overset{(12)}{=} \rho_1^{s'}(\gamma(x), \gamma sf(x), \gamma sf(x))\\
&= \alpha^{s'}(\gamma(x))\\
&= \alpha^{s'}(a)
\end{aligned}$$

and

$$\begin{aligned}
\delta(x_{n-1}) &= \rho_{n-1}^{s'}(\delta(tf)^{n-2}(x), \delta tg(x'), \delta(x'))\\
&\overset{(13)}{=} \rho_{n-1}^{s'}(\delta tg(x'), \delta tg(x'), \delta(x'))\\
&= \alpha^{s'}(\delta(x'))\\
&= \alpha^{s'}(b)
\end{aligned}$$

hold. This implies that

$$\lambda(x_1, \ldots, x_{n-1}) = (\alpha^{s'}(a), \alpha^{s'}(b)) = \alpha^{s'}(a, b)$$

and

$$\begin{aligned}
(a, b) &= \pi^{s'}(\alpha^{s'}(a, b))\\
&= \pi^{s'}(\lambda(x_1, \ldots, x_{n-1}))\\
&\in \mathrm{Im}(\lambda)_{s'}
\end{aligned}$$

which concludes the proof. □

The particular case $n = 3$ of the above theorem (i.e., for Goursat categories) already appears in [49]. For the case $n = 2$ (i.e., Mal'tsev categories), a similar characterisation also exists: We now consider the following digram in $\mathcal{C}$

$$\begin{tikzcd}
X \times_Y Z \arrow[rr, "\lambda"] \arrow[dr, "p_2"] \arrow[dd, shift right, two heads, "p_1"'] & & U \times_W V \arrow[dr, "q_2"] \arrow[dd, dotted, shift right, two heads, "q_1"'] & \\
& Z \arrow[rr, "\delta"] \arrow[dd, shift right, two heads, "g"'] & & V \arrow[dd, shift right, two heads, "k"'] \\
X \arrow[uu, shift right] \arrow[rr, dotted, "\gamma"] \arrow[dr, "f"'] & & U \arrow[uu, dotted, shift right] \arrow[dr, dotted, "h"] & \\
& Y \arrow[uu, shift right, "t"'] \arrow[rr, "\beta"'] & & W \arrow[uu, shift right, "v"']
\end{tikzcd} \tag{15}$$

in which the equalities $gt = 1_Y$, $kv = 1_W$, $h\gamma = \beta f$, $k\delta = \beta g$, $\delta t = v\beta$, $q_1\lambda = \gamma p_1$ and $q_2\lambda = \delta p_2$ hold, $(X \times_Y Z, p_1, p_2)$ is the pullback of g along f and $(U \times_W V, q_1, q_2)$ is the pullback of k along h.

Theorem 4.12. [48] The following conditions on a regular category $\mathcal{C}$ are equivalent:

1. $\mathcal{C}$ is a Mal'tsev category,
2. for any diagram (15) in $\mathcal{C}$, if γ and δ are regular epimorphisms, then λ is also a regular epimorphism.

Proof. In order to illustrate Theorem 4.8, we are only going to prove the implication 1 ⇒ 2 here. Using this embedding theorem, it is thus enough to prove it in $\mathrm{Mod}(\Gamma_{(M,X)})$ for

$$(M, X) = \left(\left(\begin{array}{ccc|c} x & y & y & x \\ x & x & y & y \end{array} \right), \varnothing \right)$$

supposing that γ and δ are surjective in each sort. So, let $s \in S_{(M,X)}$, $a \in U_s$ and $b \in V_s$ be such that $h(a) = k(b)$ and let us prove that $(a, b) \in \mathrm{Im}(\lambda)_s$. By assumption, we can find $x \in X_s$ and $z \in Z_s$ such that $\gamma(x) = a$ and $\delta(z) = b$. Let $z' = \rho_1^s(tf(x), tg(z), z) \in Z_{(s,0)}$. Since the equalities

$$g(z') = \rho_1^s(f(x), g(z), g(z)) = \alpha^s(f(x)) = f(\alpha^s(x))$$

hold, we can consider $(\alpha^s(x), z') \in (X \times_Y Z)_{(s,0)}$. Moreover, since

$$\begin{aligned} \delta(z') &= \rho_1^s(\delta tf(x), \delta tg(z), \delta(z)) \\ &= \rho_1^s(vh\gamma(x), vk\delta(z), \delta(z)) \\ &= \rho_1^s(vh(a), vk(b), b) \\ &= \rho_1^s(vk(b), vk(b), b) \\ &= \alpha^s(b), \end{aligned}$$

we know that $\lambda(\alpha^s(x), z') = (\gamma(\alpha^s(x)), \delta(z')) = (\alpha^s(a), \alpha^s(b)) = \alpha^s(a,b)$. Therefore, $(a,b) = \pi^s(\alpha^s(a,b)) = \pi^s(\lambda(\alpha^s(x), z')) \in \mathrm{Im}(\lambda)_s$. ☐

Our last example is the proof of the implication 1 ⇒ 3 in Theorem 2.49 for the particular case of Goursat categories (i.e., $n = 3$).

Lemma 4.13. [27] Let $R \rightarrowtail A \times B$ be a binary relation in a regular Goursat category $\mathcal{C}$. Then, $RR^{\mathrm{op}}RR^{\mathrm{op}} \leqslant RR^{\mathrm{op}}$.

Proof. Our embedding theorem again tells us it is enough to prove this lemma in $\mathrm{Mod}(\Gamma_{(M,X)})$ with

$$(M, X) = \left(\left(\begin{array}{ccc|cc} x & y & y & x & z \\ x & x & y & z & y \end{array} \right), \{z\} \right).$$

With Remark 4.9 in mind, we can assume without loss of generality that

$$(RR^{\mathrm{op}})_s = \{(b,b') \in B_s \times B_s \mid \exists a \in A_s \text{ such that } aRb, aRb'\}$$

for each $s \in S_{(M,X)}$. In the same way, we can assume without loss of generality that

$$\begin{aligned} (RR^{\mathrm{op}}RR^{\mathrm{op}})_s = \{(b,b'') \in B_s \times B_s \mid \exists a, a' \in A_s, b' \in B_s \\ \text{such that } aRb, aRb', a'Rb', a'Rb''\} \end{aligned}$$

for each $s \in S_{(M,X)}$. So, let $s \in S_{(M,X)}$, $(b,b'') \in (RR^{\mathrm{op}}RR^{\mathrm{op}})_s$ and we want to prove that $(b,b'') \in (RR^{\mathrm{op}})_s$. Since $(a,b) \in R_s$, $(a,b') \in R_s$ and $(a',b') \in R_s$ (for some a, a', b'), we know that

$$(\rho_1^s(a,a,a'), \rho_1^s(b,b',b')) = (\kappa_1^s(a,a'), \alpha^s(b)) \in R_{(s,0)}.$$

Moreover, since $(a, b') \in R_s$, $(a', b') \in R_s$ and $(a', b'') \in R_s$, we also know that

$$(\rho_2^s(a, a', a'), \rho_2^s(b', b', b'')) = (\kappa_1^s(a, a'), \alpha^s(b'')) \in R_{(s,0)}.$$

Therefore,

$$\alpha^s(b, b'') = (\alpha^s(b), \alpha^s(b'')) \in (RR^{\mathrm{op}})_{(s,0)}$$

which implies

$$(b, b'') = \pi^s(\alpha^s(b, b'')) \in (RR^{\mathrm{op}})_s. \qquad \square$$

4.3.4 The exact case

The aim of this subsection is to prove a similar result to Theorem 4.8 in the exact context. Since $\mathrm{Mod}(\Gamma_{(M,X)})$ is a priori not an exact category, we need to turn it into an exact one. This can be realised with the exact completion of a regular category, introduced in [77] (see also [98, 73]). Let us recall it here. Let $\mathcal{C}$ be a well-powered regular category. We define its *exact completion* $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ as the following category:

- objects of $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ are pairs (A, R) where A is an object of $\mathcal{C}$ and R an equivalence relation on A,
- a morphism $T\colon (A, R) \to (B, S)$ is a relation $T \rightarrowtail A \times B$ satisfying
  1. $STR = T$
  2. $TT^{\mathrm{op}} \leqslant S$
  3. $R \leqslant T^{\mathrm{op}}T$
- the identity on (A, R) is R itself,
- composition is the composition of relations.

We then get a functor

$$\begin{aligned} i\colon \mathcal{C} &\longrightarrow \mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}} \\ A &\longmapsto (A, \Delta_A) \\ f\colon A \to B &\longmapsto (1_A, f)\colon A \rightarrowtail A \times B. \end{aligned}$$

Proposition 4.14. [77] Let $\mathcal{C}$ be a well-powered regular category. Then, $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ is exact and $i\colon \mathcal{C} \hookrightarrow \mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ is full, faithful and regular. It is the exact completion of $\mathcal{C}$ in the sense that, for each regular functor $F\colon \mathcal{C} \to \mathcal{D}$ to an exact category $\mathcal{D}$, there exists a unique (up to isomorphism) regular functor $\overline{F}\colon \mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}} \to \mathcal{D}$ such that $\overline{F}i$ is isomorphic to F.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "i"] \arrow[dr, "\forall F"'] \arrow[dr, phantom, shift left=4, "\cong"] & \mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}} \arrow[d, dotted, "\exists!\,\overline{F}"] \\
& \mathcal{D}
\end{tikzcd}$$

Now, if we consider a $\mathcal{T}$-enrichment on $\mathcal{C}$ for a Lawvere theory $\mathcal{T}$, we can build one on $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$. Indeed, for each n-ary term t of $\mathcal{T}$ and object (A,R) of $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$, we consider the map

$$(A,R)^n = (A^n, R^n) \xrightarrow{R\circ(1_{A^n}, t^A)} (A,R)$$

where R^n denotes here the equivalence relation given by the product

$$R \times \cdots \times R \rightarrowtail A^2 \times \cdots \times A^2 \cong A^n \times A^n.$$

One can prove this defines a $\mathcal{T}$-enrichment on $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ such that $i\colon \mathcal{C} \hookrightarrow \mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ is a $\mathcal{T}$-functor. Moreover, this makes $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ the exact $\mathcal{T}$-completion of $\mathcal{C}$, in the sense that, with the notations of Proposition 4.14, if $\mathcal{D}$ and F are $\mathcal{T}$-enriched, $\overline{F}$ is also $\mathcal{T}$-enriched. We now need a few results in order to get our embedding theorem in the exact context.

Lemma 4.15. [59] Let (M,X) be an extended matrix of terms in the Lawvere theory $\mathcal{T}$ as in (6). Let also $r\colon R \rightarrowtail A^a$ be an a-ary relation in the regular $\mathcal{T}$-category $\mathcal{C}$. If $p\colon B \twoheadrightarrow A$ is a regular epimorphism and if we consider the pullback

$$\begin{tikzcd}
S \arrow[r, two heads, "q"] \arrow[d, tail, "s"'] \arrow[dr, phantom, very near start, "\lrcorner"] & R \arrow[d, tail, "r"] \\
B^a \arrow[r, two heads, "p^a"'] & A^a
\end{tikzcd}$$

then R is (M,X)-closed if and only if S is (M,X)-closed.

Proof. We are going to use Proposition 2.43. Let us first suppose that

R is (M, X)-closed and let $(y_1, \dots, y_l)\colon Y \to B^l$ be such that

$$(t_{1j}(y_1, \dots, y_l), \dots, t_{aj}(y_1, \dots, y_l)) = sv_j\colon Y \to B^a$$

for some $v_1, \dots, v_b\colon Y \to S$. Thus,

$$\begin{aligned} &(t_{1j}(py_1, \dots, py_l), \dots, t_{aj}(py_1, \dots, py_l)) \\ &= p^a(t_{1j}(y_1, \dots, y_l), \dots, t_{aj}(y_1, \dots, y_l)) \\ &= p^a sv_j \\ &= rqv_j \end{aligned}$$

factors through r. Since R is (M, X)-closed, there is a regular epimorphism $p'\colon Z \twoheadrightarrow Y$ and morphisms $z_{l+1}, \dots, z_k\colon Z \to A$ such that

$$\begin{aligned} &(u_{1j}(py_1p', \dots, py_lp', z_{l+1}, \dots, z_k), \dots, u_{aj}(py_1p', \dots, py_lp', z_{l+1}, \dots, z_k)) \\ &= rw_j \end{aligned}$$

for some $w_1, \dots, w_{b'}\colon Z \to R$. Now, we consider the pullback

$$\begin{tikzcd} Z' \arrow[r, "(z'_{l+1}, \dots, z'_k)"] \arrow[d, two heads, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^{k-l} \arrow[d, two heads, "p^{k-l}"] \\ Z \arrow[r, "(z_{l+1}, \dots, z_k)"'] & A^{k-l} \end{tikzcd}$$

and we prove that the required property is satisfied with the regular epimorphism $p'q'\colon Z' \twoheadrightarrow Y$ and the morphisms $z'_{l+1}, \dots, z'_k\colon Z' \to B$. In view of the definition of s, we only have to notice that

$$\begin{aligned} &p^a(u_{1j}(y_1p'q', \dots, y_lp'q', z'_{l+1}, \dots, z'_k), \dots \\ &\qquad\qquad \dots, u_{aj}(y_1p'q', \dots, y_lp'q', z'_{l+1}, \dots, z'_k)) \\ &= (u_{1j}(py_1p'q', \dots, py_lp'q', z_{l+1}q', \dots, z_kq'), \dots \\ &\qquad\qquad \dots, u_{aj}(py_1p'q', \dots, py_lp'q', z_{l+1}q', \dots, z_kq')) \\ &= rw_jq' \end{aligned}$$

factors through r for each $j \in \{1, \dots, b'\}$.

Conversely, let us suppose S is (M, X)-closed and consider a morphism $(y_1, \dots, y_l)\colon Y \to A^l$ such that

$$(t_{1j}(y_1, \dots, y_l), \dots, t_{aj}(y_1, \dots, y_l)) = rv_j\colon Y \to A^a$$

for some $v_1, \dots, v_b\colon Y \to R$. We also consider the following pullback.

$$\begin{tikzcd}
Y' \arrow[r, "{(y'_1,\dots,y'_l)}"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^l \arrow[d, "p^l"] \\
Y \arrow[r, "{(y_1,\dots,y_l)}"'] & A^l
\end{tikzcd}$$

Since

$$\begin{aligned}
&p^a(t_{1j}(y'_1, \dots, y'_l), \dots, t_{aj}(y'_1, \dots, y'_l)) \\
&= (t_{1j}(y_1q', \dots, y_lq'), \dots, t_{aj}(y_1q', \dots, y_lq')) \\
&= rv_jq'
\end{aligned}$$

factors through r, the morphism

$$(t_{1j}(y'_1, \dots, y'_l), \dots, t_{aj}(y'_1, \dots, y'_l))$$

factors through s for each $j \in \{1, \dots, b\}$. But S is (M, X)-closed, so there exists a regular epimorphism $p'\colon Z \twoheadrightarrow Y'$ and some morphisms $z_{l+1}, \dots, z_k\colon Z \to B$ such that

$$\begin{aligned}
&(u_{1j}(y'_1p', \dots, y'_lp', z_{l+1}, \dots, z_k), \dots, u_{aj}(y'_1p', \dots, y'_lp', z_{l+1}, \dots, z_k)) \\
&= sw_j
\end{aligned}$$

for some $w_1, \dots, w_{b'}\colon Z \to S$. Now, the required property is satisfied with the regular epimorphism $q'p'\colon Z \twoheadrightarrow Y$ and the morphisms $pz_{l+1}, \dots, pz_k\colon Z \to A$. Indeed,

$$\begin{aligned}
&(u_{1j}(y_1q'p', \dots, y_lq'p', pz_{l+1}, \dots, pz_k), \dots \\
&\qquad \dots, u_{aj}(y_1q'p', \dots, y_lq'p', pz_{l+1}, \dots, pz_k)) \\
&= p^a(u_{1j}(y'_1p', \dots, y'_lp', z_{l+1}, \dots, z_k), \dots, u_{aj}(y'_1p', \dots, y'_lp', z_{l+1}, \dots, z_k))
\end{aligned}$$

$$= p^a s w_j$$
$$= r q w_j$$

factors through r for each $j \in \{1, \dots, b'\}$. $\square$

Lemma 4.16. [59] Let (M, X) be an extended matrix of terms in the Lawvere theory $\mathcal{T}$ as in (6). Let also $r \colon R \rightarrowtail A^a$ be an a-ary relation in the well-powered regular $\mathcal{T}$-enriched category $\mathcal{C}$. This gives an a-ary relation $i(r) \colon i(R) \rightarrowtail i(A^a) \cong i(A)^a$ in $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$. Then, R is (M, X)-closed if and only if $i(R)$ is (M, X)-closed.

Proof. This comes from the fact that $i \colon \mathcal{C} \hookrightarrow \mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ is $\mathcal{T}$-enriched, conservative and regular. $\square$

It is proved in [46] that if $\mathcal{C}$ is a regular well-powered Mal'tsev category, then its exact completion $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ is also a Mal'tsev category. We now generalise this result for matrix conditions.

Proposition 4.17. [59] Let n be a natural number and (M_1, X_1),$\dots$, (M_n, X_n) and (M, X) be extended matrices of terms in the Lawvere theory $\mathcal{T}$ with the same number a of lines. Let also $\mathcal{C}$ be a well-powered regular $\mathcal{T}$-category. If every a-ary relation in $\mathcal{C}$ which is (M_i, X_i)-closed for each $i \in \{1, \dots, n\}$ is also (M, X)-closed, then the same occurs in $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$.

Proof. Let $r \colon R \rightarrowtail A^a$ be an a-ary relation in $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ which is (M_i, X_i)-closed for each $i \in \{1, \dots, n\}$. It is proved in [98] that there exists an object $B \in \mathcal{C}$ and a regular epimorphism $p \colon i(B) \twoheadrightarrow A$ in $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$. So, we can consider the following pullback.

$$\begin{tikzcd}
\bullet \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "r"] \\
i(B)^a \arrow[r, two heads, "p^a"'] & A^a
\end{tikzcd}$$

Moreover, it is also shown in [98], that with the embedding $i \colon \mathcal{C} \hookrightarrow \mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$, $\mathcal{C}$ is closed under subobjects in $\mathcal{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ (up to isomorphism).

Hence, we have the following pullback

$$\begin{tikzcd}
i(S) \arrow[r, two heads, "q"] \arrow[d, tail, "i(s)"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "r"] \\
i(B^a) \arrow[r, two heads, "p^a"'] & A^a
\end{tikzcd}$$

for some a-ary relation $s\colon S \rightarrowtail B^a$ in $\mathcal{C}$. Now, by Lemma 4.15, $i(S)$ is (M_i, X_i)-closed for each $i \in \{1, \dots, n\}$. By Lemma 4.16, S is (M_i, X_i)-closed for each $i \in \{1, \dots, n\}$. Thus, by the assumption on $\mathcal{C}$, S is (M, X)-closed. Again by Lemma 4.16, $i(S)$ is (M, X)-closed and finally by Lemma 4.15, R is (M, X)-closed. $\square$

Corollary 4.18. [59] Let $\mathcal{T}$ be a commutative Lawvere theory and (M, X) an extended matrix of terms in $\mathcal{T}$. Then $\mathrm{Mod}(\Gamma_{(M,X)})_{\mathrm{ex/reg}}$ is an exact $\mathcal{T}$-category with (M, X)-closed relations.

Proof. Since subobjects of $\mathrm{Mod}(\Gamma_{(M,X)})$ are represented by its submodels, it is well-powered. Then $\mathrm{Mod}(\Gamma_{(M,X)})_{\mathrm{ex/reg}}$ is exact from Proposition 4.14 and has (M, X)-closed relations by Propositions 4.7 and 4.17. $\square$

With this in mind, we can state our embedding theorem in the exact context.

Theorem 4.19. [59] Let $\mathcal{T}$ be a commutative Lawvere theory, (M, X) an extended matrix of terms in $\mathcal{T}$ and $\mathcal{C}$ a small exact $\mathcal{T}$-category with (M, X)-closed relations. Let 1 be the terminal object of $\mathcal{C}$. Then, there exists a regular conservative $\mathcal{T}$-enriched embedding

$$\phi\colon \mathcal{C} \hookrightarrow (\mathrm{Mod}(\Gamma_{(M,X)})_{\mathrm{ex/reg}})^{\mathrm{Sub}(1)}$$

which preserves coequalisers of equivalence relations.

Proof. We just have to compose the embedding of Theorem 4.8 with the embedding $i^{\mathrm{Sub}(1)}\colon \mathrm{Mod}(\Gamma_{(M,X)})^{\mathrm{Sub}(1)} \hookrightarrow (\mathrm{Mod}(\Gamma_{(M,X)})_{\mathrm{ex/reg}})^{\mathrm{Sub}(1)}$. Notice that an equivalence relation is the kernel pair of its coequaliser in an exact category. This implies that the embedding preserves coequalisers of equivalence relations since it preserves kernel pairs and regular epimorphisms. □

Remark 4.20. Theorem 4.19 is stated in a way which characterises exact $\mathcal{T}$-categories with (M, X)-closed relations among small $\mathcal{T}$-categories with finite limits and coequalisers of equivalence relations. In an analogous way, Theorem 4.8 characterises regular $\mathcal{T}$-categories with (M, X)-closed relations among small $\mathcal{T}$-categories with finite limits and coequalisers of kernel pairs.

4.4 Embedding for protomodular categories

Protomodular categories have been introduced by D. Bourn in [17] as categories whose change of base functors $v^*\colon \operatorname{Pt}_I(\mathcal{C}) \to \operatorname{Pt}_J(\mathcal{C})$ of the fibration of points are conservative (see for instance [15] for a detailed account on the topic). As we will see, it is a key property to define homological [15] and semi-abelian [62] categories which are known to provide good contexts to develop homological algebra. In this section, we recall some well-known characterisations of protomodular categories and syntactically describe protomodular essentially algebraic categories. We then prove an embedding theorem for regular protomodular categories, in a similar way we did for regular $\mathcal{T}$-categories with (M, X)-closed relations. However, in order to prove protomodularity is a Th[Set]-unconditional exactness property, we need to assume the existence of some colimits, which will then be a hypothesis in our embedding theorem.

4.4.1 Protomodular categories

Definition 4.21. [17] A *protomodular category* is a category $\mathcal{C}$ with pullbacks of split epimorphisms along arbitrary morphisms such that, for each morphism $v\colon J \to I$ in $\mathcal{C}$, the change of base functor $v^*\colon \operatorname{Pt}_I(\mathcal{C}) \to \operatorname{Pt}_J(\mathcal{C})$ of the fibration of points is conservative.

Example 4.22. The categories Gp, Ab, LieAlg_k (for a field k), Heyt and $\mathrm{Set}^{\mathrm{op}}$ are protomodular, while Mon and Set are not.

Proposition 4.23. [17] Let $\mathcal{C}$ be a pointed category with finite limits. Then $\mathcal{C}$ is protomodular if and only if the Split Short Five Lemma holds in $\mathcal{C}$. This means for any diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "k"] \arrow[d, "f"'] & A \arrow[r, two heads, "q"] \arrow[d, "g"'] & Q \arrow[r] \arrow[l, bend left, "s"] \arrow[d, "h"] & 0 \\
0 \arrow[r] & K' \arrow[r, tail, "k'"'] & A' \arrow[r, two heads, "q'"] & Q' \arrow[r] \arrow[l, bend left, "s'"] & 0
\end{tikzcd}$$

in $\mathcal{C}$ where $gk = k'f$, $hq = q'g$, $gs = s'h$, $qs = 1_Q$, $q's' = 1_{Q'}$ and k (resp. k') is the kernel of q (resp. q'), if f and h are isomorphisms, then so is g.

In a regular context, this is even equivalent to the Regular Short Five Lemma.

Definition 4.24. [15] A *homological category* is a pointed regular protomodular category.

Theorem 4.25. [15] A pointed regular category $\mathcal{C}$ is homological if and only if the Regular Short Five Lemma holds in $\mathcal{C}$. This means that, given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "k"] \arrow[d, "f"'] & A \arrow[r, two heads, "q"] \arrow[d, "g"'] & Q \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & K' \arrow[r, tail, "k'"'] & A' \arrow[r, two heads, "q'"'] & Q' \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{C}$ where q and q' are regular epimorphisms with k and k' their respective kernel, if f and h are isomorphisms, then so is g.

The above characterisations do not help to 'make protomodularity look like a Th[Set]-unconditional exactness property'. The following one gets it closer.

Proposition 4.26. [19] Let $\mathcal{C}$ be a finitely complete category. Then $\mathcal{C}$ is protomodular if and only if, for each morphism $(u, v)\colon (p, s) \to (p', s')$

in $\mathrm{Pt}(\mathcal{C})$ for which the square $p'u = vp$ is a pullback,

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, shift right, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, shift right, "p'"'] \\
I \arrow[u, shift right, "s"'] \arrow[r, "v"'] & I' \arrow[u, shift right, "s'"']
\end{tikzcd}$$

the morphisms u and s' are jointly strongly epimorphic.

If $\mathcal{C}$ has binary coproducts, this is equivalent to saying that the morphism $\left(\begin{smallmatrix} u \\ s' \end{smallmatrix}\right) : A + I' \to A'$ is a strong epimorphism. Or, if the pushout of s along v exists and if q denotes the factorisation of the pair (u, s') through it, then this pair is jointly strongly epimorphic if and only if q is a strong epimorphism.

$$\begin{tikzcd}
A \arrow[rr, "u"] \arrow[dr] & & A' \\
& Q \arrow[ur, "q"'] & \\
B \arrow[uu, "s"] \arrow[rr, "v"'] \arrow[ur, phantom, "\llcorner", near end] & & B' \arrow[uu, "s'"'] \arrow[ul]
\end{tikzcd}$$

Indeed, q factors through a subobject of A' if and only if u and s' simultaneously do. If moreover $\mathcal{C}$ is regular, this means in both cases that a morphism is required to be a regular epimorphism. This discussion leads us to the following proposition.

Proposition 4.27. Being 'regular protomodular with binary coproducts' and 'regular protomodular with pushouts along split monomorphisms' are Th[Set]-unconditional exactness properties.

Let us now give a syntactic characterisation of essentially algebraic categories. We first recall the one-sorted finitary algebraic case.

Theorem 4.28. [20] Let $\mathcal{T}$ be a Lawvere theory. Then $\mathcal{T}$-Alg is protomodular if and only if there exist in $\mathcal{T}$, for some natural number $n \geqslant 0$,

- n nullary terms $w_1, \dots, w_n$,
- for each $1 \leqslant i \leqslant n$, a binary term $d_i(x, y)$ such that $d_i(x, x) = w_i$ is a theorem of $\mathcal{T}$,

- an $(n+1)$-ary term π such that $\pi(d_1(x,y),\dots,d_n(x,y),y)=x$ is a theorem of $\mathcal{T}$.

Theorem 4.29. [57] Let Γ be an essentially algebraic theory. Then $\mathrm{Mod}(\Gamma)$ is protomodular if and only if, for each $s\in S$, there exists in Γ

- a term $\pi^s\colon \left(\prod_{i\in I} s_i\right)\times s\to s$,
- for each $i\in I$, an everywhere-defined term $d_i : s^2\to s_i$,
- for each $i\in I$, an everywhere-defined constant term w_i of sort s_i

such that

1. $d_i(x,x)=w_i$ is a theorem of Γ for each $i\in I$,
2. the term $s^2\to s$
$$\pi^s((d_i(x,y))_{i\in I},y)$$
is everywhere-defined,
3. the theorem
$$\pi^s((d_i(x,y))_{i\in I},y)=x$$
holds in Γ.

Proof. Firstly, let us suppose that the conditions in the statement hold in Γ, and let us prove $\mathrm{Mod}(\Gamma)$ is protomodular. So, we consider a morphism f in $\mathrm{Pt}_B(\mathrm{Mod}(\Gamma))$. This yields a diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, shift right, "p"'] & & A' \arrow[dl, shift left, "q"] \\
& B \arrow[ul, shift right, "t"'] \arrow[ur, shift left, "u"] &
\end{tikzcd}$$

with $pt=1_B=qu$, $qf=p$ and $ft=u$. We also consider a morphism $v\colon B'\to B$ such that the image f' of f by the change of base functor v^*

$$\begin{tikzcd}
A\times_{p,v}B' \arrow[rr, "f'"] \arrow[dr, shift right, "p'"'] & & A'\times_{q,v}B' \arrow[dl, shift left, "q'"] \\
& B' \arrow[ul, shift right, "t'"'] \arrow[ur, shift left, "u'"] &
\end{tikzcd}$$

is an isomorphism. We have to prove that f is also an isomorphism. Let us first prove it is a monomorphism. So, let $s \in S$ and $a, a' \in A_s$ be such that $f(a) = f(a')$. We also consider the terms given in the statement for s. For each $i \in I$, we have

$$\begin{aligned} p(d_i(a,a')) &= q(d_i(f(a), f(a'))) \\ &= q(d_i(f(a), f(a))) \\ &= q(w_i) \\ &= w_i \end{aligned}$$

and $(d_i(a,a'), w_i) \in (A \times_{p,v} B')_{s_i}$. Moreover,

$$\begin{aligned} f'(d_i(a,a'), w_i) &= (f(d_i(a,a')), w_i) \\ &= (d_i(f(a), f(a')), w_i) \\ &= (w_i, w_i) \\ &= f'(w_i, w_i) \end{aligned}$$

and $d_i(a,a') = w_i = d_i(a', a')$ since f'_{s_i} is injective. Therefore, we have

$$\begin{aligned} a &= \pi^s((d_i(a,a'))_{i \in I}, a') \\ &= \pi^s((d_i(a',a'))_{i \in I}, a') \\ &= a' \end{aligned}$$

and f_s is injective. Now, we show that $\mathrm{Im}(f)_s = A'_s$. So, let $c \in A'_s$. For each $i \in I$, we know that

$$\begin{aligned} q(d_i(c, ftq(c))) &= d_i(q(c), qftq(c)) \\ &= d_i(q(c), q(c)) \\ &= w_i \end{aligned}$$

from which $(d_i(c, ftq(c)), w_i) \in (A' \times_{q,v} B')_{s_i}$. Since f'_{s_i} is bijective, there exists an element $a_i \in A_{s_i}$ such that $(a_i, w_i) \in (A \times_{p,v} B')_{s_i}$ (i.e., $p(a_i) = w_i$) and $f(a_i) = d_i(c, ftq(c))$. Therefore, we can say that

$$c = \pi^s((d_i(c, ftq(c)))_{i \in I}, ftq(c))$$

$$\begin{aligned} &= \pi^s((f(a_i))_{i\in I}, ftq(c)) \\ &\in \operatorname{Im}(f)_s \end{aligned}$$

and f is an isomorphism.

Conversely, let us suppose $\operatorname{Mod}(\Gamma)$ is protomodular and let $s \in S$. Let also X and Y be the S-sorted sets defined by $X_s = \{x_1, x_2\}$, $Y_s = \{y\}$ and $X_{s'} = \varnothing = Y_{s'}$ for each $s' \neq s$. We consider the diagram

$$\begin{tikzcd}
& \operatorname{Fr}_\Gamma(X) \times_{p,!} \operatorname{Fr}_\Gamma(\varnothing) \arrow[r, two heads, "p_2"] \arrow[dl, two heads] \arrow[dd, "p_1"] \arrow[ddr, phantom, "\lrcorner", very near start] & \operatorname{Fr}_\Gamma(\varnothing) \arrow[dd, "!"] \\
\operatorname{Im}(p_1) \arrow[dr, tail] & & \\
& \operatorname{Fr}_\Gamma(X) \arrow[r, two heads, shift right=1, "p"'] & \operatorname{Fr}_\Gamma(Y) \arrow[l, shift right=1, "t"']
\end{tikzcd}$$

where the square is a pullback and p and t are defined by $p(x_1) = p(x_2) = y$ and $t(y) = x_2$. Since $pt = 1_{\operatorname{Fr}_\Gamma(Y)}$, t is a monomorphism and we can see $\operatorname{Fr}_\Gamma(Y)$ as a submodel of $\operatorname{Fr}_\Gamma(X)$. We write $\operatorname{Im}(p_1) \vee \operatorname{Fr}_\Gamma(Y)$ for the smallest submodel of $\operatorname{Fr}_\Gamma(X)$ which contains $\operatorname{Im}(p_1) \cup \operatorname{Fr}_\Gamma(Y)$. It is routine to prove it is described by

$$\begin{aligned} &(\operatorname{Im}(p_1) \vee \operatorname{Fr}_\Gamma(Y))_{s'} = \\ &\Big\{\tau((p_1(z_i))_{i\in I}, x_2) \,|\, \tau\colon \Big(\prod_{i\in I} s_i\Big) \times s \to s' \text{ is a term in } \Gamma, \\ &\qquad\qquad z_i \in (\operatorname{Fr}_\Gamma(X) \times_{p,!} \operatorname{Fr}_\Gamma(\varnothing))_{s_i} \\ &\qquad\qquad \text{and } \tau((p_1(z_i))_{i\in I}, x_2) \text{ is defined in } \operatorname{Fr}_\Gamma(X)\Big\} \end{aligned}$$

for each $s' \in S$. We have thus a morphism of points in the fibre over $\operatorname{Fr}_\Gamma(Y)$ in $\operatorname{Mod}(\Gamma)$:

$$\begin{tikzcd}
\operatorname{Im}(p_1) \vee \operatorname{Fr}_\Gamma(Y) \arrow[rr, hook, "i"] \arrow[dr, two heads, shift right=1, "pi"'] & & \operatorname{Fr}_\Gamma(X) \arrow[dl, two heads, shift left=1, "p"] \\
& \operatorname{Fr}_\Gamma(Y) \arrow[ul, shift right=1, "t"'] \arrow[ur, shift left=1, "t"] &
\end{tikzcd}$$

By construction, its image by the change of base functor along the unique morphism $!\colon \mathrm{Fr}_\Gamma(\varnothing) \to \mathrm{Fr}_\Gamma(Y)$ is the pullback of i along p_1, which is an isomorphism since p_1 factors through i. $\mathrm{Mod}(\Gamma)$ being protomodular, i is an isomorphism as well and $x_1 \in (\mathrm{Im}(p_1) \vee \mathrm{Fr}_\Gamma(Y))_s$. In view of the description of $(\mathrm{Im}(p_1) \vee \mathrm{Fr}_\Gamma(Y))_s$, we have a term

$$\pi^s\colon \left(\prod_{i\in I} s_i\right) \times s \to s$$

and elements $z_i \in (\mathrm{Fr}_\Gamma(X) \times_{p,!} \mathrm{Fr}_\Gamma(\varnothing))_{s_i}$ (for $i \in I$) such that

$$\pi^s((p_1(z_i))_{i\in I}, x_2)$$

is defined in $\mathrm{Fr}_\Gamma(X)$ and equal to x_1. Now, considering the description of $\mathrm{Fr}_\Gamma(X) \times_{p,!} \mathrm{Fr}_\Gamma(\varnothing)$, there exist, for each $i \in I$, two everywhere-defined terms $d_i\colon s^2 \to s_i$ and $w_i\colon 1 \to s_i$ such that $z_i = (d_i, w_i)$ and $p(d_i) = !(w_i)$. We thus got all the terms and theorems we were looking for. $\square$

Let us make the above theorem explicit in the case where $\mathrm{Mod}(\Gamma)$ is pointed (see Corollary 1.108).

Corollary 4.30. Let Γ be an essentially algebraic theory such that $\mathrm{Mod}(\Gamma)$ is pointed. Then $\mathrm{Mod}(\Gamma)$ is protomodular if and only if, for each $s \in S$, there exists a term $\pi^s\colon \left(\prod_{i\in I} s_i\right) \times s \to s$ in Γ and, for each $i \in I$, an everywhere-defined term $d_i\colon s^2 \to s_i$ such that

1. $d_i(x,x) = 0^{s_i}$ is a theorem of Γ for each $i \in I$,
2. the term $\pi^s((d_i(x,y))_{i\in I}, y)$ is everywhere-defined,
3. $\pi^s((d_i(x,y))_{i\in I}, y) = x$ is a theorem of Γ.

Again, if Γ is finitary, the term π^s from Theorem 4.29 and Corollary 4.30 can be supposed to be finitary.

Analogously to the case of (M,X)-closed relations, we can characterise protomodularity using approximate co-operations. The pointed case has been proved in [22] while the general case is from [23].

Theorem 4.31. [23] Let $\mathcal{C}$ be a finitely complete category with finite coproducts. Then $\mathcal{C}$ is protomodular if and only if, for each $Y \in \mathcal{C}$, the morphism

$$\left(\begin{smallmatrix} d^Y \\ \iota_2 \end{smallmatrix}\right) : W(Y) + Y \to Y + Y$$

is a strong epimorphism where the square

$$\begin{tikzcd}
W(Y) \arrow[r, "d^Y"] \arrow[d, "w^Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y + Y \arrow[d, "\left(\begin{smallmatrix} 1_Y \\ 1_Y \end{smallmatrix}\right)"] \\
0 \arrow[r, "!"'] & Y
\end{tikzcd}$$

is a pullback, 0 the initial object and $\iota_2 \colon Y \to Y + Y$ the second coproduct injection.

4.4.2 Construction of $\Gamma^{\mathcal{T}}_{\text{proto}}$

As we did for matrix conditions, we now construct a finitary essentially algebraic theory whose category of models will be our 'representative regular protomodular category'. In order to encompass at the same time the homological case, we do it in the $\mathcal{T}$-enriched context. So let $\mathcal{T}$ be a commutative Lawvere theory. As before, we suppose it is of the form $\mathcal{T}_{(\Sigma,E)}$ for some one-sorted finitary algebraic theory (Σ, E). An operation symbol (resp. an axiom) of $\mathcal{T}$ is thus an element of Σ (resp. E). If r is a natural number, we denote by $\Sigma^{\mathcal{T}}_r$ the set of r-ary operation symbols of $\mathcal{T}$. We are going to construct recursively a series of finitary essentially algebraic theories

$$\Gamma^0 \subseteq \Delta^1 \subseteq \cdots \subseteq \Gamma^n \subseteq \Delta^{n+1} \subseteq \cdots$$

and a $\mathcal{T}$-enrichment on the corresponding categories of models. Let us first define $\Gamma^0 = (S^0, \Sigma^0, E^0, \Sigma^0_t, \mathrm{Def}^0)$:

- $S^0 = \{\star\}$,
- $\Sigma^0 = \Sigma^0_t = \{\tau^\star \colon \star^r \to \star \,|\, r \in \mathbb{N}, \tau \in \Sigma^{\mathcal{T}}_r\}$,
- $E^0 = \{$all axioms from $\mathcal{T}$ for the $\tau^\star$'s$\}$.

We consider the obvious $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma^0) \cong \mathcal{T}$-Alg. Now, let us suppose we have defined

$$\Gamma^0 \subseteq \Delta^1 \subseteq \cdots \subseteq \Delta^n \subseteq \Gamma^n$$

and the $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma^n)$ (with $\Gamma^n = (S^n, \Sigma^n, E^n, \Sigma^n_t, \mathrm{Def}^n)$). We are going to construct

$$\Delta^{n+1} = (S'^{n+1}, \Sigma'^{n+1}, E'^{n+1}, \Sigma'^{n+1}_t, \mathrm{Def}'^{n+1})$$

first (below $\overline{S}^0 = S^0$ and $\overline{S}^n = S^n \setminus S^{n-1}$ if $n \geqslant 1$):

$$S'^{n+1} = S^n \cup \{(s,0),(s,1) \,|\, s \in \overline{S}^n\} \cong S^n \sqcup \overline{S}^n \sqcup \overline{S}^n,$$

$$\begin{aligned}\Sigma'^{n+1}_t = \Sigma^n_t &\cup \{\tau^{(s,0)} \colon (s,0)^r \to (s,0) \,|\, r \in \mathbb{N}, \tau \in \Sigma^{\mathcal{T}}_r, s \in \overline{S}^n\}\\ &\cup \{\tau^{(s,1)} \colon (s,1)^r \to (s,1) \,|\, r \in \mathbb{N}, \tau \in \Sigma^{\mathcal{T}}_r, s \in \overline{S}^n\}\\ &\cup \{\delta^s \colon s^2 \to (s,0) \,|\, s \in \overline{S}^n\}\\ &\cup \{\omega^{(s,0)} \text{ constant operation symbol on } (s,0) \,|\, s \in \overline{S}^n\}\\ &\cup \{\eta^s, \varepsilon^s \colon (s,0) \times s \to (s,1) \,|\, s \in \overline{S}^n\},\end{aligned}$$

$$\Sigma'^{n+1} = \Sigma^n \cup \Sigma'^{n+1}_t \cup \{\pi^s \colon (s,0) \times s \to s \,|\, s \in \overline{S}^n\},$$

$$\begin{aligned}&E'^{n+1} =\\ &E^n \cup \{\delta^s(x,x) = \omega^{(s,0)} \,|\, s \in \overline{S}^n\}\\ &\quad \cup \{\eta^s(\delta^s(x,y),y) = \varepsilon^s(\delta^s(x,y),y) \,|\, s \in \overline{S}^n\}\\ &\quad \cup \{\pi^s(\delta^s(x,y),y) = x \,|\, s \in \overline{S}^n\}\\ &\quad \cup \{\delta^s(\pi^s(x,y),y) = x \,|\, s \in \overline{S}^n\}\\ &\quad \cup \{\text{all axioms from } \mathcal{T} \text{ for the } \tau^{(s,0)}\text{'s and the } \tau^{(s,1)}\text{'s} \,|\, s \in \overline{S}^n\}\\ &\quad \cup \{\tau^{(s,0)}(\delta^s(x_1,y_1),\dots,\delta^s(x_r,y_r))\\ &\qquad = \delta^s(\tau^s(x_1,\dots,x_r),\tau^s(y_1,\dots,y_r)) \,|\, r \in \mathbb{N}, \tau \in \Sigma^{\mathcal{T}}_r, s \in \overline{S}^n\}\\ &\quad \cup \{\tau^{(s,0)}(\omega^{(s,0)},\dots,\omega^{(s,0)}) = \omega^{(s,0)} \,|\, r \in \mathbb{N}, \tau \in \Sigma^{\mathcal{T}}_r, s \in \overline{S}^n\}\\ &\quad \cup \{\tau^{(s,1)}(\eta^s(x_1,y_1),\dots,\eta^s(x_r,y_r))\\ &\qquad = \eta^s(\tau^{(s,0)}(x_1,\dots,x_r),\tau^s(y_1,\dots,y_r)) | \, r \in \mathbb{N}, \tau \in \Sigma^{\mathcal{T}}_r, s \in \overline{S}^n\}\end{aligned}$$

$$\begin{aligned}
&\cup \{\tau^{(s,1)}(\varepsilon^s(x_1,y_1),\dots,\varepsilon^s(x_r,y_r))\\
&\qquad = \varepsilon^s(\tau^{(s,0)}(x_1,\dots,x_r),\tau^s(y_1,\dots,y_r)) \,|\, r\in\mathbb{N},\tau\in\Sigma_r^{\mathcal{T}}, s\in\overline{S}^n\}\\
&\cup \{\tau^{s}(\pi^s(x_1,y_1),\dots,\pi^s(x_r,y_r))\\
&\qquad = \pi^s(\tau^{(s,0)}(x_1,\dots,x_r),\tau^s(y_1,\dots,y_r)) \,|\, r\in\mathbb{N},\tau\in\Sigma_r^{\mathcal{T}}, s\in\overline{S}^n\}
\end{aligned}$$

and
$$\begin{cases}
\mathrm{Def}'^{n+1}(\sigma) = \mathrm{Def}^n(\sigma) \text{ if } \sigma\in\Sigma^n\setminus\Sigma_t^n\\
\mathrm{Def}'^{n+1}(\pi^s) = \{\eta^s(x)=\varepsilon^s(x)\} \text{ for } s\in\overline{S}^n.
\end{cases}$$

Hence, we have $\Gamma^n \subseteq \Delta^{n+1}$ and we consider the obvious $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Delta^{n+1})$.

Let now T^{n+1} be the set of finitary terms $\theta\colon \prod_{i=1}^m s_i \to s$ of Σ'^{n+1} which are not terms of Σ'^n (where we consider $\Sigma'^0=\varnothing$). We then define Γ^{n+1} as:

$$S^{n+1} = S'^{n+1} \cup \{s_\theta, s'_\theta \,|\, \theta\in T^{n+1}\} \cong S'^{n+1} \sqcup T^{n+1} \sqcup T^{n+1},$$

$$\begin{aligned}
\Sigma_t^{n+1} = \Sigma_t'^{n+1} &\cup \{\tau^{s_\theta}\colon s_\theta^r \to s_\theta \,|\, r\in\mathbb{N},\tau\in\Sigma_r^{\mathcal{T}},\theta\in T^{n+1}\}\\
&\cup \{\tau^{s'_\theta}\colon (s'_\theta)^r \to s'_\theta \,|\, r\in\mathbb{N},\tau\in\Sigma_r^{\mathcal{T}},\theta\in T^{n+1}\}\\
&\cup \{\alpha_\theta\colon s\to s_\theta \,|\, \theta\colon \prod_{i=1}^m s_i \to s\in T^{n+1}\}\\
&\cup \{\mu_\theta\colon \prod_{i=1}^m s_i \to s_\theta \,|\, \theta\colon \prod_{i=1}^m s_i \to s\in T^{n+1}\}\\
&\cup \{\eta_\theta,\varepsilon_\theta\colon s_\theta\to s'_\theta \,|\, \theta\in T^{n+1}\},
\end{aligned}$$

$$\Sigma^{n+1} = \Sigma'^{n+1}\cup\Sigma_t^{n+1}\cup\{\pi_\theta\colon s_\theta\to s \,|\, \theta\colon \prod_{i=1}^m s_i\to s\in T^{n+1}\},$$

$$\begin{aligned}
&E^{n+1} =\\
&E'^{n+1} \cup \{\eta_\theta(\alpha_\theta(x)) = \varepsilon_\theta(\alpha_\theta(x)) \,|\, \theta\in T^{n+1}\}\\
&\qquad \cup \{\pi_\theta(\alpha_\theta(x)) = x \,|\, \theta\in T^{n+1}\}\\
&\qquad \cup \{\alpha_\theta(\pi_\theta(x)) = x \,|\, \theta\in T^{n+1}\}
\end{aligned}$$

$$
\begin{aligned}
&\cup \{\alpha_\theta(\theta(x_1,\dots,x_m)) = \mu_\theta(x_1,\dots,x_m) \,|\, \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\}\\
&\cup \{\text{all axioms from } \mathcal{T} \text{ for the } \tau^{s_\theta}\text{'s and the } \tau^{s'_\theta}\text{'s} \,|\, \theta \in T^{n+1}\}\\
&\cup \{\tau^{s_\theta}(\alpha_\theta(x_1),\dots,\alpha_\theta(x_r)) = \alpha_\theta(\tau^s(x_1,\dots,x_r)) \,|\\
&\qquad\qquad\qquad r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\}\\
&\cup \{\tau^{s_\theta}(\mu_\theta(x_{11},\dots,x_{1m}),\dots,\mu_\theta(x_{r1},\dots,x_{rm}))\\
&\quad = \mu_\theta(\tau^{s_1}(x_{11},\dots,x_{r1}),\dots,\tau^{s_m}(x_{1m},\dots,x_{rm})) \,|\\
&\qquad\qquad\qquad r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\}\\
&\cup \{\tau^{s'_\theta}(\eta_\theta(x_1),\dots,\eta_\theta(x_r)) = \eta_\theta(\tau^{s_\theta}(x_1,\dots,x_r)) \,|\\
&\qquad\qquad\qquad\qquad r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, \theta \in T^{n+1}\}\\
&\cup \{\tau^{s'_\theta}(\varepsilon_\theta(x_1),\dots,\varepsilon_\theta(x_r)) = \varepsilon_\theta(\tau^{s_\theta}(x_1,\dots,x_r)) \,|\\
&\qquad\qquad\qquad\qquad r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, \theta \in T^{n+1}\}\\
&\cup \{\tau^{s}(\pi_\theta(x_1),\dots,\pi_\theta(x_r)) = \pi_\theta(\tau^{s_\theta}(x_1,\dots,x_r)) \,|\\
&\qquad\qquad\qquad r \in \mathbb{N}, \tau \in \Sigma_r^{\mathcal{T}}, \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\}
\end{aligned}
$$

and

$$
\begin{cases}
\mathrm{Def}^{n+1}(\sigma) = \mathrm{Def}'^{n+1}(\sigma) \text{ if } \sigma \in \Sigma'^{n+1} \setminus \Sigma_t'^{n+1}\\
\mathrm{Def}^{n+1}(\pi_\theta) = \{\eta_\theta(x) = \varepsilon_\theta(x)\} \text{ for } \theta \in T^{n+1}.
\end{cases}
$$

Thus, we have $\Delta^{n+1} \subseteq \Gamma^{n+1}$ and we consider the obvious $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma^{n+1})$. This ends the recursive definition of the series

$$\Gamma^0 \subseteq \Delta^1 \subseteq \Gamma^1 \subseteq \cdots$$

and we set $\Gamma_{\mathrm{proto}}^{\mathcal{T}}$ to be the union of these finitary essentially algebraic theories. We provide $\mathrm{Mod}(\Gamma_{\mathrm{proto}}^{\mathcal{T}})$ with the $\mathcal{T}$-enrichment coming from the $\mathcal{T}$-enrichments on the $\mathrm{Mod}(\Gamma^n)$'s. Since they will be the most important cases, we denote $\Gamma_{\mathrm{proto}}^{\mathrm{Th}[\mathrm{Set}]}$ simply by Γ_{proto} and $\Gamma_{\mathrm{proto}}^{\mathrm{Th}[\mathrm{Set}_*]}$ by Γ_{homo}.

Proposition 4.32. [57] Let $\mathcal{T}$ be a commutative Lawvere theory. The $\mathcal{T}$-category $\mathrm{Mod}(\Gamma_{\mathrm{proto}}^{\mathcal{T}})$ is regular and protomodular.

Proof. The 'Δ part' of the construction makes $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\text{proto}})$ protomodular. Indeed, considering $\pi^s\colon (s,0)\times s \to s$, $d_1 = \delta^s$ and $w_1 = \omega^{(s,0)}$, $\Gamma^{\mathcal{T}}_{\text{proto}}$ satisfies the conditions of Theorem 4.29.

On the other hand, the 'Γ ingredient' of the construction ensures that $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\text{proto}})$ is a regular category since each finitary term θ of $\Sigma^{\mathcal{T}}_{\text{proto}}$ is in T^{n+1} for some $n \in \mathbb{N}$, which makes the conditions of Theorem 1.88 hold. $\square$

4.4.3 Proof of the embedding theorem

This subsection is devoted to the proof of our embedding theorem for protomodular categories. Since it is similar to the one for categories with (M,X)-closed relations, we only sketch it.

Theorem 4.33. [57] Let $\mathcal{T}$ be a commutative Lawvere theory and $\mathcal{C}$ a small regular protomodular $\mathcal{T}$-category with binary coproducts. Let 1 be the terminal object of $\mathcal{C}$. Then, there exists a regular conservative $\mathcal{T}$-enriched embedding $\phi\colon \mathcal{C} \hookrightarrow \mathrm{Mod}(\Gamma^{\mathcal{T}}_{\text{proto}})^{\mathrm{Sub}(1)}$. Moreover, for each morphism $f\colon C \to C'$ in $\mathcal{C}$, each $I \in \mathrm{Sub}(1)$ and each $s \in S^{\mathcal{T}}_{\text{proto}}$,

$$(\mathrm{Im}\,\phi(f)_I)_s = \{(\phi(f)_I)_s(x) \mid x \in (\phi(C)_I)_s\}.$$

Proof. By Proposition 4.27 and Theorem 3.21, $\widetilde{\mathcal{C}}$ is regular and protomodular. The proof is very similar to the one of Theorem 4.19, up to two differences. Borrowing its notations and the ones from Theorem 4.31, we now define, for $s \in \overline{S}^n$, $P_{(s,0)}$ as $\widehat{W(P_s)}$.

$$\begin{tikzcd}
P_{(s,0)} = \widehat{W(P_s)} \arrow[r, two heads, "e_{W(P_s)}"] & W(P_s) \arrow[r, "d^{P_s}"] \arrow[d, "w^{P_s}"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_s + P_s \arrow[d, "\left(\begin{smallmatrix} 1_{P_s} \\ 1_{P_s} \end{smallmatrix}\right)"] \\
 & 0 \arrow[r, "!"'] & P_s
\end{tikzcd}$$

We also define

$$\begin{aligned} \delta^s\colon \widetilde{\mathcal{C}}(P_s, C)^2 &\longrightarrow \widetilde{\mathcal{C}}(P_{(s,0)}, C) \\ (f,g) &\longmapsto \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) d^{P_s} e_{W(P_s)} \end{aligned}$$

and $\omega^{(s,0)} \in \widetilde{\mathcal{C}}(P_{(s,0)}, C)$ is the composite

$$P_{(s,0)} = \widehat{W(P_s)} \overset{e_{W(P_s)}}{\twoheadrightarrow} W(P_s) \xrightarrow{w^{P_s}} 0 \xrightarrow{!} C.$$

By Theorem 4.31, we can consider the regular epimorphism

$$p_s \colon P_{(s,0)} + P_s \overset{e_{W(P_s)}+1_{P_s}}{\twoheadrightarrow} W(P_s) + P_s \overset{\left(\begin{smallmatrix} d^{P_s} \\ \iota_2 \end{smallmatrix}\right)}{\twoheadrightarrow} P_s + P_s$$

and define

$$\begin{aligned} \pi^s \colon \widetilde{\mathcal{C}}(P_{(s,0)}, C) \times \widetilde{\mathcal{C}}(P_s, C) &\longrightarrow \widetilde{\mathcal{C}}(P_s, C) \\ (f,g) &\longmapsto h\iota_1 \end{aligned}$$

where $h \colon P_s + P_s \to C$ is the unique morphism such that $hp_s = \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ (i.e., $hd^{P_s}e_{W(P_s)} = f$ and $h\iota_2 = g$). We define this π^s if and only if such an h exists. In order to construct $P_{(s,1)}$, we consider the kernel pair (r_1, r_2) of p_s:

$$\begin{tikzcd} P_{(s,1)} = \widehat{R} \arrow[r, two heads, "e_R"] & R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & P_{(s,0)} + P_s \arrow[r, two heads, "p_s"] & P_s + P_s \end{tikzcd}$$

and set $P_{(s,1)} = \widehat{R}$. We then define

$$\begin{aligned} \eta^s \colon \widetilde{\mathcal{C}}(P_{(s,0)}, C) \times \widetilde{\mathcal{C}}(P_s, C) &\longrightarrow \widetilde{\mathcal{C}}(P_{(s,1)}, C) \\ (f,g) &\longmapsto \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) r_1 e_R \end{aligned}$$

and

$$\begin{aligned} \varepsilon^s \colon \widetilde{\mathcal{C}}(P_{(s,0)}, C) \times \widetilde{\mathcal{C}}(P_s, C) &\longrightarrow \widetilde{\mathcal{C}}(P_{(s,1)}, C) \\ (f,g) &\longmapsto \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) r_2 e_R. \end{aligned}$$

The second difference with the proof of Theorem 4.19 is the following. If $\pi^s \colon (s,0) \times s \to s \in \Sigma'^{n+1} \setminus \Sigma_t'^{n+1}$ and $\theta_1 \colon \prod_{i=1}^m s_i \to (s,0)$, $\theta_2 \colon \prod_{i=1}^m s_i \to s$ are finitary terms of Σ'^{n+1} for which $l_{\mu_{\theta_1}}$, $l_{\mu_{\theta_2}}$, $l_{\alpha_{\theta_1}}$ and $l_{\alpha_{\theta_2}}$ have been constructed, we define l_{μ_θ} and l_{α_θ} for the term

$$\theta = \pi^s(\theta_1, \theta_2) \colon \prod_{i=1}^m s_i \to s$$

as follows. We consider the diagram below where the rectangle is a pullback.

$$\begin{tikzcd}
P_{s_\theta} = \widehat{U} \arrow[r, two heads, "e_U"] & U \arrow[r, "u_1"] \arrow[dd, two heads, "u_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & P_{s_{\theta_1}} + P_{s_{\theta_2}} \arrow[r, "\begin{pmatrix} l_{\mu_{\theta_1}} \\ l_{\mu_{\theta_2}} \end{pmatrix}"] \arrow[d, two heads, "l_{\alpha_{\theta_1}} + l_{\alpha_{\theta_2}}"] & P_{s_1} + \cdots + P_{s_m} \\
& & P_{(s,0)} + P_s \arrow[d, two heads, "p_s"] & \\
& P_s \arrow[r, "\iota_1"'] & P_s + P_s &
\end{tikzcd}$$

We then set $P_{s_\theta} = \widehat{U}$, $l_{\alpha_\theta} = u_2 e_U$ and

$$l_{\mu_\theta} = \begin{pmatrix} l_{\mu_{\theta_1}} \\ l_{\mu_{\theta_2}} \end{pmatrix} u_1 e_U .$$

Let us prove it satisfies the equality

$$\alpha_\theta(\theta(f_1, \ldots, f_m)) = \mu_\theta(f_1, \ldots, f_m)$$

for any cospan $(f_i \colon P_{s_i} \to C)_{i \in \{1,\ldots,m\}}$ for which $\theta(f_1, \ldots, f_m)$ is defined, assuming the similar property for θ_1 and θ_2. Thus, for such a cospan, $\theta_1(f_1, \ldots, f_m) \colon P_{(s,0)} \to C$ and $\theta_2(f_1, \ldots, f_m) \colon P_s \to C$ are defined. Therefore,

$$\begin{aligned}
\theta_1(f_1, \ldots, f_m) l_{\alpha_{\theta_1}} &= \alpha_{\theta_1}(\theta_1(f_1, \ldots, f_m)) \\
&= \mu_{\theta_1}(f_1, \ldots, f_m) \\
&= \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} l_{\mu_{\theta_1}}
\end{aligned}$$

and

$$\begin{aligned}
\theta_2(f_1, \ldots, f_m) l_{\alpha_{\theta_2}} &= \alpha_{\theta_2}(\theta_2(f_1, \ldots, f_m)) \\
&= \mu_{\theta_2}(f_1, \ldots, f_m) \\
&= \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} l_{\mu_{\theta_2}} .
\end{aligned}$$

Moreover, since $\pi^s(\theta_1(f_1,\dots,f_m),\theta_2(f_1,\dots,f_m))$ is defined, there exists $h\colon P_s+P_s\to C$ such that

$$hp_s = \begin{pmatrix} \theta_1(f_1,\dots,f_m) \\ \theta_2(f_1,\dots,f_m) \end{pmatrix}.$$

It remains to compute

$$\begin{aligned}
\alpha_\theta(\theta(f_1,\dots,f_m)) &= \pi^s(\theta_1(f_1,\dots,f_m),\theta_2(f_1,\dots,f_m))l_{\alpha_\theta} \\
&= h\iota_1 u_2 e_U \\
&= hp_s(l_{\alpha_{\theta_1}} + l_{\alpha_{\theta_2}})u_1 e_U \\
&= \begin{pmatrix} \theta_1(f_1,\dots,f_m) \\ \theta_2(f_1,\dots,f_m) \end{pmatrix} (l_{\alpha_{\theta_1}} + l_{\alpha_{\theta_2}})u_1 e_U \\
&= \begin{pmatrix} \theta_1(f_1,\dots,f_m)l_{\alpha_{\theta_1}} \\ \theta_2(f_1,\dots,f_m)l_{\alpha_{\theta_2}} \end{pmatrix} u_1 e_U \\
&= \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \begin{pmatrix} l_{\mu_{\theta_1}} \\ l_{\mu_{\theta_2}} \end{pmatrix} u_1 e_U \\
&= \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} l_{\mu_\theta} \\
&= \mu_\theta(f_1,\dots,f_m).
\end{aligned}$$

The rest of the proof goes as before. □

The assumption about binary coproducts in Theorem 4.33 is only used to prove that $\widetilde{\mathcal{C}}$ is also protomodular. If one has another condition on the small regular protomodular $\mathcal{T}$-category $\mathcal{C}$ which also implies that $\widetilde{\mathcal{C}}$ is protomodular, such an embedding will also exist. In view of Proposition 4.27, we thus also have the following theorem.

Theorem 4.34. [57] Let $\mathcal{T}$ be a commutative Lawvere theory and $\mathcal{C}$ a small regular protomodular $\mathcal{T}$-category with pushouts along split monomorphisms. Let 1 be the terminal object of $\mathcal{C}$. Then, there exists a regular conservative $\mathcal{T}$-enriched embedding $\phi\colon \mathcal{C} \hookrightarrow \mathrm{Mod}(\Gamma^{\mathcal{T}}_{\mathrm{proto}})^{\mathrm{Sub}(1)}$. Moreover, for each morphism $f\colon C\to C'$ in $\mathcal{C}$, each $I\in\mathrm{Sub}(1)$ and each $s\in S^{\mathcal{T}}_{\mathrm{proto}}$,

$$(\mathrm{Im}\,\phi(f)_I)_s = \{(\phi(f)_I)_s(x)\,|\,x\in(\phi(C)_I)_s\}.$$

4.4.4 The semi-abelian case

Since for a pointed regular category $\mathcal{C}$, $\mathrm{Sub}(0)$ is reduced to the singleton, one gets from Theorem 4.33 the following corollary.

Corollary 4.35. [57] Let $\mathcal{C}$ be a small homological category with binary coproducts. Then, there exists a regular conservative embedding $\mathcal{C} \hookrightarrow \mathrm{Mod}(\Gamma_{\mathrm{homo}})$.

Proof. Let $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$ in Theorem 4.33. ☐

This corollary is very close to being an embedding theorem for semi-abelian categories. The only missing part is the exactness, which can be brought in via the exact completion as in Subsection 4.3.4.

Definition 4.36. [62] A *semi-abelian category* is an exact homological category with binary coproducts.

Theorem 4.37. [57] The category $\mathrm{Mod}(\Gamma_{\mathrm{homo}})_{\mathrm{ex\,/\,reg}}$ is exact and homological. Moreover, each small semi-abelian category $\mathcal{C}$ admits a regular conservative embedding $\mathcal{C} \hookrightarrow \mathrm{Mod}(\Gamma_{\mathrm{homo}})_{\mathrm{ex\,/\,reg}}$.

Proof. It is shown in [46] that the exact completion of a regular well-powered protomodular category is also protomodular. Therefore, the category $\mathrm{Mod}(\Gamma_{\mathrm{homo}})_{\mathrm{ex\,/\,reg}}$ is protomodular. In the pointed context, this can be proved using Proposition 4.17: We know from [66] that a finitely complete pointed category is protomodular if and only if each binary relation which is $\left(\begin{array}{c|c} x & x \\ y & x \end{array}\right)$-closed and $\left(\begin{array}{c|c} 0 & y \\ y & 0 \end{array}\right)$-closed is also $\left(\begin{array}{c|c} x & y \\ y & x \end{array}\right)$-closed. Proposition 4.17 implies thus that $\mathrm{Mod}(\Gamma_{\mathrm{homo}})_{\mathrm{ex\,/\,reg}}$ is protomodular and so it is exact and homological.

Then, to prove such an embedding exists, it suffices to compose the embedding $\mathcal{C} \hookrightarrow \mathrm{Mod}(\Gamma_{\mathrm{homo}})$ from Corollary 4.35 with the full embedding $i\colon \mathrm{Mod}(\Gamma_{\mathrm{homo}}) \hookrightarrow \mathrm{Mod}(\Gamma_{\mathrm{homo}})_{\mathrm{ex\,/\,reg}}$ given by the exact completion. ☐

However, this is not yet a good embedding theorem for semi-abelian categories since we do not know if $\mathrm{Mod}(\Gamma_{\mathrm{homo}})_{\mathrm{ex\,/\,reg}}$ has binary coproducts.

4.5 Embedding for categories with M-closed strong relations

In this section we prove that the category of sets equipped with a partial Mal'tsev operation is a weakly Mal'tsev category. Moreover, for each small finitely complete weakly Mal'tsev category, the Yoneda embedding fully embeds it into a power of this category of partial Mal'tsev algebras. On one hand, as usual, this implies it is enough to prove some statements about finite limits in this category in order to prove them for any weakly Mal'tsev category. Moreover, as we will see, this technique also works to prove results which are true for Mal'tsev categories (but not necessarily for weakly Mal'tsev categories).

On the other hand, this embedding theorem better explains a fundamental difference between regular Mal'tsev and weakly Mal'tsev categories. For the former, they embed in a power of a category of partial algebras in which the Mal'tsev term $p = \pi^s \circ \rho_1^s \colon s^3 \to s$ is defined for triples satisfying an everywhere-defined equation. Therefore, for each monomorphism f, if $p(f(x), f(y), f(z))$ is defined, then so is $p(x, y, z)$. On the contrary, not all monomorphisms satisfy this property in the category of partial algebras used in the embedding theorem for weakly Mal'tsev categories (but strong monomorphisms do).

Mal'tsev categories sit somewhere between weakly Mal'tsev and regular Mal'tsev categories. It would then be a major step in the understanding of Mal'tsev categories to know 'exactly where' by proving a corresponding embedding theorem. However, as far as we know now, a 'representing Mal'tsev category' leading to such an embedding theorem looks very hard to find. The reason is that 'being Mal'tsev' seems to be not 'as algebraic as being weakly or regular Mal'tsev'. Indeed, the weakly Mal'tsev property can be characterised (see Proposition 2.37) by the condition that a pair (l, r) of morphisms is jointly epimorphic. This can be written as

$$(\forall x)(\forall y)(xl = yl \wedge xr = yr \Rightarrow x = y).$$

The regular Mal'tsev property (with binary coproducts) is characterised by the condition that a morphism d is a regular epimorphism (see Theorem 2.30). Denoting (r_1, r_2) its kernel pair, this can be represented as

$$(\forall x)(xr_1 = xr_2 \Rightarrow (\exists!\, y)(yd = x)).$$

These two properties already look very much algebraic. On the contrary, Mal'tsev categories are characterised using a pair of morphisms (l, r) which should be jointly strongly epimorphic. This last property is (for now) far from being algebraic since it can not be expressed using 'generalised elements' in an algebraic way. This is the reason why we do not think our present knowledge and understanding of Mal'tsev categories can lead to an embedding theorem for them.

As before, for the sake of generality, we develop this section in the context of categories with M-closed strong relations for a simple extended matrix of terms in a commutative Lawvere theory.

4.5.1 The category of partial M-algebras

Definition 4.38. [56] Let $\mathcal{T}$ be a commutative Lawvere theory and

$$M = \left(\begin{array}{ccc|c} t_{11} & \cdots & t_{1b} & u_1 \\ \vdots & & \vdots & \vdots \\ t_{a1} & \cdots & t_{ab} & u_a \end{array} \right)$$

a simple extended matrix of terms in $\mathcal{T}$. A *partial M-algebra* is a $\mathcal{T}$-algebra A equipped with a partial operation $p\colon A^b \to A$ such that

1. for each $i \in \{1, \dots, a\}$ and all $a_1, \dots, a_k \in A$,

   $$p(t_{i1}(a_1, \dots, a_k), \dots, t_{ib}(a_1, \dots, a_k))$$

   is defined and

   $$p(t_{i1}(a_1, \dots, a_k), \dots, t_{ib}(a_1, \dots, a_k)) = u_i(a_1, \dots, a_k);$$

2. for any $n \in \mathbb{N}$, any n-ary term t of $\mathcal{T}$ and any family of elements $(a_j^{j'} \in A)_{j \in \{1,\dots,b\}, j' \in \{1,\dots,n\}}$ such that $p(a_1^{j'}, \dots, a_b^{j'})$ is defined for each $j' \in \{1, \dots, n\}$, $p(t(a_1^1, \dots, a_1^n), \dots, t(a_b^1, \dots, a_b^n))$ is defined

> and the equality
>
> $$\begin{aligned}&p(t(a_1^1,\ldots,a_1^n),\ldots,t(a_b^1,\ldots,a_b^n))\\&\quad= t(p(a_1^1,\ldots,a_b^1),\ldots,p(a_1^n,\ldots,a_b^n))\end{aligned}$$
>
> holds. Note that if $\mathcal{T}$ is on the form $\mathcal{T}_{(\Sigma,E)}$ for some finitary one-sorted algebraic theory (Σ, E), it is equivalent to require it only for simple terms as $\sigma(x_1,\ldots,x_n)$ where $\sigma \in \Sigma$.

A *homomorphism* $f\colon A \to B$ *of partial* M*-algebras* is a $\mathcal{T}$-homomorphism such that, for all $a_1,\ldots,a_b \in A$ for which $p(a_1,\ldots,a_b)$ is defined in A, $p(f(a_1),\ldots,f(a_b))$ is defined in B and

$$p(f(a_1),\ldots,f(a_b)) = f(p(a_1,\ldots,a_b)).$$

We denote by Part_M the corresponding category.

We have a $\mathcal{T}$-enrichment on Part_M: if t is an n-ary term of $\mathcal{T}$ and $f_1,\ldots,f_n\colon A \to B$ are homomorphisms of partial M-algebras, we define $t(f_1,\ldots,f_n)\colon A \to B$ by

$$t(f_1,\ldots,f_n)(a') = t(f_1(a'),\ldots,f_n(a'))$$

for all $a' \in A$. Since $\mathcal{T}$ is commutative, this is a homomorphism of $\mathcal{T}$-algebras. Moreover, if $a_1,\ldots,a_b \in A$ are such that $p(a_1,\ldots,a_b)$ is defined, for each $j' \in \{1,\ldots,n\}$, $p(f_{j'}(a_1),\ldots,f_{j'}(a_b))$ is also defined. This implies

$$\begin{aligned}&p(t(f_1,\ldots,f_n)(a_1),\ldots,t(f_1,\ldots,f_n)(a_b))\\&\quad= p(t(f_1(a_1),\ldots,f_n(a_1)),\ldots,t(f_1(a_b),\ldots,f_n(a_b)))\end{aligned}$$

is defined as well and equal to

$$\begin{aligned}&t(p(f_1(a_1),\ldots,f_1(a_b)),\ldots,p(f_n(a_1),\ldots,f_n(a_b)))\\&\quad= t(f_1(p(a_1,\ldots,a_b)),\ldots,f_n(p(a_1,\ldots,a_b)))\\&\quad= t(f_1,\ldots,f_n)(p(a_1,\ldots,a_b))\end{aligned}$$

in view of the second condition in the definition of partial M-algebras.

This proves $t(f_1, \dots, f_n)$ is indeed a homomorphism of partial M-algebras.

Let us now describe small limits in Part_M. For that purpose, we consider a small diagram $G\colon \mathcal{D} \to \mathrm{Part}_M$. Let

$$(\lambda_D\colon L \to U_{\mathcal{T}}G(D))_{D\in\mathcal{D}}$$

be the limit of $U_{\mathcal{T}}G$ in $\mathcal{T}$-Alg, where $U_{\mathcal{T}}\colon \mathrm{Part}_M \to \mathcal{T}$-Alg is the forgetful functor. So L is given by

$$L = \{(a_D)_{D\in\mathcal{D}} \in \prod_{D\in\mathcal{D}} G(D) \,|\, G(d)(a_D) = a_{D'}\ \forall d\colon D \to D' \in \mathcal{D}\}$$

with

$$t((a_D^1)_{D\in\mathcal{D}}, \dots, (a_D^n)_{D\in\mathcal{D}}) = (t(a_D^1, \dots, a_D^n))_{D\in\mathcal{D}}$$

for each n-ary term t of $\mathcal{T}$. Now, for $(a_D^1)_{D\in\mathcal{D}}, \dots, (a_D^b)_{D\in\mathcal{D}} \in L$, we define $p((a_D^1)_{D\in\mathcal{D}}, \dots, (a_D^b)_{D\in\mathcal{D}})$ if and only if $p(a_D^1, \dots, a_D^b)$ is defined for each $D \in \mathcal{D}$. In this case, we set

$$p((a_D^1)_{D\in\mathcal{D}}, \dots, (a_D^b)_{D\in\mathcal{D}}) = (p(a_D^1, \dots, a_D^b))_{D\in\mathcal{D}}.$$

This makes L a partial M-algebra. Indeed, for each $i \in \{1, \dots, a\}$ and each $(a_D^1)_{D\in\mathcal{D}}, \dots, (a_D^k)_{D\in\mathcal{D}} \in L$,

$$\begin{aligned}&p(t_{i1}((a_D^1)_{D\in\mathcal{D}}, \dots, (a_D^k)_{D\in\mathcal{D}}), \dots, t_{ib}((a_D^1)_{D\in\mathcal{D}}, \dots, (a_D^k)_{D\in\mathcal{D}}))\\ &= p((t_{i1}(a_D^1, \dots, a_D^k))_{D\in\mathcal{D}}, \dots, (t_{ib}(a_D^1, \dots, a_D^k))_{D\in\mathcal{D}})\end{aligned}$$

is defined since $p(t_{i1}(a_D^1, \dots, a_D^k), \dots, t_{ib}(a_D^1, \dots, a_D^k))$ is for each $D \in \mathcal{D}$ and it is equal to

$$\begin{aligned}&(p(t_{i1}(a_D^1, \dots, a_D^k), \dots, t_{ib}(a_D^1, \dots, a_D^k)))_{D\in\mathcal{D}}\\ &= (u_i(a_D^1, \dots, a_D^k))_{D\in\mathcal{D}}\\ &= u_i((a_D^1)_{D\in\mathcal{D}}, \dots, (a_D^k)_{D\in\mathcal{D}}).\end{aligned}$$

We check the second condition analogously: Let t be an n-ary term of $\mathcal{T}$ and, for each $j' \in \{1, \dots, n\}$, $(a_D^{1,j'})_{D\in\mathcal{D}}, \dots, (a_D^{b,j'})_{D\in\mathcal{D}}$ elements of L such that $p((a_D^{1,j'})_{D\in\mathcal{D}}, \dots, (a_D^{b,j'})_{D\in\mathcal{D}})$ is defined (i.e., $p(a_D^{1,j'}, \dots, a_D^{b,j'})$

is defined for each $D \in \mathcal{D}$). This implies

$$p(t(a_D^{1,1}, \ldots, a_D^{1,n}), \ldots, t(a_D^{b,1}, \ldots, a_D^{b,n}))$$

is defined and equal to

$$t(p(a_D^{1,1}, \ldots, a_D^{b,1}), \ldots, p(a_D^{1,n}, \ldots, a_D^{b,n}))$$

for each $D \in \mathcal{D}$. Thus

$$\begin{aligned} &p(t((a_D^{1,1})_{D\in\mathcal{D}}, \ldots, (a_D^{1,n})_{D\in\mathcal{D}}), \ldots, t((a_D^{b,1})_{D\in\mathcal{D}}, \ldots, (a_D^{b,n})_{D\in\mathcal{D}})) \\ &= p((t(a_D^{1,1}, \ldots, a_D^{1,n}))_{D\in\mathcal{D}}, \ldots, (t(a_D^{b,1}, \ldots, a_D^{b,n}))_{D\in\mathcal{D}}) \end{aligned}$$

is also defined in L and equal to

$$\begin{aligned} &(t(p(a_D^{1,1}, \ldots, a_D^{b,1}), \ldots, p(a_D^{1,n}, \ldots, a_D^{b,n})))_{D\in\mathcal{D}} \\ &= t(p((a_D^{1,1})_{D\in\mathcal{D}}, \ldots, (a_D^{b,1})_{D\in\mathcal{D}}), \ldots, p((a_D^{1,n})_{D\in\mathcal{D}}, \ldots, (a_D^{b,n})_{D\in\mathcal{D}})), \end{aligned}$$

which shows that L is a partial M-algebra. Moreover, given a cone $(\mu_D \colon A \to G(D))_{D\in\mathcal{D}}$ over G, let f be the unique $\mathcal{T}$-homomorphism

$$\begin{aligned} f \colon A &\longrightarrow L \\ a' &\longmapsto (\mu_D(a'))_{D\in\mathcal{D}} \end{aligned}$$

such that $\lambda_D f = \mu_D$ for each $D \in \mathcal{D}$. If $a_1, \ldots, a_b \in A$ are such that $p(a_1, \ldots, a_b)$ is defined in A, $p(\mu_D(a_1), \ldots, \mu_D(a_b))$ is defined in $G(D)$ for each $D \in \mathcal{D}$. Thus, $p(f(a_1), \ldots, f(a_b))$ is also defined and equal to

$$\begin{aligned} p((\mu_D(a_1))_{D\in\mathcal{D}}, \ldots, (\mu_D(a_b))_{D\in\mathcal{D}}) &= (p(\mu_D(a_1), \ldots, \mu_D(a_b)))_{D\in\mathcal{D}} \\ &= (\mu_D(p(a_1, \ldots, a_b)))_{D\in\mathcal{D}} \\ &= f(p(a_1, \ldots, a_b)), \end{aligned}$$

which proves that f is a homomorphism of partial M-algebras and the cone $(\lambda_D \colon L \to G(D))_{D\in\mathcal{D}}$ the limit of G. Therefore, Part_M is complete and $U_{\mathcal{T}} \colon \mathrm{Part}_M \to \mathcal{T}$-Alg preserves small limits, but it does not reflect them in general. Indeed, one could have defined p on a smaller subset of L^b in order to make L a partial M-algebra, but this would not have made it a limit in Part_M. This means $U_{\mathcal{T}}$ is not conservative in general. Here is a simple counterexample.

Counterexample 4.39. Let $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$ and $M = \left(\begin{array}{cc|c} x & 0 & x \\ 0 & x & x \end{array}\right)$. Let A be the pointed set $\{0, x\}$ endowed with the structure of a partial M-algebra given by $p(0,0) = 0$, $p(x,0) = x = p(0,x)$ and $p(x,x)$ undefined. Let also B be the partial M-algebra on $\{0,x\}$ given by $p(0,0) = 0$ and $p(x,0) = x = p(0,x) = p(x,x)$. Then, the identity map $A \to B$ is a bijective homomorphism but not an isomorphism in Part_M.

We turn now our attention to strong monomorphisms in Part_M. In order to understand them better, we need to construct a left adjoint to the forgetful functor $U\colon \mathrm{Part}_M \to \mathrm{Set}$. As an intermediate step, we consider the category b-Part where objects are sets X equipped with a partial b-ary operation $p\colon X^b \to X$ and morphisms are functions $f\colon X \to Y$ such that if $p(x_1, \dots, x_b)$ is defined for some $x_1, \dots, x_b \in X$, then $p(f(x_1), \dots, f(x_b))$ is also defined and equal to $f(p(x_1, \dots, x_b))$. The forgetful functor $U\colon \mathrm{Part}_M \to \mathrm{Set}$ thus factors as $\mathrm{Part}_M \to b\text{-Part} \to \mathrm{Set}$.

Proposition 4.40. [56] Let $\mathcal{T}$ be a commutative Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$ as in (5). The forgetful functor $U'\colon \mathrm{Part}_M \to b\text{-Part}$ has a left adjoint.

Proof. Let X be an object of b-Part. Let us add the constant operation symbols c_x for all $x \in X$ to the algebraic theory $(\Sigma_{\mathcal{T}}, E_{\mathcal{T}})$ to form the algebraic theory (Σ', E') and the corresponding Lawvere theory $\mathcal{T}' = \mathcal{T}_{(\Sigma', E')}$. We denote by I the set

$$\begin{aligned} I &= \{1, \dots, a\} \sqcup \{(x_1, \dots, x_b) \in X^b \,|\, p(x_1, \dots, x_b) \text{ is defined}\} \\ &= \{1, \dots, a\} \sqcup \mathrm{dom}(p) \end{aligned}$$

and, for each $i = (x_1, \dots, x_b) \in \mathrm{dom}(p)$, $t_{ij}(y_1, \dots, y_k)$ is the k-ary term c_{x_j} of $\mathcal{T}'$ for each $j \in \{1, \dots, b\}$ and $u_i(y_1, \dots, y_k)$ the k-ary term $c_{p(i)}$ of $\mathcal{T}'$. Let $\mathcal{Q}$ be the quasivariety of $\mathcal{T}'$-algebras satisfying, for all n-ary (resp. n'-ary) terms τ and τ' of $\mathcal{T}$ and all indices $i_1, \dots, i_n, i'_1, \dots, i'_{n'}$

in I, the following implication: if, for each $j \in \{1, \dots, b\}$,

$$\begin{aligned} &\tau(t_{i_1 j}(y_{11}, \dots, y_{1k}), \dots, t_{i_n j}(y_{n1}, \dots, y_{nk})) \\ &\quad = \tau'(t_{i'_1 j}(y'_{11}, \dots, y'_{1k}), \dots, t_{i'_{n'} j}(y'_{n'1}, \dots, y'_{n'k})) \end{aligned}$$

then

$$\begin{aligned} &\tau(u_{i_1}(y_{11}, \dots, y_{1k}), \dots, u_{i_n}(y_{n1}, \dots, y_{nk})) \\ &\quad = \tau'(u_{i'_1}(y'_{11}, \dots, y'_{1k}), \dots, u_{i'_{n'}}(y'_{n'1}, \dots, y'_{n'k})). \end{aligned}$$

For an object A in $\mathcal{Q}$, we define p in A via the equalities

$$\begin{aligned} &p(\tau(t_{i_1 1}(a_{11}, \dots, a_{1k}), \dots, t_{i_n 1}(a_{n1}, \dots, a_{nk})), \dots \\ &\qquad\qquad \dots, \tau(t_{i_1 b}(a_{11}, \dots, a_{1k}), \dots, t_{i_n b}(a_{n1}, \dots, a_{nk}))) \\ &\quad = \tau(u_{i_1}(a_{11}, \dots, a_{1k}), \dots, u_{i_n}(a_{n1}, \dots, a_{nk})) \end{aligned}$$

for all n-ary terms τ of $\mathcal{T}$, all indices $i_1, \dots, i_n \in I$ and all families of elements $(a_{j'i'} \in A)_{j' \in \{1,\dots,n\}, i' \in \{1,\dots,k\}}$. We do not define p for any other elements of A^b. In view of the implications defining $\mathcal{Q}$, this partial operation p is well-defined. We see that the first condition defining partial M-algebras is satisfied by choosing τ to be the identity term $\tau(y) = y$. The second condition is also satisfied: Let t be an n-ary term of $\mathcal{T}$, $\tau^{j'}$ an $r^{j'}$-ary term of $\mathcal{T}$ for each $j' \in \{1, \dots, n\}$, $i^{j'}_{j''} \in I$ an index for each $j' \in \{1, \dots, n\}$ and each $j'' \in \{1, \dots, r^{j'}\}$, and $a^{j'}_{j''i'}$ an element of A for all $j' \in \{1, \dots, n\}$, $j'' \in \{1, \dots, r^{j'}\}$ and $i' \in \{1, \dots, k\}$. Then,

$$p((t((\tau^{j'}((t_{i^{j'}_{j''} j}(a^{j'}_{j''1}, \dots, a^{j'}_{j''k}))^{r^{j'}}_{j''=1}))^{n}_{j'=1}))^{b}_{j=1})$$

is defined in view of the $(r^1 + \dots + r^n)$-ary term

$$t(\tau^1(y_{11}, \dots, y_{1r^1}), \dots, \tau^n(y_{n1}, \dots, y_{nr^n}))$$

of $\mathcal{T}$. Moreover, it is equal to

$$\begin{aligned} &t((\tau^{j'}((u_{i^{j'}_{j''}}(a^{j'}_{j''1}, \dots, a^{j'}_{j''k}))^{r^{j'}}_{j''=1}))^{n}_{j'=1}) \\ &\quad = t((p((\tau^{j'}((t_{i^{j'}_{j''} j}(a^{j'}_{j''1}, \dots, a^{j'}_{j''k}))^{r^{j'}}_{j''=1}))^{b}_{j=1}))^{n}_{j'=1}) \end{aligned}$$

as required. So A has been endowed with a structure of partial M-algebra. We consider the function

$$\begin{aligned} f\colon X &\longrightarrow U'(A) \\ x &\longmapsto c_x. \end{aligned}$$

It is a morphism in b-Part. Indeed, if $i = (x_1, \ldots, x_b) \in \mathrm{dom}(p)$, choosing τ to be the identity term $\tau(y) = y$ and $i_1 = i$, we have

$$\begin{aligned} p(f(x_1), \ldots, f(x_b)) &= p(c_{x_1}, \ldots, c_{x_b}) \\ &= p(t_{i1}(a_1, \ldots, a_k), \ldots, t_{ib}(a_1, \ldots, a_k)) \\ &= u_i(a_1, \ldots, a_k) \\ &= c_{p(i)} \\ &= f(p(x_1, \ldots, x_b)). \end{aligned}$$

If $g\colon A \to A'$ is a morphism in $\mathcal{Q}$, it can be considered as a homomorphism of partial M-algebras making the triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f'"'] & U'(A) \arrow[d, "g"] \\ & U'(A') \end{tikzcd}$$

commutative. Indeed, the above triangle commutes since g is a $\mathcal{T}'$-homomorphism and when

$$\begin{aligned} &p(\tau(t_{i_1 1}(a_{11}, \ldots, a_{1k}), \ldots, t_{i_n 1}(a_{n1}, \ldots, a_{nk})), \ldots \\ &\qquad \ldots, \tau(t_{i_1 b}(a_{11}, \ldots, a_{1k}), \ldots, t_{i_n b}(a_{n1}, \ldots, a_{nk}))) \end{aligned}$$

is defined in A,

$$\begin{aligned} &p(g(\tau(t_{i_1 1}(a_{11}, \ldots, a_{1k}), \ldots, t_{i_n 1}(a_{n1}, \ldots, a_{nk}))), \ldots \\ &\qquad \ldots, g(\tau(t_{i_1 b}(a_{11}, \ldots, a_{1k}), \ldots, t_{i_n b}(a_{n1}, \ldots, a_{nk})))) \\ &= p(\tau(t_{i_1 1}(g(a_{11}), \ldots, g(a_{1k})), \ldots, t_{i_n 1}(g(a_{n1}), \ldots, g(a_{nk}))), \ldots \\ &\qquad \ldots, \tau(t_{i_1 b}(g(a_{11}), \ldots, g(a_{1k})), \ldots, t_{i_n b}(g(a_{n1}), \ldots, g(a_{nk})))) \end{aligned}$$

is defined in A' and equal to

$$\begin{aligned}&\tau(u_{i_1}(g(a_{11}),\dots,g(a_{1k})),\dots,u_{i_n}(g(a_{n1}),\dots,g(a_{nk})))\\&\quad=g(\tau(u_{i_1}(a_{11},\dots,a_{1k}),\dots,u_{i_n}(a_{n1},\dots,a_{nk}))).\end{aligned}$$

We have thus define a functor $F'\colon \mathcal{Q}\to(X\downarrow U')$.

On the other hand, if $f\colon X\to U'(A)$ is an object of $(X\downarrow U')$, A admits a $\mathcal{T}'$-algebra structure considering $c_x=f(x)$ for each $x\in X$. Moreover, for each $i\in I$ and $a_1,\dots,a_k\in A$,

$$p(t_{i1}(a_1,\dots,a_k),\dots,t_{ib}(a_1,\dots,a_k))=u_i(a_1,\dots,a_k).$$

So, if τ is an n-ary term of $\mathcal{T}$, $i_1,\dots,i_n\in I$ and $a_{j'i'}\in A$ for each $j'\in\{1,\dots,n\}$ and each $i'\in\{1,\dots,k\}$,

$$\begin{aligned}&p(\tau(t_{i_11}(a_{11},\dots,a_{1k}),\dots,t_{i_n1}(a_{n1},\dots,a_{nk})),\dots\\&\qquad\qquad\dots,\tau(t_{i_1b}(a_{11},\dots,a_{1k}),\dots,t_{i_nb}(a_{n1},\dots,a_{nk})))\\&=\tau(u_{i_1}(a_{11},\dots,a_{1k}),\dots,u_{i_n}(a_{n1},\dots,a_{nk}))\end{aligned}$$

since A is a partial M-algebra. Hence, A satisfies the implications defining $\mathcal{Q}$ and this makes $G'\colon(X\downarrow U')\to\mathcal{Q}$ a functor.

Since the equality above holds in A for any object $f\colon X\to U'(A)$ of $(X\downarrow U')$, the identity map on A defines a morphism $\varepsilon_f\colon F'G'(f)\to f$ in $(X\downarrow U')$. This gives a natural transformation $\varepsilon\colon F'G'\Rightarrow 1_{(X\downarrow U')}$. Moreover, $G'F'=1_{\mathcal{Q}}$ and we have constructed an adjunction $F'\dashv G'$. But $\mathcal{Q}$ is a quasivariety, so it is locally presentable (see Proposition 1.79) and has an initial object. Therefore, $(X\downarrow U')$ has also an initial object which is the reflection of X along U'. □

To construct the reflection of the set X along the forgetful functor b-Part $\to$ Set is much easier. It suffices to consider the identity map $1_X\colon X\to X$ where the partial operation p on X is nowhere defined. This gives a left adjoint Set $\to$ b-Part. Composed with the left adjoint b-Part $\to$ Part_M given by the above proposition, we have constructed the left adjoint $F\colon$ Set $\to\mathrm{Part}_M$ to the forgetful functor $U\colon\mathrm{Part}_M\to$ Set. We remark that in the particular case $X=\varnothing$, the quasivariety $\mathcal{Q}$

described above is the quasivariety $\mathcal{Q}_M$ of $\mathcal{T}$-algebras satisfying, for all n-ary (resp. n'-ary) terms τ and τ' of $\mathcal{T}$ and all indices $i_1, \ldots, i_n, i'_1, \ldots, i'_{n'}$ in $\{1, \ldots, a\}$, the following implication: if

$$\begin{aligned}&\tau(t_{i_1 j}(a_{11}, \ldots, a_{1k}), \ldots, t_{i_n j}(a_{n1}, \ldots, a_{nk}))\\ &\quad = \tau'(t_{i'_1 j}(a'_{11}, \ldots, a'_{1k}), \ldots, t_{i'_{n'} j}(a'_{n'1}, \ldots, a'_{n'k}))\end{aligned}$$

for each $j \in \{1, \ldots, b\}$, then

$$\begin{aligned}&\tau(u_{i_1}(a_{11}, \ldots, a_{1k}), \ldots, u_{i_n}(a_{n1}, \ldots, a_{nk}))\\ &\quad = \tau'(u_{i'_1}(a'_{11}, \ldots, a'_{1k}), \ldots, u_{i'_{n'}}(a'_{n'1}, \ldots, a'_{n'k})).\end{aligned}$$

The functor $F' \colon \mathcal{Q} \to (X \downarrow U')$ is then nothing but the left adjoint $\mathcal{Q}_M \to \mathrm{Part}_M$ to the forgetful functor $\mathrm{Part}_M \to \mathcal{Q}_M$. The left adjoint $F \colon \mathrm{Set} \to \mathrm{Part}_M$ can thus be also obtained by composing $F' \colon \mathcal{Q}_M \to \mathrm{Part}_M$ with the free functor $\mathrm{Set} \to \mathcal{Q}_M$.

We now consider the case $X = \{1, \ldots, b+1\}$ with p defined only by $p(1, \ldots, b) = b + 1$. We denote by $X \to U'(F_M)$ its reflection along $U' \colon \mathrm{Part}_M \to b\text{-Part}$ and g its restriction $g \colon \{1, \ldots, b\} \hookrightarrow X \to U(F_M)$. The function g is such that $p(g(1), \ldots, g(b))$ is defined in F_M and universal with that property, i.e., if $h \colon \{1, \ldots, b\} \to U(A)$ is a function to a partial M-algebra A where $p(h(1), \ldots, h(b))$ is defined, there exists a unique homomorphism of partial M-algebras $\overline{h} \colon F_M \to A$ such that $U(\overline{h}) \circ g = h$.

$$\begin{tikzcd}
\{1, \ldots, b\} \arrow[rr, "g"] \arrow[dr, "\forall h"'] & & U(F_M) \arrow[dl, dotted, "\exists!\, U(\overline{h})"] \\
& U(A) &
\end{tikzcd}$$

Since $U \colon \mathrm{Part}_M \to \mathrm{Set}$ preserves kernel pairs, monomorphisms in Part_M are exactly the injective homomorphisms. Let now $f \colon A \rightarrowtail B$ be a strong monomorphism in Part_M. Consider also the homomorphism $e \colon F(\{1, \ldots, b\}) \to F_M$ given by the universal property of $F(\{1, \ldots, b\})$ and the function $g \colon \{1, \ldots, b\} \to U(F_M)$. If $\overline{h}, \overline{k} \colon F_M \rightrightarrows C$ are homomorphisms of partial M-algebras such that $\overline{h}e = \overline{k}e$, then $\overline{h}g = \overline{k}g$ and $\overline{h} = \overline{k}$. Thus e is actually an epimorphism in Part_M. If $a_1, \ldots, a_b \in A$

are such that $p(f(a_1), \dots, f(a_b))$ is defined, we can construct a commutative square as below with $k(j) = a_j$ and $\overline{h}(g(j)) = f(a_j)$ for each $j \in \{1, \dots, b\}$.

$$\begin{tikzcd}
F(\{1,\dots,b\}) \arrow[r, two heads, "e"] \arrow[d, "k"'] & F_M \arrow[d, "\overline{h}"] \arrow[ld, dotted] \\
A \arrow[r, tail, "f"'] & B
\end{tikzcd}$$

Since f is supposed to be a strong monomorphism, $\overline{h}$ factors through f. Hence, $p(a_1, \dots, a_b)$ is defined as well. Therefore, strong monomorphisms in Part_M reflect the b-tuples where p is defined, i.e., if $p(f(a_1), \dots, f(a_b))$ is defined, then so is $p(a_1, \dots, a_b)$.

Definition 4.41. [53] Let $\mathcal{T}$ be a commutative Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$ as in (5). A homomorphism $f \colon A \to B$ in Part_M is said to be *closed* if, given $a_1, \dots, a_b \in A$, $p(a_1, \dots, a_b)$ is defined in A if and only if $p(f(a_1), \dots, f(a_b))$ is defined in B.

Such homomorphisms are also called 'strong' in [51]. The above discussion leads us to the following proposition.

Proposition 4.42. [56] Let $\mathcal{T}$ be a commutative Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$. Strong monomorphisms in Part_M are closed.

The homomorphism from Counterexample 4.39 is an example of a bijective homomorphism of partial M-algebras which is not closed. Note that isomorphisms in Part_M are exactly the bijective closed homomorphisms. Indeed, in view of the next lemma, closedness of a bijective homomorphism $f \colon B \to C$ is exactly what we need to prove the inverse map $f^{-1} \colon C \to B$ is a homomorphism of partial M-algebras.

Lemma 4.43. Let $\mathcal{T}$ be a commutative Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$. Let also $g \colon A \to B$ be a function between partial M-algebras and $f \colon B \rightarrowtail C$ a closed monomorphism in Part_M. If fg is a homomorphism of partial M-algebras, then so is g.

Proof. Let t be an n-ary term of $\mathcal{T}$ and $a_1, \dots, a_n \in A$. Since

$$\begin{aligned} f(g(t(a_1, \dots, a_n))) &= t(fg(a_1), \dots, fg(a_n)) \\ &= f(t(g(a_1), \dots, g(a_n))) \end{aligned}$$

and f is injective, g is a $\mathcal{T}$-homomorphism.

Besides, if $a_1, \dots, a_b \in A$ are such that $p(a_1, \dots, a_b)$ are defined in A, $p(fg(a_1), \dots, fg(a_b))$ is defined in C and $p(g(a_1), \dots, g(a_b))$ is defined in B since f is closed. We can also compute

$$\begin{aligned} f(p(g(a_1), \dots, g(a_b))) &= p(fg(a_1), \dots, fg(a_b)) \\ &= fg(p(a_1, \dots, a_b)), \end{aligned}$$

which implies

$$p(g(a_1), \dots, g(a_b)) = g(p(a_1, \dots, a_b))$$

since f is injective. □

We now want to prove that, for some M, closed monomorphisms in Part_M are exactly the strong monomorphisms. To achieve this, we need to study the properties of closed monomorphisms.

Proposition 4.44. Let $\mathcal{T}$ be a commutative Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$. Closed monomorphisms in Part_M are stable under pullbacks.

Proof. We consider a closed monomorphism $f\colon A \rightarrowtail B$ in Part_M and its pullback along $g\colon C \to B$.

$$\begin{tikzcd} P \arrow[r, tail, "f'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, tail, "f"'] & B \end{tikzcd}$$

If $(a_1, c_1), \dots, (a_b, c_b) \in P$, $p((a_1, c_1), \dots, (a_b, c_b))$ is defined if and only if $p(a_1, \dots, a_b)$ and $p(c_1, \dots, c_b)$ are defined. But if $p(c_1, \dots, c_b)$ is defined, $p(g(c_1), \dots, g(c_b)) = p(f(a_1), \dots, f(a_b))$ is also defined. Since f

is closed, this further implies $p(a_1, \ldots, a_b)$ and so $p((a_1, c_1), \ldots, (a_b, c_b))$ are defined. Thus, f' is a closed monomorphism. $\square$

Let us recall the following well-known proposition, which will be used in the particular case $\mathcal{C} = \mathrm{Part}_M$ and $\mathcal{R}$ the class of closed monomorphisms.

Proposition 4.45. Let $\mathcal{R}$ be a class of monomorphisms in the finitely complete category $\mathcal{C}$ which is stable under pullbacks and contains regular monomorphisms (i.e., equalisers). A morphism e in $\mathcal{C}$ is orthogonal to all elements of $\mathcal{R}$ if and only if, when e factors as fg with $f \in \mathcal{R}$, then f is an isomorphism. In this case, e is an epimorphism.

Proposition 4.46. [56] Let $\mathcal{T}$ be a commutative Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$. If $\mathcal{R}$ denotes the class of closed monomorphisms in Part_M and

$$\mathcal{R}^{\perp} = \{e \in \mathrm{ar}(\mathrm{Part}_M) \,|\, e \perp m \, \forall m \in \mathcal{R}\}$$

its orthogonal class, $(\mathcal{R}^{\perp}, \mathcal{R})$ is a factorisation system.

Proof. Since $\mathcal{R}$ contains regular monomorphisms, is stable under pullbacks and closed under composition, it remains to prove that each homomorphism $f \colon A \to B$ of partial M-algebras factors as an element of $\mathcal{R}^{\perp}$ followed by a closed monomorphism. Let C be the smallest subset of B satisfying the conditions:

- $f(a') \in C$ for each $a' \in A$,
- C is a sub-$\mathcal{T}$-algebra of B (in the sense of Definition 1.83),
- if $c_1, \ldots, c_b \in C$ are such that $p(c_1, \ldots, c_b)$ is defined in B, then $p(c_1, \ldots, c_b) \in C$.

We consider the unique structure of partial M-algebra on C making the inclusion $i \colon C \hookrightarrow B$ a closed monomorphism. Then, f factors as if' with $f' \colon A \to C$ a homomorphism of partial M-algebras by Lemma 4.43. Moreover, if $f' = f''g$ with f'' a closed monomorphism, the image of f'' contains C by definition of C. Thus f'' is surjective and so an isomorphism. By Proposition 4.45, this means $f' \in \mathcal{R}^{\perp}$. $\square$

Epimorphisms in Part_M thus factor as an epimorphism orthogonal to closed monomorphisms followed by a closed monomorphism (which is also an epimorphism). Therefore, to prove that epimorphisms are orthogonal to closed monomorphisms (i.e., that closed monomorphisms are strong monomorphisms), it suffices to prove that closed epimorphisms are surjective. Indeed, in that case, this would imply that the only epimorphisms which are closed monomorphisms are the isomorphisms. This will be true for some particular M's.

Proposition 4.47. [56] Let M be a simple extended matrix of terms in $\mathrm{Th}[\mathrm{Set}_*]$. Closed epimorphisms in Part_M are surjective.

Proof. Firstly, we notice that all partial M-algebras with one element are isomorphic (since $p(0,\dots,0)$ has to be defined). If this partial M-algebra is the unique one, the result is trivial. Hence, we suppose that there exists a partial M-algebra C with a non-zero element $c \in C$. Now, we also suppose we have a closed epimorphism $f\colon A \twoheadrightarrow B$ in Part_M which is not surjective. Let $\mathrm{Im}(f)$ be the set-theoretical image of f and $D = D' = B \setminus \mathrm{Im}(f) \neq \varnothing$. Notice that $0 \in \mathrm{Im}(f)$. We define a partial b-ary operation p on

$$\mathrm{Im}(f) \sqcup D \sqcup D'$$

in the following way:

1. $p(t_{i1}(x_1,\dots,x_k),\dots,t_{ib}(x_1,\dots,x_k))$ is defined as $u_i(x_1,\dots,x_k)$ for all $i \in \{1,\dots,a\}$ and all $x_1,\dots,x_k \in \mathrm{Im}(f) \sqcup D \sqcup D'$,

2. p restricted on $(\mathrm{Im}(f) \sqcup D)^b$ is defined as in B via the isomorphism of pointed sets $\mathrm{Im}(f) \sqcup D \cong B$,

3. p restricted on $(\mathrm{Im}(f) \sqcup D')^b$ is defined as in B via the isomorphism of pointed sets $\mathrm{Im}(f) \sqcup D' \cong B$,

4. p is defined nowhere else than required by one of the above conditions.

Let us prove this p is well-defined. There is no problem with condition 1 alone. Indeed, let us suppose by contradiction there exist

$i, i' \in \{1, \dots, a\}$ and $x_1, \dots, x_k, x'_1, \dots, x'_k \in \mathrm{Im}(f) \sqcup D \sqcup D'$ satisfying $t_{ij}(x_1, \dots, x_k) = t_{i'j}(x'_1, \dots, x'_k)$ for each $j \in \{1, \dots, b\}$, whereas $u_i(x_1, \dots, x_k) \neq u_{i'}(x'_1, \dots, x'_k)$. Without loss of generality, we can suppose $u_i(x_1, \dots, x_k) \neq 0$. We consider any morphism of pointed sets $g \colon \mathrm{Im}(f) \sqcup D \sqcup D' \to C$ which sends $u_i(x_1, \dots, x_k)$ to c and $u_{i'}(x'_1, \dots, x'_k)$ to 0. Then,

$$t_{ij}(g(x_1), \dots, g(x_k)) = t_{i'j}(g(x'_1), \dots, g(x'_k))$$

for each $j \in \{1, \dots, b\}$ and therefore

$$\begin{aligned} c &= g(u_i(x_1, \dots, x_k)) \\ &= u_i(g(x_1), \dots, g(x_k)) \\ &= p(t_{i1}(g(x_1), \dots, g(x_k)), \dots, t_{ib}(g(x_1), \dots, g(x_k))) \\ &= p(t_{i'1}(g(x'_1), \dots, g(x'_k)), \dots, t_{i'b}(g(x'_1), \dots, g(x'_k))) \\ &= u_{i'}(g(x'_1), \dots, g(x'_k)) \\ &= g(u_{i'}(x'_1, \dots, x'_k)) \\ &= 0, \end{aligned}$$

which is a contradiction.

Since B is a (well-defined) partial M-algebra, there is no problem with condition 2 alone nor with condition 3 alone. The cohabitation of conditions 2 and 3 does not cause any problem neither. Indeed, the only way it could, is to have $x_1, \dots, x_b \in \mathrm{Im}(f)$ such that $p(x_1, \dots, x_b)$ is defined but does not belong to $\mathrm{Im}(f)$. If we write $x_i = f(a_i)$ for some $a_i \in A$, this means $p(f(a_1), \dots, f(a_b))$ is defined. But since f is closed, it implies $p(a_1, \dots, a_b)$ is defined and

$$p(x_1, \dots, x_b) = p(f(a_1), \dots, f(a_b)) = f(p(a_1, \dots, a_b)) \in \mathrm{Im}(f).$$

By symmetry, it remains to check there is no problem with the cohabitation of conditions 1 and 2. If there is one, it means there exist $x_1, \dots, x_k \in \mathrm{Im}(f) \sqcup D \sqcup D'$ and $i \in \{1, \dots, a\}$ such that

$$t_{ij}(x_1, \dots, x_k) \in \mathrm{Im}(f) \sqcup D$$

for each $j \in \{1, \dots, b\}$, but $p(t_{i1}(x_1, \dots, x_k), \dots, t_{ib}(x_1, \dots, x_k))$ defined as in B (via $\mathrm{Im}(f) \sqcup D \cong B$) is not $u_i(x_1, \dots, x_k)$. We denote by

$$q \colon \mathrm{Im}(f) \sqcup D \sqcup D' \to \mathrm{Im}(f) \sqcup D$$

the homomorphism of pointed sets which coequalises the two copies of D. This implies

$$t_{ij}(x_1, \dots, x_k) = q(t_{ij}(x_1, \dots, x_k)) = t_{ij}(q(x_1), \dots, q(x_k))$$

for each $j \in \{1, \dots, b\}$. Since we have already shown there is no problem with condition 1 alone, we can write using this condition

$$\begin{aligned} u_i(x_1, \dots, x_k) &= p(t_{i1}(x_1, \dots, x_k), \dots, t_{ib}(x_1, \dots, x_k)) \\ &= p(t_{i1}(q(x_1), \dots, q(x_k)), \dots, t_{ib}(q(x_1), \dots, q(x_k))) \\ &= u_i(q(x_1), \dots, q(x_k)). \end{aligned}$$

But since B is a partial M-algebra, if we compute using condition 2, we also get

$$\begin{aligned} & p(t_{i1}(x_1, \dots, x_k), \dots, t_{ib}(x_1, \dots, x_k)) \\ & = p(t_{i1}(q(x_1), \dots, q(x_k)), \dots, t_{ib}(q(x_1), \dots, q(x_k))) \\ & = u_i(q(x_1), \dots, q(x_k)). \end{aligned}$$

This discussion proves p is well defined.

The first condition of Definition 4.38 is satisfied by $\mathrm{Im}(f) \sqcup D \sqcup D'$ in view of condition 1. In the case $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$, the second one resumes to $p(0, \dots, 0) = 0$ which is true since it holds in B. Thus $\mathrm{Im}(f) \sqcup D \sqcup D'$ is a partial M-algebra. Now, we consider the two obvious homomorphisms of partial M-algebras $g_1, g_2 \colon B \rightrightarrows \mathrm{Im}(f) \sqcup D \sqcup D'$. They satisfy $g_1 f = g_2 f$ but $g_1 \neq g_2$ since $D = D' \neq \varnothing$. This is a contradiction since f was supposed to be an epimorphism. $\square$

If $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$, there are two partial M-algebras with at most one element, i.e., the empty partial M-algebra and the singleton one $\{\star\}$ (in which $p(\star, \dots, \star)$ has to be defined since $a \geqslant 1$). Therefore, the first argument in the previous proof does not hold if we replace Th[Set$_*$] by Th[Set]. For instance, if $M = \left(\begin{array}{c|c} x & y \end{array} \right)$, the category Part$_M$ is equivalent to the arrow category $0 \to 1$. With this M, the unique homomorphism of partial M-algebras $\varnothing \to \{\star\}$ is an injective closed epimorphism, but not an isomorphism. However, if M is such that there exists a partial M-algebra with at least two elements, the same proof can be repeated to get the following proposition.

Proposition 4.48. [56] Let M be a simple extended matrix of terms in Th[Set] such that there exists a partial M-algebra with at least two elements. Closed epimorphisms in Part$_M$ are surjective.

Counterexample 4.49. If $\mathcal{T}$ = Th[ComMon] and M is the trivial matrix $\left(\begin{array}{c|c} x & x \end{array} \right)$, Part$_M$ is isomorphic to the category ComMon of commutative monoids. There, the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ is an injective closed epimorphism but not an isomorphism.

As explained above, Propositions 4.47 and 4.48 admit the following corollary.

Corollary 4.50. [56] If M is the simple extended matrix of Example 2.7, 2.12, 2.19 or 2.22, then closed monomorphisms coincide with strong monomorphisms in Part$_M$.

We now prove that Part$_M$ has M-closed strong relations.

Proposition 4.51. [56] Let $\mathcal{T}$ be a commutative Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$ as in (5). Every a-ary relation $r \colon R \rightarrowtail A_1 \times \cdots \times A_a$ which is a closed monomorphism in Part$_M$ is strictly M-closed. In particular, Part$_M$ has M-closed strong relations.

Proof. Consider a family of morphisms $(y_{ii'} \colon Y \to A_i)_{i \in \{1,\dots,a\}, i' \in \{1,\dots,k\}}$ in Part$_M$ for which the morphism

$$(t_{1j}(y_{11}, \dots, y_{1k}), \dots, t_{aj}(y_{a1}, \dots, y_{ak})) \colon Y \to A_1 \times \cdots \times A_a$$

factors as rw_j for each $j \in \{1, \dots, b\}$.

$$\begin{tikzcd}
 & R \arrow[d, tail, "r"] \\
Y \arrow[ur, "w_j"] \arrow[r, "{(t_{1j}(y_{11},\dots,y_{1k}),\dots,t_{aj}(y_{a1},\dots,y_{ak}))}"'] & A_1 \times \cdots \times A_a
\end{tikzcd}$$

We know that, for each $y \in Y$ and each $i \in \{1, \dots, a\}$,

$$p(t_{i1}(y_{i1}(y), \dots, y_{ik}(y)), \dots, t_{ib}(y_{i1}(y), \dots, y_{ik}(y)))$$

is defined and equal to $u_i(y_{i1}(y), \dots, y_{ik}(y))$. Using the description of small products in Part_M, we can say that $p(rw_1(y), \dots, rw_b(y))$ is defined for any $y \in Y$ and equal to

$$(u_1(y_{11}(y), \dots, y_{1k}(y)), \dots, u_a(y_{a1}(y), \dots, y_{ak}(y))).$$

Since r is closed, $p(w_1(y), \dots, w_b(y))$ is defined in R and we can consider the function $w\colon Y \to R\colon y \mapsto p(w_1(y), \dots, w_b(y))$ which satisfies

$$rw = (u_1(y_{11}, \dots, y_{1k}), \dots, u_a(y_{a1}, \dots, y_{ak})).$$

Finally, Lemma 4.43 tells us w is a homomorphism of partial M-algebras since rw is and r is a closed monomorphism, which concludes the proof. □

4.5.2 Proof of the embedding theorems

As announced at the beginning of this section, we can prove an embedding theorem for small categories with M-closed strong relations, but we also have one for small categories with M-closed relations (which is however not as good since Part_M does not have M-closed relations in general). In order to prove both at the same time, we are going to use a set of monomorphisms, closed under composition, stable under pullbacks and which contains regular monomorphisms.

Theorem 4.52. [56] Let $\mathcal{T}$ be a commutative Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$ as in (5). Let also $\mathcal{R}$ be a set of monomorphisms in the small finitely complete $\mathcal{T}$-category $\mathcal{C}$ such that $\mathcal{R}$ is closed under composition, stable under pullbacks and contains regular monomorphisms. Suppose also that all a-ary relations $R \rightarrowtail A^a$ in $\mathcal{R}$ are M-closed in $\mathcal{C}$. Then, there exists a full and faithful $\mathcal{T}$-enriched embedding $\phi\colon \mathcal{C} \hookrightarrow \mathrm{Part}_M^{\mathcal{C}^{\mathrm{op}}}$ which preserves and reflects finite limits. Moreover, for each monomorphism $f\colon A \rightarrowtail B$ in $\mathcal{R}$ and each $X \in \mathcal{C}^{\mathrm{op}}$, $\phi(f)_X$ is a closed monomorphism in Part_M.

Proof. We would like to factorise the $\mathcal{T}$-enriched Yoneda embedding $Y_{\mathcal{T}}\colon \mathcal{C} \to \mathcal{T}\text{-}\mathrm{Alg}^{\mathcal{C}^{\mathrm{op}}}$ through $\mathrm{Part}_M^{\mathcal{C}^{\mathrm{op}}}$.

$$\begin{tikzcd}
& \mathrm{Part}_M^{\mathcal{C}^{\mathrm{op}}} \arrow[d, "U_{\mathcal{T}}^{\mathcal{C}^{\mathrm{op}}}"] \\
\mathcal{C} \arrow[ur, dotted, "\phi"] \arrow[r, "Y_{\mathcal{T}}"'] & \mathcal{T}\text{-}\mathrm{Alg}^{\mathcal{C}^{\mathrm{op}}}
\end{tikzcd}$$

In order to do so, let us provide $\mathcal{C}(X, Y)$ with a structure of partial M-algebra, for all objects $X, Y \in \mathcal{C}$. Thus, let $f_1, \ldots, f_b\colon X \to Y$ be morphisms in $\mathcal{C}$. We define $p(f_1, \ldots, f_b)$ if and only if there exist morphisms $x_1, \ldots, x_k\colon X \to W$, a relation $r\colon Z \rightarrowtail W^a$ in $\mathcal{R}$, and morphisms $g_1, \ldots, g_b\colon X \to Z$ and $f\colon Z \to Y$ such that, for each $j \in \{1, \ldots, b\}$, $fg_j = f_j$ and $rg_j = (t_{1j}(x_1, \ldots, x_k), \ldots, t_{aj}(x_1, \ldots, x_k))$.

$$\begin{tikzcd}
& X \arrow[dl, bend right, "{(t_{1j}(x_1,\ldots,x_k),\ldots,t_{aj}(x_1,\ldots,x_k))}"'] \arrow[d, "g_j"] \arrow[dr, bend left, "f_j"] & \\
W^a & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

In this case, since r is M-closed, there exists a unique morphism $h\colon X \to Z$ such that $rh = (u_1(x_1, \ldots, x_k), \ldots, u_a(x_1, \ldots, x_k))$ and we define $p(f_1, \ldots, f_b) = fh$.

$$\begin{tikzcd}
& Z \arrow[d, tail, "r"] \\
X \arrow[ur, dotted, "\exists!\, h"] \arrow[r, "{(u_1(x_1,\ldots,x_k),\ldots,u_a(x_1,\ldots,x_k))}"'] & W^a
\end{tikzcd}$$

Let us first prove the independence of the choices. Given $x'_1, \ldots, x'_k$:

$X \to W'$, $r' \colon Z' \rightarrowtail W'^a$, $g'_1, \dots, g'_b \colon X \to Z'$, $f' \colon Z' \to Y$ and $h' \colon X \to Z'$ which also satisfy the above conditions, let us prove $fh = f'h'$. We consider the following pullback

$$\begin{tikzcd}
Z_1 \arrow[r, "q_1"] \arrow[d, tail, "r_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, tail, "r"] \\
(W \times W')^a \arrow[r, "\pi_1^a"'] & W^a
\end{tikzcd}$$

where $\pi_1 \colon W \times W' \to W$ is the first projection. We also consider the unique morphisms $l_1^1, \dots, l_1^b \colon X \to Z_1$ such that $q_1 l_1^j = g_j$ and

$$r_1 l_1^j = (t_{1j}((x_1, x'_1), \dots, (x_k, x'_k)), \dots, t_{aj}((x_1, x'_1), \dots, (x_k, x'_k)))$$

for each $j \in \{1, \dots, b\}$. Let also $h_1 \colon X \to Z_1$ be the unique morphism such that $q_1 h_1 = h$ and

$$r_1 h_1 = (u_1((x_1, x'_1), \dots, (x_k, x'_k)), \dots, u_a((x_1, x'_1), \dots, (x_k, x'_k))).$$

Similarly, we define Z_2, r_2, q_2, $l_2^1, \dots, l_2^b$ and h_2 using the pullback of r' along π_2^a where $\pi_2 \colon W \times W' \to W'$ is the second projection. Since $\mathcal{R}$ is stable under pullbacks, $r_1, r_2 \in \mathcal{R}$. We also construct their intersection,

$$\begin{tikzcd}
P \arrow[r, tail, "r_3"] \arrow[d, tail, "r_4"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z_2 \arrow[d, tail, "r_2"] \\
Z_1 \arrow[r, tail, "r_1"'] & (W \times W')^a
\end{tikzcd}$$

the unique morphism $h_3 \colon X \to P$ such that $r_3 h_3 = h_2$ and $r_4 h_3 = h_1$ and, for each $j \in \{1, \dots, b\}$, the unique morphism $l_3^j \colon X \to P$ such that $r_3 l_3^j = l_2^j$ and $r_4 l_3^j = l_1^j$. Again, $r_3, r_4 \in \mathcal{R}$. Finally, we consider the following equaliser diagram.

$$\begin{tikzcd}
E \arrow[r, tail, "e"] & P \arrow[r, shift left, "f q_1 r_4"] \arrow[r, shift right, "f' q_2 r_3"'] & Y
\end{tikzcd}$$

For each $j \in \{1, \dots, b\}$, l_3^j factors as $e l_4^j = l_3^j$ since

$$f q_1 r_4 l_3^j = f q_1 l_1^j = f g_j = f_j = f' g'_j = f' q_2 l_2^j = f' q_2 r_3 l_3^j.$$

Hence, for each such j, the morphism

$$(t_{1j}((x_1,x_1'),\dots,(x_k,x_k')),\dots,t_{aj}((x_1,x_1'),\dots,(x_k,x_k')))$$

factors as $r_1r_4el_4^j$. But since the relation $r_1r_4e\colon E\rightarrowtail (W\times W')^a$ is in $\mathcal{R}$, it is M-closed and so there exists a unique morphism $l_5\colon X\to E$ such that

$$r_1r_4el_5=(u_1((x_1,x_1'),\dots,(x_k,x_k')),\dots,u_a((x_1,x_1'),\dots,(x_k,x_k'))).$$

The equalities $r_1r_4h_3=r_1h_1=r_1r_4el_5$ imply that $h_3=el_5$ and it remains to compute

$$\begin{aligned} fh &= fq_1h_1=fq_1r_4h_3=fq_1r_4el_5\\ &= f'q_2r_3el_5=f'q_2r_3h_3=f'q_2h_2\\ &= f'h'. \end{aligned}$$

Now that we have shown p is well-defined, let us prove it makes $\mathcal{C}(X,Y)$ a partial M-algebra. If $i\in\{1,\dots,a\}$ and $x_1,\dots,x_k\in\mathcal{C}(X,Y)$, we can set $W=Y$, $r=1_{Y^a}$,

$$g_j=(t_{1j}(x_1,\dots,x_k),\dots,t_{aj}(x_1,\dots,x_k)),$$

$f=\pi_i\colon Y^a\to Y$ the i-th projection and

$$h=(u_1(x_1,\dots,x_k),\dots,u_a(x_1,\dots,x_k)).$$

This shows that $p(t_{i1}(x_1,\dots,x_k),\dots,t_{ib}(x_1,\dots,x_k))$ is defined and equal to $fh=u_i(x_1,\dots,x_k)$.

Let now t be an n-ary term of $\mathcal{T}$ with $n>0$ and

$$(f_j^{j'}\in\mathcal{C}(X,Y))_{j\in\{1,\dots,b\},j'\in\{1,\dots,n\}}$$

be families of morphisms such that $p(f_1^{j'},\dots,f_b^{j'})$ is defined for each

$j' \in \{1, \ldots, n\}$ using the diagrams below.

$$\begin{tikzcd}
& X \arrow[dl, bend right, "{(t_{1j}(x_1^{j'},\ldots,x_k^{j'}),\ldots,t_{aj}(x_1^{j'},\ldots,x_k^{j'}))}"'] \arrow[d, "g_j^{j'}"] \arrow[dr, bend left, "f_j^{j'}"] & \\
(W_{j'})^a & Z_{j'} \arrow[l, tail, "r^{j'}"] \arrow[r, "f^{j'}"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
& Z_{j'} \arrow[d, tail, "r^{j'}"] \\
X \arrow[ur, "h^{j'}"] \arrow[r, "{(u_1(x_1^{j'},\ldots,x_k^{j'}),\ldots,u_a(x_1^{j'},\ldots,x_k^{j'}))}"'] & (W_{j'})^a
\end{tikzcd}$$

We consider the pullbacks

$$\begin{tikzcd}
S_{j'} \arrow[r, "q^{j'}"] \arrow[d, tail, "s^{j'}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z_{j'} \arrow[d, tail, "r^{j'}"] \\
(W_1 \times \cdots \times W_n)^a \arrow[r, "\pi^a_{j'}"'] & (W_{j'})^a
\end{tikzcd}$$

where $\pi_{j'} \colon W_1 \times \cdots \times W_n \to W_{j'}$ is the j'-th projection as usual. We denote by $l_j^{j'}$ the unique morphism $X \to S_{j'}$ such that $q^{j'} l_j^{j'} = g_j^{j'}$ and

$$s^{j'} l_j^{j'} = (t_{1j}(x_1, \ldots, x_k), \ldots, t_{aj}(x_1, \ldots, x_k))$$

where $x_{i'}$ is the factorisation $(x_{i'}^1, \ldots, x_{i'}^n) \colon X \to W_1 \times \cdots \times W_n$. Let also $h_1^{j'} : X \to S_{j'}$ be the unique morphism satisfying $q^{j'} h_1^{j'} = h^{j'}$ and

$$s^{j'} h_1^{j'} = (u_1(x_1, \ldots, x_k), \ldots, u_a(x_1, \ldots, x_k)).$$

We now consider the intersection of the $s^{j'}$'s

$$\begin{tikzcd}
& Z \arrow[dl, tail, "v^1"'] \arrow[dr, tail, "v^n"] & \\
S^1 \arrow[dr, tail, "s^1"'] & \cdots & S^n \arrow[dl, tail, "s^n"] \\
& (W^1 \times \cdots \times W^n)^a &
\end{tikzcd}$$

and the unique morphisms $l_j, h\colon X \to Z$ such that $v^{j'} l_j = l_j^{j'}$ and $v^{j'} h = h_1^{j'}$. Since this intersection can be built using pullbacks and compositions, $s^1 v^1 = \cdots = s^n v^n \in \mathcal{R}$. Thus, we end up with the commutative diagrams

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(x_1,\dots,x_k),\dots,t_{aj}(x_1,\dots,x_k))}"'] \arrow[d, "l_j"] \arrow[dr, bend left, "{t(f_j^1,\dots,f_j^n)}"] & \\
(W^1 \times \cdots \times W^n)^a & Z \arrow[l, tail, "s^1 v^1"] \arrow[r, "{t(f^1 q^1 v^1,\dots,f^n q^n v^n)}"'] & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & Z \arrow[d, tail, "s^1 v^1"] \\
X \arrow[ur, "h"] \arrow[r, "{(u_1(x_1,\dots,x_k),\dots,u_a(x_1,\dots,x_k))}"'] & (W^1 \times \cdots \times W^n)^a
\end{tikzcd}$$

proving that $p(t(f_1^1, \dots, f_1^n), \dots, t(f_b^1, \dots, f_b^n))$ is defined and equal to

$$t(f^1 q^1 v^1 h, \dots, f^n q^n v^n h) = t(p(f_1^1, \dots, f_b^1), \dots, p(f_1^n, \dots, f_b^n)).$$

If $n = 0$, we also have $p(t, \dots, t) = t$. To see it, we can use for instance the commutative diagram below.

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(1_X,\dots,1_X),\dots,t_{aj}(1_X,\dots,1_X))}"'] \arrow[d] \arrow[dr, bend left, "t"] & \\
X^a & X^a \arrow[l, tail, "1_{X^a}"] \arrow[r, "t"'] & Y
\end{tikzcd}$$

We have therefore provided $\mathcal{C}(X,Y)$ with a structure of partial M-algebra.

In view of the definition of a $\mathcal{T}$-enrichment, if $x\colon X' \to X$ and $y\colon Y \to Y'$ are morphisms in $\mathcal{C}$,

$$- \circ x\colon \mathcal{C}(X,Y) \to \mathcal{C}(X',Y)$$

and

$$y \circ -\colon \mathcal{C}(X,Y) \to \mathcal{C}(X,Y')$$

are homomorphisms of $\mathcal{T}$-algebras. Let us prove they are actually homomorphisms of partial M-algebras. So let $f_1, \dots, f_b\colon X \to Y$ be morphisms of $\mathcal{C}$ such that $p(f_1, \dots, f_b)$ is defined via the following diagrams.

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(x_1,\dots,x_k),\dots,t_{aj}(x_1,\dots,x_k))}"'] \arrow[d, "g_j"] \arrow[dr, bend left, "f_j"] & \\
W^a & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
 & Z \arrow[d, tail, "r"] \\
X \arrow[ur, "h"] \arrow[r, "{(u_1(x_1,\dots,x_k),\dots,u_a(x_1,\dots,x_k))}"'] & W^a
\end{tikzcd}$$

Thus, in view of the commutative diagrams

$$\begin{tikzcd}
 & X' \arrow[dl, bend right, "{(t_{1j}(x_1x,\dots,x_kx),\dots,t_{aj}(x_1x,\dots,x_kx))}"'] \arrow[d, "g_jx"] \arrow[dr, bend left, "f_jx"] & \\
W^a & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & Z \arrow[d, tail, "r"] \\
X' \arrow[ur, "hx"] \arrow[r, "{(u_1(x_1x,\dots,x_kx),\dots,u_a(x_1x,\dots,x_kx))}"'] & W^a
\end{tikzcd}$$

$p(f_1x, \dots, f_bx)$ is defined and equal to

$$fhx = p(f_1, \dots, f_b)x,$$

which shows that $- \circ x$ is a homomorphism of partial M-algebras. Besides, since the diagram

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(x_1,\dots,x_k),\dots,t_{aj}(x_1,\dots,x_k))}"'] \arrow[d, "g_j"] \arrow[dr, bend left, "yf_j"] & \\
W^a & Z \arrow[l, tail, "r"] \arrow[r, "yf"'] & Y'
\end{tikzcd}$$

commutes, $p(yf_1, \dots, yf_b)$ is defined and equal to

$$yfh = yp(f_1, \dots, f_b),$$

which proves that $y \circ -$ is a homomorphism of partial M-algebras. We have thus constructed a functor $\phi\colon \mathcal{C} \to \mathrm{Part}_M^{\mathcal{C}^{\mathrm{op}}}$ as announced.

$$\begin{tikzcd}
 & \mathrm{Part}_M^{\mathcal{C}^{\mathrm{op}}} \arrow[d, "U_{\mathcal{T}}^{\mathcal{C}^{\mathrm{op}}}"] \\
\mathcal{C} \arrow[ur, "\phi"] \arrow[r, "Y_{\mathcal{T}}"'] & \mathcal{T}\text{-}\mathrm{Alg}^{\mathcal{C}^{\mathrm{op}}}
\end{tikzcd}$$

This ϕ preserves $\mathcal{T}$-enrichment since $Y_{\mathcal{T}}$ and $U_{\mathcal{T}}$ do and $U_{\mathcal{T}}$ is faithful. It is full and faithful since $Y_{\mathcal{T}}$ is full and faithful and $U_{\mathcal{T}}$ is faithful.

Since ϕ is full and faithful, it reflects isomorphisms. Thus, it will reflect finite limits if it preserves them. So, let $(\lambda_D\colon L \to G(D))_{D\in\mathcal{D}}$ be the limit of $G\colon \mathcal{D} \to \mathcal{C}$ with $\mathcal{D}$ a finite category. We would like to prove that, for each $X \in \mathcal{C}$,

$$(\phi(\lambda_D)_X\colon \mathcal{C}(X, L) \to \mathcal{C}(X, G(D)))_{D\in\mathcal{D}}$$

is a limit in Part_M. But since $Y_{\mathcal{T}}$ preserves limits, and in view of the description of small limits in Part_M, we only have to prove that, if $f_1, \ldots, f_b\colon X \to L$ are such that $p(\lambda_D f_1, \ldots, \lambda_D f_b)$ is defined for every $D \in \mathcal{D}$, then $p(f_1, \ldots, f_b)$ is also defined.

Thus, to prove that ϕ preserves the terminal object, we have to show that $p(!, \ldots, !)$ is defined where ! is the unique morphism $X \to 1$. This is obvious in view of the diagram below.

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(!,\ldots,!),\ldots,t_{aj}(!,\ldots,!))}"'] \arrow[d, "!"] \arrow[dr, bend left, "!"] & \\
1^a = 1 & 1 \arrow[l, tail, "1_1"] \arrow[r, "1_1"'] & 1
\end{tikzcd}$$

Moreover, ϕ preserves the binary product $Y \times Y'$.

$$\begin{tikzcd}
 & Y \times Y' \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Y & & Y'
\end{tikzcd}$$

Indeed, suppose $f_1, \ldots, f_b\colon X \to Y$ and $f'_1, \ldots, f'_b\colon X \to Y'$ are such that

$p(f_1, \ldots, f_b)$ and $p(f'_1, \ldots, f'_b)$ are defined using the following diagrams.

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(x_1,\ldots,x_k),\ldots,t_{aj}(x_1,\ldots,x_k))}"'] \arrow[d, "g_j"] \arrow[dr, bend left, "f_j"] & \\
W^a & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(x'_1,\ldots,x'_k),\ldots,t_{aj}(x'_1,\ldots,x'_k))}"'] \arrow[d, "g'_j"] \arrow[dr, bend left, "f'_j"] & \\
W'^a & Z' \arrow[l, tail, "r'"] \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

We consider again the pullback

$$\begin{tikzcd}
Z_1 \arrow[r, "q_1"] \arrow[d, tail, "r_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, tail, "r"] \\
(W \times W')^a \arrow[r, "\pi_1^a"'] & W^a
\end{tikzcd}$$

and the unique morphisms $l_1^1, \ldots, l_1^b \colon X \to Z_1$ such that $q_1 l_1^j = g_j$ and

$$r_1 l_1^j = (t_{1j}((x_1, x'_1), \ldots, (x_k, x'_k)), \ldots, t_{aj}((x_1, x'_1), \ldots, (x_k, x'_k)))$$

for each $j \in \{1, \ldots, b\}$. We define similarly Z_2, r_2, q_2 and $l_2^1, \ldots, l_2^b$. We also consider the intersection

$$\begin{tikzcd}
P \arrow[r, tail, "r_3"] \arrow[d, tail, "r_4"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z_2 \arrow[d, tail, "r_2"] \\
Z_1 \arrow[r, tail, "r_1"'] & (W \times W')^a
\end{tikzcd}$$

and the unique morphisms $l_3^1, \ldots, l_3^b \colon X \to P$ such that $r_4 l_3^j = l_1^j$ and $r_3 l_3^j = l_2^j$ for each $j \in \{1, \ldots, b\}$. Then, since the diagram below is commutative,

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}((x_1,x'_1),\ldots,(x_k,x'_k)),\ldots,t_{aj}((x_1,x'_1),\ldots,(x_k,x'_k)))}"'] \arrow[d, "l_3^j"] \arrow[dr, bend left, "{(f_j,f'_j)}"] & \\
(W \times W')^a & P \arrow[l, tail, "r_1 r_4"] \arrow[r, "{(f q_1 r_4, f' q_2 r_3)}"'] & Y \times Y'
\end{tikzcd}$$

$p((f_1,f'_1),\ldots,(f_b,f'_b))$ is also defined and ϕ preserves finite products.

Finally, to prove that ϕ preserves equalisers, it is enough to show that $\phi(e)_X = e \circ - \colon \mathcal{C}(X,Y) \to \mathcal{C}(X,Y')$ is a closed homomorphism for each $X \in \mathcal{C}^{\mathrm{op}}$ and each regular monomorphism $e\colon Y \rightarrowtail Y'$. To conclude the proof, we are going to prove the more general fact that $\phi(e)_X$ is a closed homomorphism for each $e\colon Y \rightarrowtail Y'$ in $\mathcal{R}$ and each $X \in \mathcal{C}^{\mathrm{op}}$. So, let $f_1,\ldots,f_b\colon X \to Y$ be such that $p(ef_1,\ldots,ef_b)$ is defined using the diagram below.

$$\begin{tikzcd}
& X \arrow[dl, bend right, "{(t_{1j}(x_1,\ldots,x_k),\ldots,t_{aj}(x_1,\ldots,x_k))}"'] \arrow[d, "g_j"] \arrow[dr, bend left, "ef_j"] & \\
W^a & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y'
\end{tikzcd}$$

We consider the pullback of e along f

$$\begin{tikzcd}
Z' \arrow[r, "f'"] \arrow[d, tail, "r'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, tail, "e"] \\
Z \arrow[r, "f"'] & Y'
\end{tikzcd}$$

and the unique morphisms $g'_1,\ldots,g'_b\colon X \to Z'$ satisfying $f'g'_j = f_j$ and $r'g'_j = g_j$ for each $j \in \{1,\ldots,b\}$. Then, considering the diagram

$$\begin{tikzcd}
& X \arrow[dl, bend right, "{(t_{1j}(x_1,\ldots,x_k),\ldots,t_{aj}(x_1,\ldots,x_k))}"'] \arrow[d, "g'_j"] \arrow[dr, bend left, "f_j"] & \\
W^a & Z' \arrow[l, tail, "rr'"] \arrow[r, "f'"'] & Y
\end{tikzcd}$$

we see that $p(f_1,\ldots,f_b)$ is defined, which concludes the proof. □

Taking $\mathcal{R}$ to be the whole set of monomorphisms in $\mathcal{C}$, we immediately get the following corollary.

Corollary 4.53. [56] Let $\mathcal{T}$ be a commutative Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$. Let also $\mathcal{C}$ be a small finitely complete $\mathcal{T}$-category with M-closed relations. There exists a full and faithful $\mathcal{T}$-enriched embedding $\phi\colon \mathcal{C} \hookrightarrow \mathrm{Part}_M^{\mathcal{C}^{\mathrm{op}}}$ which preserves and reflects finite limits. Moreover, for each monomorphism $f\colon A \rightarrowtail B$ and each $X \in \mathcal{C}^{\mathrm{op}}$, $\phi(f)_X$ is a closed monomorphism in Part_M.

And now with $\mathcal{R}$ the set of strong monomorphisms.

Corollary 4.54. [56] Let $\mathcal{T}$ be a commutative Lawvere theory and M a simple extended matrix of terms in $\mathcal{T}$. Let also $\mathcal{C}$ be a small finitely complete $\mathcal{T}$-category with M-closed strong relations. There exists a full and faithful $\mathcal{T}$-enriched embedding $\phi\colon \mathcal{C} \hookrightarrow \mathrm{Part}_M^{\mathcal{C}^{\mathrm{op}}}$ which preserves and reflects finite limits. Moreover, for each strong monomorphism $f\colon A \rightarrowtail B$ and each $X \in \mathcal{C}^{\mathrm{op}}$, $\phi(f)_X$ is a closed monomorphism in Part_M.

Remark 4.55. Notice that Corollaries 4.53 and 4.54 characterise categories with M-closed relations (resp. with M-closed strong relations) among all small finitely complete $\mathcal{T}$-categories. Indeed, if we have such an embedding, to prove that a (strong) relation $r\colon R \rightarrowtail A^a$ in $\mathcal{C}$ is M-closed, it is enough to show that $\phi(r)_X$ is M-closed in Part_M for each $X \in \mathcal{C}^{\mathrm{op}}$, which is true by Proposition 4.51.

4.5.3 Applications

As usual, the Embedding Theorem 4.54 gives a way to use elements and partial operations to prove statements about finite limits in finitely complete $\mathcal{T}$-categories with M-closed strong relations. Suppose we are given a commutative Lawvere theory $\mathcal{T}$, a simple extended matrix M of terms in $\mathcal{T}$ and a statement of the form $P \Rightarrow Q$ where P and Q are conjunctions of properties which can be expressed as

1. some finite diagram is commutative,
2. some finite diagram is a limit diagram,
3. the equality $t(f_1, \ldots, f_n) = g$ holds for an n-ary term t of $\mathcal{T}$ and parallel morphisms $f_1, \ldots, f_n, g$,
4. some morphism is a monomorphism,
5. some morphism is an isomorphism,
6. some morphism factors through a given monomorphism.

Then, this statement $P \Rightarrow Q$ is valid in all finitely complete $\mathcal{V}$-$\mathcal{T}$-categories with M-closed strong relations (for all universes $\mathcal{V}$) if and only if it is valid in $\mathcal{V}\text{-Part}_M$ (for all universes $\mathcal{V}$).

Moreover, due to Corollary 4.53, if we have to prove this statement $P \Rightarrow Q$ in all finitely complete $\mathcal{T}$-categories with M-closed relations, it is enough to prove it in $\mathcal{V}\text{-Part}_M$ (for a bigger universe $\mathcal{V} \ni \mathcal{U}$) supposing that each monomorphism considered in the given statement is closed.

To conclude this chapter, we illustrate those two techniques in concrete examples. The first one takes place in the 'weakly strongly unital context', i.e., for pointed finitely complete categories with M-closed strong relations where

$$M = \left(\begin{array}{ccc|c} x & 0 & 0 & x \\ x & x & y & y \end{array}\right)$$

(see Example 2.19). This lemma has been proved in [15] in the strongly unital case, we now slightly improve it.

Lemma 4.56. Consider the following diagram in a pointed finitely complete category with $\left(\begin{array}{ccc|c} x & 0 & 0 & x \\ x & x & y & y \end{array}\right)$-closed strong relations

$$\begin{tikzcd}
 & X \times R \arrow[d, shift right, "1_X \times r_1"'] \arrow[d, shift left, "1_X \times r_2"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\
X \arrow[r, "{(1_X, 0)}"] \arrow[dr, shift left, "h"] & X \times Y \arrow[d, "\psi"] & Y \arrow[l, "{(0, 1_Y)}"'] \arrow[dl, "f"] \\
 & Z \arrow[ul, shift left, "g"] &
\end{tikzcd}$$

where $\psi(1_X, 0) = h$, $\psi(0, 1_Y) = f$, $gh = 1_X$, $gf = 0$ and (r_1, r_2) is the kernel pair of f. Then $(1_X \times r_1, 1_X \times r_2)$ is the kernel pair of ψ.

Proof. The above discussion tells us it is enough to prove it in Part_M for $M = \left(\begin{array}{ccc|c} x & 0 & 0 & x \\ x & x & y & y \end{array}\right)$. First of all, let us compute, for all $x \in X$ and all $y \in Y$,

$$\begin{aligned}\psi(x,y) &= \psi(p(x,0,0),p(0,0,y))\\ &= \psi(p((x,0),(0,0),(0,y)))\\ &= p(\psi(x,0),\psi(0,0),\psi(0,y))\\ &= p(h(x),0,f(y))\end{aligned}$$

which is always defined. Next, let $x, x' \in X$ and $y, y' \in Y$ be such that $\psi(x,y) = \psi(x',y')$. We have

$$\begin{aligned}x &= p(x,0,0) = p(gh(x),0,gf(y)) = g(\psi(x,y))\\ &= g(\psi(x',y')) = p(gh(x'),0,gf(y')) = p(x',0,0)\\ &= x'\end{aligned}$$

and

$$\begin{aligned}f(y) &= \psi(0,y) = \psi(p(x,x,0),p(y,0,0))\\ &= \psi(p((x,y),(x,0),(0,0))) = p(\psi(x,y),\psi(x,0),\psi(0,0))\\ &= p(\psi(x',y'),\psi(x',0),\psi(0,0)) = \psi(p(x',x',0),p(y',0,0))\\ &= \psi(0,y') = f(y').\end{aligned}$$

Then,

$$X \times R = \{(x,y_1,y_2) \in X \times Y \times Y \mid f(y_1) = f(y_2)\}$$

in which p is defined componentwise. For an element (x,y_1,y_2) in $X \times R$, we thus have

$$\begin{aligned}\psi(x,y_1) &= p(h(x),0,f(y_1))\\ &= p(h(x),0,f(y_2))\\ &= \psi(x,y_2).\end{aligned}$$

The kernel pair of ψ is given by

$$\{(x,y,x',y') \in X \times Y \times X \times Y \mid \psi(x,y) = \psi(x',y')\}$$

in which p is also defined componentwise. It is thus isomorphic to $X \times R$

via the mutually inverse homomorphisms $(x, y_1, y_2) \mapsto (x, y_1, x, y_2)$ and $(x, y, x', y') \mapsto (x, y, y')$. $\square$

To conclude, we prove a well-known fact in Mal'tsev categories.

Proposition 4.57. [29] In a Mal'tsev category, every internal category can be extended to a groupoid.

Proof. If

$$\begin{tikzcd} \mathbb{A} = (A_1 \times_{c,d} A_1 \arrow[r, "m"] & A_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & A_0 \arrow[l, bend left, "e"]) \end{tikzcd}$$

is an internal category in any finitely complete category $\mathcal{C}$, we can construct its object of isomorphisms $\mathrm{Iso}(\mathbb{A})$ via the finite limit below.

$$\begin{tikzcd}
 & & \mathrm{Iso}(\mathbb{A}) \arrow[ddll, dotted, "\lambda_1"'] \arrow[d, dotted, "\lambda_3"'] \arrow[ddrr, dotted, "\lambda_2"] \arrow[ddd, dotted, bend left, "\lambda_4"] & & \\
 & & A_1 \arrow[dddl, "d"'] \arrow[dddr, "c"] & & \\
A_1 \times_{c,d} A_1 \arrow[urr, "\pi_1"] \arrow[drr, "\pi_2"'] \arrow[dd, "m"'] & & & & A_1 \times_{c,d} A_1 \arrow[ull, "\pi_2"'] \arrow[dll, "\pi_1"] \arrow[dd, "m"] \\
 & & A_1 \arrow[dl, "c"] \arrow[dr, "d"'] & & \\
A_1 & A_0 \arrow[l, "e"] & & A_0 \arrow[r, "e"'] & A_1
\end{tikzcd}$$

Then, $\lambda_4 \colon \mathrm{Iso}(\mathbb{A}) \rightarrowtail A_1$ is a monomorphism and $\mathbb{A}$ extends to a groupoid if and only if λ_4 is an isomorphism (with $i = \lambda_3 \lambda_4^{-1} \colon A_1 \to A_1$). Moreover, there is at most one way to extend $\mathbb{A}$ to a groupoid. All these statements can be easily proved in Set and generalised to $\mathcal{C}$ by the Yoneda embedding.

Now, if $\mathcal{C}$ is a Mal'tsev category, we have to prove λ_4 is always an isomorphism. This is thus enough to prove it in Part_M with

$$M = \left(\begin{array}{ccc|c} x & y & y & x \\ x & x & y & y \end{array} \right).$$

Let us first prove that

$$A_1 \times_{c,d} A_1 \xrightarrow{(\pi_2,m)} A_1 \times A_1$$

is a monomorphism. So let

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

be morphisms in $\mathbb{A}$ such that $m(f,g) = m(f',g)$. Then

$$\begin{aligned} f &= m(f, 1_Y) \\ &= m(p(f, 1_Y, 1_Y), p(g, g, 1_Y)) \\ &= p(m(f,g), m(1_Y, g), m(1_Y, 1_Y)) \\ &= p(m(f',g), m(1_Y, g), m(1_Y, 1_Y)) \\ &= f' \end{aligned}$$

and (π_2, m) is a monomorphism. We can therefore suppose it is a closed homomorphism (using the last part in Corollary 4.53).

Let us now prove that every map $f \colon X \to Y$ in $\mathbb{A}$ is invertible (i.e., that λ_4 is surjective). We know that

$$p((1_Y, 1_Y), (f, 1_Y), (1_X, f)) \in A_1 \times_{c,d} A_1$$

is defined since $p(1_Y, 1_Y, f)$ and $p(1_Y, f, f)$ are and (π_2, m) is a closed monomorphism. Thus, applying π_1, we deduce that $p(1_Y, f, 1_X)$ is defined. It remains to compute

$$d(p(1_Y, f, 1_X)) = p(Y, X, X) = Y,$$

$$c(p(1_Y, f, 1_X)) = p(Y, Y, X) = X,$$

$$\begin{aligned} m(f, p(1_Y, f, 1_X)) &= m(p(f, 1_X, 1_X), p(1_Y, f, 1_X)) \\ &= p(m(f, 1_Y), m(1_X, f), m(1_X, 1_X)) \\ &= p(f, f, 1_X) \\ &= 1_X \end{aligned}$$

and similarly for $m(p(1_Y, f, 1_X), f) = 1_Y$. Therefore, λ_4 is bijective and can also be supposed to be closed. This means it is an isomorphism. $\square$

Chapter 5

Bicategory of fractions for weak equivalences

Let $F\colon \mathbb{A} \to \mathbb{B}$ be an internal functor between internal groupoids in Gp and $U(F)\colon U(\mathbb{A}) \to U(\mathbb{B})$ its underlying ordinary functor. If F is a weak equivalence (i.e., essentially surjective, full and faithful), $U(F)$ is also a weak equivalence and so it is an equivalence. Moreover, its pseudo-inverse $U(F)^*\colon U(\mathbb{B}) \to U(\mathbb{A})$ is a monoidal functor with respect to the tensor product given by the group laws in $\mathbb{A}$, but in general it fails to be an internal functor in Gp. Denoting by MON the 2-category of internal groupoids in Gp, monoidal functors and monoidal natural transformations, this phenomenon can be formalised [102] as the fact that the inclusion

$$\mathrm{Grpd}(\mathrm{Gp}) \hookrightarrow \mathrm{MON}$$

is the bicategory of fractions of Grpd(Gp) with respect to weak equivalences. A similar result has also been proved in [102] if we replace the category Gp by LieAlg_k for a field k and MON by its analogue for Lie algebras. Both Gp and LieAlg_k are monadic over regular categories (Set and Vect_k respectively) in which the Axiom of Choice holds (i.e., every regular epimorphism admits a section). As a generalisation, in [61], for any monad $\mathbb{T}$ on a regular category $\mathcal{C}$ satisfying the Axiom of Choice,

$\mathbb{T}$-monoidal functors are defined and the corresponding inclusion

$$\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) \hookrightarrow \mathbb{T}\text{-MON}$$

is proved to be the bicategory of fractions with respect to weak equivalences.

To get an idea on how $\mathbb{T}$-monoidal functors are defined, we can first restrict our attention to the case where the functor part $T\colon \mathcal{C} \to \mathcal{C}$ of the monad preserves pullbacks. In that case, $\mathbb{T}$ induces a pseudo-monad $\mathbf{T}$ on $\mathrm{Grpd}(\mathcal{C})$ whose 2-category $\mathrm{Alg}(\mathbf{T})$ of strict algebras and strict morphisms is isomorphic to $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$. Then, we define the 2-category $\mathbb{T}$-MON as the 2-category $\mathbf{T}$-MON of strict algebras and pseudo-morphisms of $\mathbf{T}$.

In the case where T does not preserve pullbacks, such a pseudo-monad $\mathbf{T}$ does not exist because T destroys the internal composition of an internal groupoid $\mathbb{A}$, so that $\mathbf{T}(\mathbb{A})$ is a reflexive graph but not an internal groupoid. Nevertheless, we can still define pseudo-morphisms since, for doing that, only internal natural transformations of the form

$$\begin{tikzcd}\mathbf{T}(\mathbb{A}) \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] & \mathbb{B} \arrow[Rightarrow, from=U, to=D, "\alpha"]\end{tikzcd}$$

are needed, and, to express the naturality of α, one uses the internal composition in $\mathbb{B}$, not in $\mathbf{T}(\mathbb{A})$.

This last chapter is devoted to the definition of $\mathbb{T}$-MON, its connections with the theory of pseudo-monads and the proof that it is the announced bicategory of fractions. We also compare it to the already known cases $\mathcal{C} = \mathrm{Gp}$ and $\mathcal{C} = \mathrm{LieAlg}_k$ from [102]. Observe that, since Gp and LieAlg_k are semi-abelian categories, the bicategories of fractions of Grpd(Gp) and of $\mathrm{Grpd}(\mathrm{LieAlg}_k)$ with respect to weak equivalences can also be described using 'butterflies', see [94] and [1].

5.1 2-dimensional category theory

In this section, we recall many definitions of 2-dimensional category theory we need in this chapter. The reader may consult [12, 74] or [14] for a more detailed account on the topic. We start with bicategories, a generalisation of 2-categories introduced by Bénabou [12]. Roughly speaking, these are 'categories for which the axioms are satisfied up to isomorphisms'.

Definition 5.1. [12] A *bicategory* $\mathcal{C}$ consists of

- a class of objects,
- for each pair A, B of objects, a small category $\mathcal{C}(A, B)$ (whose objects $f\colon A \to B$ are called *1-cells*, and morphisms *2-cells*),
- for any objects A, B, C, a composition law
$$\begin{aligned} \mathcal{C}(A,B) \times \mathcal{C}(B,C) &\longrightarrow \mathcal{C}(A,C) \\ (f,g) &\longmapsto g \circ f = gf, \end{aligned}$$
- for any 1-cells $\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & C \end{tikzcd}$, a *horizontal composition*
$$\begin{aligned} \mathcal{C}(A,B)(f,f') \times \mathcal{C}(B,C)(g,g') &\longrightarrow \mathcal{C}(A,C)(gf, g'f') \\ (\alpha, \beta) &\longmapsto \beta \star \alpha, \end{aligned}$$
- for each object A, a 1-cell $1_A\colon A \to A$,
- for each 1-cell $f\colon A \to B$, two invertible 2-cells, natural in f, $l_f\colon 1_B \circ f \Rightarrow f$ and $r_f\colon f \circ 1_A \Rightarrow f$,
- for any 1-cells $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$, an invertible 2-cell, natural in f, g and h, $a_{h,g,f}\colon (hg)f \Rightarrow h(gf)$

satisfying

1. $1_g \star 1_f = 1_{gf}$ for all 1-cells $f\colon A \to B$ and $g\colon B \to C$,
2. $(\delta \star \beta)(\gamma \star \alpha) = (\delta\gamma) \star (\beta\alpha)$ for any diagram $\begin{tikzcd} A \arrow[r, bend left=70, "f", ""{name=F, below}] \arrow[r, "f'" description, ""{name=F1}] \arrow[r, bend right=70, "f''"', ""{name=F2, above}] & B \arrow[r, bend left=70, "g", ""{name=G, below}] \arrow[r, "g'" description, ""{name=G1}] \arrow[r, bend right=70, "g''"', ""{name=G2, above}] & C \arrow[Rightarrow, from=F, to=F1, "\alpha"] \arrow[Rightarrow, from=F1, to=F2, "\beta"] \arrow[Rightarrow, from=G, to=G1, "\gamma"] \arrow[Rightarrow, from=G1, to=G2, "\delta"] \end{tikzcd}$,

3. the Pentagon Axiom: $(1_k \star a_{h,g,f})a_{k,hg,f}(a_{k,h,g} \star 1_f) = a_{k,h,gf}a_{kh,g,f}$ for any 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$,
4. the Triangle Axiom: $(1_g \star l_f)a_{g,1_B,f} = r_g \star 1_f$ for any 1-cells $f\colon A \to B$ and $g\colon B \to C$.

As for 2-categories, an invertible 2-cell is called a *2-isomorphism*. Besides the horizontal composition, the composition of 2-cells in a category $\mathcal{C}(A,B)$ is called the *vertical composition*. The bicategory is said to be *small* if there is only a set of objects. Homomorphisms between bicategories (as named in [12]) are called pseudo-functors in [14].

Definition 5.2. [12] Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. A *pseudo-functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of:

- for each object $A \in \mathcal{C}$, an object $F(A) \in \mathcal{D}$,
- for any pair A, B of objects, a functor

$$F\colon \mathcal{C}(A,B) \to \mathcal{D}(F(A),F(B)),$$

- for each object $A \in \mathcal{C}$, a 2-isomorphism $\varphi^F_A\colon 1_{F(A)} \Rightarrow F(1_A)$,
- for any 1-cells $f\colon A \to B$ and $g\colon B \to C$, a 2-isomorphism, natural in f and g, $\varphi^F_{g,f}\colon F(g)F(f) \Rightarrow F(gf)$

such that

1. $\varphi^F_{h,gf}(1_{F(h)} \star \varphi^F_{g,f})a_{F(h),F(g),F(f)} = F(a_{h,g,f})\varphi^F_{hg,f}(\varphi^F_{h,g} \star 1_{F(f)})$ for any 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$,
2. $F(r_f)\varphi^F_{f,1_A}(1_{F(f)} \star \varphi^F_A) = r_{F(f)}$ and $F(l_f)\varphi^F_{1_B,f}(\varphi^F_B \star 1_{F(f)}) = l_{F(f)}$ for any 1-cell $f\colon A \to B$.

Small bicategories and pseudo-functors form a category which we denote by BiCat.

Example 5.3. A bicategory where all the isomorphisms $a_{h,g,f}$, l_f and r_f are identities is exactly a 2-category. A pseudo-functor $F\colon \mathcal{C} \to \mathcal{D}$ between 2-categories for which all the isomorphisms φ^F_A and $\varphi^F_{f,g}$ are identities is called a *2-functor*.

Example 5.4. A bicategory $\mathcal{C}$ with one object $*$ can be seen as a small monoidal category when looking at the category $\mathcal{C}(*,*)$. A pseudo-functor between one-object-bicategories is then nothing but a monoidal functor.

We now recall the notions of a pseudo-natural transformation, a modification and a biequivalence.

Definition 5.5. Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be pseudo-functors between bicategories. A *pseudo-natural transformation* $\theta\colon F \Rightarrow G$ consists of

- for each object $A \in \mathcal{C}$, a 1-cell $\theta_A\colon F(A) \to G(A)$ in $\mathcal{D}$,
- for each 1-cell $f\colon A \to B$ in $\mathcal{C}$, a 2-isomorphism, natural in f, $\tau^\theta_f\colon G(f)\theta_A \Rightarrow \theta_B F(f)$

such that

1. $\tau^\theta_{1_A}(\varphi^G_A \star 1_{\theta_A}) = (1_{\theta_A} \star \varphi^F_A) r^{-1}_{\theta_A} l_{\theta_A}$ for each object $A \in \mathcal{C}$,
2. for any 1-cells $f\colon A \to B$ and $g\colon B \to C$ in $\mathcal{C}$,

$$\begin{aligned}&(1_{\theta_C} \star \varphi^F_{g,f}) a_{\theta_C, F(g), F(f)} (\tau^\theta_g \star 1_{F(f)}) a^{-1}_{G(g),\theta_B,F(f)} (1_{G(g)} \star \tau^\theta_f)\\ &= \tau^\theta_{gf}(\varphi^G_{g,f} \star 1_{\theta_A}) a^{-1}_{G(g),G(f),\theta_A}.\end{aligned}$$

If $\mathcal{C}$ and $\mathcal{D}$ are 2-categories, F and G 2-functors and if τ^θ_f is the identity for each f, θ is then called a *2-natural transformation*. It is thus a natural transformation between the underlying functors such that $1_{\theta_B} \star F(\alpha) = G(\alpha) \star 1_{\theta_A}$ for each 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$.

Definition 5.6. Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be pseudo-functors between bicategories and $\theta, \psi\colon F \Rightarrow G$ two pseudo-natural transformations. A *modification* $\Xi\colon \theta \rightsquigarrow \psi$ is a family of 2-cells $(\Xi_A\colon \theta_A \Rightarrow \psi_A)_{A \in \mathcal{C}}$ in $\mathcal{D}$ such that for each 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$ in $\mathcal{C}$, $(\Xi_B \star F(\alpha))\tau^\theta_f = \tau^\psi_g(G(\alpha) \star \Xi_A)$.

A modification Ξ for which Ξ_A is a 2-isomorphism for each object $A \in \mathcal{C}$ is called an *isomodification*. If $\mathcal{C}$ and $\mathcal{D}$ are bicategories with

$\mathcal{C}$ small, pseudo-functors $\mathcal{C} \to \mathcal{D}$, pseudo-natural transformations and modifications form a bicategory denoted by $\mathrm{PsFunct}(\mathcal{C}, \mathcal{D})$.

Definition 5.7. A pseudo-functor $F \colon \mathcal{C} \to \mathcal{D}$ between bicategories is a *biequivalence* if there exists

- a pseudo functor $G \colon \mathcal{D} \to \mathcal{C}$,
- four pseudo-natural transformations $\theta_1 \colon GF \Rightarrow 1_{\mathcal{C}}$, $\theta_2 \colon 1_{\mathcal{C}} \Rightarrow GF$, $\theta_3 \colon FG \Rightarrow 1_{\mathcal{D}}$ and $\theta_4 \colon 1_{\mathcal{D}} \Rightarrow FG$,
- four isomodifications $\Xi_1 \colon \theta_1\theta_2 \rightsquigarrow 1_{1_{\mathcal{C}}}$, $\Xi_2 \colon \theta_2\theta_1 \rightsquigarrow 1_{GF}$, $\Xi_3 \colon \theta_3\theta_4 \rightsquigarrow 1_{1_{\mathcal{D}}}$ and $\Xi_4 \colon \theta_4\theta_3 \rightsquigarrow 1_{FG}$.

We can also define equivalences internally to a bicategory.

Definition 5.8. Let $\mathcal{C}$ be a bicategory. A 1-cell $f \colon A \to B$ in $\mathcal{C}$ is an *equivalence* if there exists a 1-cell $g \colon B \to A$ and two 2-isomorphisms $\eta \colon 1_A \Rightarrow gf$ and $\varepsilon \colon fg \Rightarrow 1_B$.

If such 2-isomorphisms exist, one can always choose them such that they satisfy the triangular identities $(\varepsilon \star 1_f)a^{-1}_{f,g,f}(1_f \star \eta)r^{-1}_f = l^{-1}_f$ and $(1_g \star \varepsilon)a_{g,f,g}(\eta \star 1_g)l^{-1}_g = r^{-1}_g$.

In [96], Pronk defined bicategories of fractions as the 2-dimensional analogue to the categories of fractions introduced by Gabriel and Zisman in [45]. The idea is to universally send some 1-cells to equivalences.

Definition 5.9. [96] Let $\mathcal{C}$ be a bicategory and W a class of 1-cells in $\mathcal{C}$. A *bicategory of fractions of $\mathcal{C}$ with respect to W* is a bicategory $\mathcal{C}[W^{-1}]$ together with a pseudo-functor $P_W \colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ satisfying the following conditions

1. $P_W(w)$ is an equivalence for each $w \in W$,
2. for any bicategory $\mathcal{D}$, the $\mathcal{V}$-pseudo-functor (for a bigger universe $\mathcal{V} \ni \mathcal{U}$)

$$- \circ P_W \colon \mathrm{PsFunct}(\mathcal{C}[W^{-1}], \mathcal{D}) \to \mathrm{PsFunct}_W(\mathcal{C}, \mathcal{D})$$

acting by precomposition with P_W is a biequivalence, where $\mathrm{PsFunct}_W(\mathcal{C}, \mathcal{D})$ is the $\mathcal{V}$-bicategory of pseudo-functors $\mathcal{C} \to \mathcal{D}$

which sends elements of W to equivalences, pseudo-natural transformations and modifications.

This definition is independent of the bigger universe $\mathcal{V}$ we choose. Moreover, the bicategory of fractions is unique up to biequivalence. In the same way, admitting a right calculus of fractions for a class of 1-cells in a bicategory is the 2-dimensional version of the 1-dimensional case.

Definition 5.10. [96] Let $\mathcal{C}$ be a bicategory and W a class of 1-cells in $\mathcal{C}$. We say that W *admits a right calculus of fractions* if it satisfies the following conditions:

1. all equivalences are in W,
2. W is stable by composition,
3. W is closed under 2-isomorphisms,
4. if $w\colon A \to B$ is in W and $f\colon C \to B$ is a 1-cell of $\mathcal{C}$, there exists a 1-cell $h\colon D \to A$ of $\mathcal{C}$ and an element $v\colon D \to C$ of W such that wh is 2-isomorphic to fv,
5. if $f, g\colon A \rightrightarrows B$ are 1-cells of $\mathcal{C}$, $w\colon B \to C$ an element of W and $\alpha\colon wf \Rightarrow wg$ a 2-cell (resp. a 2-isomorphism) of $\mathcal{C}$, there exists $v\colon D \to A$ in W and a 2-cell (resp. a 2-isomorphism) $\beta\colon fv \Rightarrow gv$ such that $a_{w,g,v}(\alpha \star 1_v) = (1_w \star \beta)a_{w,f,v}$. If $(v\colon D \to A, \beta\colon fv \Rightarrow gv)$ and $(v'\colon D' \to A, \beta'\colon fv' \Rightarrow gv')$ are two such pairs, we also require the existence of 1-cells $u\colon E \to D$, $u'\colon E \to D'$ and a 2-isomorphism $\varepsilon\colon vu \Rightarrow v'u'$ in $\mathcal{C}$ such that $vu \in W$ and
$$a^{-1}_{g,v',u'}(1_g \star \varepsilon)a_{g,v,u}(\beta \star 1_u) = (\beta' \star 1_{u'})a^{-1}_{f,v',u'}(1_f \star \varepsilon)a_{f,v,u}.$$

Proposition 5.11. [96] Let $\mathcal{C}$ be a bicategory and W a class of 1-cells in $\mathcal{C}$ which admits a right calculus of fractions. Then, the bicategory of fractions $\mathcal{C} \to \mathcal{C}[W^{-1}]$ exists. Moreover, consider a pseudo-functor $F\colon \mathcal{C} \to \mathcal{D}$ which sends elements of W to equivalences and let $\overline{F}\colon \mathcal{C}[W^{-1}] \to \mathcal{D}$ be its extension. Suppose the following conditions hold.

1. F is essentially surjective on objects (i.e., for each object $D \in \mathcal{D}$, there exists an object $A \in \mathcal{C}$ and an equivalence $F(A) \to D$),

2. F is full and faithful on 2-cells (i.e., each functor $F\colon \mathcal{C}(A,B) \to \mathcal{D}(F(A),F(B))$ is full and faithful),
3. for each 1-cell $f\colon F(A) \to F(B)$ in $\mathcal{D}$, there exist 1-cells $g\colon C \to B$ in $\mathcal{C}$ and $w\colon C \to A$ in W such that $F(g)$ is 2-isomorphic to $fF(w)$.

Then, $\overline{F}$ is a biequivalence and $F\colon \mathcal{C} \to \mathcal{D}$ the bicategory of fractions of $\mathcal{C}$ with respect to W.

We end this section by recalling the notion of a strong homotopy-pullback.

Definition 5.12. The diagram

$$\begin{tikzcd} P \arrow[r, "\pi_C"] \arrow[d, "\pi_A"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] \arrow[ur, Rightarrow, "\sim", "\omega"'] & B \end{tikzcd}$$

in a bicategory $\mathcal{C}$ (where $\omega\colon f\pi_A \Rightarrow g\pi_C$ is a 2-isomorphism) is a *strong homotopy-pullback* if

1. for any diagram

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "h"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] \arrow[ur, Rightarrow, "\sim", "\theta"'] & B \end{tikzcd}$$

   with $\theta\colon fh \Rightarrow gk$ being a 2-isomorphism, there exists a unique 1-cell $l\colon X \to P$ such that $\pi_A l = h$, $\pi_C l = k$ and $a_{g,\pi_C,l}(\omega \star 1_l) = \theta a_{f,\pi_A,l}$,

2. given two 1-cells $l, l'\colon X \rightrightarrows P$ and two 2-cells (resp. two 2-isomorphisms) $\alpha\colon \pi_A l \Rightarrow \pi_A l'$, $\beta\colon \pi_C l \Rightarrow \pi_C l'$ satisfying

$$a^{-1}_{g,\pi_C,l'}(1_g \star \beta)a_{g,\pi_C,l}(\omega \star 1_l) = (\omega \star 1_{l'})a^{-1}_{f,\pi_A,l'}(1_f \star \alpha)a_{f,\pi_A,l},$$

   there exists a unique 2-cell (resp. 2-isomorphism) $\gamma\colon l \Rightarrow l'$ such that $1_{\pi_A} \star \gamma = \alpha$ and $1_{\pi_C} \star \gamma = \beta$.

When only condition 1 is satisfied, P is usually called a homotopy-pullback. Compare for example with [50], or with [102] where bipullbacks are considered. Note also that strong homotopy-pullbacks are unique up to equivalence.

5.2 Weak equivalences

With classical categories, a functor is an equivalence if and only if it is essentially surjective, full and faithful. This is not the case any more with internal categories. These functors are then called weak equivalences.

Definition 5.13. Let $\mathcal{C}$ be a category with finite limits and $\mathbb{A}$ and $\mathbb{B}$ two internal categories (resp. internal groupoids) in $\mathcal{C}$. An *internal functor* $F\colon \mathbb{A} \to \mathbb{B}$ is given by two morphisms $F_0\colon A_0 \to B_0$ and $F_1\colon A_1 \to B_1$ such that the downward squares

$$\begin{tikzcd}
A_1 \times_{c,d} A_1 \arrow[r, "m"] \arrow[d, "F_1 \times_{c,d} F_1"'] & A_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] \arrow[d, "F_1"'] & A_0 \arrow[l, bend left, "e"] \arrow[d, "F_0"] \\
B_1 \times_{c,d} B_1 \arrow[r, "m"'] & B_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & B_0 \arrow[l, bend left, "e"]
\end{tikzcd}$$

commute.

In the case where $\mathbb{A}$ and $\mathbb{B}$ are groupoids, this implies that $F_1 i = i F_1$.

Example 5.14. If $\mathcal{C} = \mathrm{Gp}$, internal categories in Gp can be seen as particular cases of monoidal categories, where the tensor product is given by the group laws on the sets of objects and of arrows. Then, internal functors can also be seen as particular cases of monoidal functors. While internal functors $\mathbb{A} \to \mathbb{B}$ preserve the group structures of A_0 and A_1, a monoidal functor $F\colon \mathbb{A} \to \mathbb{B}$ preserves the group structure of A_0 only up to (coherent) isomorphisms. This means that for each pair of objects $X, Y \in A_0$, we have an isomorphism $F(X) + F(Y) \cong F(X + Y)$ in $\mathbb{B}$ satisfying some commutativity axioms. When these isomorphisms are identities, the monoidal functor is actually an internal functor.

Definition 5.15. Let $\mathcal{C}$ be a category with finite limits and $F, G\colon \mathbb{A} \rightrightarrows \mathbb{B}$ two internal functors in $\mathcal{C}$. An *internal natural transformation* $\alpha\colon F \Rightarrow G$

is given by a morphism $\alpha\colon A_0 \to B_1$ making the diagrams

$$\begin{tikzcd}
 & A_0 \arrow[dl, "\alpha"'] \arrow[d, "F_0"] \\
B_1 \arrow[r, "d"'] & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A_0 \arrow[dl, "\alpha"'] \arrow[d, "G_0"] \\
B_1 \arrow[r, "c"'] & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "{(F_1,\alpha c)}"] \arrow[d, "{(\alpha d,G_1)}"'] & B_1 \times_{c,d} B_1 \arrow[d, "m"] \\
B_1 \times_{c,d} B_1 \arrow[r, "m"'] & B_1
\end{tikzcd}$$

commutative.

Together with $1_F = eF_0$, $\beta\alpha = m(\alpha,\beta)\colon A_0 \to B_1$ for $\alpha\colon F \Rightarrow G$, $\beta\colon G \Rightarrow H$ and $\alpha' \star \alpha = m(F_1'\alpha, \alpha' G_0)\colon A_0 \to C_1$ for

$$\begin{tikzcd}
\mathbb{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"] & \mathbb{B} \arrow[r, bend left, "F'"{name=U2}] \arrow[r, bend right, "G'"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\alpha'"] & \mathbb{C}
\end{tikzcd},$$

this forms the 2-categories Cat($\mathcal{C}$) and Grpd($\mathcal{C}$) of internal categories (resp. internal groupoids) in $\mathcal{C}$, internal functors and internal natural transformations. Note that every 2-cell in Grpd($\mathcal{C}$) is a 2-isomorphism.

Definition 5.16. [26] Let $\mathcal{C}$ be a category with finite limits and $F\colon \mathbb{A} \to \mathbb{B}$ an internal functor between internal groupoids in $\mathcal{C}$.

- We say that F is *full and faithful* if the diagram

$$\begin{tikzcd}
 & & A_1 \arrow[dll, "d"'] \arrow[d, "F_1"] \arrow[drr, "c"] & & \\
A_0 \arrow[dr, "F_0"'] & & B_1 \arrow[dl, "d"] \arrow[dr, "c"'] & & A_0 \arrow[dl, "F_0"] \\
 & B_0 & & B_0 &
\end{tikzcd}$$

  is a limit in $\mathcal{C}$.

Moreover, if $\mathcal{C}$ is regular,

- F is said to be *essentially surjective* if

$$A_0 \times_{F_0,d} B_1 \xrightarrow{t_2} B_1 \xrightarrow{c} B_0$$

is a regular epimorphism, where t_2 is the pullback of F_0 along d.

$$\begin{tikzcd}
A_0 \times_{F_0,d} B_1 \arrow[r, "t_2"] \arrow[d, "t_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d, "d"] \\
A_0 \arrow[r, "F_0"'] & B_0
\end{tikzcd}$$

- F is a *weak equivalence* if it is essentially surjective, full and faithful.

We notice that if $\mathcal{C} = \text{Set}$, this corresponds to the usual notion of a fully faithful and essentially surjective functor between small groupoids. In general, one needs the Axiom of Choice in $\mathcal{C}$ to build an inverse to those functors.

Definition 5.17. The *Axiom of Choice* holds in a regular category when every regular epimorphism is a split epimorphism.

Lemma 5.18. [102] Let $\mathcal{C}$ be a finitely complete category. An internal functor $F\colon \mathbb{A} \to \mathbb{B}$ between internal groupoids in $\mathcal{C}$ is an equivalence if and only if it is full and faithful and the morphism

$$A_0 \times_{F_0,d} B_1 \xrightarrow{t_2} B_1 \xrightarrow{c} B_0$$

from Definition 5.16 is a split epimorphism.

This lemma immediately implies the following proposition.

Proposition 5.19. [102] An internal functor $F\colon \mathbb{A} \to \mathbb{B}$ between internal groupoids in a regular category where the Axiom of Choice holds is an equivalence if and only if it is a weak equivalence.

Our goal is to describe the bicategory of fractions of $\text{Grpd}(\mathcal{C})$ with respect to weak equivalences (for some monadic category $\mathcal{C}$). In order to apply Proposition 5.11, we need a right calculus of fractions.

Proposition 5.20. [102] Let $\mathcal{C}$ be a regular category and W the class of weak equivalences in $\text{Grpd}(\mathcal{C})$. Then W admits a right calculus of fractions.

It is also worthy to remark that a pullback preserving functor $U\colon \mathcal{C} \to \mathcal{D}$ between finitely complete categories induces a 2-functor (also denoted U by abuse of notations)

$$\begin{aligned} U\colon\ \mathrm{Grpd}(\mathcal{D}) &\longrightarrow \mathrm{Grpd}(\mathcal{C}) \\ \mathbb{A} &\longmapsto U(\mathbb{A}) = (U(A_0), U(A_1), U(d), U(c), U(e), U(m), U(i)) \\ F &\longmapsto U(F) = (U(F_0), U(F_1)) \\ \alpha &\longmapsto U(\alpha). \end{aligned}$$

Lemma 5.21. Let $U\colon \mathcal{D} \to \mathcal{C}$ be a pullback preserving functor between finitely complete categories. Let also $F\colon \mathbb{A} \to \mathbb{B}$ be an internal functor between internal groupoids in $\mathcal{D}$. Then,

1. if U reflects pullbacks, F is full and faithful if and only if $U(F)$ is,
2. if $\mathcal{C}$ and $\mathcal{D}$ are regular and if U preserves and reflects regular epimorphisms, then F is essentially surjective if and only if $U(F)$ is.

Proof. The 'only if parts' follow from the preserving hypotheses while the 'if parts' follow from the reflecting hypotheses. ☐

5.3 $\mathbb{T}$-monoidal functors

As already written in Example 5.14, monoidal functors $F\colon \mathbb{A} \to \mathbb{B}$ between internal groupoids in Gp can be seen as 'pseudo-internal functors'. Indeed, since for all $X, Y \in \mathbb{A}$ we have only an isomorphism $F(X) + F(Y) \cong F(X+Y)$, F is not an internal functor in Gp, but only an 'internal functor up to isomorphisms'. The aim of this section is to generalise this notion of a 'pseudo-internal functor' replacing Gp by any monadic category.

So, we are given a monad $\mathbb{T} = (T, \eta, \mu)$ on a finitely complete category $\mathcal{C}$. By Proposition 1.59, the forgetful functor $U^{\mathbb{T}}\colon \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ preserves, reflects and creates finite limits. It thus induces a 2-functor $U^{\mathbb{T}}\colon\ \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) \to \mathrm{Grpd}(\mathcal{C})$. If $\mathbb{A}$ is a groupoid in $\mathcal{C}^{\mathbb{T}}$ and if the groupoid

$U^{\mathbb{T}}(\mathbb{A})$ in $\mathcal{C}$ is given by

$$\begin{tikzcd}
A_1 \times_{c,d} A_1 \arrow[r, "m"] & A_1 \arrow[loop above, "i"] \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & A_0 \arrow[l, bend left=40, "e"]
\end{tikzcd}$$

then $\mathbb{A}$ is of the form

$$\begin{tikzcd}
(A_1 \times_{c,d} A_1, (a_1T(\pi_1), a_1T(\pi_2))) \arrow[r, "m"] & (A_1, a_1) \arrow[loop above, "i"] \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & (A_0, a_0) \arrow[l, bend left=40, "e"]
\end{tikzcd}$$

where $a_1\colon T(A_1) \to A_1$ and $a_0\colon T(A_0) \to A_0$ are $\mathbb{T}$-algebras and d, c, e, m, i are $\mathbb{T}$-homomorphisms. In particular, this means that the square

$$\begin{tikzcd}
T(A_1 \times_{c,d} A_1) \arrow[r, "T(m)"] \arrow[d, "{(a_1T(\pi_1), a_1T(\pi_2))}"'] & T(A_1) \arrow[d, "a_1"] \\
A_1 \times_{c,d} A_1 \arrow[r, "m"'] & A_1
\end{tikzcd}$$

commutes since the left downward morphism is the arrow part of $(A_1, a_1) \times_{c,d} (A_1, a_1)$. We can now define $\mathbb{T}$-monoidal functors.

Definition 5.22. [61] Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a finitely complete category $\mathcal{C}$. We define the 2-category $\mathbb{T}$-MON as follows:

- Objects are internal groupoids in $\mathcal{C}^{\mathbb{T}}$.

- 1-cells are $\mathbb{T}$-*monoidal functors* $(F, \varphi)\colon \mathbb{A} \to \mathbb{B}$. These are internal functors $F\colon U^{\mathbb{T}}(\mathbb{A}) \to U^{\mathbb{T}}(\mathbb{B})$ in $\mathcal{C}$ together with a morphism $\varphi\colon T(A_0) \to B_1$ such that the five diagrams

$$\begin{tikzcd}
T(A_0) \arrow[r, "\varphi"] \arrow[d, "T(F_0)"'] & B_1 \arrow[d, "d"] & & T(A_0) \arrow[r, "\varphi"] \arrow[d, "a_0"'] & B_1 \arrow[d, "c"] \\
T(B_0) \arrow[r, "b_0"'] & B_0 & & A_0 \arrow[r, "F_0"'] & B_0
\end{tikzcd}$$

$$\begin{tikzcd}
T(A_1) \arrow[r, "{(b_1T(F_1),\varphi T(c))}"] \arrow[d, "{(\varphi T(d),F_1a_1)}"'] & B_1 \times_{c,d} B_1 \arrow[d, "m"] \\
B_1 \times_{c,d} B_1 \arrow[r, "m"'] & B_1
\end{tikzcd}$$

$$\begin{tikzcd}
A_0 \arrow[r, "\eta_{A_0}"] \arrow[d, "e"'] & T(A_0) \arrow[d, "\varphi"] & \text{and} & TT(A_0) \arrow[r, "\mu_{A_0}"] \arrow[d, "{(b_1T(\varphi),\varphi T(a_0))}"'] & T(A_0) \arrow[d, "\varphi"] \\
A_1 \arrow[r, "F_1"'] & B_1 & & B_1 \times_{c,d} B_1 \arrow[r, "m"'] & B_1
\end{tikzcd}$$

commute.

- $1_{\mathbb{A}} = (1_{U^{\mathbb{T}}(\mathbb{A})}, ea_0)$.

- The composition of $\mathbb{A} \xrightarrow{(F,\varphi)} \mathbb{B} \xrightarrow{(G,\psi)} \mathbb{C}$ is $(GF, m(\psi T(F_0), G_1\varphi))$.

- 2-cells $\alpha\colon (F,\varphi) \Rightarrow (F',\varphi')\colon \mathbb{A} \to \mathbb{B}$ are $\mathbb{T}$-*monoidal natural transformations*. These are internal natural transformations $\alpha\colon F \Rightarrow F'$ in $\mathcal{C}$ such that the square

$$\begin{tikzcd}
T(A_0) \arrow[r, "{(\varphi,\alpha a_0)}"] \arrow[d, "{(b_1T(\alpha),\varphi')}"'] & B_1 \times_{c,d} B_1 \arrow[d, "m"] \\
B_1 \times_{c,d} B_1 \arrow[r, "m"'] & B_1
\end{tikzcd}$$

commutes.

- Identities, vertical and horizontal compositions of 2-cells are computed as in Grpd($\mathcal{C}$).

Using the Yoneda embedding, it is easy (but lengthy) to prove that $\mathbb{T}$-MON is actually a 2-category. Moreover, each of its 2-cells is a 2-isomorphism since if $\alpha\colon (F,\varphi) \Rightarrow (F',\varphi')$ is $\mathbb{T}$-monoidal, then so is α^{-1}.

Diagrams involved in Definition 5.22 might be thought as unintuitive at a first glance. The example of the free group monad on Set is treated in Section 5.5 while an explanation where these axioms come from can be found in Section 5.7 in the context of strict algebras for a pseudo-monad.

Remark that we have two 2-functors

$$\begin{array}{rcl}
I\colon \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) & \hookrightarrow & \mathbb{T}\text{-MON} \\
\mathbb{A} & \longmapsto & \mathbb{A} \\
F & \longmapsto & (U^{\mathbb{T}}(F), eb_0T(F_0)) \\
\alpha & \longmapsto & U^{\mathbb{T}}(\alpha)
\end{array}
\qquad \text{and} \qquad
\begin{array}{rcl}
J\colon \mathbb{T}\text{-MON} & \longrightarrow & \mathrm{Grpd}(\mathcal{C}) \\
\mathbb{A} & \longmapsto & U^{\mathbb{T}}(\mathbb{A}) \\
(F,\varphi) & \longmapsto & F \\
\alpha & \longmapsto & \alpha.
\end{array}$$

Thus, by abuse of notation, we say that a $\mathbb{T}$-monoidal functor (F, φ) : $\mathbb{A} \to \mathbb{B}$ is internal in $\mathcal{C}^{\mathbb{T}}$ when $\varphi = eb_0T(F_0)$. We will often identify an internal functor F in $\mathcal{C}^{\mathbb{T}}$ with $(U^{\mathbb{T}}(F), eb_0T(F_0))$.

It is a well-known fact that, if a monoidal functor between monoidal categories has a pseudo-inverse, then this pseudo-inverse can be equipped with a monoidal structure. The next proposition asserts that the same occurs for $\mathbb{T}$-monoidal functors.

Proposition 5.23. [61] Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a finitely complete category $\mathcal{C}$. A $\mathbb{T}$-monoidal functor $(F, \varphi)\colon \mathbb{A} \to \mathbb{B}$ is an equivalence in $\mathbb{T}$-MON if and only if $F\colon U^{\mathbb{T}}(\mathbb{A}) \to U^{\mathbb{T}}(\mathbb{B})$ is an equivalence in Grpd($\mathcal{C}$).

Proof. The 'only if part' is clear. Let us prove the 'if part'. Suppose we have an internal functor $G\colon \mathbb{B} \to \mathbb{A}$ and internal natural isomorphisms $\alpha\colon GF \Rightarrow 1_{\mathbb{A}}$ and $\beta\colon FG \Rightarrow 1_{\mathbb{B}}$ in $\mathcal{C}$. With the remark after Definition 5.8 in mind, we can assume without loss of generality that the triangular identities $\beta \star 1_F = 1_F \star \alpha$ and $1_G \star \beta = \alpha \star 1_G$ hold. Since F is full and faithful, there exists a unique $\psi\colon T(B_0) \to A_1$ such that $d\psi = a_0T(G_0)$, $c\psi = G_0b_0$ and $F_1\psi = m(i\varphi T(G_0), m(b_1T(\beta), i\beta b_0))$. This makes (G, ψ) and β $\mathbb{T}$-monoidal. Moreover, since α and β satisfy the triangular identities, α is also $\mathbb{T}$-monoidal. Therefore, (F, φ) is an equivalence in $\mathbb{T}$-MON. □

Let us notice here that, even if $\varphi = eb_0T(F_0)$, this does not imply that $\psi = ea_0T(G_0)$. So, an internal functor in $\mathcal{C}^{\mathbb{T}}$ can be an equivalence in $\mathbb{T}$-MON without being an equivalence in Grpd($\mathcal{C}^{\mathbb{T}}$).

5.4 $\mathbb{T}$-MON as a bicategory of fractions

We show in this section that, for a regular category $\mathcal{C}$ where the Axiom of Choice holds and a monad $\mathbb{T}$ on it, $I\colon \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) \hookrightarrow \mathbb{T}\text{-MON}$ is the bicategory of fractions for $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$ with respect to weak equivalences. As in [102], the key lemma to achieve this is the fact that the strong homotopy-pullback of two $\mathbb{T}$-monoidal functors exists and the legs can be chosen to be in $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$. Firstly, from [102], we know that, for a finitely complete category $\mathcal{C}$, the 2-category $\mathrm{Grpd}(\mathcal{C})$ has strong homotopy-pullbacks, constructed as follows. Given

$$\begin{tikzcd}
 & \mathbb{C} \arrow[d, "G"] \\
\mathbb{A} \arrow[r, "F"'] & \mathbb{B}
\end{tikzcd}$$

in $\mathrm{Grpd}(\mathcal{C})$, we construct the pullback in $\mathcal{C}$

$$\begin{tikzcd}
\vec{B}_1 \arrow[r, "m_2"] \arrow[d, "m_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \times_{c,d} B_1 \arrow[d, "m"] \\
B_1 \times_{c,d} B_1 \arrow[r, "m"'] & B_1
\end{tikzcd}$$

and define P_0 and P_1 to be the limits below.

$$\begin{tikzcd}
 & & P_0 \arrow[dll, "G'_0"'] \arrow[d, "\omega"] \arrow[drr, "F'_0"] & & \\
A_0 \arrow[dr, "F_0"'] & & B_1 \arrow[dl, "d"] \arrow[dr, "c"'] & & C_0 \arrow[dl, "G_0"] \\
 & B_0 & & B_0 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & P_1 \arrow[dll, "G'_1"'] \arrow[d, "\omega_1"] \arrow[drr, "F'_1"] & & \\
A_1 \arrow[dr, "F_1"'] & & \vec{B}_1 \arrow[dl, "\pi_1 m_2"] \arrow[dr, "\pi_2 m_1"'] & & C_1 \arrow[dl, "G_1"] \\
 & B_1 & & B_1 &
\end{tikzcd}$$

It turns out we can build an internal groupoid $\mathbb{P}$ in $\mathcal{C}$ on P_0 and P_1 and that

$$\omega_1 = ((\omega d, G_1 F_1'), (F_1 G_1', \omega c)).$$

Finally, the strong homotopy-pullback of F and G is given by the following diagram.

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "F'"] \arrow[d, "G'"'] & \mathbb{C} \arrow[d, "G"] \\
\mathbb{A} \arrow[r, "F"'] \arrow[ur, Rightarrow, "\omega"'] & \mathbb{B}
\end{tikzcd}$$

As far as its universal property is concerned, if

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "K"] \arrow[d, "H"'] & \mathbb{C} \arrow[d, "G"] \\
\mathbb{A} \arrow[r, "F"'] \arrow[ur, Rightarrow, "\theta"'] & \mathbb{B}
\end{tikzcd}$$

is another square in $\mathrm{Grpd}(\mathcal{C})$, the unique internal functor $L \colon \mathbb{X} \to \mathbb{P}$ satisfying the required properties is constructed using the limit definition of P_0 and P_1 via the equalities $G'L = H$, $F'L = K$, $\omega L_0 = \theta$ and $\omega_1 L_1 = ((\theta d, G_1 K_1), (F_1 H_1, \theta c))$. Similarly, given two internal functors $L, L' \colon \mathbb{X} \rightrightarrows \mathbb{P}$ and two internal natural transformations $\alpha \colon G'L \Rightarrow G'L'$ and $\beta \colon F'L \Rightarrow F'L'$ such that $m(\omega L_0, G_1\beta) = m(F_1\alpha, \omega L_0')$, the unique $\gamma \colon L \Rightarrow L'$ satisfying the required properties is obtained via the equalities $G_1'\gamma = \alpha$, $\omega_1\gamma = ((\omega L_0, G_1\beta), (F_1\alpha, \omega L_0'))$ and $F_1'\gamma = \beta$. Knowing that, we can prove $\mathbb{T}$-MON has strong homotopy-pullbacks.

Lemma 5.24. [61] Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a finitely complete category $\mathcal{C}$. The 2-category $\mathbb{T}$-MON has strong homotopy-pullbacks. Moreover, given $\mathbb{T}$-monoidal functors $(F, \varphi) \colon \mathbb{A} \to \mathbb{B}$ and $(G, \psi) \colon \mathbb{C} \to \mathbb{B}$, it is possible to choose a strong homotopy-pullback of (F, φ) and (G, ψ)

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "F'"] \arrow[d, "G'"'] & \mathbb{C} \arrow[d, "{(G,\psi)}"] \\
\mathbb{A} \arrow[r, "{(F,\varphi)}"'] \arrow[ur, Rightarrow, "\omega"'] & \mathbb{B}
\end{tikzcd}$$

in such a way that F' and G' are internal functors in $\mathcal{C}^{\mathbb{T}}$.

Proof. Let

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "F'"] \arrow[d, "G'"'] & \mathbb{C} \arrow[d, "G"] \\
\mathbb{A} \arrow[r, "F"'] \arrow[ur, Rightarrow, "\omega"'] & \mathbb{B}
\end{tikzcd}$$

be the strong homotopy-pullback in Grpd($\mathcal{C}$) described above. We now turn $\mathbb{P}$ into an internal groupoid in $\mathcal{C}^{\mathbb{T}}$ in the following way. In view of the limit defining P_0, there exists a unique morphism $p_0 \colon T(P_0) \to P_0$ such that $G'_0 p_0 = a_0 T(G'_0)$, $\omega p_0 = m(i\varphi T(G'_0), m(b_1 T(\omega), \psi T(F'_0)))$ and $F'_0 p_0 = c_0 T(F'_0)$. Moreover, P_1 being also a limit, there exists a unique morphism $p_1 \colon T(P_1) \to P_1$ such that $G'_1 p_1 = a_1 T(G'_1)$,

$$\omega_1 p_1 = ((\omega p_0 T(d), G_1 c_1 T(F'_1)), (F_1 a_1 T(G'_1), \omega p_0 T(c)))$$

and $F'_1 p_1 = c_1 T(F'_1)$. This makes $\mathbb{P}$ an internal groupoid in $\mathcal{C}^{\mathbb{T}}$, F' and G' internal functors in $\mathcal{C}^{\mathbb{T}}$ and ω a $\mathbb{T}$-monoidal natural transformation.

If

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "{(K,k)}"] \arrow[d, "{(H,h)}"'] & \mathbb{C} \arrow[d, "{(G,\psi)}"] \\
\mathbb{A} \arrow[r, "{(F,\varphi)}"'] \arrow[ur, Rightarrow, "\theta"'] & \mathbb{B}
\end{tikzcd}$$

is also a square in $\mathbb{T}$-MON, the unique internal functor $L \colon \mathbb{X} \to \mathbb{P}$ (in $\mathcal{C}$) defined above for the square

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "K"] \arrow[d, "H"'] & \mathbb{C} \arrow[d, "G"] \\
\mathbb{A} \arrow[r, "F"'] \arrow[ur, Rightarrow, "\theta"'] & \mathbb{B}
\end{tikzcd}$$

in Grpd($\mathcal{C}$) can be turned in a $\mathbb{T}$-monoidal functor. Indeed, it suffices to set $l \colon T(X_0) \to P_1$ as the unique morphism such that $G'_1 l = h$, $\omega_1 l = ((\omega p_0 T(L_0), G_1 k), (F_1 h, \omega L_0 x_0))$ and $F'_1 l = k$. This is the only way it can be done.

Finally, if $(L, l), (L', l') \colon \mathbb{X} \rightrightarrows \mathbb{P}$ are two $\mathbb{T}$-monoidal functors and $\alpha \colon (G'L, G'_1 l) \Rightarrow (G'L', G'_1 l')$, $\beta \colon (F'L, F'_1 l) \Rightarrow (F'L', F'_1 l')$ two $\mathbb{T}$-monoidal natural transformations satisfying $m(\omega L_0, G_1 \beta) = m(F_1 \alpha, \omega L'_0)$, the unique internal natural transformation (in $\mathcal{C}$) $\gamma\colon L \Rightarrow L'$ as above is also a 2-cell in $\mathbb{T}$-MON. To see this, it suffices to prove $m(p_1T(\gamma), l') = m(l, \gamma x_0)$, which can be done by composing with the legs of the limits involved. □

We are now able to prove the main theorem of this chapter.

Theorem 5.25. [61] Let $\mathcal{C}$ be a regular category where the Axiom of Choice holds and $\mathbb{T} = (T, \eta, \mu)$ a monad on $\mathcal{C}$. The inclusion 2-functor

$$I\colon\ \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) \hookrightarrow \mathbb{T}\text{-MON}$$

is the bicategory of fractions of $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$ with respect to the class of weak equivalences.

Proof. Since regular epimorphisms in $\mathcal{C}$ are split, T preserves them. By Lemma 1.67, the forgetful functor $U^{\mathbb{T}}\colon \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ preserves and reflects regular epimorphisms and $\mathcal{C}^{\mathbb{T}}$ is regular. Let W be the class of weak equivalences in $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$. Then, we know from Proposition 5.20 that W admits a right calculus of fractions.

Now, let $F \in W$. By Lemma 5.21, we know that $U^{\mathbb{T}}(F) \in \mathrm{Grpd}(\mathcal{C})$ is a weak equivalence. Since the Axiom of Choice holds in $\mathcal{C}$, Proposition 5.19 tells us $U^{\mathbb{T}}(F)$ is actually an equivalence. Thus, by Proposition 5.23, $I(F)$ is an equivalence and I sends elements of W to equivalences.

Therefore, it remains to prove that I satisfies conditions 1, 2 and 3 of Proposition 5.11. The first one is obvious and the second one is the fact that, between internal functors in $\mathcal{C}^{\mathbb{T}}$, $\mathbb{T}$-monoidal natural transformations are exactly internal natural transformations in $\mathcal{C}^{\mathbb{T}}$. Let us prove the last one. Given $(F, \varphi)\colon \mathbb{A} \to \mathbb{B}$ in $\mathbb{T}$-MON, consider the strong homotopy-pullback

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "G"] \arrow[d, "V"'] & \mathbb{B} \arrow[d, "1_{\mathbb{B}}"] \arrow[dl, phantom, "\overset{\omega}{\Nearrow}"] \\
\mathbb{A} \arrow[r, "{(F,\varphi)}"'] & \mathbb{B}
\end{tikzcd}$$

given by Lemma 5.24, in such a way that G and V are internal functors in $\mathcal{C}^{\mathbb{T}}$. Thus, $G \cong (F, \varphi) \circ V$. Since strong homotopy-pullbacks preserve equivalences, V is an equivalence in $\mathbb{T}$-MON and thus in $\mathrm{Grpd}(\mathcal{C})$. By Lemma 5.21 again, this implies that $V \in W$. □

Corollary 5.26. [61] Let $\mathcal{C}$ be a regular category where the Axiom of Choice holds and $G\colon \mathcal{D} \to \mathcal{C}$ a monadic functor. Denote by $\mathbb{T} = (T, \eta, \mu)$ the monad induced by the adjunction $F \dashv G$ on $\mathcal{C}$ and $K\colon \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ the comparison functor given by Proposition 1.60. Then, the composite

$$\begin{tikzcd}\mathrm{Grpd}(\mathcal{D}) \arrow[r, "K"] & \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) \arrow[r, hook, "I"] & \mathbb{T}\text{-MON}\end{tikzcd}$$

is the bicategory of fractions of $\mathrm{Grpd}(\mathcal{D})$ with respect to the class of weak equivalences.

Proof. Since G is monadic, $K\colon \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is an equivalence and $K :$ $\mathrm{Grpd}(\mathcal{D}) \to \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$ a biequivalence of 2-categories. In addition, by Lemma 5.21, $F \in \mathrm{Grpd}(\mathcal{D})$ is a weak equivalence if and only if $K(F) \in \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$ is. Thus, IK satisfies conditions 1, 2 and 3 of Proposition 5.11 since I does. □

5.5 The case of groups

We study in this section the particular case of the monadic forgetful functor $U_{\mathcal{T}}\colon \mathrm{Gp} \to \mathrm{Set}$, where $\mathcal{T}$ is the Lawvere theory corresponding to groups. We denote by $\mathbb{T}$ the corresponding monad on Set. Since we work under the axioms ZFCU, the Axiom of Choice holds in the regular category Set. Therefore, $\mathbb{T}$-MON is the bicategory of fractions of Grpd(Gp) with respect to W, the class of weak equivalences. In order to explain the axioms of Definition 5.22, we make them explicit in this context. Let $\mathbb{A}$ and $\mathbb{B}$ be two groupoids in $\mathrm{Set}^{\mathbb{T}}$ (i.e., in Gp by the biequivalence K). A $\mathbb{T}$-monoidal functor $(F, \varphi)\colon \mathbb{A} \to \mathbb{B}$ is a functor $F\colon \mathbb{A} \to \mathbb{B}$ between the underlying categories, together with a function $\varphi\colon \mathrm{Fr}_{\mathcal{T}}(A_0) \to B_1$ satisfying the following axioms:

1 and 2: For all objects $a_1, \ldots, a_n \in A_0$ and $i_1, \ldots, i_n \in \{-1, 1\}$, $\varphi(a_1^{i_1} \cdots a_n^{i_n})$ is an arrow

$$F(a_1)^{i_1} + \cdots + F(a_n)^{i_n} \xrightarrow{\varphi(a_1^{i_1} \cdots a_n^{i_n})} F(a_1^{i_1} + \cdots + a_n^{i_n}).$$

3: For all arrows $f_1, \dots, f_n \in A_1$ (with $f_i \colon a_i \to a'_i$ for each $i \in \{1, \dots, n\}$) and $i_1, \dots, i_n \in \{-1, 1\}$, the diagram

$$\begin{tikzcd}
F(a_1)^{i_1} + \cdots + F(a_n)^{i_n} \arrow[r, "F(f_1)^{i_1}+\cdots+F(f_n)^{i_n}"] \arrow[d, "\varphi(a_1^{i_1}\cdots a_n^{i_n})"'] & F(a'_1)^{i_1} + \cdots + F(a'_n)^{i_n} \arrow[d, "\varphi(a'^{i_1}_1\cdots a'^{i_n}_n)"] \\
F(a_1^{i_1} + \cdots + a_n^{i_n}) \arrow[r, "F(f_1^{i_1}+\cdots+f_n^{i_n})"'] & F(a'^{i_1}_1 + \cdots + a'^{i_n}_n)
\end{tikzcd}$$

commutes.

4: For each $a \in A_0$, $\varphi(a) = 1_{F(a)} \colon F(a) \to F(a)$.

5: For all $a_{11}, \dots, a_{1n_1}, \dots, a_{k1}, \dots, a_{kn_k} \in A_0$, the diagram

$$\begin{tikzcd}
F(a_{11}) + \cdots + F(a_{kn_k}) \arrow[r, "\varphi(a_{11}\cdots a_{kn_k})"] \arrow[d, "\varphi(a_{11}\cdots a_{1n_1})+\cdots+\varphi(a_{k1}\cdots a_{kn_k})"'] & F(a_{11} + \cdots + a_{kn_k}) \\
F(a_{11} + \cdots + a_{1n_1}) + \cdots + F(a_{k1} + \cdots + a_{kn_k}) \arrow[ur, "\varphi((a_{11}+\cdots+a_{1n_1})\cdots(a_{k1}+\cdots+a_{kn_k}))"'] &
\end{tikzcd}$$

commutes (for the sake of simplicity, we only express axiom 5 with exponents 1).

The bicategory of fractions $\mathrm{Grpd(Gp)}[W^{-1}]$ has another description in [102].

Definition 5.27. The 2-category MON is defined as follows:

- Objects are internal groupoids in Gp.
- 1-cells are monoidal functors $(F, F_2) \colon \mathbb{A} \to \mathbb{B}$. We recall they are given in this case by a functor $F \colon \mathbb{A} \to \mathbb{B}$ between the underlying groupoids in Set and a family of arrows in $\mathbb{B}$

$$F_2 = (F_2^{a,a'} \colon F(a) + F(a') \to F(a + a'))_{a,a' \in A_0}$$

natural in a and a' and such that the rectangle

$$\begin{tikzcd}
F(a) + F(a') + F(a'') \arrow[r, "1_{F(a)}+F_2^{a',a''}"] \arrow[d, "F_2^{a,a'}+1_{F(a'')}"'] & F(a) + F(a' + a'') \arrow[d, "F_2^{a,a'+a''}"] \\
F(a + a') + F(a'') \arrow[r, "F_2^{a+a',a''}"'] & F(a + a' + a'')
\end{tikzcd}$$

commutes for all objects $a, a', a'' \in A_0$. The suitable morphism $F_0\colon 0 \to F(0)$ is then determined uniquely.

- 2-cells are monoidal natural transformations $\alpha\colon (F, F_2) \Rightarrow (G, G_2)$. In this case, these are natural transformations $\alpha\colon F \Rightarrow G$ such that the diagram

$$\begin{tikzcd}
F(a) + F(a') \arrow[r, "F_2^{a,a'}"] \arrow[d, "\alpha_a + \alpha_{a'}"'] & F(a+a') \arrow[d, "\alpha_{a+a'}"] \\
G(a) + G(a') \arrow[r, "G_2^{a,a'}"'] & G(a+a')
\end{tikzcd}$$

  commutes for all $a, a' \in A_0$.

Theorem 5.28. [102] The inclusion 2-functor

$$\mathrm{Grpd(Gp)} \hookrightarrow \mathrm{MON}$$

is the bicategory of fractions for Grpd(Gp) with respect to weak equivalences.

Since they are both the bicategory of fractions with respect to weak equivalences of Grpd(Gp), MON and $\mathbb{T}$-MON are biequivalent. This biequivalence $\widetilde{K}\colon$ MON $\to$ $\mathbb{T}$-MON makes the diagram

$$\begin{tikzcd}
\mathrm{Grpd(Gp)} \arrow[rr, hook] \arrow[d, equal] & & \mathrm{MON} \arrow[r] \arrow[d, "\widetilde{K}"', "\simeq"] & \mathrm{Grpd(Set)} \arrow[d, equal] \\
\mathrm{Grpd(Gp)} \arrow[r, "\simeq", "K"'] & \mathrm{Grpd(Set^{\mathbb{T}})} \arrow[r, hook, "I"'] & \mathbb{T}\text{-MON} \arrow[r, "J"'] & \mathrm{Grpd(Set)}
\end{tikzcd}$$

commutative. Moreover, it can be described by

$$\begin{aligned}
\widetilde{K}\colon \mathrm{MON} &\longrightarrow \mathbb{T}\text{-MON} \\
\mathbb{A} &\longmapsto K(\mathbb{A}) \\
\mathbb{A} \xrightarrow{(F,F_2)} \mathbb{B} &\longmapsto K(\mathbb{A}) \xrightarrow{(F,\varphi)} K(\mathbb{B}) \\
\alpha &\longmapsto \alpha
\end{aligned}$$

where $\varphi\colon \mathrm{Fr}_{\mathcal{T}}(A_0) \to B_1$ is defined on the word $a_1^{i_1} \cdots a_n^{i_n}$ ($a_k \in A_0$ and

$i_k \in \{-1, 1\}$) to be the arrow part of the sum

$$\big(F(a_1), 1_{F(a_1)}, a_1\big)^{i_1} + \cdots + \big(F(a_n), 1_{F(a_n)}, a_n\big)^{i_n}.$$

This sum is calculated in the group of triples

$$(b \in B_0, f\colon b \to F(a), a \in A_0)$$

and defined by

$$(b, f, a) + (b', f', a') =$$
$$\left(b + b',\ b + b' \xrightarrow{f+f'} F(a) + F(a') \xrightarrow{F_2^{a,a'}} F(a + a')\ , a + a'\right).$$

5.6 The case of Lie algebras

The aim of this section is to make the link between a result in [102] and our Corollary 5.26 for the particular monadic forgetful functor U : $\mathrm{LieAlg}_k \to \mathrm{Vect}_k$ for a fixed field k. Now, $\mathbb{T}$ denotes the corresponding monad on Vect_k. This category is regular and the Axiom of Choice holds in it since every vector space is free (admits a basis). Thus, we can apply Corollary 5.26 to deduce that $\mathbb{T}$-MON is the bicategory of fractions of $\mathrm{Grpd}(\mathrm{LieAlg}_k)$ with respect to weak equivalences.

This bicategory of fractions has also been described as LIE_k in [102].

Definition 5.29. [8] Let k be a field. The 2-category LIE_k is defined as follows.

- Its objects are the internal groupoids in LieAlg_k.
- 1-cells are *homomorphisms* $(F, F_2)\colon \mathbb{A} \to \mathbb{B}$. These are given by an internal functor $F\colon \mathbb{A} \to \mathbb{B}$ between the underlying groupoids in Vect_k and a family of morphisms in $\mathbb{B}$

  $$F_2 = (F_2^{a,a'} : [F(a), F(a')] \to F([a, a']))_{a,a' \in A_0}$$

  natural in a and a', bilinear, skew-symmetric and such that the diagram

$$\begin{tikzcd}
{[F(a),[F(a'),F(a'')]]} \arrow[r, equal] \arrow[d, "{[1_{F(a)},F_2^{a,a'}]}"'] & {[[F(a),F(a')],F(a'')]+[F(a'),[F(a),F(a'')]]} \arrow[d, "{[F_2^{a,a'},1_{F(a'')}]+[1_{F(a')},F_2^{a,a''}]}"] \\
{[F(a),F([a',a''])]} \arrow[d, "{F_2^{a,[a',a'']}}"'] & {[F([a,a']),F(a'')]+[F(a'),F([a,a''])]} \arrow[d, "{F_2^{[a,a'],a''}+F_2^{a',[a,a'']}}"] \\
{F([a,[a',a'']])} \arrow[r, equal] & {F([[a,a'],a''])+F([a',[a,a'']])}
\end{tikzcd}$$

commutes for all $a, a', a'' \in A_0$.

- 2-cells $\alpha\colon (F, F_2) \Rightarrow (G, G_2)$ are *2-homomorphisms.* These are internal natural transformations $\alpha\colon F \Rightarrow G$ in Vect_k such that the diagram

$$\begin{tikzcd}
{[F(a),F(a')]} \arrow[r, "{F_2^{a,a'}}"] \arrow[d, "{[\alpha_a,\alpha_{a'}]}"'] & {F([a,a'])} \arrow[d, "{\alpha_{[a,a']}}"] \\
{[G(a),G(a')]} \arrow[r, "{G_2^{a,a'}}"'] & {G([a,a'])}
\end{tikzcd}$$

commutes for all $a, a' \in A_0$.

Theorem 5.30. [102] Let k be a field. The inclusion 2-functor

$$\mathrm{Grpd}(\mathrm{LieAlg}_k) \hookrightarrow \mathrm{LIE}_k$$

is the bicategory of fractions for $\mathrm{Grpd}(\mathrm{LieAlg}_k)$ with respect to weak equivalences.

Therefore, the 2-categories $\mathbb{T}$-MON and LIE_k are biequivalent. As for groups, this biequivalence $\widetilde{K}\colon \mathrm{LIE}_k \to \mathbb{T}\text{-MON}$ makes the diagram

$$\begin{tikzcd}
\mathrm{Grpd}(\mathrm{LieAlg}_k) \arrow[rr, hook] \arrow[d, equal] & & \mathrm{LIE}_k \arrow[r] \arrow[d, "\widetilde{K}"', "\simeq"] & \mathrm{Grpd}(\mathrm{Vect}_k) \arrow[d, equal] \\
\mathrm{Grpd}(\mathrm{LieAlg}_k) \arrow[r, "\simeq", "K"'] & \mathrm{Grpd}(\mathrm{Vect}_k^{\mathbb{T}}) \arrow[r, hook, "I"'] & \mathbb{T}\text{-MON} \arrow[r, "J"'] & \mathrm{Grpd}(\mathrm{Vect}_k)
\end{tikzcd}$$

commutative. Moreover, it can be described by

$$\begin{aligned} \widetilde{K}\colon \mathrm{LIE}_k &\longrightarrow \mathbb{T}\text{-MON} \\ \mathbb{A} &\longmapsto K(\mathbb{A}) \\ \mathbb{A} \xrightarrow{(F,F_2)} \mathbb{B} &\longmapsto K(\mathbb{A}) \xrightarrow{(F,\varphi)} K(\mathbb{B}) \\ \alpha &\longmapsto \alpha \end{aligned}$$

where $\varphi\colon T(A_0) \to B_1$ is defined as follows. $T(A_0)$ is the free Lie algebra of the underlying vector space of A_0. It is actually the Lie subalgebra generated by A_0 of the tensor algebra

$$k \oplus A_0 \oplus (A_0 \otimes A_0) \oplus (A_0 \otimes A_0 \otimes A_0) \oplus \cdots = \bigoplus_{n\in\mathbb{N}} A_0^{\otimes n}$$

(considered as a Lie algebra with $[v,w] = v \cdot w - w \cdot v$). To each element $v \in T(A_0)$, we associate a triple

$$\hat{\varphi}(v) = (b \in B_0, f\colon b \to F(a), a \in A_0)$$

by recursion:

1. if $a \in A_0$, $\hat{\varphi}(a) = (F(a), 1_{F(a)}, a)$,
2. if $x \in k$ and $v \in T(A_0)$, $\hat{\varphi}(xv) = (xb, xf, xa)$ where $(b, f, a) = \hat{\varphi}(v)$,
3. if $v_1, v_2 \in T(A_0)$, $\hat{\varphi}(v_1 + v_2) = (b_1 + b_2, f_1 + f_2, a_1 + a_2)$ where $(b_i, f_i, a_i) = \hat{\varphi}(v_i)$ for $i \in \{1, 2\}$,
4. if $v_1, v_2 \in T(A_0)$, $\hat{\varphi}([v_1, v_2]) = ([b_1, b_2], F_2^{a_1,a_2}[f_1, f_2], [a_1, a_2])$ where $(b_i, f_i, a_i) = \hat{\varphi}(v_i)$ for $i \in \{1, 2\}$.

Then, $\varphi(v) = f$ is the arrow part of this triple $\hat{\varphi}(v) = (b, f, a)$.

5.7 Pseudo-algebras

In this last section, we give some intuition where the axioms of Definition 5.22 of $\mathbb{T}$-monoidal functors come from. We will see that these axioms are actually particular cases of coherence axioms defining pseudo-morphisms between strict algebras for a pseudo-monad. We adopt the following definitions from [89] and [78].

Definition 5.31. Let $\mathcal{C}$ be a 2-category. A *pseudo-monad* $\mathbf{T}$ on $\mathcal{C}$ consists of

- a 2-functor $T\colon \mathcal{C} \to \mathcal{C}$,
- two pseudo-natural transformations $\eta\colon 1_{\mathcal{C}} \Rightarrow T$ and $\mu\colon T^2 \Rightarrow T$,
- three isomodifications m, l and r

$$\begin{tikzcd}
T^3 \arrow[r, "\mu_T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \arrow[dl, rightsquigarrow, "m"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta_T"] \arrow[dr, "1_T"'] & T^2 \arrow[d, "\mu"] \arrow[dl, rightsquigarrow, shorten >=10pt, "l"] \arrow[dr, rightsquigarrow, shorten >=10pt, "r"'] & T \arrow[l, "T\eta"'] \arrow[dl, "1_T"] \\
& T &
\end{tikzcd}$$

such that the two diagrams

$$\begin{tikzcd}
\mu\,\mu_T\,T\eta_T \arrow[rr, "m\star 1_{T\eta_T}"] \arrow[dr, "1_\mu\star r_T"'] & & \mu\,T\mu\,T\eta_T \arrow[dl, "1_\mu\star Tl"] \\
& \mu &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mu\,\mu_T\,\mu_{T^2} \arrow[r, "1_\mu\star m_T"] \arrow[d, "m\star 1_{\mu_{T^2}}"'] & \mu\,\mu_T\,T\mu_T \arrow[r, "m\star 1_{T\mu_T}"] & \mu\,T\mu\,T\mu_T \arrow[d, "1_\mu\star Tm"] \\
\mu\,T\mu\,\mu_{T^2} \arrow[r, "1_\mu\star\tau^\mu_\mu"'] & \mu\,\mu_T\,T^2\mu \arrow[r, "m\star 1_{T^2\mu}"'] & \mu\,T\mu\,T^2\mu
\end{tikzcd}$$

commute.

Definition 5.32. Let $\mathcal{C}$ be a 2-category and $\mathbf{T}$ a pseudo-monad on it. We define the 2-category $\mathrm{PsAlg}(\mathbf{T})$ as follows:

- The objects are the *pseudo-algebras* of $\mathbf{T}$. These are quadruples (A, a, a_*, a_2) where A is an object of $\mathcal{C}$, $a\colon T(A) \to A$ a 1-cell and

a_*, a_2 two 2-isomorphisms

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & T(A) \arrow[d, "a"] \arrow[dl, Rightarrow, shorten >=12pt, "a_*"] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2(A) \arrow[r, "\mu_A"] \arrow[d, "T(a)"'] & T(A) \arrow[d, "a"] \arrow[dl, Rightarrow, "a_2"] \\
T(A) \arrow[r, "a"'] & A
\end{tikzcd}$$

such that the diagrams

$$\begin{tikzcd}
a\ \mu_A\ T(\eta_A) \arrow[rr, "a_2\star 1_{T(\eta_A)}"] \arrow[dr, "1_a\star r_A"'] & & a\ T(a)\ T(\eta_A) \arrow[dl, "1_a\star T(a_*)"] \\
 & a &
\end{tikzcd}$$

and

$$\begin{tikzcd}
a\ \mu_A\ \mu_{T(A)} \arrow[r, "1_a\star m_A"] \arrow[d, "a_2\star 1_{\mu_{T(A)}}"'] & a\ \mu_A\ T(\mu_A) \arrow[r, "a_2\star 1_{T(\mu_A)}"] & a\ T(a)\ T(\mu_A) \arrow[d, "1_a\star T(a_2)"] \\
a\ T(a)\ \mu_{T(A)} \arrow[r, "1_a\star\tau^\mu_a"'] & a\ \mu_A\ T^2(a) \arrow[r, "a_2\star 1_{T^2(a)}"'] & a\ T(a)\ T^2(a)
\end{tikzcd}$$

commute.

- 1-cells $(A, a, a_*, a_2) \to (B, b, b_*, b_2)$ are *pseudo-morphisms* between pseudo-algebras, i.e., pairs (f, φ) where $f\colon A \to B$ is a 1-cell and φ a 2-isomorphism

$$\begin{tikzcd}
T(A) \arrow[r, "T(f)"] \arrow[d, "a"'] & T(B) \arrow[d, "b"] \arrow[dl, Rightarrow, "\varphi"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

such that the diagrams

$$\begin{tikzcd}
b\ T(f)\ \eta_A \arrow[r, "1_b\star\tau^\eta_f"] \arrow[d, "\varphi\star 1_{\eta_A}"'] & b\ \eta_B\ f \arrow[d, "b_*\star 1_f"] \\
f\ a\ \eta_A \arrow[r, "1_f\star a_*"'] & f
\end{tikzcd}$$

and

$$\begin{tikzcd}
b\,T(f)\,\mu_A \arrow[r, "\varphi\star 1_{\mu_A}"] \arrow[d, "1_b\star\tau^{\mu}_f"'] & f\;a\;\mu_A \arrow[r, "1_f\star a_2"] & f\;a\;T(a) \\
b\;\mu_B\;T^2(f) \arrow[r, "b_2\star 1_{T^2(f)}"'] & b\;T(b)\;T^2(f) \arrow[r, "1_b\star T(\varphi)"'] & b\;T(f)\;T(a) \arrow[u, "\varphi\star 1_{T(a)}"']
\end{tikzcd}$$

commute.

- 2-cells $\alpha\colon (f,\varphi)\Rightarrow(g,\psi)$ are 2-cells $\alpha\colon f\Rightarrow g$ such that the square

$$\begin{tikzcd}
b\,T(f) \arrow[r, "1_b\star T(\alpha)"] \arrow[d, "\varphi"'] & b\,T(g) \arrow[d, "\psi"] \\
fa \arrow[r, "\alpha\star 1_a"'] & ga
\end{tikzcd}$$

commutes.

Similarly to Proposition 5.23, a pseudo-morphism (f,φ) is an equivalence in $\mathrm{PsAlg}(\mathbf{T})$ if and only if f is an equivalence in $\mathcal{C}$.

We are now going to see how pseudo-morphisms are linked with $\mathbb{T}$-monoidal functors. Firstly, if $\mathbf{T}$ is a pseudo-monad, we define $\mathbf{T}$-MON to be the full sub-2-category of $\mathrm{PsAlg}(\mathbf{T})$ whose objects are the *strict algebras*, i.e., the pseudo-algebras (A,a,a_*,a_2) with a_* and a_2 being identities. Moreover, we denote by $\mathrm{Alg}(\mathbf{T})$ the sub-2-category of $\mathbf{T}$-MON in which we only consider *strict morphisms* of strict algebras, i.e., the pseudo-morphisms (f,φ) where φ is the identity.

$$\mathrm{Alg}(\mathbf{T}) \hookrightarrow \mathbf{T}\text{-MON} \hookrightarrow \mathrm{PsAlg}(\mathbf{T})$$

Now, suppose $\mathbb{T}=(T,\eta,\mu)$ is a (1-dimensional) monad on a finitely complete category $\mathcal{C}$. If $T\colon\mathcal{C}\to\mathcal{C}$ preserves pullbacks, then $\mathbb{T}$ induces a pseudo-monad $\mathbf{T}$ on the 2-category $\mathrm{Grpd}(\mathcal{C})$. Moreover, $\mathbf{T}$ is such that η and μ are 2-natural transformations, given by $\eta_{\mathbb{A}}=(\eta_{A_0},\eta_{A_1})$ and $\mu_{\mathbb{A}}=(\mu_{A_0},\mu_{A_1})$ for each $\mathbb{A}\in\mathrm{Grpd}(\mathcal{C})$. For this pseudo-monad, we know that the modifications m, l and r are identities and that the two coherence axioms become trivial. Moreover, we have an isomorphism of

2-categories

$$\begin{aligned} \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) &\longrightarrow \mathrm{Alg}(\mathbf{T}) \\ \mathbb{A} &\longmapsto (U^{\mathbb{T}}(\mathbb{A}), a = (a_0, a_1)) \\ F &\longmapsto U^{\mathbb{T}}(F) \\ \alpha &\longmapsto U^{\mathbb{T}}(\alpha) \end{aligned}$$

where $a_i \colon T(A_i) \to A_i$ are the $\mathbb{T}$-algebra structures for $i \in \{0,1\}$. Note that $a = (a_0, a_1)$ is an internal functor since d, c, e, m and i are $\mathbb{T}$-homomorphisms. Notice also that the fact that $U^{\mathbb{T}}(\alpha)$ satisfies the coherence axiom for the definition of 2-cells in PsAlg($\mathbf{T}$) corresponds to the fact that α is a $\mathbb{T}$-homomorphism.

With this particular pseudo-monad on Grpd($\mathcal{C}$), we remark that, if we extend this isomorphism, $\mathbf{T}$-MON becomes the following 2-category:

- Objects are internal groupoids in $\mathcal{C}^{\mathbb{T}}$.
- A 1-cell $(F, \varphi) \colon \mathbb{A} \to \mathbb{B}$ consists of an internal functor $F \colon U^{\mathbb{T}}(\mathbb{A}) \to U^{\mathbb{T}}(\mathbb{B})$ in $\mathcal{C}$ together with an internal natural isomorphism

  $$\begin{tikzcd} TU^{\mathbb{T}}(\mathbb{A}) \arrow[r, "T(F)"] \arrow[d, "a"'] & TU^{\mathbb{T}}(\mathbb{B}) \arrow[d, "b"] \arrow[dl, Rightarrow, "\varphi"] \\ U^{\mathbb{T}}(\mathbb{A}) \arrow[r, "F"'] & U^{\mathbb{T}}(\mathbb{B}) \end{tikzcd}$$

  in $\mathcal{C}$ such that $\varphi{\star}1_{\eta_{U^{\mathbb{T}}(\mathbb{A})}} = 1_F$ and $\varphi{\star}1_{\mu_{U^{\mathbb{T}}(\mathbb{A})}} = (\varphi{\star}1_{T(a)})(1_b{\star}T(\varphi))$.

  $$\begin{tikzcd} b\,T(F)\,\eta_{U^{\mathbb{T}}(\mathbb{A})} \arrow[r, equal] \arrow[d, "\varphi\star 1_{\eta_{U^{\mathbb{T}}(\mathbb{A})}}"'] & b\,\eta_{U^{\mathbb{T}}(\mathbb{B})}\,F \arrow[d, equal] \\ F\,a\,\eta_{U^{\mathbb{T}}(\mathbb{A})} \arrow[r, equal] & F \end{tikzcd}$$

  $$\begin{tikzcd} b\,T(F)\,\mu_{U^{\mathbb{T}}(\mathbb{A})} \arrow[r, "\varphi\star 1_{\mu_{U^{\mathbb{T}}(\mathbb{A})}}"] \arrow[d, equal] & F\,a\,\mu_{U^{\mathbb{T}}(\mathbb{A})} \arrow[r, equal] & F\,a\,T(a) \\ b\,\mu_{U^{\mathbb{T}}(\mathbb{B})}\,T^2(F) \arrow[r, equal] & b\,T(b)\,T^2(F) \arrow[r, "1_b\star T(\varphi)"'] & b\,T(F)\,T(a) \arrow[u, "\varphi\star 1_{T(a)}"'] \end{tikzcd}$$

- A 2-cell $\alpha\colon (F,\varphi) \Rightarrow (G,\psi)\colon \mathbb{A} \to \mathbb{B}$ is an internal natural transformation $\alpha\colon F \Rightarrow G$ in $\mathcal{C}$ such that the square

$$\begin{tikzcd}
b\,T(F) \arrow[r, "1_b \star T(\alpha)"] \arrow[d, "\varphi"'] & b\,T(G) \arrow[d, "\psi"] \\
F\,a \arrow[r, "\alpha \star 1_a"'] & G\,a
\end{tikzcd}$$

commutes.

We notice that these are exactly the axioms of Definition 5.22 of $\mathbb{T}$-MON. Indeed, the first three axioms defining a $\mathbb{T}$-monoidal functor are the fact that the 2-cell $\varphi\colon bT(F) \Rightarrow Fa$ is an internal natural transformation in $\mathcal{C}$, while the last two are the above ones. In other words, if $\mathbf{T}$ is the pseudo-monad on $\mathrm{Grpd}(\mathcal{C})$ induced by a pullback preserving monad $\mathbb{T}$ on $\mathcal{C}$ (i.e., in which the functor part T of $\mathbb{T}$ preserves pullbacks), the 2-categories $\mathbf{T}$-MON and $\mathbb{T}$-MON coincide. What makes it possible to define $\mathbb{T}$-MON even if the monad $\mathbb{T}$ does not preserve pullbacks is the fact that, to express the naturality of $\varphi\colon bT(F) \Rightarrow Fa\colon TU^{\mathbb{T}}(\mathbb{A}) \to U^{\mathbb{T}}(\mathbb{B})$, one only needs the composition in the codomain category $U^{\mathbb{T}}(\mathbb{B})$ and not in the domain $TU^{\mathbb{T}}(\mathbb{A})$.

Analogously to Theorem 5.25, we are now going to prove that, under some hypotheses, $\mathrm{Alg}(\mathbf{T}) \hookrightarrow \mathbf{T}$-MON is the bicategory of fractions of $\mathrm{Alg}(\mathbf{T})$ with respect to a certain class of 1-cells W. The following lemma is analogous to Lemma 5.24.

Lemma 5.33. [61] Let $\mathcal{C}$ be a 2-category where every 2-cell is invertible and $\mathbf{T} = (T, \eta, \mu, m, l, r)$ a pseudo-monad on $\mathcal{C}$ such that η and μ are 2-natural transformations. If $\mathcal{C}$ has strong homotopy-pullbacks, so has $\mathbf{T}$-MON. In this case, given pseudo-morphisms of strict algebras $(f,\varphi)\colon (A,a) \to (B,b)$ and $(g,\psi)\colon (C,c) \to (B,b)$, it is possible to choose a strong homotopy-pullback of (f,φ) and (g,ψ)

$$\begin{tikzcd}
(P,p) \arrow[r, "(\pi_C, 1_{\pi_C p})"] \arrow[d, "(\pi_A, 1_{\pi_A p})"'] & (C,c) \arrow[d, "(g,\psi)"] \\
(A,a) \arrow[r, "(f,\varphi)"'] \arrow[ur, Rightarrow, "\omega"'] & (B,b)
\end{tikzcd}$$

in such a way that $(\pi_A, 1_{\pi_A p})$ and $(\pi_C, 1_{\pi_C p})$ are strict morphisms.

Proof. Consider the strong homotopy-pullback

$$\begin{tikzcd}
P \arrow[r, "\pi_C"] \arrow[d, "\pi_A"'] & C \arrow[d, "g"] \\
A \arrow[r, "f"'] \arrow[ur, Rightarrow, "\omega"'] & B
\end{tikzcd}$$

in $\mathcal{C}$. There exists a unique 1-cell $p\colon T(P) \to P$ such that $\pi_A p = aT(\pi_A)$, $\pi_C p = cT(\pi_C)$ and $(\omega \star 1_p)(\varphi \star 1_{T(\pi_A)}) = (\psi \star 1_{T(\pi_C)})(1_b \star T(\omega))$. It is routine to check that this makes (P, p) a strict algebra and that we have constructed the announced strong homotopy-pullback. □

As for Theorem 5.25, this lemma is the key point to prove the next proposition.

Proposition 5.34. [61] Let $\mathcal{C}$ be a 2-category where every 2-cell is invertible and which has strong homotopy-pullbacks. Let also $\mathbf{T} = (T, \eta, \mu, m, l, r)$ be a pseudo-monad on $\mathcal{C}$ such that η and μ are 2-natural transformations. If W is the class of 1-cells $(f, 1)$ of $\mathrm{Alg}(\mathbf{T})$ such that f is an equivalence in $\mathcal{C}$, then

$$\mathrm{Alg}(\mathbf{T}) \hookrightarrow \mathbf{T}\text{-MON}$$

is the bicategory of fractions of $\mathrm{Alg}(\mathbf{T})$ with respect to W.

Proof. In view of Proposition 5.2 in [102], we know that W has a right calculus of fractions. The rest of the proof is similar to the one of Theorem 5.25 using Lemma 5.33. □

Index

Bibliography

[1] O. Abbad, S. Mantovani, G. Metere and E.M. Vitale, Butterflies in a semi-abelian context, *Advances Math.* **238** (2013), 140–183.

[2] J. Adámek, On quasivarieties and varieties as categories, *Studia Logica: Intern. J. Symb. Logic* **78** No. 1 (2004), 7–33.

[3] J. Adámek, M. Hébert and J. Rosický, On essentially algebraic theories and their generalizations, *Algebra Univers.* **41** (1999), 213–227.

[4] J. Adámek and J. Rosický, Locally presentable and accessible categories, *London Math. Soc.* **189** (1994).

[5] J. Adámek, J. Rosický and E.M. Vitale, Algebraic theories: A categorical introduction to general algebra, *Cambridge Tracts Math. (Cambridge Uni. Press)* **184** (2011).

[6] J. Adámek, J. Rosický and E.M. Vitale, Birkhoff's variety theorem in many sorts, *Algebra Univers.* **68** (2012), 39–42.

[7] M. Artin, A. Grothendieck and J.L. Verdier, Séminaire de géométrie algébrique du Bois Marie 1963–1964, Théorie des topos et cohomologie étale des schémas (SGA4), *Springer Lect. Notes Math.* **269** (1972) (in French).

[8] J.C. Baez and A.S. Crans, Higher-dimensional algebra VI: Lie 2-algebras, *Theory Appl. Categ.* **12** (2004), 492–538.

[9] M. Barr, Exact categories, *Springer Lect. Notes Math.* **236** (1971), 1–120.

[10] M. Barr, Representation of categories, *J. Pure Appl. Algebra*, **41** (1986), 113–137.

[11] M. Barr and C. Wells, Toposes, triples and theories, *Springer Grund. Math. Wiss.* **278** (1985).

[12] J. Bénabou, Introduction to bicategories, *Springer Lect. Notes Math.* **40** (1967), 1–77.

[13] G. Birkhoff, On the structure of abstract algebras, *Proc. Cambridge Phil. Soc.* **31** (1935), 433–454.

[14] F. Borceux, Handbook of Categorical Algebra, *Encycl. Math. Appl. (Cambridge Uni. Press)* **50, 51, 52** (1994).

[15] F. Borceux and D. Bourn, Mal'cev, protomodular, homological and semi-abelian categories, *Kluwer Acad. Publ.* (2004).

[16] F. Borceux and M.C. Pedicchio, Left exact presheaves on a small pretopos, *J. Pure Appl. Algebra*, **135** (1999), 9–22.

[17] D. Bourn, Normalization equivalence, kernel equivalence and affine categories, *Springer Lect. Notes Math.* **1488** (1991), 43–62.

[18] D. Bourn, Mal'cev categories and fibration of pointed objects, *Appl. Categ. Struct.* **4** (1996), 307–327.

[19] D. Bourn, 3×3 lemma and protomodularity, *J. Algebra* **236** (2001), 778–795.

[20] D. Bourn and G. Janelidze, Characterization of protomodular varieties of universal algebras, *Theory Appl. Categ.*, **11** (2003), 143–147.

[21] D. Bourn and Z. Janelidze, Approximate Mal'tsev operations, *Theory Appl. Categ.* **21** (2008), 152–171.

[22] D. Bourn and Z. Janelidze, Pointed protomodularity via natural imaginary subtractions, *J. Pure Appl. Algebra*, **213** (2009), 1835–1851.

[23] D. Bourn and Z. Janelidze, Categorical (binary) difference terms and protomodularity, *Algebra Univers.* **66** (2011), 277–316.

[24] D. Bourn and Z. Janelidze, A note on the abelianization functor, *Comm. Algebra* **44** (2016), 2009–2033.

[25] D. A. Buchsbaum, Exact categories and duality, *Trans. Amer. Math. Soc.* **80** No. 1 (1955), 1–34.

[26] M. Bunge and R. Paré, Stacks and equivalence of indexed categories, *Cah. Topol. Géom. Differ. Catég.* **20** (1979), 373–399.

[27] A. Carboni, G.M. Kelly and M.C. Pedicchio, Some remarks on Maltsev and Goursat categories, *Appl. Categ. Struct.* **1** (1993), 385–421.

[28] A. Carboni, J. Lambek and M.C. Pedicchio, Diagram chasing in Mal'cev categories, *J. Pure Appl. Algebra* **69** (1990), 271–284.

[29] A. Carboni, M.C. Pedicchio and N. Pirovano, Internal graphs and internal groupoids in Mal'tsev categories, *Canad. Math. Soc. Conf. Proc.* **13** (1992), 97–109.

[30] A. Carboni, M.C. Pedicchio and J. Rosický, Syntactic characterizations of various classes of locally presentable categories, *J. Pure Appl. Algebra* **161** (2001), 65–90.

[31] D. Chikhladze, Barr's embedding theorem for enriched categories, *J. Pure Appl. Algebra* **215** (2011), 2148–2153.

[32] M.M. Clementino and D. Hofmann, Topological features of lax algebras, *Appl. Categ. Struct.* **11** (2003), 267–286.

[33] B. Day and R. Street, Localisation of locally presentable categories, *J. Pure Appl. Algebra*, **58** (1989), 227–233.

[34] B. Day and R. Street, Localisation of locally presentable categories II, *J. Pure Appl. Algebra*, **63** (1990), 225–229.

[35] J. Duskin, Variations on Beck's tripleability theorem, *Springer Lect. Notes Math.* **106** (1969), 74–129.

[36] C. Ehresmann, Esquisses et types des structures algébriques, *Bul. Inst. Politehn. Iasi* **14** (1968), 1–14 (in French).

[37] S. Eilenberg and S. Mac Lane, General theory of natural equivalences, *Trans. Amer. Math. Soc.* **58** No. 2 (1945), 231–294.

[38] S. Eilenberg and J.C. Moore, Adjoint functors and triples, *Illinois J. Math.* **9** (1965), 381–398.

[39] L. Euler, Solutio problematis ad geometriam situs pertinentis, *Comment. Acad. Sci. U. Petrop.* **8** (1736), 128–140 (in Latin).

[40] E. Faro and G.M. Kelly, On the canonical algebraic structure of a category, *J. Pure Appl. Algebra* **154** (2000), 159–176.

[41] A.A. Fraenkel, Y. Bar-Hillel and A. Levy, Foundations of set theory (2nd ed.), *Stud. Logic Found. Math.* **67** (1973).

[42] P.J. Freyd, Abelian categories: An introduction to the theory of functors, *Harper and Row* (1964).

[43] P.J. Freyd and G.M. Kelly, Categories of continuous functors I, *J. Pure Appl. Algebra* **2** (1972), 169–191.

[44] P. Gabriel and F. Ulmer, Lokal präsentierbare kategorien, *Springer Lect. Notes Math.* **221** (1971) (in German).

[45] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory, *Ergebnisse Math. Grenzgebiete (Springer)* **35** (1967).

[46] M. Gran and S. Lack, Semi-localizations of semi-abelian categories, *J. Algebra* **454** (2016), 206–232.

[47] M. Gran and M.C. Pedicchio, n-Permutable locally finitely presentable categories, *Theory Appl. Categ.* **8** (2001), 1–15.

[48] M. Gran and D. Rodelo, The Cuboid Lemma and Mal'tsev categories, *Appl. Categ. Struct.* **22** (2014), 805–816.

[49] M. Gran and D. Rodelo, Beck-Chevalley condition and Goursat categories, *in preparation.*

[50] M. Grandis, Simplicial homotopical algebras and satellites, *Appl. Categ. Struct.* **5** (1997), 75–97.

[51] G. Grätzer, Universal Algebra, *D.van Nostrand Co.* (1968), 2nd edition: *Springer-Verlag* (1979).

[52] A. Grothendieck, Sur quelques points d'algèbre homologique, *Tohôku Math. J.* **2** (1957), 199–221 (in French).

[53] H. Höft, A characterization of strong homomorphisms, *Coll. Math.* **28** (1973), 189–193.

[54] K. Hrbacek and T. Jech, Introduction to set theory (3rd ed.), *Monogr. Text. Pure Appl. Math. (Marcel Dekker)* **220** (1999).

[55] P.-A. Jacqmin, An embedding theorem for regular Mal'tsev categories, *submitted to J. Pure Appl. Algebra.*

[56] P.-A. Jacqmin, Partial Mal'tsev algebras and an embedding theorem for (weakly) Mal'tsev categories, *submitted to Cah. Topol. Géom. Differ. Catég.*

[57] P.-A. Jacqmin, Syntactic characterisation of protomodular essentially algebraic categories, *in preparation.*

[58] P.-A. Jacqmin and Z. Janelidze, Unconditional exactness properties, *in preparation.*

[59] P.-A. Jacqmin and Z. Janelidze, Closedness properties of internal relations VII: Embedding theorems, *in preparation.*

[60] P.-A. Jacqmin, D. Rodelo and T. Van der Linden, Stability properties characterising n-permutable categories, *in preparation.*

[61] P.-A. Jacqmin and E.M. Vitale, Bicategories of fractions for groupoids in monadic categories, *Tbilisi Math. J.* **8** (2015), 85–105.

[62] G. Janelidze, L. Márki and W. Tholen, Semi-abelian categories, *J. Pure Appl. Algebra* **168** (2002), 367–386.

[63] Z. Janelidze, Subtractive categories, *Appl. Categ. Struct.* **13** (2005), 343–350.

[64] Z. Janelidze, Closedness properties of internal relations I: A unified approach to Mal'tsev, unital and subtractive categories, *Theory Appl. Categ.* **16** (2006), 236–261.

[65] Z. Janelidze, Closedness properties of internal relations II: Bourn localization, *Theory Appl. Categ.* **16** (2006), 262–282.

[66] Z. Janelidze, Closedness properties of internal relations III: Pointed protomodular categories, *Appl. Categ. Struct.* **15** (2007), 325–338.

[67] Z. Janelidze, Closedness properties of internal relations V: Linear Mal'tsev conditions, *Algebra Univers.* **58** (2008), 105–117.

[68] Z. Janelidze, Closedness properties of internal relations VI: Approximate operations, *Cah. Topol. Géom. Différ. Catég.* **50** (2009), 298–319.

[69] Z. Janelidze and N. Martins-Ferreira, Weakly Mal'tsev categories and strong relations, *Theory Appl. Categ.* **27** (2012), 65–79.

[70] Z. Janelidze, D. Rodelo and T. Van der Linden, Hagemann's theorem for regular categories, *J. Homotopy Related Struct.* **9** (2014), 55–66.

[71] B. Jónsson and A. Tarski, Direct decompositions of finite algebraic systems, *Notre Dame Math. Lect.*, Notre Dame, Indiana (1947).

[72] G.M. Kelly, Basics concepts of enriched category theory, *London Math. Soc. Lecture Note Ser.* **64** (1982), 2nd edition: *Repr. Theory Appl. Categ.* **10** (2005).

[73] S. Lack, A note on the exact completion of a regular category, and its infinitary generalizations, *Theory Appl. Categ.* **5** (1999), 70–80.

[74] S. Lack, A 2-categories companion, *Towards Higher Categories, Springer* (2009), 105–191.

[75] J. Lambek, Goursat's theorem and the Zassenhaus lemma, *Canad. J. Math.* **10** (1957), 45–56.

[76] F.W. Lawvere, Functorial semantics of algebraic theories, *PhD thesis, Columbia Uni.* (1963).

[77] F.W. Lawvere, Category theory over a base topos, *mimeographed notes*, Università di Perugia (1972–1973).

[78] I.J. Le Creurer, F. Marmolejo and E.M. Vitale, Beck's theorem for pseudo-monads, *J. Pure Appl. Algebra* **173** (2002), 293–313.

[79] F.E.J. Linton, Autonomous equational categories, *J. Math. Mech.* **15** (1966), 637–642.

[80] S. Lubkin, Imbedding of abelian categories, *Trans. Amer. Math. Soc.* **97** (1960), 410–417.

[81] S. Mac Lane, Duality for groups, *Bull. Amer. Math. Soc.* **56** (1950), 485-516.

[82] S. Mac Lane, Categories for the working mathematician, *Springer Grad. Texts Math.* **5** (1971), 2nd edition: *Springer* (1998).

[83] S. Mac Lane, The Yoneda lemma, *Math. Japon.* **47** (1998), 156.

[84] M. Mahmoudi, C. Schubert and W. Tholen, Universality of coproducts in categories of lax algebras, *Appl. Categ. Struct.* **14** (2006), 243–249.

[85] M. Makkai, Strong conceptual completeness for first order logic, *Ann. Pure Appl. Logic* **40** (1988), 167–215.

[86] M. Makkai and R. Paré, Accessible categories: The foundations of categorical model theory, *Contemporary Mathematics (Amer. Math. Soc.)* **104** (1989).

[87] A. I. Mal'tsev, On the general theory of algebraic systems, *Mat. Sbornik*, N.S. **35 (77)** (1954), 3–20 (in Russian); English translation: *Amer. Math. Soc. Trans.* (2) **27** (1963), 125–142.

[88] E.G. Manes, Algebraic Theories, *Springer Grad. Texts Math.* **26** (1976).

[89] F. Marmolejo, Distributive laws for pseudo-monads, *Theory Appl. Categ.* **5** (1999), 91–147.

[90] N. Martins-Ferreira, Low-dimensional internal categorical structures in weakly Mal'cev sesquicategories, *PhD thesis, UCT* (2008).

[91] N. Martins-Ferreira, Weakly Mal'cev categories, *Theory Appl. Categ.* **21** (2008), 91–117.

[92] N. Martins-Ferreira, New wide classes of weakly Mal'tsev categories, *Appl. Categ. Struct.* **23** (2015), 741–751.

[93] B. Mitchell, The full imbedding theorem, *Amer. J. Math.* **86** (1964), 619–637.

[94] B. Noohi, Notes on 2-groupoids, 2-groups and crossed modules, *Homology, Homotopy Appl.* **9** (2007), 75–106.

[95] R. Paré, On absolute colimits, *J. Algebra* **19** (1971), 80–95.

[96] D. Pronk, Etendues and stacks as bicategories of fractions, *Compositio Math.* **102** (1996), 243–303.

[97] D. Rodelo and T. Van der Linden, Approximate Hagemann-Mitschke co-operations, *Appl. Categ. Struct.* **22** (2014), 1009–1020.

[98] R. Succi Cruciani, La teoria delle relazioni nello studio di categorie regolari e di categorie esatte, *Riv. Mat. Univ. Parma (4)* **1** (1975), 143–158 (in Italian).

[99] M. Trefois, Topics in combinatorial matrix theory, *PhD thesis, UCL* (2016).

[100] A. Ursini, On subtractive varieties, I, *Algebra Univers.* **31** (1994), 204–222.

[101] E.M. Vitale, On the characterization of monadic categories over Set, *Cah. Topol. Géom. Différ. Catég.* **35** (1994), 351–358.

[102] E.M. Vitale, Bipullbacks and calculus of fractions, *Cah. Topol. Géom. Différ. Catég.* **51** (2010), 83–113.