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PARTIAL ALGEBRAS AND EMBEDDING THEOREMS FOR (WEAKLY) MAL'TSEV CATEGORIES AND MATRIX CONDITIONS

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Résumé. Il est montré que la catégorie des ensembles munis d'une opération partielle de Mal'tsev est faiblement de Mal'tsev. De plus, pour toute petite catégorie faiblement de Mal'tsev et finiment complète, le foncteur de Yoneda la plonge pleinement dans une puissance de cette catégorie des algèbres partielles de Mal'tsev. Ces résultats sont en fait prouvés en utilisant le langage des 'conditions matricielles' de Z. Janelidze afin d'obtenir des théorèmes de plongement pour les catégories faiblement de Mal'tsev, de Mal'tsev, faiblement unitaires, unitaires, fortement unitaires et soustractives.

Abstract. We prove that the category of sets equipped with a partial Mal'tsev operation is a weakly Mal'tsev category. Moreover, for each small finitely complete weakly Mal'tsev category, the Yoneda embedding fully embeds it into a power of this category of partial Mal'tsev algebras. We actually prove these results using the language of 'matrix conditions' from Z. Janelidze, getting in this way embedding theorems for weakly Mal'tsev, Mal'tsev, weakly unital, unital, strongly unital and subtractive categories.

Keywords. embedding theorem, (weakly) Mal'tsev category, (weakly) unital category, partial algebra, closed homomorphism.

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1. Introduction

Mal'tsev categories have been defined in [4] as finitely complete categories in which every binary relation is difunctional. This generalises the notion of a regular Mal'tsev category from [3]. A variety of universal algebras is a Mal'tsev category if and only if its corresponding theory contains a ternary term $p(x,y,z)$ satisfying the identities $p(x,x,y) = y = p(y,x,x)$ [13]. There are many characterisations of Mal'tsev categories in the literature. For instance, a finitely complete category is a Mal'tsev category if and only if, for any pullback of split epimorphisms,

$$\begin{tikzcd} P \arrow[r, shift right, two heads] \arrow[d, shift right, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, shift right, "r_Y"'] \arrow[d, shift left, two heads] \\ X \arrow[u, shift right, "l_X"] \arrow[r, shift left, two heads] & Z \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd}$$

the induced morphisms l_X and r_Y are jointly strongly epimorphic [2]. In [15], N. Martins-Ferreira generalises this notion defining a weakly Mal'tsev category as a category in which the pullbacks as above exist and the morphisms l_X and r_Y are jointly epimorphic.

For a small category $\mathbb{C}$, the full Yoneda embedding $\mathbb{C} \to \mathrm{Set}^{\mathbb{C}^{\mathrm{op}}}$ preserves limits. This allows one to reduce the proofs of some statements about limits in any category to the particular case of Set, the category of sets. The aim of this paper is to construct a weakly Mal'tsev category $\mathbb{M}$ for which, if $\mathbb{C}$ is a small weakly Mal'tsev finitely complete category, the Yoneda embedding factors through $\mathbb{M}^{\mathbb{C}^{\mathrm{op}}}$.

$$\begin{tikzcd} & \mathbb{M}^{\mathbb{C}^{\mathrm{op}}} \arrow[d] \\ \mathbb{C} \arrow[ur, "\phi"] \arrow[r, "Y"'] & \mathrm{Set}^{\mathbb{C}^{\mathrm{op}}} \end{tikzcd}$$

This functor ϕ is a full and faithful embedding which preserves and reflects finite limits. Up to a change of universe, it is then enough to prove some statements about finite limits in $\mathbb{M}$ in order to prove them in all weakly Mal'tsev finitely complete categories.

An object in this category $\mathbb{M}$ is a set A equipped with a partial operation $p\colon A^3 \to A$ which is defined (at least) for all triples of the form (x,x,y) and

(y, x, x) and which satisfies the axioms $p(x, x, y) = y = p(y, x, x)$. A homomorphism between such partial Mal'tsev algebras is a function $f\colon A \to B$ such that, if $p(x, y, z) \in A$ is defined, then $p(f(x), f(y), f(z)) \in B$ is also defined and equal to $f(p(x, y, z))$. In general, they fail to satisfy the converse property: if $p(f(x), f(y), f(z))$ is defined in B, then $p(x, y, z)$ is defined in A. Homomorphisms satisfying this additional property are said to be closed [7] (also called strong homomorphisms in [6]). We prove that the monomorphisms in $\mathbb{M}$ are exactly the injective homomorphisms and strong monomorphisms are exactly the injective closed homomorphisms. With this notion of a closed monomorphism, we get a similar embedding theorem for Mal'tsev categories: any small Mal'tsev category $\mathbb{C}$ admits a full and faithful embedding $\phi\colon \mathbb{C} \to \mathbb{M}^{\mathbb{C}^{\mathrm{op}}}$ which preserves and reflects finite limits and such that for each monomorphism f and each object $X \in \mathbb{C}^{\mathrm{op}}$, $\phi(f)_X$ is a closed monomorphism.

In [8], an embedding theorem for the smaller collection of regular Mal'tsev categories has been proved. More precisely, a regular Mal'tsev category $\mathbb{M}'$ has been constructed such that each small regular Mal'tsev category has a regular conservative embedding into a power of $\mathbb{M}'$. That category $\mathbb{M}'$ is also constructed using a partial ternary operation p satisfying the Mal'tsev identities. But one of the main differences between the embedding theorem of [8] and the ones of this paper is the fact that, in $\mathbb{M}'$, the domain of definition of p is determined as the solution set of some totally defined equation. Therefore, all monomorphisms in $\mathbb{M}'$ are closed, which is not the case in $\mathbb{M}$.

In order to establish at the same time embedding theorems for weakly Mal'tsev, Mal'tsev, weakly unital [14], unital [2], strongly unital [2] and subtractive [9] categories, we use the 'matrix conditions' introduced in [10]. For each extended matrix M of terms in a commutative algebraic theory, we construct the category of partial M-algebras Part_M (being $\mathbb{M}$ when M is the Mal'tsev matrix). This category Part_M has M-closed strong relations, its monomorphisms are exactly the injective homomorphisms and its strong monomorphisms are closed. Moreover, for some particular M's, closed epimorphisms are surjective and closed monomorphisms actually coincide with strong monomorphisms (see Propositions 3.8, 3.9 and Corollary 3.11). For a general M, we then prove an embedding theorem for small categories with M-closed relations and their 'weakly version', the categories with M-closed strong relations.

The paper is divided as follows. In Section 2, we recall the notions of a category with M-closed relations and with M-closed strong relations. In Section 3, we construct the category Part_M and study its closed monomorphisms. Section 4 is devoted to the proof of our embedding theorems, while in Section 5 we give some examples how to use these embedding theorems to make proofs using elements in the above contexts.

2. Categories with M-closed (strong) relations

In order to recall the general treatment of unital, strongly unital, Mal'tsev and subtractive categories introduced in [10], we first need to recall the notion of a $\mathcal{T}$-enrichment.

2.1 $\mathcal{T}$-enrichments

Let $\mathcal{T}$ be an algebraic theory (by that we will always mean a finitary one-sorted algebraic theory). An *internal $\mathcal{T}$-algebra* in a category $\mathbb{C}$ is an object A of $\mathbb{C}$ equipped with a structure of (ordinary) $\mathcal{T}$-algebra on $Y(A)$, where $Y\colon \mathbb{C} \to \mathrm{Set}^{\mathbb{C}^{\mathrm{op}}}$ is the Yoneda embedding. An *internal homomorphism of internal $\mathcal{T}$-algebras* is a morphism $f\colon A \to B$ in $\mathbb{C}$ such that $Y(f)$ is an ordinary homomorphism of algebras. This forms the category $\mathrm{Alg}_{\mathcal{T}}\,\mathbb{C}$ of internal $\mathcal{T}$-algebras.

A *$\mathcal{T}$-enrichment* on $\mathbb{C}$ is a section of the forgetful functor $\mathrm{Alg}_{\mathcal{T}}\,\mathbb{C} \to \mathbb{C}$. In order words, it is the assignment of an internal $\mathcal{T}$-algebra structure for each object A of $\mathbb{C}$ in such a way that every morphism is an internal $\mathcal{T}$-algebra homomorphism. A *$\mathcal{T}$-enriched category* is a category $\mathbb{C}$ with a fixed $\mathcal{T}$-enrichment. Thus, a $\mathcal{T}$-enriched category is a category $\mathbb{C}$ equipped with a factorisation $\mathrm{Hom}_{\mathbb{C}}$ of the functor $\mathrm{hom}_{\mathbb{C}}$ through $\mathrm{Alg}_{\mathcal{T}}$, the category of $\mathcal{T}$-algebras.

$$\begin{tikzcd} & \mathrm{Alg}_{\mathcal{T}} \arrow[d, "U"] \\ \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \arrow[ur, "\mathrm{Hom}_{\mathbb{C}}"] \arrow[r, "\mathrm{hom}_{\mathbb{C}}"'] & \mathrm{Set} \end{tikzcd}$$

A *$\mathcal{T}$-enriched functor* between the $\mathcal{T}$-enriched categories $\mathbb{C}$ and $\mathbb{D}$ is a functor $F\colon \mathbb{C} \to \mathbb{D}$ such that for all $A, B \in \mathbb{C}$,

$$F\colon \operatorname{Hom}_{\mathbb{C}}(A,B) \to \operatorname{Hom}_{\mathbb{D}}(F(A),F(B))$$

is a homomorphism of $\mathcal{T}$-algebras.

If $\mathcal{K}$ is another algebraic theory, $\mathcal{T}$-enrichments of $\mathrm{Alg}_{\mathcal{K}}$ are in one-to-one correspondence with central morphisms $\mathcal{T} \to \mathcal{K}$ of algebraic theories. These are morphisms such that for every term t from $\mathcal{T}$, its interpretation t^ι as a term of $\mathcal{K}$ commutes with every term q of $\mathcal{K}$ in the sense that

$$\begin{aligned} &t^\iota(q(x_{11},\dots,x_{1m}),\dots,q(x_{n1},\dots,x_{nm})) \\ &= q(t^\iota(x_{11},\dots,x_{n1}),\dots,t^\iota(x_{1m},\dots,x_{nm})) \end{aligned}$$

is a theorem in $\mathcal{K}$ (where n and m are the arities of t and q respectively) (see [5]). The theory $\mathcal{T}$ is said to be *commutative* [12] if the identity $\mathcal{T} \to \mathcal{T}$ is a central morphism, i.e., if every two operations in $\mathcal{T}$ commute with each other.

Notice that if $\mathbb{C}$ is a $\mathcal{T}$-enriched category and $\mathbb{P}$ a small category, then the equalities

$$t(\alpha_1,\dots,\alpha_n)_P = t(\alpha_{1,P},\dots,\alpha_{n,P})$$

for all n-ary terms t of $\mathcal{T}$, $P \in \mathbb{P}$ and natural transformations $\alpha_1,\dots,\alpha_n : F \Rightarrow G$ define a $\mathcal{T}$-enrichment on the functor category $\mathbb{C}^{\mathbb{P}}$. If $\mathcal{T}$ is commutative and $\mathbb{C}$ small, the Yoneda embedding factors through $\mathrm{Alg}_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}}$ as a $\mathcal{T}$-enriched functor $Y_{\mathcal{T}}\colon \mathbb{C} \to \mathrm{Alg}_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}}$.

$$\begin{tikzcd} & \mathrm{Alg}_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}} \arrow[d, "U^{\mathbb{C}^{\mathrm{op}}}"] \\ \mathbb{C} \arrow[ur, "Y_{\mathcal{T}}"] \arrow[r, "Y"'] & \mathrm{Set}^{\mathbb{C}^{\mathrm{op}}} \end{tikzcd}$$

2.2 Categories with M-closed relations

Let again $\mathcal{T}$ be an algebraic theory. An *extended matrix of terms in $\mathcal{T}$* [10] is a matrix

$$M = \left(\begin{array}{ccc|c} t_{11} & \cdots & t_{1m} & u_1 \\ \vdots & & \vdots & \vdots \\ t_{n1} & \cdots & t_{nm} & u_n \end{array} \right) \qquad (1)$$

where the t_{ij}'s and the u_i's are terms of $\mathcal{T}$ in the variables $x_1, \dots, x_k$ with $n \geqslant 1$, $m \geqslant 0$ and $k \geqslant 0$.

Let $r = (r_i \colon R \to A)_{i\in\{1,\dots,n\}}$ be an n-ary relation in a $\mathcal{T}$-enriched category $\mathbb{C}$. We say that r is *M-closed* when, for all object X in $\mathbb{C}$ and morphisms $x_1, \dots, x_k \colon X \to A$, if for each $j \in \{1, \dots, m\}$, the span $(t_{ij}(x_1, \dots, x_k) \colon X \to A)_{i\in\{1,\dots,n\}}$ factors through r then so does the span $(u_i(x_1, \dots, x_k) \colon X \to A)_{i\in\{1,\dots,n\}}$.

Now, if $r = (r_i \colon R \to A_i)_{i\in\{1,\dots,n\}}$ is an n-ary relation in $\mathbb{C}$, we say that this relation r is *strictly M-closed* when, for all object X in $\mathbb{C}$ and families of morphisms $(x_{ii'} \colon X \to A_i)_{i\in\{1,\dots,n\},i'\in\{1,\dots,k\}}$, if for each $j \in \{1, \dots, m\}$, the span $(t_{ij}(x_{i1}, \dots, x_{ik}) \colon X \to A_i)_{i\in\{1,\dots,n\}}$ factors through r then so does the span $(u_i(x_{i1}, \dots, x_{ik}) \colon X \to A_i)_{i\in\{1,\dots,n\}}$.

Here is the link between M-closedness and strict M-closedness.

Theorem 2.1. *(Theorem 5.5 in [10]) Let $\mathcal{T}$ be an algebraic theory, M an extended matrix of terms in $\mathcal{T}$ as in (1) and $\mathbb{C}$ a finitely complete $\mathcal{T}$-enriched category. Then, the following conditions are equivalent:*

1. *Every relation $r \colon R \rightarrowtail A^n$ in $\mathbb{C}$ is M-closed.*
2. *Every relation $r \colon R \rightarrowtail A_1 \times \cdots \times A_n$ in $\mathbb{C}$ is strictly M-closed.*

If the above conditions are satisfied, we say that $\mathbb{C}$ *has M-closed relations*. This matrix notation allows an easy characterisation in the varietal context.

Theorem 2.2. *(Theorem 3.2 in [10]) Let $\mathcal{T} \to \mathcal{K}$ be a central morphism of algebraic theories. Let also M be an extended matrix of terms in $\mathcal{T}$ as in (1). Then, the $\mathcal{T}$-enriched category $\mathrm{Alg}_{\mathcal{K}}$ has M-closed relations if and only if there exists an m-ary term p in $\mathcal{K}$ such that*

$$p(t^{\iota}_{i1}(x_1, \dots, x_k), \dots, t^{\iota}_{im}(x_1, \dots, x_k)) = u^{\iota}_i(x_1, \dots, x_k)$$

is a theorem of $\mathcal{K}$ for each $i \in \{1, \dots, n\}$ (where t^{ι} is the interpretation in $\mathcal{K}$ of the term t in $\mathcal{T}$ induced by the morphism $\mathcal{T} \to \mathcal{K}$).

Example 2.3. Let $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$ be the theory of sets, $\mathbb{C}$ a finitely complete category and M_{Mal} the extended matrix

$$M_{\mathrm{Mal}} = \left(\begin{array}{ccc|c} x & y & y & x \\ x & x & y & y \end{array} \right)$$

of terms in $\mathrm{Th}[\mathrm{Set}]$. Then $\mathbb{C}$ has M_{Mal}-closed relations if and only if $\mathbb{C}$ is a Mal'tsev category [4, 10].

If $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$ is the theory of pointed sets and $\mathbb{C}$ a finitely complete pointed category, then the following equivalences hold:

- $\mathbb{C}$ has M_{Uni}-closed relations if and only if $\mathbb{C}$ is unital [2, 10], where M_{Uni} is the extended matrix
$$M_{\mathrm{Uni}} = \left(\begin{array}{cc|c} x & 0 & x \\ 0 & x & x \end{array}\right).$$
- $\mathbb{C}$ has M_{StrUni}-closed relations if and only if $\mathbb{C}$ is strongly unital [2, 10], where M_{StrUni} is the extended matrix
$$M_{\mathrm{StrUni}} = \left(\begin{array}{ccc|c} x & 0 & 0 & x \\ y & y & x & x \end{array}\right).$$
- $\mathbb{C}$ has M_{Subt}-closed relations if and only if $\mathbb{C}$ is subtractive [9, 10], where M_{Subt} is the extended matrix
$$M_{\mathrm{Subt}} = \left(\begin{array}{cc|c} x & 0 & x \\ x & x & 0 \end{array}\right).$$

2.3 Categories with M-closed strong relations

We now weaken this notion of a category with M-closed relations, considering only strong relations. We recall that in a finitely complete category $\mathbb{C}$, a morphism m is said to be a *strong monomorphism* if it is orthogonal to any epimorphism e. This means that for any commutative square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d] & B \arrow[d] \arrow[dl, dotted, "d"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

with e an epimorphism, there exists a (unique) diagonal d making the two triangles commutative. It is easy to see that since $\mathbb{C}$ has pullbacks, it implies that m is a monomorphism and even an extremal monomorphism. Strong monomorphisms are closed under composition and stable under pullbacks. Regular monomorphisms (i.e., equalisers) are strong monomorphisms. We say that a span $(r_i\colon R \to A_i)_{i\in\{1,\dots,n\}}$ is a *strong relation* if the induced morphism $r = (r_1,\dots,r_n)\colon R \to A_1 \times \cdots \times A_n$ is a strong monomorphism.

Theorem 2.4. *Let $\mathcal{T}$ be an algebraic theory, M an extended matrix of terms in $\mathcal{T}$ as in (1) and $\mathbb{C}$ a finitely complete $\mathcal{T}$-enriched category. Then, the following conditions are equivalent:*

1. *Every strong relation $r\colon R \rightarrowtail A^n$ in $\mathbb{C}$ is M-closed.*
2. *Every strong relation $r\colon R \rightarrowtail A_1 \times \cdots \times A_n$ in $\mathbb{C}$ is strictly M-closed.*

Proof. $2 \Rightarrow 1$ being trivial, let us prove $1 \Rightarrow 2$. So, let us consider a strong relation $r\colon R \rightarrowtail A_1 \times \cdots \times A_n$ in $\mathbb{C}$. Since r is strong, its pullback along $\pi_1 \times \cdots \times \pi_n$ is also strong, where $\pi_i\colon A_1 \times \cdots \times A_n \to A_i$ is the i-th projection.

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "r"] \\
(A_1 \times \cdots \times A_n)^n \arrow[r, "\pi_1 \times \cdots \times \pi_n"'] & A_1 \times \cdots \times A_n
\end{tikzcd}$$

We conclude the proof by Proposition 1.9 in [10] which says that r is strictly M-closed if and only if s is M-closed. $\square$

If the above conditions are satisfied, we say that $\mathbb{C}$ *has M-closed strong relations*. In view of the following examples, we could also have written that $\mathbb{C}$ is 'weakly with M-closed relations'.

Example 2.5. If $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$ and $\mathbb{C}$ is a finitely complete category, $\mathbb{C}$ has M_{Mal}-closed strong relations if and only if $\mathbb{C}$ is a weakly Mal'tsev category. Let us recall that $\mathbb{C}$ is *weakly Mal'tsev* [15] if for every pullback of split epimorphisms,

$$\begin{tikzcd} X \times_Z Y \arrow[r, two heads, "p_Y"] \arrow[d, two heads, "p_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, bend left, "r_Y"] \arrow[d, two heads, "g"'] \\ X \arrow[r, two heads, "f"] \arrow[u, bend right, "l_X"'] & Z \arrow[l, bend left, "s"] \arrow[u, bend right, "t"'] \end{tikzcd}$$

the induced morphisms $l_X = (1_X, tf)$ and $r_Y = (sg, 1_Y)$ are jointly epimorphic. Such a characterisation holds because a binary relation is strictly M_{Mal}-closed precisely when it is difunctional [10] and by Corollary 5.1 in [11], $\mathbb{C}$ is weakly Mal'tsev if and only if every binary strong relation in $\mathbb{C}$ is difunctional.

Example 2.6. If $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$ and $\mathbb{C}$ is a finitely complete pointed category, $\mathbb{C}$ has M_{Uni}-closed strong relations if and only if $\mathbb{C}$ is weakly unital. We recall that $\mathbb{C}$ is *weakly unital* [14] if for all objects X and Y in $\mathbb{C}$, the product injections

$$X \xrightarrow{(1_X,0)} X \times Y \xleftarrow{(0,1_Y)} Y$$

are jointly epimorphic. In that case, if $r\colon R \rightarrowtail A^2$ is a strong relation and $x\colon X \to A$ a morphism such that $(x,0)\colon X \to A^2$ and $(0,x)\colon X \to A^2$ factor through r, we consider the pullback s of r along x^2.

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "r"] \\ X^2 \arrow[r, "x^2"'] & A^2 \end{tikzcd}$$

The relation s is strong, $(1_X, 0)$ and $(0, 1_X)\colon X \to X^2$ factor through it and we only have to prove that $(1_X, 1_X)$ also factors through s. But since $(1_X, 0)$ and $(0, 1_X)$ are jointly epimorphic, s is an epimorphism. Together with the fact that it is also a strong monomorphism, s is an isomorphism and so $(1_X, 1_X)$ factors through it.

Conversely, suppose that $\mathbb{C}$ has M_{Uni}-closed strong relations and let $f, g :$ $X \times Y \to Z$ be morphisms such that $f(1_X, 0) = g(1_X, 0)$ and $f(0, 1_Y) = g(0, 1_Y)$. Their equaliser $e\colon E \rightarrowtail X \times Y$ is a strong relation through which $(p_X, 0)$ and $(0, p_Y)\colon X \times Y \to X \times Y$ factor. Thus, by assumption, $1_{X\times Y} = (p_X, p_Y)\colon X \times Y \to X \times Y$ also factors through it, so that e is an isomorphism and $f = g$.

3. The category of partial M-algebras

3.1 Part_M and its limits

We suppose from now on that $\mathcal{T}$ is a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$ as in (1). A *partial M-algebra* is a $\mathcal{T}$-algebra A equipped with a partial operation $p\colon A^m \to A$ such that

- for each $i \in \{1, \dots, n\}$ and all $a_1, \dots, a_k \in A$,
$$p(t_{i1}(a_1, \dots, a_k), \dots, t_{im}(a_1, \dots, a_k))$$
is defined and
$$p(t_{i1}(a_1, \dots, a_k), \dots, t_{im}(a_1, \dots, a_k)) = u_i(a_1, \dots, a_k);$$
- for each r-ary operation symbol σ of $\mathcal{T}$ and all families of elements $(a_j^{j'} \in A)_{j\in\{1,\dots,m\}, j'\in\{1,\dots,r\}}$ such that $p(a_1^{j'}, \dots, a_m^{j'})$ is defined for each $j' \in \{1, \dots, r\}$, $p(\sigma(a_1^1, \dots, a_1^r), \dots, \sigma(a_m^1, \dots, a_m^r))$ is defined and the equality
$$\begin{aligned} &p(\sigma(a_1^1, \dots, a_1^r), \dots, \sigma(a_m^1, \dots, a_m^r)) \\ &= \sigma(p(a_1^1, \dots, a_m^1), \dots, p(a_1^r, \dots, a_m^r)) \end{aligned}$$
holds.

A *homomorphism $f\colon A \to B$ of partial M-algebras* is a homomorphism between the corresponding $\mathcal{T}$-algebras such that, for all $a_1, \dots, a_m \in A$ for which $p(a_1, \dots, a_m)$ is defined, $p(f(a_1), \dots, f(a_m))$ is also defined and
$$p(f(a_1), \dots, f(a_m)) = f(p(a_1, \dots, a_m)).$$
We denote by Part_M the corresponding category. We have a $\mathcal{T}$-enrichment on Part_M: if σ is an r-ary operation symbol of $\mathcal{T}$ and $f_1, \dots, f_r\colon A \to B$ are homomorphisms of partial M-algebras, we define $\sigma(f_1, \dots, f_r)\colon A \to B$ by
$$\sigma(f_1, \dots, f_r)(a) = \sigma(f_1(a), \dots, f_r(a))$$

for all $a \in A$. Since $\mathcal{T}$ is commutative, $\mathrm{Alg}_{\mathcal{T}}$ has a $\mathcal{T}$-enrichment computed as above and so $\sigma(f_1, \ldots, f_r)$ is a homomorphism of $\mathcal{T}$-algebras. Moreover, if $a_1, \ldots, a_m \in A$ are such that $p(a_1, \ldots, a_m)$ is defined, for each $j' \in \{1, \ldots, r\}$, $p(f_{j'}(a_1), \ldots, f_{j'}(a_m))$ is also defined. This implies

$$\begin{aligned}&p(\sigma(f_1, \ldots, f_r)(a_1), \ldots, \sigma(f_1, \ldots, f_r)(a_m))\\ &= p(\sigma(f_1(a_1), \ldots, f_r(a_1)), \ldots, \sigma(f_1(a_m), \ldots, f_r(a_m)))\end{aligned}$$

is defined as well and equal to

$$\begin{aligned}&\sigma(p(f_1(a_1), \ldots, f_1(a_m)), \ldots, p(f_r(a_1), \ldots, f_r(a_m)))\\ &= \sigma(f_1(p(a_1, \ldots, a_m)), \ldots, f_r(p(a_1, \ldots, a_m)))\\ &= \sigma(f_1, \ldots, f_r)(p(a_1, \ldots, a_m))\end{aligned}$$

in view of the second condition in the definition of partial M-algebras. This proves $\sigma(f_1, \ldots, f_r)$ is indeed a homomorphism of partial M-algebras.

Let us now describe small limits in Part_M. In order to do so, we consider a small diagram $D\colon \mathbb{J} \to \mathrm{Part}_M$. Let $(\lambda_j \colon L \to U_{\mathcal{T}}D(j))_{j\in\mathbb{J}}$ be the limit of $U_{\mathcal{T}}D$ in $\mathrm{Alg}_{\mathcal{T}}$, where $U_{\mathcal{T}}\colon \mathrm{Part}_M \to \mathrm{Alg}_{\mathcal{T}}$ is the forgetful functor. So L is given by

$$L = \{(a_j)_{j\in\mathbb{J}} \in \prod_{j\in\mathbb{J}} D(j) \mid D(d)(a_j) = a_{j'} \ \forall d\colon j \to j' \in \mathbb{J}\}$$

with

$$\sigma((a_j^1)_{j\in\mathbb{J}}, \ldots, (a_j^r)_{j\in\mathbb{J}}) = (\sigma(a_j^1, \ldots, a_j^r))_{j\in\mathbb{J}}$$

for each r-ary operation symbol σ of $\mathcal{T}$. Now, if $(a_j^1)_{j\in\mathbb{J}}, \ldots, (a_j^m)_{j\in\mathbb{J}} \in L$, we define $p((a_j^1)_{j\in\mathbb{J}}, \ldots, (a_j^m)_{j\in\mathbb{J}})$ if and only if $p(a_j^1, \ldots, a_j^m)$ is defined for all $j \in \mathbb{J}$. In this case, we set

$$p((a_j^1)_{j\in\mathbb{J}}, \ldots, (a_j^m)_{j\in\mathbb{J}}) = (p(a_j^1, \ldots, a_j^m))_{j\in\mathbb{J}}.$$

This makes L a partial M-algebra. Indeed, for each $i \in \{1, \ldots, n\}$ and each $(a_j^1)_{j\in\mathbb{J}}, \ldots, (a_j^k)_{j\in\mathbb{J}} \in L$,

$$\begin{aligned}&p(t_{i1}((a_j^1)_{j\in\mathbb{J}}, \ldots, (a_j^k)_{j\in\mathbb{J}}), \ldots, t_{im}((a_j^1)_{j\in\mathbb{J}}, \ldots, (a_j^k)_{j\in\mathbb{J}}))\\ &= p((t_{i1}(a_j^1, \ldots, a_j^k))_{j\in\mathbb{J}}, \ldots, (t_{im}(a_j^1, \ldots, a_j^k))_{j\in\mathbb{J}})\end{aligned}$$

is defined since $p(t_{i1}(a_j^1,\dots,a_j^k),\dots,t_{im}(a_j^1,\dots,a_j^k))$ is for each $j\in\mathbb{J}$ and it is equal to

$$\begin{aligned}(p(t_{i1}(a_j^1,\dots,a_j^k),\dots,t_{im}(a_j^1,\dots,a_j^k)))_{j\in\mathbb{J}} &= (u_i(a_j^1,\dots,a_j^k))_{j\in\mathbb{J}}\\ &= u_i((a_j^1)_{j\in\mathbb{J}},\dots,(a_j^k)_{j\in\mathbb{J}}).\end{aligned}$$

We check the second condition analogously: Let σ be an r-ary operation symbol of $\mathcal{T}$ and for each $j'\in\{1,\dots,r\}$, $(a_j^{1,j'})_{j\in\mathbb{J}},\dots,(a_j^{m,j'})_{j\in\mathbb{J}}$ elements of L such that $p((a_j^{1,j'})_{j\in\mathbb{J}},\dots,(a_j^{m,j'})_{j\in\mathbb{J}})$ is defined (i.e., $p(a_j^{1,j'},\dots,a_j^{m,j'})$ is defined for each $j\in\mathbb{J}$). This implies

$$p(\sigma(a_j^{1,1},\dots,a_j^{1,r}),\dots,\sigma(a_j^{m,1},\dots,a_j^{m,r}))$$

is defined and equal to

$$\sigma(p(a_j^{1,1},\dots,a_j^{m,1}),\dots,p(a_j^{1,r},\dots,a_j^{m,r}))$$

for each $j\in\mathbb{J}$. Thus

$$\begin{aligned}&p(\sigma((a_j^{1,1})_{j\in\mathbb{J}},\dots,(a_j^{1,r})_{j\in\mathbb{J}}),\dots,\sigma((a_j^{m,1})_{j\in\mathbb{J}},\dots,(a_j^{m,r})_{j\in\mathbb{J}}))\\ &= p((\sigma(a_j^{1,1},\dots,a_j^{1,r}))_{j\in\mathbb{J}},\dots,(\sigma(a_j^{m,1},\dots,a_j^{m,r}))_{j\in\mathbb{J}})\end{aligned}$$

is also defined in L and equal to

$$\begin{aligned}&(\sigma(p(a_j^{1,1},\dots,a_j^{m,1}),\dots,p(a_j^{1,r},\dots,a_j^{m,r})))_{j\in\mathbb{J}}\\ &= \sigma(p((a_j^{1,1})_{j\in\mathbb{J}},\dots,(a_j^{m,1})_{j\in\mathbb{J}}),\dots,p((a_j^{1,r})_{j\in\mathbb{J}},\dots,(a_j^{m,r})_{j\in\mathbb{J}})),\end{aligned}$$

which shows that L is a partial M-algebra. Moreover, given a cone $(\mu_j\colon A\to D(j))_{j\in\mathbb{J}}$ over D, let f be the unique homomorphism of $\mathcal{T}$-algebras

$$\begin{aligned}f\colon A &\longrightarrow L\\ a &\longmapsto (\mu_j(a))_{j\in\mathbb{J}}\end{aligned}$$

such that $\lambda_j f=\mu_j$ for each $j\in\mathbb{J}$. If $a_1,\dots,a_m\in A$ are such that $p(a_1,\dots,a_m)$ is defined in A, $p(\mu_j(a_1),\dots,\mu_j(a_m))$ is defined in $D(j)$ for each $j\in\mathbb{J}$. Thus, $p(f(a_1),\dots,f(a_m))$ is also defined and equal to

$$\begin{aligned}p((\mu_j(a_1))_{j\in\mathbb{J}},\dots,(\mu_j(a_m))_{j\in\mathbb{J}}) &= (p(\mu_j(a_1),\dots,\mu_j(a_m)))_{j\in\mathbb{J}}\\ &= (\mu_j(p(a_1,\dots,a_m)))_{j\in\mathbb{J}}\\ &= f(p(a_1,\dots,a_m)),\end{aligned}$$

which proves that f is a homomorphism of partial M-algebras and the cone $(\lambda_j\colon L \to D(j))_{j\in\mathbb{J}}$ the limit of D. Therefore, Part_M is complete and $U_{\mathcal{T}}\colon \mathrm{Part}_M \to \mathrm{Alg}_{\mathcal{T}}$ preserves small limits, but it does not reflect them in general. Indeed, one could have defined p on a smaller subset of L^m in order to make L a partial M-algebra, but this would not have made it a limit in Part_M. This means $U_{\mathcal{T}}$ is not conservative in general. Here is a simple counterexample.

Counterexample 3.1. Let $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$ and $M = M_{\mathrm{Uni}}$ from Example 2.3. Let A be the pointed set $\{0, x\}$ endowed with the structure of a partial M_{Uni}-algebra given by $p(0,0) = 0$, $p(x,0) = x = p(0,x)$ and $p(x,x)$ undefined. Let also B be the partial M_{Uni}-algebra on $\{0, x\}$ given by $p(0,0) = 0$ and $p(x,0) = x = p(0,x) = p(x,x)$. Then, the identity map $A \to B$ is a bijective homomorphism but not an isomorphism in $\mathrm{Part}_{M_{\mathrm{Uni}}}$.

3.2 Strong monomorphisms in Part_M

In order to understand strong monomorphisms in Part_M, we need to construct a left adjoint to the forgetful functor $U\colon \mathrm{Part}_M \to \mathrm{Set}$. As an intermediate step, we consider the category m-Part where objects are sets X equipped with a partial operation $p\colon X^m \to X$ and morphisms are functions $f\colon X \to Y$ such that if $p(x_1,\dots,x_m)$ is defined for some $x_1,\dots,x_m \in X$, then $p(f(x_1),\dots,f(x_m))$ is also defined and equal to $f(p(x_1,\dots,x_m))$. The forgetful functor $U\colon \mathrm{Part}_M \to \mathrm{Set}$ thus factors as $\mathrm{Part}_M \to m$-$\mathrm{Part} \to \mathrm{Set}$.

Proposition 3.2. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$ as in (1). The forgetful functor*

$$U'\colon \mathrm{Part}_M \to m\text{-}\mathrm{Part}$$

has a left adjoint.

Proof. Let X be an object of m-Part. Let us add the constant operation symbols c_x for all $x \in X$ in $\mathcal{T}$ to form the theory $\mathcal{T}'$. We denote by I the set

$$\begin{aligned} I &= \{1,\dots,n\} \sqcup \{(x_1,\dots,x_m) \in X^m \mid p(x_1,\dots,x_m) \text{ is defined}\} \\ &= \{1,\dots,n\} \sqcup \mathrm{dom}(p) \end{aligned}$$

and, for each $i = (x_1, \dots, x_m) \in \mathrm{dom}(p)$, $t_{ij}(y_1, \dots, y_k)$ is the k-ary term c_{x_j} of $\mathcal{T}'$ for each $j \in \{1, \dots, m\}$ and $u_i(y_1, \dots, y_k)$ the k-ary term $c_{p(i)}$ of $\mathcal{T}'$. Let $\mathcal{Q}$ be the quasivariety of $\mathcal{T}'$-algebras satisfying, for all r-ary (respectively r'-ary) terms τ and τ' of $\mathcal{T}$ and all indices $i_1, \dots, i_r, i'_1, \dots, i'_{r'}$ in I, the following implication: if

$$\begin{aligned}&\tau(t_{i_1 j}(y_{11}, \dots, y_{1k}), \dots, t_{i_r j}(y_{r1}, \dots, y_{rk}))\\ &= \tau'(t_{i'_1 j}(y'_{11}, \dots, y'_{1k}), \dots, t_{i'_{r'} j}(y'_{r'1}, \dots, y'_{r'k}))\end{aligned}$$

for each $j \in \{1, \dots, m\}$, then

$$\begin{aligned}&\tau(u_{i_1}(y_{11}, \dots, y_{1k}), \dots, u_{i_r}(y_{r1}, \dots, y_{rk}))\\ &= \tau'(u_{i'_1}(y'_{11}, \dots, y'_{1k}), \dots, u_{i'_{r'}}(y'_{r'1}, \dots, y'_{r'k})).\end{aligned}$$

For an object A of the quasivariety $\mathcal{Q}$, we define p in A via the equalities

$$\begin{aligned}&p(\tau(t_{i_1 1}(a_{11}, \dots, a_{1k}), \dots, t_{i_r 1}(a_{r1}, \dots, a_{rk})), \dots\\ &\qquad\qquad \dots, \tau(t_{i_1 m}(a_{11}, \dots, a_{1k}), \dots, t_{i_r m}(a_{r1}, \dots, a_{rk})))\\ &= \tau(u_{i_1}(a_{11}, \dots, a_{1k}), \dots, u_{i_r}(a_{r1}, \dots, a_{rk}))\end{aligned}$$

for all r-ary terms τ of $\mathcal{T}$, all indices $i_1, \dots, i_r \in I$ and all families of elements $(a_{j'i'} \in A)_{j' \in \{1,\dots,r\}, i' \in \{1,\dots,k\}}$. We do not define p for any other elements of A^m. In view of the implications defining $\mathcal{Q}$, this partial operation p is well-defined. We see that the first condition defining partial M-algebras is satisfied by choosing τ to be the identity term $\tau(y) = y$. The second condition is also satisfied: Let σ be an r-ary term of $\mathcal{T}$, $\tau^{j'}$ an $r^{j'}$-ary term of $\mathcal{T}$ for each $j' \in \{1, \dots, r\}$, $i^{j'}_{j''} \in I$ for all $j' \in \{1, \dots, r\}$ and $j'' \in \{1, \dots, r^{j'}\}$, and $a^{j'}_{j''i'} \in A$ for all $j' \in \{1, \dots, r\}$, $j'' \in \{1, \dots, r^{j'}\}$ and $i' \in \{1, \dots, k\}$. Then,

$$p((\sigma((\tau^{j'}((t_{i^{j'}_{j''} j}(a^{j'}_{j''1}, \dots, a^{j'}_{j''k}))^{r^{j'}}_{j''=1}))^{r}_{j'=1}))^{m}_{j=1})$$

is defined in view of the $(r^1 + \dots + r^r)$-ary term

$$\sigma(\tau^1(y_{11}, \dots, y_{1r^1}), \dots, \tau^r(y_{r1}, \dots, y_{rr^r}))$$

of $\mathcal{T}$. Moreover, it is equal to

$$\begin{aligned}
&\sigma((\tau^{j'}((u_{i^{j'}_{j''}}(a^{j'}_{j''1},\dots,a^{j'}_{j''k}))^{r^{j'}}_{j''=1}))^{r}_{j'=1})\\
&=\sigma((p((\tau^{j'}((t_{i^{j'}_{j''}j}(a^{j'}_{j''1},\dots,a^{j'}_{j''k}))^{r^{j'}}_{j''=1}))^{m}_{j=1}))^{r}_{j'=1})
\end{aligned}$$

as required. So A has been endowed with a structure of partial M-algebra. We consider the function $f\colon X \to U'(A)\colon x \mapsto c_x$. It is a morphism in m-Part. Indeed, if $i = (x_1,\dots,x_m) \in \mathrm{dom}(p)$, choosing τ to be the identity term $\tau(y) = y$ and $i_1 = i$, we have

$$\begin{aligned}
p(f(x_1),\dots,f(x_m)) &= p(c_{x_1},\dots,c_{x_m})\\
&= p(t_{i1}(a_1,\dots,a_k),\dots,t_{im}(a_1,\dots,a_k))\\
&= u_i(a_1,\dots,a_k)\\
&= c_{p(i)}\\
&= f(p(x_1,\dots,x_m)).
\end{aligned}$$

If $g\colon A \to A'$ is a morphism in $\mathcal{Q}$, it can be considered as a homomorphism of partial M-algebras making the triangle

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "f'"'] & U'(A) \arrow[d, "g"] \\
& U'(A')
\end{tikzcd}$$

commutative. Indeed, the above triangle commutes since g is a $\mathcal{T}'$-homomorphism and when

$$\begin{aligned}
&p(\tau(t_{i_11}(a_{11},\dots,a_{1k}),\dots,t_{i_r1}(a_{r1},\dots,a_{rk})),\dots\\
&\qquad\dots,\tau(t_{i_1m}(a_{11},\dots,a_{1k}),\dots,t_{i_rm}(a_{r1},\dots,a_{rk})))
\end{aligned}$$

is defined in A,

$$\begin{aligned}
&p(g(\tau(t_{i_11}(a_{11},\dots,a_{1k}),\dots,t_{i_r1}(a_{r1},\dots,a_{rk}))),\dots\\
&\qquad\dots,g(\tau(t_{i_1m}(a_{11},\dots,a_{1k}),\dots,t_{i_rm}(a_{r1},\dots,a_{rk}))))\\
&=p(\tau(t_{i_11}(g(a_{11}),\dots,g(a_{1k})),\dots,t_{i_r1}(g(a_{r1}),\dots,g(a_{rk}))),\dots\\
&\qquad\dots,\tau(t_{i_1m}(g(a_{11}),\dots,g(a_{1k})),\dots,t_{i_rm}(g(a_{r1}),\dots,g(a_{rk}))))
\end{aligned}$$

is defined in A' and equal to

$$\begin{aligned}&\tau(u_{i_1}(g(a_{11}),\ldots,g(a_{1k})),\ldots,u_{i_r}(g(a_{r1}),\ldots,g(a_{rk})))\\&= g(\tau(u_{i_1}(a_{11},\ldots,a_{1k}),\ldots,u_{i_r}(a_{r1},\ldots,a_{rk}))).\end{aligned}$$

We have thus defined a functor $F'\colon \mathcal{Q} \to (X \downarrow U')$ where $(X \downarrow U')$ is the comma category of morphisms $X \to U'(A)$ in *m*-Part.

On the other hand, if $f\colon X \to U'(A)$ is an object of $(X \downarrow U')$, A admits a $\mathcal{T}'$-algebra structure considering $c_x = f(x)$ for each $x \in X$. Moreover, for each $i \in I$ and $a_1,\ldots,a_k \in A$,

$$p(t_{i1}(a_1,\ldots,a_k),\ldots,t_{im}(a_1,\ldots,a_k)) = u_i(a_1,\ldots,a_k).$$

So, if τ is an r-ary term of $\mathcal{T}$, $i_1,\ldots,i_r \in I$ and $(a_{j'i'} \in A)_{j'\in\{1,\ldots,r\},i'\in\{1,\ldots,k\}}$,

$$\begin{aligned}&p(\tau(t_{i_11}(a_{11},\ldots,a_{1k}),\ldots,t_{i_r1}(a_{r1},\ldots,a_{rk})),\ldots\\&\qquad\qquad\ldots,\tau(t_{i_1m}(a_{11},\ldots,a_{1k}),\ldots,t_{i_rm}(a_{r1},\ldots,a_{rk})))\\&= \tau(u_{i_1}(a_{11},\ldots,a_{1k}),\ldots,u_{i_r}(a_{r1},\ldots,a_{rk}))\end{aligned}$$

since A is a partial M-algebra. Hence, A satisfies the implications defining $\mathcal{Q}$ and this makes $G'\colon (X \downarrow U') \to \mathcal{Q}$ a functor.

Since the equality above holds in A for any object $f\colon X \to U'(A)$ of $(X \downarrow U')$, the identity map on A defines a morphism $\varepsilon_f\colon F'G'(f) \to f$ in $(X \downarrow U')$. This gives a natural transformation $\varepsilon\colon F'G' \Rightarrow 1_{(X\downarrow U')}$. Moreover, $G'F' = 1_{\mathcal{Q}}$ and we have constructed an adjunction $F' \dashv G'$. But $\mathcal{Q}$ is a quasivariety, so it has an initial object. Therefore $(X \downarrow U')$ has also an initial object which is the reflection of X along U'. $\square$

To construct the reflection of the set X along the forgetful functor *m*-$\mathrm{Part} \to \mathrm{Set}$ is much easier. It suffices to consider the identity map $1_X\colon X \to X$ where the partial operation p on X is nowhere defined. This gives a left adjoint $\mathrm{Set} \to$ *m*-Part. Composed with the left adjoint *m*-$\mathrm{Part} \to \mathrm{Part}_M$ given by the above proposition, we have constructed the left adjoint $F\colon \mathrm{Set} \to \mathrm{Part}_M$ to the forgetful functor $U\colon \mathrm{Part}_M \to \mathrm{Set}$. We remark that in the particular case $X = \varnothing$, the quasivariety $\mathcal{Q}$ described above is the quasivariety $\mathcal{Q}_M$ of $\mathcal{T}$-algebras satisfying, for all r-ary (respectively r'-ary) terms τ and τ' of $\mathcal{T}$ and all indices $i_1,\dots,i_r,i'_1,\dots,i'_{r'}$ in $\{1,\dots,n\}$, the following implication: if

$$\begin{aligned}&\tau(t_{i_1 j}(a_{11},\dots,a_{1k}),\dots,t_{i_r j}(a_{r1},\dots,a_{rk}))\\ &\quad=\tau'(t_{i'_1 j}(a'_{11},\dots,a'_{1k}),\dots,t_{i'_{r'} j}(a'_{r'1},\dots,a'_{r'k}))\end{aligned}$$

for each $j\in\{1,\dots,m\}$, then

$$\begin{aligned}&\tau(u_{i_1}(a_{11},\dots,a_{1k}),\dots,u_{i_r}(a_{r1},\dots,a_{rk}))\\ &\quad=\tau'(u_{i'_1}(a'_{11},\dots,a'_{1k}),\dots,u_{i'_{r'}}(a'_{r'1},\dots,a'_{r'k})).\end{aligned}$$

The functor $F'\colon \mathcal{Q}\to(X\downarrow U')$ is then nothing but the left adjoint $\mathcal{Q}_M\to\mathrm{Part}_M$ to the forgetful functor $\mathrm{Part}_M\to\mathcal{Q}_M$. The left adjoint $F\colon \mathrm{Set}\to\mathrm{Part}_M$ can thus be also obtained by composing $F'\colon \mathcal{Q}_M\to\mathrm{Part}_M$ with the usual free functor $\mathrm{Set}\to\mathcal{Q}_M$.

We now consider the case $X=\{1,\dots,m+1\}$ with p defined only by $p(1,\dots,m)=m+1$. We denote by $X\to U'(F_M)$ its reflection along $U'\colon \mathrm{Part}_M\to m\text{-Part}$ and g its restriction $g\colon\{1,\dots,m\}\hookrightarrow X\to U(F_M)$. The function g is such that $p(g(1),\dots,g(m))$ is defined in F_M and universal with that property, i.e., if $h\colon\{1,\dots,m\}\to U(A)$ is a function to a partial M-algebra A where $p(h(1),\dots,h(m))$ is defined, there exists a unique homomorphism of partial M-algebras $\overline{h}\colon F_M\to A$ such that $U(\overline{h})\circ g=h$.

$$\begin{tikzcd}\{1,\dots,m\} \arrow[rr, "g"] \arrow[dr, "\forall h"'] & & U(F_M) \arrow[dl, dotted, "\exists!\, U(\overline{h})"] \\ & U(A) & \end{tikzcd}$$

Since $U\colon \mathrm{Part}_M\to\mathrm{Set}$ preserves kernel pairs, monomorphisms in Part_M are exactly the injective homomorphisms. We can now study strong monomorphisms: Let $f\colon A\rightarrowtail B$ be such a monomorphism. Consider also the homomorphism $e\colon F(\{1,\dots,m\})\to F_M$ given by the universal property of $F(\{1,\dots,m\})$ and the function $g\colon\{1,\dots,m\}\to U(F_M)$. If $\overline{h},\overline{k}\colon F_M\to C$ are homomorphisms of partial M-algebras such that $\overline{h}e=\overline{k}e$, then $\overline{h}g=\overline{k}g$ and $\overline{h}=\overline{k}$. Thus e is actually an epimorphism in Part_M. If $a_1,\dots,a_m\in A$ are such that $p(f(a_1),\dots,f(a_m))$ is defined, we can construct a commutative square as below with $k(j)=a_j$ and

$\overline{h}(g(j)) = f(a_j)$ for each $j \in \{1, \dots, m\}$.

$$\begin{tikzcd} F(\{1,\dots,m\}) \arrow[r, two heads, "e"] \arrow[d, "k"'] & F_M \arrow[d, "\overline{h}"] \arrow[dl, dotted] \\ A \arrow[r, tail, "f"'] & B \end{tikzcd}$$

Since f is supposed to be a strong monomorphism, $\overline{h}$ factors through f. Hence, $p(a_1, \dots, a_m)$ is defined as well. Therefore, strong monomorphisms in Part_M reflect the m-uples where p is defined, i.e., if $p(f(a_1), \dots, f(a_m))$ is defined, then so is $p(a_1, \dots, a_m)$. Following the terminology of [7] in universal partial algebra, homomorphisms in Part_M which reflect the m-uples where p is defined are said to be *closed*. This leads us to the following proposition.

Proposition 3.3. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$. Strong monomorphisms in Part_M are closed.*

The homomorphism from Counterexample 3.1 is an example of a bijective homomorphism of partial M-algebras which is not closed. Note that isomorphisms in Part_M are exactly the bijective closed homomorphisms. Indeed, in view of the next lemma, closedness of a bijective homomorphism $f\colon B \to C$ is exactly what we need to prove the inverse map $f^{-1}\colon C \to B$ is a homomorphism of partial M-algebras.

Lemma 3.4. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$. Let also $g\colon A \to B$ be a function between partial M-algebras and $f\colon B \rightarrowtail C$ a closed monomorphism in Part_M. If fg is a homomorphism of partial M-algebras, then so is g.*

Proof. Let σ be an r-ary operation symbol of $\mathcal{T}$ and $a_1, \dots, a_r \in A$. Since

$$\begin{aligned} f(g(\sigma(a_1, \dots, a_r))) &= \sigma(fg(a_1), \dots, fg(a_r)) \\ &= f(\sigma(g(a_1), \dots, g(a_r))) \end{aligned}$$

and f is injective, g is a homomorphism of $\mathcal{T}$-algebras.

Besides, if $a_1, \dots, a_m \in A$ are such that $p(a_1, \dots, a_m)$ are defined in A, $p(fg(a_1), \dots, fg(a_m))$ is defined in C and $p(g(a_1), \dots, g(a_m))$ is defined in

B since f is closed. We can also compute

$$\begin{aligned} f(p(g(a_1),\dots,g(a_m))) &= p(fg(a_1),\dots,fg(a_m)) \\ &= fg(p(a_1,\dots,a_m)), \end{aligned}$$

which implies

$$p(g(a_1),\dots,g(a_m)) = g(p(a_1,\dots,a_m))$$

since f is injective. □

We now want to prove that, in some cases, closed monomorphisms in Part_M are exactly the strong monomorphisms. To achieve this, we need to study the properties of closed monomorphisms.

Proposition 3.5. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$. Closed monomorphisms in Part_M are stable under pullbacks.*

Proof. We consider a closed monomorphism $f\colon A \rightarrowtail B$ in Part_M and its pullback along $g\colon C \to B$.

$$\begin{tikzcd} P \arrow[r, tail, "f'"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, "g"] \\ A \arrow[r, tail, "f"'] & B \end{tikzcd}$$

If $(a_1,c_1),\dots,(a_m,c_m) \in P$, $p((a_1,c_1),\dots,(a_m,c_m))$ is defined if and only if $p(a_1,\dots,a_m)$ and $p(c_1,\dots,c_m)$ are defined. But if $p(c_1,\dots,c_m)$ is defined, $p(g(c_1),\dots,g(c_m)) = p(f(a_1),\dots,f(a_m))$ is also defined. Since f is closed, this further implies $p(a_1,\dots,a_m)$ and so $p((a_1,c_1),\dots,(a_m,c_m))$ are defined. Thus f' is a closed monomorphism. □

Let us recall the following well-known proposition, which will be used in the particular case $\mathbb{C} = \mathrm{Part}_M$ and $\mathcal{R}$ the class of closed monomorphisms.

Proposition 3.6. *Let $\mathcal{R}$ be a class of monomorphisms in the finitely complete category $\mathbb{C}$ which is stable under pullbacks and contains regular monomorphisms. A morphism e in $\mathbb{C}$ is orthogonal to all elements of $\mathcal{R}$ if and only if, when e factors as fg with $f \in \mathcal{R}$, then f is an isomorphism. In this case, e is an epimorphism.*

Proposition 3.7. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$. If $\mathcal{R}$ denotes the class of closed monomorphisms in Part_M and $\mathcal{R}^\perp$ its orthogonal class, $(\mathcal{R}^\perp, \mathcal{R})$ is a factorisation system.*

Proof. Since $\mathcal{R}$ contains regular monomorphisms, is stable under pullbacks and closed under composition, we only have to prove that each homomorphism $f\colon A \to B$ of partial M-algebras factors as an element of $\mathcal{R}^\perp$ followed by a closed monomorphism. Let B' be the smallest sub-$\mathcal{T}$-algebra of B satisfying the conditions:

- $f(a) \in B'$ for each $a \in A$,
- if $b_1, \ldots, b_m \in B'$ are such that $p(b_1, \ldots, b_m)$ is defined in B, then $p(b_1, \ldots, b_m) \in B'$.

We consider the unique structure of partial M-algebra on B' making the inclusion $i\colon B' \hookrightarrow B$ a closed monomorphism. Then, f factors as if' with $f'\colon A \to B'$ a homomorphism of partial M-algebras by Lemma 3.4. Moreover, if $f' = f''g$ with f'' a closed monomorphism, the image of f'' contains B' by definition of B'. Thus f'' is surjective and so an isomorphism. By Proposition 3.6, $f' \in \mathcal{R}^\perp$. $\square$

Epimorphisms in Part_M thus factor as an epimorphism orthogonal to closed monomorphisms followed by a closed monomorphism (which is also an epimorphism). Therefore, to prove that closed monomorphisms are orthogonal to epimorphisms (i.e., are strong monomorphisms), it suffices to prove that closed epimorphisms are surjective. Indeed, in that case, this would imply that the only epimorphisms which are closed monomorphisms are the isomorphisms. This will be true for some particular M's.

Proposition 3.8. *Let M be an extended matrix of terms in $\mathrm{Th}[\mathrm{Set}_*]$. Closed epimorphisms in Part_M are surjective.*

Proof. Firstly, we notice that all partial M-algebras with one element are isomorphic (since $p(0, \ldots, 0)$ has to be defined). If this partial M-algebra is the unique one, the result is trivial. Hence, we suppose that there exists a partial M-algebra C with a non-zero element $c \in C$. Now, we also suppose we have a closed epimorphism $f\colon A \twoheadrightarrow B$ in Part_M which is not surjective.

Let $\mathrm{Im}(f)$ be the set-theoretical image of f and $D = D' = B \setminus \mathrm{Im}(f) \neq \varnothing$. Notice that $0 \in \mathrm{Im}(f)$. We define a partial m-ary operation p on

$$\mathrm{Im}(f) \sqcup D \sqcup D'$$

in the following way:

1. $p(t_{i1}(x_1, \dots, x_k), \dots, t_{im}(x_1, \dots, x_k))$ is defined as $u_i(x_1, \dots, x_k)$ for all $i \in \{1, \dots, n\}$ and all $x_1, \dots, x_k \in \mathrm{Im}(f) \sqcup D \sqcup D'$;
2. p restricted on $(\mathrm{Im}(f) \sqcup D)^m$ is defined as in B via the isomorphism of pointed sets $\mathrm{Im}(f) \sqcup D \cong B$;
3. p restricted on $(\mathrm{Im}(f) \sqcup D')^m$ is defined as in B via the isomorphism of pointed sets $\mathrm{Im}(f) \sqcup D' \cong B$;
4. p is defined nowhere else than required by one of the above conditions.

Let us prove this p is well-defined. There is no problem with condition 1 alone. Indeed, let us suppose by contradiction there exist $i, i' \in \{1, \dots, n\}$ and $x_1, \dots, x_k, x'_1, \dots, x'_k \in \mathrm{Im}(f) \sqcup D \sqcup D'$ satisfying $t_{ij}(x_1, \dots, x_k) = t_{i'j}(x'_1, \dots, x'_k)$ for all $j \in \{1, \dots, m\}$, but $u_i(x_1, \dots, x_k) \neq u_{i'}(x'_1, \dots, x'_k)$. Without loss of generality, we can suppose $u_i(x_1, \dots, x_k) \neq 0$. We consider any homomorphism of pointed sets $g\colon \mathrm{Im}(f) \sqcup D \sqcup D' \to C$ which sends $u_i(x_1, \dots, x_k)$ to c and $u_{i'}(x'_1, \dots, x'_k)$ to 0. Then

$$t_{ij}(g(x_1), \dots, g(x_k)) = t_{i'j}(g(x'_1), \dots, g(x'_k))$$

for each $j \in \{1, \dots, m\}$ and therefore

$$\begin{aligned}
c &= g(u_i(x_1, \dots, x_k)) \\
&= u_i(g(x_1), \dots, g(x_k)) \\
&= p(t_{i1}(g(x_1), \dots, g(x_k)), \dots, t_{im}(g(x_1), \dots, g(x_k))) \\
&= p(t_{i'1}(g(x'_1), \dots, g(x'_k)), \dots, t_{i'm}(g(x'_1), \dots, g(x'_k))) \\
&= u_{i'}(g(x'_1), \dots, g(x'_k)) \\
&= g(u_{i'}(x'_1, \dots, x'_k)) \\
&= 0,
\end{aligned}$$

which is a contradiction.

Since B is a (well-defined) partial M-algebra, there is no problem with condition 2 alone nor with condition 3 alone. The cohabitation of conditions 2 and 3 does not cause any problem neither. Indeed, the only way it could, is to have $b_1, \dots, b_m \in \mathrm{Im}(f)$ such that $p(b_1, \dots, b_m)$ is defined but does not belong to $\mathrm{Im}(f)$. If we write $b_i = f(a_i)$ for some $a_i \in A$, this means $p(f(a_1), \dots, f(a_m))$ is defined. But since f is closed, it implies $p(a_1, \dots, a_m)$ is defined and

$$p(b_1, \dots, b_m) = p(f(a_1), \dots, f(a_m)) = f(p(a_1, \dots, a_m)) \in \mathrm{Im}(f).$$

By symmetry, it remains to check there is no problem with the cohabitation of conditions 1 and 2. If there is one, it means there exist $x_1, \dots, x_k \in \mathrm{Im}(f) \sqcup D \sqcup D'$ and $i \in \{1, \dots, n\}$ such that $t_{ij}(x_1, \dots, x_k) \in \mathrm{Im}(f) \sqcup D$ for each $j \in \{1, \dots, m\}$ but $p(t_{i1}(x_1, \dots, x_k), \dots, t_{im}(x_1, \dots, x_k))$ defined as in B (via $\mathrm{Im}(f) \sqcup D \cong B$) is not $u_i(x_1, \dots, x_k)$. We denote by

$$q\colon\ \mathrm{Im}(f) \sqcup D \sqcup D' \to \mathrm{Im}(f) \sqcup D$$

the homomorphism of pointed sets which coequalises the two copies of D. This implies

$$t_{ij}(x_1, \dots, x_k) = q(t_{ij}(x_1, \dots, x_k)) = t_{ij}(q(x_1), \dots, q(x_k))$$

for each $j \in \{1, \dots, m\}$. Since we have already shown there is no problem with condition 1 alone, we can write using this condition

$$\begin{aligned} u_i(x_1, \dots, x_k) &= p(t_{i1}(x_1, \dots, x_k), \dots, t_{im}(x_1, \dots, x_k)) \\ &= p(t_{i1}(q(x_1), \dots, q(x_k)), \dots, t_{im}(q(x_1), \dots, q(x_k))) \\ &= u_i(q(x_1), \dots, q(x_k)). \end{aligned}$$

But since B is a partial M-algebra, if we compute using condition 2, we also get

$$\begin{aligned} &p(t_{i1}(x_1, \dots, x_k), \dots, t_{im}(x_1, \dots, x_k)) \\ &= p(t_{i1}(q(x_1), \dots, q(x_k)), \dots, t_{im}(q(x_1), \dots, q(x_k))) \\ &= u_i(q(x_1), \dots, q(x_k)). \end{aligned}$$

This discussion proves p is well defined.

The first condition to be a partial M-algebra is satisfied by $\mathrm{Im}(f) \sqcup D \sqcup D'$ in view of condition 1. In the case $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$, the second one resumes to $p(0, \ldots, 0) = 0$ which is true since it holds in B. Thus $\mathrm{Im}(f) \sqcup D \sqcup D'$ is a partial M-algebra. Now we consider the two obvious homomorphisms of partial M-algebras $g_1, g_2 \colon B \to \mathrm{Im}(f) \sqcup D \sqcup D'$. They satisfy $g_1 f = g_2 f$ but $g_1 \neq g_2$ since $D = D' \neq \varnothing$. This is a contradiction since f was supposed to be an epimorphism. $\square$

If $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$, there are two partial M-algebras with at most one element, i.e., the empty partial M-algebra and the singleton one $\{\star\}$ (in which $p(\star, \ldots, \star)$ has to be defined since $n \geqslant 1$). Therefore, the first argument in the previous proof does not hold if we replace $\mathrm{Th}[\mathrm{Set}_*]$ by $\mathrm{Th}[\mathrm{Set}]$. For instance, if $M = \left(\begin{array}{c|c} x & y \end{array} \right)$, the category Part_M is equivalent to the arrow category $0 \to 1$. With this M, the unique homomorphism of partial M-algebras $\varnothing \to \{\star\}$ is an injective closed epimorphism, but not an isomorphism. However, if M is such that there exists a partial M-algebra with at least two elements, the same proof can be repeated to get the following proposition.

Proposition 3.9. *Let M be an extended matrix of terms in* $\mathrm{Th}[\mathrm{Set}]$ *such that there exists a partial M-algebra with at least two elements. Closed epimorphisms in* Part_M *are surjective.*

Counterexample 3.10. If $\mathcal{T}$ is the theory of commutative monoids and M is the trivial matrix $\left(\begin{array}{c|c} x & x \end{array} \right)$, Part_M is isomorphic to the category of commutative monoids. There, the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ is an injective closed epimorphism but not an isomorphism.

As explained above, Propositions 3.8 and 3.9 admit the following corollary.

Corollary 3.11. *In* $\mathrm{Part}_{M_{\mathrm{Mal}}}$, $\mathrm{Part}_{M_{\mathrm{Uni}}}$, $\mathrm{Part}_{M_{\mathrm{StrUni}}}$ *and* $\mathrm{Part}_{M_{\mathrm{Subt}}}$, *closed monomorphisms coincide with strong monomorphisms.*

We now prove that Part_M has M-closed strong relations.

Proposition 3.12. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$ as in (1). Every relation $r \colon R \rightarrowtail A_1 \times \cdots \times A_n$ which is a closed monomorphism in* Part_M *is strictly M-closed. In particular,* Part_M *has M-closed strong relations.*

Proof. Consider a family of morphisms $(x_{ii'}\colon X \to A_i)_{i\in\{1,\dots,n\},i'\in\{1,\dots,k\}}$ in Part_M for which the morphism

$$(t_{1j}(x_{11},\dots,x_{1k}),\dots,t_{nj}(x_{n1},\dots,x_{nk}))\colon X \to A_1\times\cdots\times A_n$$

factors as rw_j for each $j \in \{1,\dots,m\}$.

$$\begin{tikzcd}
 & R \arrow[d, tail, "r"] \\
X \arrow[ur, "w_j"] \arrow[r, "(t_{1j}(x_{11},\dots,x_{1k}),\dots,t_{nj}(x_{n1},\dots,x_{nk}))"'] & A_1\times\cdots\times A_n
\end{tikzcd}$$

We know that for all $x \in X$ and each $i \in \{1,\dots,n\}$,

$$p(t_{i1}(x_{i1}(x),\dots,x_{ik}(x)),\dots,t_{im}(x_{i1}(x),\dots,x_{ik}(x)))$$

is defined and equal to $u_i(x_{i1}(x),\dots,x_{ik}(x))$. Using the description of small products in Part_M, we can say that, for all $x \in X$, $p(rw_1(x),\dots,rw_m(x))$ is defined and equal to

$$(u_1(x_{11}(x),\dots,x_{1k}(x)),\dots,u_n(x_{n1}(x),\dots,x_{nk}(x))).$$

Since r is closed, $p(w_1(x),\dots,w_m(x))$ is defined in R and we can consider the function $w\colon X \to R\colon x \mapsto p(w_1(x),\dots,w_m(x))$ which satisfies

$$rw = (u_1(x_{11},\dots,x_{1k}),\dots,u_n(x_{n1},\dots,x_{nk})).$$

Finally, Lemma 3.4 tells us w is a homomorphism of partial M-algebras since rw is and r is a closed monomorphism, which concludes the proof. □

4. The embedding theorems

Now that the preliminary work on Part_M has been done, we can prove our embedding theorems for small categories with M-closed relations and for small categories with M-closed strong relations. In order to prove both at the same time, we are going to use a set of monomorphisms, closed under composition, stable under pullbacks and which contains regular monomorphisms.

Theorem 4.1. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$ as in (1). Let also $\mathcal{R}$ be a set of monomorphisms in the small finitely complete $\mathcal{T}$-enriched category $\mathbb{C}$ such that $\mathcal{R}$ is closed under composition, stable under pullbacks and contains regular monomorphisms. Suppose also that all n-ary relations $R \rightarrowtail A^n$ in $\mathcal{R}$ are M-closed in $\mathbb{C}$. Then, there exists a full and faithful $\mathcal{T}$-enriched embedding $\phi\colon \mathbb{C} \hookrightarrow \mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}}$ which preserves and reflects finite limits. Moreover, for each monomorphism $f\colon A \rightarrowtail B$ in $\mathcal{R}$ and each $X \in \mathbb{C}^{\mathrm{op}}$, $\phi(f)_X$ is a closed monomorphism in Part_M.*

Proof. We would like to factorise the $\mathcal{T}$-enriched Yoneda embedding $Y_{\mathcal{T}}$: $\mathbb{C} \to \mathrm{Alg}_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}}$ through $\mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}}$.

$$\begin{tikzcd} & \mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}} \arrow[d, "U_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}}"] \\ \mathbb{C} \arrow[ur, dotted, "\phi"] \arrow[r, "Y_{\mathcal{T}}"'] & \mathrm{Alg}_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}} \end{tikzcd}$$

In order to do so, let us provide $\mathbb{C}(X,Y)$ with a structure of partial M-algebra, for all objects $X, Y \in \mathbb{C}$. Thus, let $f_1, \ldots, f_m : X \to Y$ be morphisms in $\mathbb{C}$. We define $p(f_1, \ldots, f_m)$ if and only if there exist morphisms $x_1, \ldots, x_k\colon X \to W$, a relation $r\colon Z \rightarrowtail W^n$ in $\mathcal{R}$, and morphisms $g_1, \ldots, g_m\colon X \to Z$ and $f\colon Z \to Y$ such that, for all $j \in \{1, \ldots, m\}$, $fg_j = f_j$ and $rg_j = (t_{1j}(x_1, \ldots, x_k), \ldots, t_{nj}(x_1, \ldots, x_k))$.

$$\begin{tikzcd} & X \arrow[dl, bend right, "{(t_{1j}(x_1,\ldots,x_k),\ldots,t_{nj}(x_1,\ldots,x_k))}"'] \arrow[d, "g_j"] \arrow[dr, bend left, "f_j"] & \\ W^n & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y \end{tikzcd}$$

In this case, since r is M-closed, there exists a unique $h\colon X \to Z$ such that $rh = (u_1(x_1, \ldots, x_k), \ldots, u_n(x_1, \ldots, x_k))$ and we define $p(f_1, \ldots, f_m) = fh$.

$$\begin{tikzcd} & Z \arrow[d, tail, "r"] \\ X \arrow[ur, dotted, "\exists!\, h"] \arrow[r, "{(u_1(x_1,\ldots,x_k),\ldots,u_n(x_1,\ldots,x_k))}"'] & W^n \end{tikzcd}$$

Let us first prove the independence of the choices. So, suppose $x'_1,\dots,x'_k : X \to W'$, $r'\colon Z' \rightarrowtail W'^n$, $g'_1,\dots,g'_m\colon X \to Z'$, $f'\colon Z' \to Y$ and $h'\colon X \to Z'$ also satisfy the above conditions and let us prove $fh = f'h'$. We consider the following pullback

$$\begin{tikzcd} Z_1 \arrow[r, "q_1"] \arrow[d, tail, "r_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Z \arrow[d, tail, "r"] \\ (W \times W')^n \arrow[r, "\pi_1^n"'] & W^n \end{tikzcd}$$

where $\pi_1\colon W \times W' \to W$ is the first projection. We also consider the unique morphisms $l_1^1,\dots,l_1^m\colon X \to Z_1$ such that $q_1 l_1^j = g_j$ and

$$r_1 l_1^j = (t_{1j}((x_1,x'_1),\dots,(x_k,x'_k)),\dots,t_{nj}((x_1,x'_1),\dots,(x_k,x'_k)))$$

for each $j \in \{1,\dots,m\}$. Let also $h_1\colon X \to Z_1$ be the unique morphism such that $q_1 h_1 = h$ and

$$r_1 h_1 = (u_1((x_1,x'_1),\dots,(x_k,x'_k)),\dots,u_n((x_1,x'_1),\dots,(x_k,x'_k))).$$

Similarly, we define Z_2, r_2, q_2, $l_2^1,\dots,l_2^m$ and h_2 using the pullback of r' along π_2^n where $\pi_2\colon W \times W' \to W'$ is the second projection. Since $\mathcal{R}$ is stable under pullbacks, $r_1, r_2 \in \mathcal{R}$. We also construct their intersection,

$$\begin{tikzcd} P \arrow[r, tail, "r_3"] \arrow[d, tail, "r_4"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Z_2 \arrow[d, tail, "r_2"] \\ Z_1 \arrow[r, tail, "r_1"'] & (W \times W')^n \end{tikzcd}$$

the unique morphism $h_3\colon X \to P$ such that $r_3 h_3 = h_2$ and $r_4 h_3 = h_1$ and, for each $j \in \{1,\dots,m\}$, the unique morphism $l_3^j\colon X \to P$ such that $r_3 l_3^j = l_2^j$ and $r_4 l_3^j = l_1^j$. Again, $r_3, r_4 \in \mathcal{R}$. Finally, we consider the following equaliser diagram.

$$\begin{tikzcd} E \arrow[r, tail, "e"] & P \arrow[r, shift left, "f q_1 r_4"] \arrow[r, shift right, "f' q_2 r_3"'] & Y \end{tikzcd}$$

For each $j \in \{1,\dots,m\}$, l_3^j factors as $e l_4^j = l_3^j$ since

$$f q_1 r_4 l_3^j = f q_1 l_1^j = f g_j = f_j = f' g'_j = f' q_2 l_2^j = f' q_2 r_3 l_3^j.$$

Hence, for all such j, the morphism

$$(t_{1j}((x_1,x'_1),\dots,(x_k,x'_k)),\dots,t_{nj}((x_1,x'_1),\dots,(x_k,x'_k)))$$

factors as $r_1r_4el_4^j$. But since the relation $r_1r_4e\colon E\rightarrowtail (W\times W')^n$ is in $\mathcal{R}$, it is M-closed and so there exists a unique morphism $l_5\colon X\to E$ such that

$$r_1r_4el_5=(u_1((x_1,x'_1),\dots,(x_k,x'_k)),\dots,u_n((x_1,x'_1),\dots,(x_k,x'_k))).$$

The equalities $r_1r_4h_3=r_1h_1=r_1r_4el_5$ imply that $h_3=el_5$ and it remains to compute

$$\begin{aligned} fh &= fq_1h_1=fq_1r_4h_3=fq_1r_4el_5\\ &= f'q_2r_3el_5=f'q_2r_3h_3=f'q_2h_2\\ &= f'h'. \end{aligned}$$

Now that we have shown p is well-defined, let us prove it makes $\mathbb{C}(X,Y)$ a partial M-algebra. If $i\in\{1,\dots,n\}$ and $x_1,\dots,x_k\in\mathbb{C}(X,Y)$, we can set $W=Y$, $r=1_{Y^n}$,

$$g_j=(t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k)),$$

$f=\pi_i\colon Y^n\to Y$ the i-th projection and

$$h=(u_1(x_1,\dots,x_k),\dots,u_n(x_1,\dots,x_k)).$$

This shows that $p(t_{i1}(x_1,\dots,x_k),\dots,t_{im}(x_1,\dots,x_k))$ is defined and equal to $fh=u_i(x_1,\dots,x_k)$.

Now, let σ be an r-ary operation symbol of $\mathcal{T}$ with $r>0$ and

$$(f_j^{j'}\in\mathbb{C}(X,Y))_{j\in\{1,\dots,m\},j'\in\{1,\dots,r\}}$$

be families of morphisms such that $p(f_1^{j'},\dots,f_m^{j'})$ is defined for each $j'\in\{1,\dots,r\}$ using the diagrams below.

$$\begin{tikzcd}
& X \arrow[dl, bend right, "{(t_{1j}(x_1^{j'},\dots,x_k^{j'}),\dots,t_{nj}(x_1^{j'},\dots,x_k^{j'}))}"'] \arrow[d, "g_j^{j'}"] \arrow[dr, bend left, "f_j^{j'}"] & \\
(W_{j'})^n & Z_{j'} \arrow[l, tail, "r^{j'}"] \arrow[r, "f^{j'}"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
 & & Z_{j'} \arrow[d, tail, "r^{j'}"] \\
X \arrow[rru, "h^{j'}"] \arrow[rr, "(u_1(x_1^{j'},\dots,x_k^{j'}),\dots,u_n(x_1^{j'},\dots,x_k^{j'}))"'] & & (W_{j'})^n
\end{tikzcd}$$

We consider the pullbacks

$$\begin{tikzcd}
S_{j'} \arrow[r, "q^{j'}"] \arrow[d, tail, "s^{j'}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z_{j'} \arrow[d, tail, "r^{j'}"] \\
(W_1 \times \cdots \times W_r)^n \arrow[r, "\pi_{j'}^n"'] & (W_{j'})^n
\end{tikzcd}$$

where $\pi_{j'} \colon W_1 \times \cdots \times W_r \to W_{j'}$ is the j'-th projection as usual. We denote by $l_j^{j'}$ the unique morphism $X \to S_{j'}$ such that $q^{j'} l_j^{j'} = g_j^{j'}$ and

$$s^{j'} l_j^{j'} = (t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k))$$

where $x_{i'}$ is the factorisation $(x_{i'}^1,\dots,x_{i'}^r) \colon X \to W_1 \times \cdots \times W_r$. Let also $h_1^{j'} \colon X \to S_{j'}$ be the unique morphism satisfying $q^{j'} h_1^{j'} = h^{j'}$ and

$$s^{j'} h_1^{j'} = (u_1(x_1,\dots,x_k),\dots,u_n(x_1,\dots,x_k)).$$

We now consider the intersection of the $s^{j'}$'s

$$\begin{tikzcd}
 & Z \arrow[dl, tail, "t^1"'] \arrow[dr, tail, "t^r"] & \\
S^1 \arrow[dr, tail, "s^1"'] & \cdots & S^r \arrow[dl, tail, "s^r"] \\
 & (W^1 \times \cdots \times W^r)^n &
\end{tikzcd}$$

and the unique morphisms $l_j, h \colon X \to Z$ such that $t^{j'} l_j = l_j^{j'}$ and $t^{j'} h = h_1^{j'}$. Since this intersection can be built using pullbacks and compositions, $s^1 t^1 = \cdots = s^r t^r \in \mathcal{R}$. Thus, we end up with the commutative diagrams

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "(t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k))"'] \arrow[d, "l_j"] \arrow[dr, bend left, "\sigma(f_j^1,\dots,f_j^r)"] & \\
(W^1 \times \cdots \times W^r)^n & Z \arrow[l, tail, "s^1 t^1"] \arrow[r, "\sigma(f^1 q^1 t^1,\dots,f^r q^r t^r)"'] & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & Z \arrow[d, tail, "s^1t^1"] \\
X \arrow[ur, "h"] \arrow[r, "(u_1(x_1,\dots,x_k),\dots,u_n(x_1,\dots,x_k))"'] & (W^1 \times \cdots \times W^r)^n
\end{tikzcd}$$

proving that $p(\sigma(f_1^1,\dots,f_1^r),\dots,\sigma(f_m^1,\dots,f_m^r))$ is defined and equal to

$$\sigma(f^1q^1t^1h,\dots,f^rq^rt^rh) = \sigma(p(f_1^1,\dots,f_m^1),\dots,p(f_1^r,\dots,f_m^r)).$$

If $r = 0$, we also have $p(\sigma,\dots,\sigma) = \sigma$. To see it, we can use for instance the commutative diagram below.

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "(t_{1j}(1_X,\dots,1_X),\dots,t_{nj}(1_X,\dots,1_X))"'] \arrow[d] \arrow[dr, bend left, "\sigma"] & \\
X^n & X^n \arrow[l, tail, "1_{X^n}"] \arrow[r, "\sigma"'] & Y
\end{tikzcd}$$

We have therefore provided $\mathbb{C}(X,Y)$ with a structure of partial M-algebra.

In view of the definition of a $\mathcal{T}$-enrichment, if $x\colon X' \to X$ and $y\colon Y \to Y'$ are morphisms in $\mathbb{C}$,

$$- \circ x\colon \mathbb{C}(X,Y) \to \mathbb{C}(X',Y)$$

and

$$y \circ -\colon \mathbb{C}(X,Y) \to \mathbb{C}(X,Y')$$

are homomorphisms of $\mathcal{T}$-algebras. Let us prove they are actually homomorphisms of partial M-algebras. So let $f_1,\dots,f_m\colon X \to Y$ be morphisms of $\mathbb{C}$ such that $p(f_1,\dots,f_m)$ is defined via the following diagrams.

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "(t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k))"'] \arrow[d, "g_j"] \arrow[dr, bend left, "f_j"] & \\
W^n & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Z \arrow[d, tail, "r"] \\
X \arrow[ur, "h"] \arrow[r, "(u_1(x_1,\dots,x_k),\dots,u_n(x_1,\dots,x_k))"'] & W^n
\end{tikzcd}$$

Thus, in view of the commutative diagrams

$$\begin{tikzcd}
 & X' \arrow[dl, bend right, "(t_{1j}(x_1x,\dots,x_kx),\dots,t_{nj}(x_1x,\dots,x_kx))"'] \arrow[d, "g_jx"] \arrow[dr, bend left, "f_jx"] & \\
W^n & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
& Z \arrow[d, tail, "r"] \\
X' \arrow[ur, "hx"] \arrow[r, "{(u_1(x_1x,\dots,x_kx),\dots,u_n(x_1x,\dots,x_kx))}"'] & W^n
\end{tikzcd}$$

$p(f_1x, \dots, f_mx)$ is defined and equal to $fhx = p(f_1, \dots, f_m)x$, which shows that $- \circ x$ is a homomorphism of partial M-algebras. Besides, since the diagram

$$\begin{tikzcd}
& X \arrow[dl, bend right, "{(t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k))}"'] \arrow[d, "g_j"] \arrow[dr, bend left, "yf_j"] & \\
W^n & Z \arrow[l, tail, "r"] \arrow[r, "yf"'] & Y'
\end{tikzcd}$$

commutes, $p(yf_1, \dots, yf_m)$ is defined and equal to $yfh = yp(f_1, \dots, f_m)$, which proves that $y \circ -$ is a homomorphism of partial M-algebras. We have thus constructed a functor $\phi\colon \mathbb{C} \to \mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}}$ as announced.

$$\begin{tikzcd}
& \mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}} \arrow[d, "U_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}}"] \\
\mathbb{C} \arrow[ur, "\phi"] \arrow[r, "Y_{\mathcal{T}}"'] & \mathrm{Alg}_{\mathcal{T}}^{\mathbb{C}^{\mathrm{op}}}
\end{tikzcd}$$

This ϕ preserves $\mathcal{T}$-enrichment since $Y_{\mathcal{T}}$ and $U_{\mathcal{T}}$ do and $U_{\mathcal{T}}$ is faithful. It is full and faithful since $Y_{\mathcal{T}}$ is full and faithful and $U_{\mathcal{T}}$ is faithful.

Since ϕ is full and faithful, it reflects isomorphisms. Thus, it will reflect finite limits if it preserves them. So, let $(\lambda_j\colon L \to D(j))_{j\in\mathbb{J}}$ be the limit of $D\colon \mathbb{J} \to \mathbb{C}$ with $\mathbb{J}$ a finite category. We would like to prove that for all $X \in \mathbb{C}$,

$$(\phi(\lambda_j)_X\colon \mathbb{C}(X, L) \to \mathbb{C}(X, D(j)))_{j\in\mathbb{J}}$$

is a limit in Part_M. But since $Y_{\mathcal{T}}$ preserves limits, and in view of the description of small limits in Part_M (Section 3.1), we only have to prove that, if $f_1, \dots, f_m\colon X \to L$ are such that $p(\lambda_j f_1, \dots, \lambda_j f_m)$ is defined for all $j \in \mathbb{J}$, then $p(f_1, \dots, f_m)$ is also defined.

Thus, to prove that ϕ preserves the terminal object, we have to show that $p(!, \dots, !)$ is defined where ! is the unique morphism $X \to 1$. This is obvious in view of the diagram below.

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(!,\ldots,!),\ldots,t_{nj}(!,\ldots,!))}"'] \arrow[d, "!"] \arrow[dr, bend left, "!"] & \\
1^n = 1 & 1 \arrow[l, tail, "1_1"] \arrow[r, "1_1"'] & 1
\end{tikzcd}$$

Moreover, ϕ preserves the binary product $Y \times Y'$.

$$\begin{tikzcd}
 & Y \times Y' \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Y & & Y'
\end{tikzcd}$$

Indeed, suppose $f_1, \ldots, f_m \colon X \to Y$ and $f'_1, \ldots, f'_m \colon X \to Y'$ are such that $p(f_1, \ldots, f_m)$ and $p(f'_1, \ldots, f'_m)$ are defined using the following diagrams.

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(x_1,\ldots,x_k),\ldots,t_{nj}(x_1,\ldots,x_k))}"'] \arrow[d, "g_j"] \arrow[dr, bend left, "f_j"] & \\
W^n & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "{(t_{1j}(x'_1,\ldots,x'_k),\ldots,t_{nj}(x'_1,\ldots,x'_k))}"'] \arrow[d, "g'_j"] \arrow[dr, bend left, "f'_j"] & \\
W'^n & Z' \arrow[l, tail, "r'"] \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

We consider again the pullback

$$\begin{tikzcd}
Z_1 \arrow[r, "q_1"] \arrow[d, tail, "r_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, tail, "r"] \\
(W \times W')^n \arrow[r, "\pi_1^n"'] & W^n
\end{tikzcd}$$

and the unique morphisms $l_1^1, \ldots, l_1^m \colon X \to Z_1$ such that $q_1 l_1^j = g_j$ and

$$r_1 l_1^j = (t_{1j}((x_1, x'_1), \ldots, (x_k, x'_k)), \ldots, t_{nj}((x_1, x'_1), \ldots, (x_k, x'_k)))$$

for all $j \in \{1, \dots, m\}$. We define similarly Z_2, r_2, q_2 and $l_2^1, \dots, l_2^m$. We also consider the intersection

$$\begin{tikzcd} P \arrow[r, tail, "r_3"] \arrow[d, tail, "r_4"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z_2 \arrow[d, tail, "r_2"] \\ Z_1 \arrow[r, tail, "r_1"'] & (W \times W')^n \end{tikzcd}$$

and the unique morphisms $l_3^1, \dots, l_3^m \colon X \to P$ such that $r_4 l_3^j = l_1^j$ and $r_3 l_3^j = l_2^j$ for all $j \in \{1, \dots, m\}$. Then, since the diagram below is commutative,

$$\begin{tikzcd} & X \arrow[dl, bend right, "(t_{1j}((x_1,x_1'),\dots,(x_k,x_k')),\dots,t_{nj}((x_1,x_1'),\dots,(x_k,x_k')))"'] \arrow[d, "l_3^j"] \arrow[dr, bend left, "(f_j,f_j')"] & \\ (W \times W')^n & P \arrow[l, tail, "r_1 r_4"] \arrow[r, "(f q_1 r_4, f' q_2 r_3)"'] & Y \times Y' \end{tikzcd}$$

$p((f_1, f_1'), \dots, (f_m, f_m'))$ is also defined and ϕ preserves finite products.

Finally, to prove that ϕ preserves equalisers, it is enough to show that $\phi(e)_X = e \circ - \colon \mathbb{C}(X, Y) \to \mathbb{C}(X, Y')$ is a closed homomorphism for each $X \in \mathbb{C}^{\mathrm{op}}$ and each regular monomorphism $e \colon Y \rightarrowtail Y'$. To conclude the proof, we are going to prove the more general fact that $\phi(e)_X$ is a closed homomorphism for each $e \colon Y \rightarrowtail Y'$ in $\mathcal{R}$ and each $X \in \mathbb{C}^{\mathrm{op}}$. So let $f_1, \dots, f_m \colon X \to Y$ be such that $p(ef_1, \dots, ef_m)$ is defined using the diagram below.

$$\begin{tikzcd} & X \arrow[dl, bend right, "(t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k))"'] \arrow[d, "g_j"] \arrow[dr, bend left, "ef_j"] & \\ W^n & Z \arrow[l, tail, "r"] \arrow[r, "f"'] & Y' \end{tikzcd}$$

We consider the pullback of e along f

$$\begin{tikzcd} Z' \arrow[r, "f'"] \arrow[d, tail, "r'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, tail, "e"] \\ Z \arrow[r, "f"'] & Y' \end{tikzcd}$$

and the unique morphisms $g'_1, \dots, g'_m \colon X \to Z'$ satisfying $f'g'_j = f_j$ and $r'g'_j = g_j$ for each $j \in \{1, \dots, m\}$. Then, considering the diagram

$$\begin{tikzcd} & X \arrow[dl, bend right, "{(t_{1j}(x_1,\dots,x_k),\dots,t_{nj}(x_1,\dots,x_k))}"'] \arrow[d, "g'_j"] \arrow[dr, bend left, "f_j"] & \\ W^n & Z' \arrow[l, tail, "rr'"] \arrow[r, "f'"'] & Y \end{tikzcd}$$

we see that $p(f_1, \dots, f_m)$ is defined, which concludes the proof. $\square$

Taking $\mathcal{R}$ to be the whole set of monomorphisms in $\mathbb{C}$, we immediately get the following corollary.

Corollary 4.2. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$. Let also $\mathbb{C}$ be a small finitely complete $\mathcal{T}$-enriched category with M-closed relations. There exists a full and faithful $\mathcal{T}$-enriched embedding $\phi\colon \mathbb{C} \hookrightarrow \mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}}$ which preserves and reflects finite limits. Moreover, for each monomorphism $f\colon A \rightarrowtail B$ and each $X \in \mathbb{C}^{\mathrm{op}}$, $\phi(f)_X$ is a closed monomorphism in Part_M.*

And now with $\mathcal{R}$ the set of strong monomorphisms.

Corollary 4.3. *Let $\mathcal{T}$ be a commutative algebraic theory and M an extended matrix of terms in $\mathcal{T}$. Let also $\mathbb{C}$ be a small finitely complete $\mathcal{T}$-enriched category with M-closed strong relations. There exists a full and faithful $\mathcal{T}$-enriched embedding $\phi\colon \mathbb{C} \hookrightarrow \mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}}$ which preserves and reflects finite limits. Moreover, for each strong monomorphism $f\colon A \rightarrowtail B$ and each $X \in \mathbb{C}^{\mathrm{op}}$, $\phi(f)_X$ is a closed monomorphism in Part_M.*

Remark 4.4. Notice that Corollaries 4.2 and 4.3 characterise categories with M-closed relations (resp. with M-closed strong relations) among small finitely complete $\mathcal{T}$-enriched categories. Indeed, if we have such an embedding, to prove that a (strong) relation $r\colon R \rightarrowtail A^n$ in $\mathbb{C}$ is M-closed, it is enough to prove that $\phi(r)_X$ is M-closed in Part_M for all $X \in \mathbb{C}^{\mathrm{op}}$, which is true by Proposition 3.12.

5. Applications

Our embedding theorems give a way to reduce the proofs of some statements in finitely complete $\mathcal{T}$-enriched categories with M-closed strong relations to the particular case of Part_M as follows: Consider a statement P of the form $\psi \Rightarrow \omega$, where ψ and ω can be expressed as conjunctions of the conditions that some finite diagrams are commutative, some finite cones are limit cones and some equalities as $t(f_1, \ldots, f_r) = g$ hold where t is an r-ary term of $\mathcal{T}$ and $f_1, \ldots, f_r, g$ are parallel morphisms. Then, P is valid in all finitely complete $\mathcal{T}$-enriched $\mathcal{V}$-categories with M-closed strong relations (for all universes $\mathcal{V}$) if and only if it is valid in Part_M (for all universes). Indeed, in view of Proposition 3.12, the 'only if part' is obvious. Conversely, if $\mathbb{C}$ is a finitely complete $\mathcal{T}$-enriched category with M-closed strong relations, we can consider it is small up to a change of universe. Therefore, by Corollary 4.3, it suffices to prove P in $\mathrm{Part}_M^{\mathbb{C}^{\mathrm{op}}}$. Since every part of the statement P is 'componentwise', it is enough to prove it in Part_M. Note that the conditions that some morphisms are monomorphisms, isomorphisms, or factor through some given monomorphisms can also be expressed using finite limits.

Similarly, to prove this statement P in all finitely complete $\mathcal{T}$-enriched categories with M-closed relations, it is enough to prove it in Part_M (for all universes) supposing that each monomorphism considered in the statement P is closed.

Let us now give two concrete examples, the first one taking place in the 'weakly strongly unital context', i.e., for pointed finitely complete categories with M_{StrUni}-closed strong relations (see Example 2.3). This lemma has been proved in [1] as Lemma 1.8.18 in the strongly unital case, we now slightly improve it.

Lemma 5.1. *Consider the following diagram in a pointed finitely complete* *category with M_{StrUni}-closed strong relations*

$$\begin{tikzcd}
& X \times R \arrow[d, shift right, "1_X \times r_1"'] \arrow[d, shift left, "1_X \times r_2"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\
X \arrow[r, "{(1_X,0)}"] \arrow[dr, shift left, "h"] & X \times Y \arrow[d, "\psi"] & Y \arrow[l, "{(0,1_Y)}"'] \arrow[dl, "f"] \\
& Z \arrow[ul, shift left, two heads, "g"] &
\end{tikzcd}$$

where $\psi(1_X, 0) = h$, $\psi(0, 1_Y) = f$, $gh = 1_X$, $gf = 0$ and (r_1, r_2) is the kernel pair of f. Then $(1_X \times r_1, 1_X \times r_2)$ is the kernel pair of ψ.

Proof. By Corollary 4.3, it is enough to prove it in $\mathrm{Part}_{M_{\mathrm{StrUni}}}$. First of all, let us compute, for all $x \in X$ and $y \in Y$

$$\begin{aligned}
\psi(x, y) &= \psi(p(x, 0, 0), p(0, 0, y)) \\
&= \psi(p((x, 0), (0, 0), (0, y))) \\
&= p(\psi(x, 0), \psi(0, 0), \psi(0, y)) \\
&= p(h(x), 0, f(y))
\end{aligned}$$

which is always defined. Next, let $x, x' \in X$ and $y, y' \in Y$ be such that $\psi(x, y) = \psi(x', y')$. We have

$$\begin{aligned}
x &= p(x, 0, 0) = p(gh(x), 0, gf(y)) = g(\psi(x, y)) \\
&= g(\psi(x', y')) = p(gh(x'), 0, gf(y')) = p(x', 0, 0) \\
&= x'
\end{aligned}$$

and

$$\begin{aligned}
f(y) &= \psi(0, y) = \psi(p(x, x, 0), p(y, 0, 0)) \\
&= \psi(p((x, y), (x, 0), (0, 0))) = p(\psi(x, y), \psi(x, 0), \psi(0, 0)) \\
&= p(\psi(x', y'), \psi(x', 0), \psi(0, 0)) = \psi(p(x', x', 0), p(y', 0, 0)) \\
&= \psi(0, y') = f(y').
\end{aligned}$$

Then,

$$X \times R = \{(x, y_1, y_2) \in X \times Y \times Y \mid f(y_1) = f(y_2)\}$$

in which p is defined componentwise. If $(x, y_1, y_2) \in X \times R$, we have

$$\begin{aligned}\psi(x, y_1) &= p(h(x), 0, f(y_1)) \\ &= p(h(x), 0, f(y_2)) \\ &= \psi(x, y_2).\end{aligned}$$

The kernel pair of ψ is given by

$$\{(x, y, x', y') \in X \times Y \times X \times Y \,|\, \psi(x, y) = \psi(x', y')\}$$

in which p is also defined componentwise. It is thus isomorphic to $X \times R$ via the mutually inverse homomorphisms $(x, y_1, y_2) \mapsto (x, y_1, x, y_2)$ and $(x, y, x', y') \mapsto (x, y, y')$. $\square$

To conclude, we prove a well-known fact in Mal'tsev categories.

Proposition 5.2. *(Theorem 2.2 in [4]) In a Mal'tsev category, every internal category is a groupoid.*

Proof. If

$$A = (\ A_1 \times_{c,d} A_1 \xrightarrow{m} A_1 \underset{c}{\overset{d}{\rightrightarrows}} A_0\ ) \quad (e \colon A_0 \to A_1)$$

is an internal category, we have to prove that $\mathrm{Iso}(A) \rightarrowtail A_1$ is an isomorphism where $\mathrm{Iso}(A)$ is the object of isomorphisms of A, constructed via a limit of a finite diagram involving e, d, c and m. Thus, by Corollary 4.2, it is enough to prove this statement in $\mathrm{Part}_{M_{\mathrm{Mal}}}$.

We write π_1 and π_2 for the projections of the pullback of c along d.

$$\begin{tikzcd} A_1 \times_{c,d} A_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_1 \arrow[d, "d"] \\ A_1 \arrow[r, "c"'] & A_0 \end{tikzcd}$$

Let us first prove that

$$A_1 \times_{c,d} A_1 \xrightarrow{(\pi_2, m)} A_1 \times A_1$$

is a monomorphism. So let

$$X \underset{f'}{\overset{f}{\rightrightarrows}} Y \xrightarrow{g} Z$$

be morphisms in A such that $m(f,g) = m(f',g)$. Then

$$\begin{aligned} f &= m(f, 1_Y) \\ &= m(p(f, 1_Y, 1_Y), p(g, g, 1_Y)) \\ &= p(m(f,g), m(1_Y, g), m(1_Y, 1_Y)) \\ &= p(m(f',g), m(1_Y, g), m(1_Y, 1_Y)) \\ &= f' \end{aligned}$$

and (π_2, m) is a monomorphism. We can therefore suppose it is closed (using the last part of Corollary 4.2). Let us now prove that every map $f \colon X \to Y$ in A is invertible (i.e., that $\mathrm{Iso}(A) \rightarrowtail A_1$ is surjective). We know that

$$p((1_Y, 1_Y), (f, 1_Y), (1_X, f)) \in A_1 \times_{c,d} A_1$$

is defined since $p(1_Y, 1_Y, f)$ and $p(1_Y, f, f)$ are and (π_2, m) is a closed monomorphism. Thus, applying π_1, we deduce that $p(1_Y, f, 1_X)$ is defined. It remains to compute

$$d(p(1_Y, f, 1_X)) = p(Y, X, X) = Y,$$

$$c(p(1_Y, f, 1_X)) = p(Y, Y, X) = X,$$

$$\begin{aligned} m(f, p(1_Y, f, 1_X)) &= m(p(f, 1_X, 1_X), p(1_Y, f, 1_X)) \\ &= p(m(f, 1_Y), m(1_X, f), m(1_X, 1_X)) \\ &= p(f, f, 1_X) \\ &= 1_X \end{aligned}$$

and similarly for $m(p(1_Y, f, 1_X), f) = 1_Y$. Therefore, $\mathrm{Iso}(A) \rightarrowtail A_1$ is bijective and can also be supposed to be closed. This means it is an isomorphism. $\square$

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