Finiteness lengths of arithmetic groups over global function fields

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An abstract group is said to be of type F_n, $n \in \mathbb{N}$, if it admits a classifying space with a finite n-skeleton. It is of type F_1 if and only if it is finitely generated and of type F_2 if and only if it is finitely presented. By a classical theorem of Nagao's (see [Nag59]), the group $\mathrm{SL}_2(\mathbb{Z})$ and, in fact, each non-uniform tree lattice is not of type F_1. By Raghunathan [Rag68] and by Borel–Serre [BS76] every S-arithmetic subgroup of a reductive algebraic group over a global number field has type F_∞, i.e., it is of type F_n for every n.

In the function field case, the situation is much more difficult. It is conjectured in [Beh98, page 80] that an arithmetic lattice whose local ranks sum up to n, is of type F_{n-1} but not F_n. This conjecture is known to hold for the Iwahori subgroups of algebraic groups of classical type over global function fields with sufficiently large residue fields such as $\mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ (cf. [Abe91], [AA93], [Abr96]) and for unitary forms of split Kac–Moody groups of types $\tilde{A}_n$ or $\tilde{C}_n$ over sufficiently large fields (cf. [DGM], [GW]). All these results have been obtained by studying the geometry of orbits of the respective arithmetic lattices on their affine (twin) buildings to which Brown's criterion [Bro87] can be applied successfully.

For general arithmetic lattices Γ whose local ranks sum up to n the following partial results towards proving Behr's above rank conjecture have been achieved: By [BW07] the group Γ is not of type F_n and by [BW], if the global rank of Γ equals one, then it is of type F_{n-1}. These articles make heavy use of Morse theory, the sphericity of complexes of hemispheres in thick spherical buildings established in [Sch05], and classical reduction theory ([Har69], [Beh69]). Quite remarkably, these results are achieved without any requirement on the size of the residue fields.

In my talk I will present classical and recent examples, discuss some of the strategies for establishing finiteness properties, and state a number of open problems.

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