Reflection rigidity of 2-spherical Coxeter groups

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20th January 2005

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1 Introduction

The purpose of the present paper is to prove the following:

(1.1) Main Result: *Let* $(W, S), (W', S')$ *be Coxeter systems and let* $\varphi : W \to W'$ *be a reflection-preserving isomorphism, i.e.* $\varphi(S) \subseteq {S'}^{W'}$*. Suppose furthermore that* S *is finite, that* W *is infinite, that* (W, S) *is irreducible, and that* (W, S) *is 2-spherical (which means that for all* $s, t \in S$ *the order of* st *is finite). Then* $\varphi(S)$ *is conjugate to* S' *in* W'*.*

In the language of [2] our main result can be stated as follows:

(1.2) Rigidity-version: *Coxeter systems of finite rank which are non-spherical, irreducible and 2-spherical are strongly reflection rigid.*

Our main result follows from a slightly more general statement, which is Theorem (12.13) below. Let us record that Coxeter groups of type H_3 or $I_2(n)$ for $n = 5$ or $n \geq 7$ provide counter examples to the preceding statement if the assumption of non-sphericity is removed. Similarly, if we do not assume the Coxeter system to be 2-spherical, then the statement is false (see [2]).

(1.3) Applications: As a consequence of our main result we obtain the main result of [13] which says that the outer automorphism group of a 2-spherical Coxeter group of finite rank is finite. By a result of Richardson, there are only finitely many conjugacy classes of involutions in a Coxeter group W of finite rank and $\mathrm{Aut}(W)$ acts naturally on them. The kernel of this action is contained in the stabilizer of the set of reflections which we denote by $\mathrm{Aut}_{\mathrm{Refl}}(W)$. In [13] it is shown that $\mathrm{Inn}(W)$ has finite index in $\mathrm{Aut}_{\mathrm{Refl}}(W)$. As a consequence of our main result it turns out that $\mathrm{Aut}_{\mathrm{Refl}}(W)$ is actually equal to the group of inner automorphisms extended by the group of diagram automorphisms in the irreducible non-spherical case.

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*Aspirant du Fonds National de la Recherche Scientifique
AMS subject classification codes (2000) : 20F55, 51F15.
Keywords: Coxeter group, thin building.

As already indicated in (1.2), our main result has a rigidity version. In the context of rigidity of Coxeter groups a conjecture was formulated in [2] whose validity for the 2-spherical case is a consequence of our main result. Our main result (and the methods of its proof) can be used to prove the validity of this conjecture for a much larger class of Coxeter groups [4].

Automorphisms of nearly finite 2-spherical Coxeter groups have been also considered in [10]. Several of the results there are consequences of our main result. However, in order to prove our result we use the result of [10] in a special case (see the proof of Lemma (12.10)).

Finally, we would like to mention, that our main result yields in combination with the main result of [9] the following:

(1.4) Theorem. *Let (W, S) be as in the statement of our main result and let $R \subseteq W$ be such that (W, R) is a Coxeter system. Then $R = S^w$ for some $w \in W$.*

The details are given in [9].

(1.5) The proof of the main result is based on some refinements and modifications of the ideas and techniques used in [18]. In that paper the notion of a geometric (2-geometric, universal) set of reflections (which will be recalled in (3.2)) plays a crucial role in order to prove that certain Coxeter systems are strongly reflection rigid. One of the crucial ideas in the present paper is to prove and use the following general fact concerning these notions:

(1.6) Fact: *Any 2-geometric, universal set of reflections in a Coxeter group is geometric.*

It was pointed out to us by Bob Howlett, that the fact above could be deduced as a corollary from Theorem 1.2 in [13] which is a statement about isometries of the root system associated to a Coxeter group. It turns out that a slightly stronger version of the fact above which was proved by the first author in [5] is equivalent to that theorem. Thus, our Theorem (3.3) can be considered as the being the 'Cayley graph'-version of Theorem 1.2 in [13]. There is also a 'Kac-Moody'-version which can be found in [14], Sections 5.9 and 5.10, for Coxeter groups of spherical, affine and hyperbolic type (see also [17]). In [11] some ideas are described how the general case can be dealt by modifying the techniques of [14] and [17]. It is remarkable that different interpretations of the same fact have been proved independently by completely different approaches. In [13] the crucial idea is to use the dominance-order on the set of roots associated to a Coxeter group whereas the Kac-Moody approach is based on the consideration of the positive imaginary cone. We make use of the well-known combinatorial facts about the Cayley graph. However, it appears that all approaches need at a certain point a tedious case by case distinction ([12]). We include our combinatorial proof of the Cayley-graph-version in the paper. This enables us on the one hand to avoid the somewhat technical task of doing the 'translation' between real root systems and abstract root systems ; on the other hand it makes the paper self-contained.

2 Preliminaries

(2.1) Let I be a finite set. A **Coxeter matrix** over I is a symmetric matrix $M = (m_{ij})_{i,j \in I}$ with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ for all $i \in I$ and $m_{ij} \geq 2$ for all $i \neq j \in I$. Given a Coxeter matrix M, then we put $E(M) := \{\{i,j\} \subset I | 1 < m_{ij} \neq \infty\}$. The **Coxeter diagram** associated to M, and also denoted by M, is the graph $(I, E(M))$ whose edges are labelled by the corresponding m_{ij}. The Coxeter matrix (and the associated Coxeter diagram) is called **irreducible** of the associated diagram is connected. We shall often confuse the notions of 'Coxeter matrix' and 'Coxeter diagram'. For example, the sentence 'the Coxeter matrix M is a tree' will be a shorthand for 'the Coxeter graph assicated to the Coxeter matrix M is a tree'.

(2.2) Let M be a Coxeter matrix over I. A **Coxeter system** of type M is a pair (W, S) consisting of a group W and a set $S = \{s_i | i \in I\} \subset W$ such that S generates W and that the relations $((s_i s_j)^{m_{ij}})_{i,j \in I}$ form a presentation of W. The group W is called a **Coxeter group**. An element of W which is conjugate to an element of S is called a **reflection**. The Coxeter matrix (or diagram) M is called **spherical** when W is finite.

(2.3) Given a subset J of I then M_J denotes the restriction of M onto J. In that case, M is called an **extension** of M_J. It is called a k**-extension** if k is the cardinality of $I \backslash J$. If M is a 1-extension which is a tree, then there is a unique vertex in J which is joined to a vertex in $I \backslash J$; this vertex is then called the **extension vertex**. In the general situation, we put $W_J := \langle s_j | j \in J \rangle$; in that case $(W_J, \{s_j | j \in J\})$ is a Coxeter system of type M_J.

(2.4) Let (W, S) be a Coxeter system of type M over I. The **chamber system** associated to (W, S) is the graph $\Sigma(W, S)$ with vertex set $C := W$ and edge set $P := \{\{c, d\} | c^{-1}d \in S\}$. The vertices of $\Sigma(W, S)$ are called **chambers**, the edges are called **panels**. Since the s_i's generate W, the graph $\Sigma(W, S)$ is connected. Note that we have a natural mapping type : $P \to I$, defined by type$(\{c, d\}) = i$ if $c^{-1}d = s_i$. The group W acts from the left (via left translation) on $\Sigma(W, S)$. This action is regular on C and type-preserving on P.

Let J be a subset of I, and let $c \in C$. The J**-residue** of c is the set $R_J(c) := cW_J$. A residue is called **spherical** if it contains finitely many chambers. It is a fact that a subgroup U of W is finite if and only if it stabilizes a spherical residue. For any residue R and any chamber c of $\Sigma(W, S)$ there is a unique chamber of R at minimal distance of c in $\Sigma(W, S)$. This chamber is called the **projection** of c on R, and is denoted by $\mathrm{proj}_R(c)$.

(2.5) Given a reflection $t \in W$ we define $P(t) \subset P$ to be the set of edges of $\Sigma(W, S)$ which are fixed by t, and we put $C(t) := \bigcup_{p \in P(t)} p$. It follows from Proposition 2.6 in [19] that the graph $(W, P \backslash P(t))$ has two connected components ; these are called the **roots** associated to t. The set of all roots is denoted by $\Phi(W, S)$. Given a root α, then the reflection to which it is associated is uniquely determined and it is denoted by r_α. Moreover, we denote by $-\alpha$ the root which is associated to r_α and which is not equal to α. If Ψ is a set of roots, we put $-\Psi := \{-\psi | \psi \in \Psi\}$.

(2.6) Given a chamber c and a reflection t, then $H(t, c)$ denotes the root associated to t, which contains c. More generally, if R is a residue in $\Sigma(W, S)$ which is not stabilized by t, then we denote by $H(t, R)$ the unique root associated to t which intersects R. This makes sense because of the following fact : t stabilizes R if and only if both roots associated to t intersect R.

3 Geometric sets of roots

(3.1) Let Σ be a set, and let W be a group acting on Σ from the left. A subset $D \neq \emptyset$ of Σ is called **prefundamental** (or a **prefundamental domain**) if, for $w \in W$, we have $w = 1$ whenever $wD \cap D \neq \emptyset$. We call D **fundamental** (or a **fundamental domain**) if, moreover, we have $\bigcup_{w\in W} wD = \Sigma$.

(3.2) Let now (W,S) be a Coxeter system, T the set of its reflections, $\Phi = \Phi(W,S)$ the set of its roots and $\Sigma = \Sigma(W,S)$ be the corresponding chamber system. A set $R \subseteq T$ is called **universal** if $(\langle R\rangle, R)$ is a Coxeter system. For any $R \subset T$ we set $M(R) := (o(st))_{s,t\in R}$, where $o(g)$ denotes the order of the group element g.

Let $\Psi \subseteq \Phi$ be a set of roots. We put $R(\Psi) := \{r_\psi | \psi \in \Psi\}$ (where r_ψ is the reflection associated to ψ) and $M(\Psi) := M(R(\Psi))$. The set Ψ is called **universal** if $R(\Psi)$ is universal. It is called **2-geometric** if for all $\psi, \psi' \in \Psi$ the set $\psi \cap \psi'$ is a fundamental domain for the action of $\langle r_\psi, r_{\psi'}\rangle$ on $\Sigma(W,S)$. It is called **geometric** if it is 2-geometric and if, moreover, $\bigcap_{\psi\in\Psi}\psi$ is not empty. A pair of roots $\{\alpha,\beta\}$ is called **weakly geometric** if $\alpha\cap\beta$ or if $(-\alpha)\cap(-\beta)$ is a fundamental domain for $\langle r_\alpha, r_\beta\rangle$. A set $\Psi \subseteq \Phi$ is called **weakly 2-geometric** if each 2-subset is weakly geometric.

The set R itself is called **geometric** (resp. **2-geometric**, **weakly 2-geometric**) if there exists a geometric (resp. 2-geometric, weakly 2-geometric) set of roots $\Psi \subseteq \Phi$ such that $R = R(\Psi)$.

(3.3) Let now s and t be two reflections such that the order of st is infinite. Then $C(t)$ is completely contained in one of the two roots associated with s ; this root is denoted by $H(s,t)$. The only geometric set of roots Ψ such that $R(\Psi) = \{s,t\}$ is given by $\Psi = \{H(s,t), H(t,s)\}$; the set $(-H(s,t)) \cap (-H(t,s))$ is empty. If t' is a third reflection such that st' has infinite order and that $H(s,t) = -H(s,t')$ then tt' has infinite order (see Lemma 3.8 in [18]).

(3.4) Let D be a set of chambers in the chamber system $\Sigma(W,S)$ associated to the Coxeter system (W,S), and let $t \in W$ be a reflection. Then t is said to **border** D when no element of $P(t)$ is contained in D and $C(t) \cap D \neq \emptyset$. In particular, if D is a root, then t borders D if and only if $t = r_D$. The following result is essentially a consequence of (3.5) ; we omit the proof.

(3.5) Proposition. *Let (W_0,S_0) be a Coxeter system, let $\Sigma = \Sigma(W_0,S_0)$ be the associated chamber system, let $\Psi \subset \Phi(W_0,S_0)$ be a geometric set of roots and put $D := \bigcap_{\psi\in\Psi}\psi$.*

Then

1) *D is a fundamental domain for the action of $W := \langle R(\Psi)\rangle$ and $(W, R(\Psi))$ is a Coxeter system.*

2) *If we set $C := \{wD | w \in W\}$ and $P := \{\{vD, wD\} | v^{-1}w \in R(\Psi)\}$ then the graph (C,P) is isomorphic to $\Sigma(W, R(\Psi))$.*

3) *$R(\Psi) = \{t \mid t$ borders $D\}$.*

4) *$S_0^{W_0} \cap W = R(\Psi)^W$.*

Proof. This is essentially a consequence of Lemma 1 in [20] (see also Sections 3 and 4 of [18] for further details). □

In particular, if $t \in W$ then there is a unique root associated with t and which contains D. This root is denoted by $H(t,D)$.

4 An outline of the proof of the main result

The proof of our main result is based on the following proposition which is essentially Proposition 5.2 in [18].

(4.1) Proposition. *Let M be a Coxeter matrix. Then the following are equivalent.*

1) *If (W,S) is a Coxeter system and if $R \subseteq S^W$ is a universal set of reflections such that $M(R) = M$ then R is a geometric set of reflections.*
2) *If (W,S), (W',S') are Coxeter systems such that $M(S) = M$ and if $\varphi : W \to W'$ is an isomorphism such that $\varphi(S) \subseteq W'^{S'}$, then $\varphi(S)^{w'} = S'$ for some $w' \in W'$.*

In view of this proposition, it suffices to prove that every irreducible, universal and 2-spherical set of reflections generating an infinite group is geometric. In order to achieve this we use the following characterization of geometric sets which we already mentioned in the introduction.

(4.2) Theorem. *Any finite, universal and weakly 2-geometric set of reflections is geometric.*

The proof will be given in Section 10.

In view of this theorem, it suffices to prove that every irreducible and universal set of reflections generating an infinite group is 2-geometric. In a first step, we will prove the latter fact for any universal set of reflections whose diagram is a circuit, is geometric. This is done in Section 11. This partial result has an interesting consequence, which will be also given in Section 11. In order to explain it, we need a further notion. We call a set of reflections in a Coxeter system **sharp-angled**, if any of its 2-subsets is geometric. In view of the result for circuits, one obtains that each sharp-angled, universal and 2-spherical set of reflections is 2-geometric (see Proposition (11.7)). Hence it suffices to show that each universal and 2-spherical set of reflections generating an infinite group is sharp-angled. This will be done in Section 12.

5 Fusion

We start with a preliminary observation.

(5.1) Lemma. *Let (W_0, S_0) be a Coxeter system, let $\{\alpha, \beta\}$ be a geometric pair of roots such that the product $r_\alpha r_\beta$ has order 3. Let ψ be a root such that r_ψ commutes with r_α. Then the pair $\{H(r_\alpha r_\beta r_\alpha, \alpha \cap \beta), \psi\}$ is geometric if and only if $\{\beta, \psi\}$ is geometric.*

Proof. By hypothesis, we have $r_\alpha(\psi) = \psi$ and $r_\alpha(\beta) = H(r_\alpha r_\beta r_\alpha, \alpha \cap \beta)$. The claim follows. □

(5.2) Let M be a Coxeter diagram over a set I. A subset $F = \{f, g\}$ of I is called a **fusion edge** of M if $m_{fg} = 3$ and if $m_{fc} = 2$ or $m_{gc} = 2$ for each $c \in I \backslash F$. Given a fusion edge F, we define the Coxeter diagram M^F over the set $I^F = (I \backslash \{f, g\}) \cup \{F\}$ by setting $m^F_{xy} = m_{xy}$ if $\{x, y\} \subset I \backslash F$ and $m^F_{xF} = \max\{m_{xf}, m_{xg}\}$ for all $x \in I \backslash F$.

(5.3) Proposition. *Let (W_0, S_0) be a Coxeter system, and let Ψ be a set of roots in $\Sigma(W_0, S_0)$. Suppose that there is a geometric pair of roots $\{\alpha, \beta\} \subset \Psi$ which corresponds to a fusion edge in the diagram $M(\Psi)$. Let γ be the unique root associated with $r_\alpha r_\beta r_\alpha$ which contains $\alpha \cap \beta$. Then Ψ is a 2-geometric set of roots if and only if $(\Psi \backslash \{\alpha, \beta\}) \cup \{\gamma\}$ is a 2-geometric set of roots. If Ψ is geometric (resp. universal), then so is $(\Psi \backslash \{\alpha, \beta\}) \cup \{\gamma\}$.*

Proof. The first part is an immediate consequence of (5.1). For the second part, assume first that Ψ is geometric. In particular, the set Ψ is then 2-geometric, whence $(\Psi \backslash \{\alpha, \beta\}) \cup \{\gamma\}$ is 2-geometric by the first part. Moreover, as $\alpha \cap \beta \subset \gamma$ and $\bigcap \Psi \neq \emptyset$, it follows that $\gamma \cap (\bigcap \Psi)$ is not empty, proving that $(\Psi \backslash \{\alpha, \beta\}) \cup \{\gamma\}$ is geometric. Finally, if Ψ is universal, then we may apply the preceding statement to the geometric set of roots

$$\{\psi \in \Phi(\langle R(\Psi) \rangle, R(\Psi)) \mid 1 \in \psi,\ r_\psi \text{ borders } 1\}$$

in the chamber system $\Sigma(\langle R(\Psi) \rangle, R(\Psi))$. Since a geometric set of roots is necessarily universal (see (3.5)), the result follows. □

6 The non-irreducible case

(6.1) Proposition. *Let (W_0, S_0) be a Coxeter system, and let Ψ_1 and Ψ_2 be disjoint geometric sets of roots such that $R(\Psi_1)$ and $R(\Psi_2)$ centralize each other. For $i \in \{1, 2\}$, set $W_i = \langle R(\Psi_i) \rangle$ and $D_i = \bigcap \Psi_i$. Then W_1 stabilizes D_2, W_2 stabilizes D_1, and $D_1 \cap D_2 \neq \emptyset$.*

Proof. Set $\Sigma := \Sigma(W_0, S_0)$ and let $r \in W_2$. The proof is divided into several steps.

Claim 1: *The set rD_1 is fundamental for the action of W_1 on Σ.*

This is an easy computation, in view of the hypotheses.

Claim 2: *There exists $w \in W_1$ such that $rD_1 = wD_1$.*

Let us choose $w \in W_1$ such that $rD_1 \cap wD_1 \neq \emptyset$. If $rD_1 \not\subset wD_1$, then there exists a reflection t bordering wD_1, such that there is a panel in $P(t)$ which is contained in rD_1. Hence, we have $t(rD_1) \cap (rD_1) \neq \emptyset$. But on the other hand, t belongs to W_1 by **??**, which implies that $t = 1$ by Step 1. This is a contradiction. Thus $rD_1 \subset wD_1$. Similarly, one obtains $r^{-1}D_1 \subset w'D_1$ for some $w' \in W_1$. From $D_1 \subset ww'D_1$ it follows that $w' = w^{-1}$ since D_1 is fundamental for W_1. Finally, we obtain $r^{-1}D_1 \subset w^{-1}D_1$, whence the conclusion.

Claim 3: *We have $rD_1 = D_1$.*

It suffices to prove the claim for $r \in R(\Psi_2)$. Assume $rD_1 \neq D_1$, whence $rD_1 \cap D_1 = \emptyset$ by Step 2. Let us choose a minimal path γ joining a chamber of D_1 to a chamber of rD_1. Let π be the panel of Σ stabilized by r and crossed by γ. Since the reflection r does not belong to $R(\Psi_1)$ by hypothesis, **??** implies that π is contained in vD_1 for some $v \in W_1$. Therefore, by Step 2, we have $r(vD_1) = vD_1$, and, transforming by v^{-1}, we obtain a contradiction.

Claim 4: *We have $D_1 \cap D_2 \neq \emptyset$.*

By Claim 3, the group W_2 stabilizes D_1. Now, suppose $D_1 \cap D_2 = \emptyset$. Then, for each $w \in W_2$, $wD_1 \cap wD_2 = D_1 \cap wD_2 = \emptyset$, whence $D_1 = D_1 \cap \Sigma = \emptyset$, a contradiction.

This concludes the proof of the proposition. □

7 The "almost spherical" case

(7.1) Lemma. *Let (W_0, S_0) be a Coxeter system, and let Ψ be a geometric set of roots in $\Sigma(W_0, S_0)$. Then $M(\Psi)$ is spherical if and only if $-\Psi$ is also geometric. In that case, if we put $D := \bigcap \Psi$ and $D' := \bigcap(-\Psi)$, then $H(t, D) = -H(t, D')$ for each reflection t in $\langle R(\Psi) \rangle$.*

Proof. The 'if' part follows from Lemma 3.10 in [18]. The 'only if' part, as well as the last statement of the lemma, is a consequence of Theorem 2.15 in [19]. □

We now prove a special case of (1.6).

(7.2) Proposition. *Let $\Psi := \{\psi_0, \psi_1, \dots, \psi_n\}$ be a universal 2-geometric set of roots in the chamber system Σ associated to a Coxeter system (W', S'). Put $\Psi_0 := \Psi \backslash \{\psi_0\}$, and assume that $M(\Psi_0)$ is spherical. Then there is a sign $\epsilon \in \{+, -\}$ such that $\epsilon\Psi$ is a geometric set of roots. If $M(\Psi)$ is non-spherical, this sign is unique, and in that case, every spherical residue stabilized $\langle R(\Psi_0) \rangle$ is contained in $\epsilon\psi_0$.*

Proof. The proof is by induction on n ; the result is trivial for $n = 1$.

Let R be a spherical residue invariant under $W_0 := \langle R(\Psi_0) \rangle$, but not under $W := \langle R(\Psi) \rangle$. Such a residue exists : this follows from the geometric representation of a Coxeter group, using the universality of $R(\Psi)$ (see [1]) – actually, if $M(\Psi)$ is not spherical, no spherical residue is stabilized by W.

By the induction hypothesis together with (7.1), we know that $D := \bigcap_{i=1}^n \psi_i$ and $-D := \bigcap_{i=1}^n (-\psi_i)$ are both fundamental domains for W_0 (see (3.5)). Now, since R is invariant under W_0 it follows that $R \cap D$ and $R \cap (-D)$ are both non-empty. Now, if $\epsilon \in \{+, -\}$ is such that $H(r_{\psi_0}, R) = \epsilon\psi_0$, then $R \cap (\epsilon D) \subset (\epsilon\psi_0) \cap (\epsilon D)$, and the latter set is not empty, proving that $\epsilon\Psi$ is geometric.

Finally, assume that $M(\Psi)$ is non-spherical. In that case, the uniqueness of the sign ϵ follows from (7.1). Now, if R' is another spherical residue stabilized by W_0, and if $H(r_{\psi_0}, R') \neq H(r_{\psi_0}, R) = \epsilon\psi_0$, then we conclude as in the previous paragraph that $-\epsilon\Psi$ is geometric, which contradicts the uniqueness of ϵ. □

8 Critical extensions

(8.1) Let M be an irreducible and spherical Coxeter diagram over the set I. Let $\bar{M}$ be a 2-extension of M over the set $\bar{I} = \{a, b\} \cup I$. If the 1-extensions $\bar{M}_{I \cup \{a\}}$ and $\bar{M}_{I \cup \{b\}}$ are both non-spherical, then $\bar{M}$ is called a **critical extension** of M.

Before going further, we record the following consequence of (7.2).

(8.2) Lemma. *Let $\Psi_c = \{\alpha_1, \alpha_2\} \cup \Psi$ be a universal 2-geometric set of roots in the chamber system associated to some Coxeter system. Assume that $M(\Psi)$ is irreducible and spherical, and that $M(\Psi_c)$ is a critical extension of $M(\Psi)$. For $i = 1, 2$, let $\epsilon_i \in \{+, -\}$ be the unique sign such that $\epsilon_i(\{\alpha_i\} \cup \Psi)$ is geometric (see (7.2)). If $\epsilon_1 = \epsilon_2$, and if we denote by ϵ the common value, then $\epsilon\Psi_c$ is geometric. Conversely, if $R(\Psi_c)$ is geometric then $\epsilon_1 = \epsilon_2$.*

Proof. Let R be a spherical residue stabilized by $\langle R(\Psi) \rangle$. By (7.2), we have $R \subset (\epsilon_1\alpha_1) \cap (\epsilon_2\alpha_2)$. Now, if $\epsilon_1 = \epsilon_2 =: \epsilon$ then by (7.1) and (3.5) we have $\emptyset \neq R \cap (\bigcap_{\psi \in \Psi} \epsilon\psi) \subset (\epsilon\alpha) \cap (\epsilon\beta)$ and so $\epsilon\Psi_c$ is geometric.

Conversely, assume that $R(\Psi)$ is geometric. By Lemma 3.9 in [18], we know that the only geometric set of roots $\bar{\Psi}$ such that $R(\bar{\Psi}) = R(\Psi_c)$ is Ψ_c or $-\Psi_c$. Now, since a subset of a geometric set of roots is itself geometric, the equality $\epsilon_1 = \epsilon_2$ follows from the uniqueness of these signs. □

(8.3) A Coxeter diagram M is said to have **property (LG)** if the following holds : given any Coxeter system (W_0, S_0) and any 2-geometric and universal set of reflections T in W_0 such that $(\langle T \rangle, T)$ has type M, then T is a geometric set of reflections.

Hence, the fact (1.6) is equivalent to the statement that any finite Coxeter diagram has property (LG). Thanks to the following result, we will see that it suffices to prove the validity of that fact for critical extensions of spherical diagrams.

(8.4) Proposition. *Let $\Psi = \Psi_s \cup \Psi_1 \cup \Psi_\perp$ be a universal 2-geometric set of roots in the chamber system Σ associated to a Coxeter system (W_0, S_0). Assume that the following hypotheses are satisfied :*

(i) $M(\Psi)$ is irreducible ;
(ii) $M(\Psi_s)$ is irreducible and spherical, and every critical extension of $M(\Psi_s)$ has property (LG) ;
(iii) for each $\psi \in \Psi_1$, $M(\Psi_s \cup \{\psi\})$ is non-spherical ;
(iv) for for each $\psi' \in \Psi_\perp$, $M(\Psi_s \cup \{\psi'\})$ is reducible.

Then there is a sign $\epsilon \in \{+, -\}$ such that $\epsilon(\Psi_s \cup \Psi_1)$ is geometric. If moreover $\epsilon\Psi_\perp$ is geometric, then $\epsilon\Psi$ is geometric.

Proof. By (7.2), we may assume that Ψ_1 is not empty. By (iii) and (7.2), we know that for each $\psi \in \Psi_1$, there exists a unique sign $\epsilon_\psi \in \{+, -\}$ such that $\epsilon_\psi(\Psi_s \cup \{\psi\})$ is geometric. Now, for all $\psi, \psi' \in \Psi_1$, the diagram $M(\Psi_s \cup \{\psi, \psi'\})$ is a critical extension of $M(\Psi_s)$, and it follows from (ii) that $\epsilon_\psi = \epsilon_{\psi'}$. We may (and we do) assume that the common value of all ϵ_ψ's is the sign $+$. Hence $\Psi_s \cup \{\psi\}$ is geometric for each $\psi \in \Psi_1$.

Let now ρ be a minimal spherical residue of Σ which is stabilized by $\langle R(\Psi_s) \rangle$. It follows from (7.2), that

$$\emptyset \neq (\bigcap \Psi_s) \cap \rho \subset \psi \qquad \text{for each } \psi \in \Psi_1. \tag{1}$$

In particular, $\Psi_s \cup \Psi_1$ is a geometric set of roots.

Now, assume moreover that $\Psi_\perp$ is geometric. Thus the set $D_\perp = \bigcap \Psi_\perp$ is not empty. By the minimality of ρ and the universality of $\Psi_s \cup \Psi_\perp$, it follows that no element of $R(\Psi_\perp)$ stabilizes ρ. Therefore ρ is contained in $wD_\perp$ for some $w \in \langle R(\Psi_\perp) \rangle$. Replacing ρ by $w^{-1}\rho$, we may assume that ρ is contained in $D_\perp$. Since w centralizes $R(\Psi_s)$, it is still true that ρ is a minimal spherical residue stabilized by $\langle R(\Psi_s) \rangle$. Therefore, by (1), we finally obtain $\bigcap \Psi \neq \emptyset$. This proves the claim. □

The following two results will be used in order to prove that critical extensions of spherical diagrams have property (LG).

(8.5) Proposition. *Let M be an irreducible spherical Coxeter graph over I, and let $\bar{M}$ be a critical extension of M over $\bar{I} = \{a, b\} \cup I$. Let (W, S) be the Coxeter system of type $\bar{M}$, and suppose that there exists a reflection $t \in W_I$ such that $s_a t$ and $s_b t$ have both infinite order. Then $\bar{M}$ has property (LG).*

Proof. Let (W_0, S_0) be a Coxeter system, and let Σ be the corresponding chamber system. Let $\Psi = \{\alpha, \beta\} \cup \{\psi_i | i \in I\}$ be a universal 2-geometric set of roots such that $M(\Psi) = \bar{M}$, put $s_a := r_\alpha$, $s_b := r_\beta$, $s_i := r_{\psi_i}$ for $i \in I$ and $U := \langle s_i | i \in I \rangle$. Let R be a spherical residue which is stabilized by U. Put also $\Psi_a := \{\alpha\} \cup \{\psi_i | i \in I\}$ and $\Psi_b := \{\beta\} \cup \{\psi_i | i \in I\}$. By (7.2), we know that there is a unique sign $\epsilon_a \in \{+, -\}$ (resp. ϵ_b) such that the set $\epsilon_a \Psi_a$ (resp. $\epsilon_b \Psi_b$) is geometric. Let $D_a := \bigcap(\epsilon_a \Psi_a)$, $D_b := \bigcap(\epsilon_b \Psi_b)$ and let $t \in U$ be a reflection such that $s_a t$ and $s_b t$ have infinite order. By **??**, the sets $C(s_a) \cap D_a$ and $C(s_b) \cap D_b$ are both non-empty, and so $H(t, D_a) = H(t, s_a)$ and $H(t, D_b) = H(t, s_b)$.

Assume that $\bar{m}_{ab} \neq \infty$. This implies $H(t, s_a) = H(t, s_b)$ by (3.3), and therefore $H(t, D_a) = H(t, D_b)$. This implies that $\epsilon_a = \epsilon_b$ by (7.1), and hence $\epsilon\Psi$ is a geometric set of roots by (8.2).

Assume now that $\bar{m}_{ab} = \infty$. Since Ψ is 2-geometric, we have $H(s_a, s_b) = \alpha$, $H(s_b, s_a) = \beta$ and $(-H(s_a, s_b)) \cap (-H(s_b, s_a)) = \emptyset$ by (3.3). Since $R \subset (\epsilon_a \alpha) \cap (\epsilon_b \beta)$ by (7.2), this implies that ϵ_a and ϵ_b are not both equal to $-$. On the other hand, if $\epsilon_a \neq \epsilon_b$ then $H(t, s_a) = H(t, D_a) = -H(t, D_b) = -H(t, s_b)$ where the second equality follows from (7.1). In that situation, we have $\alpha = H(s_a, s_b) = H(s_a, t) = H(s_a, R) = \epsilon_a \alpha$ and similarly $\beta = \epsilon_b \beta$ whence $\epsilon_a = \epsilon_b = +$, a contradiction. This implies that ϵ_a and ϵ_b are both equal to $+$, and the conclusion follows from (8.2). □

The importance of the notion of fusion is due to the following result.

(8.6) Proposition. *Let M be an irreducible spherical Coxeter diagram over I and let $\bar{M}$ be a critical extension over $\{a, b\} \cup I$. Suppose there is a fusion edge $F \subset I$ such that $\bar{M}^F$ is a critical extension of M^F. If $\bar{M}^F$ has property (LG), then $\bar{M}$ has property (LG).*

Proof. Let (W_0, S_0) be a Coxeter system, and let Σ be the corresponding chamber system. Let $\Psi = \{\alpha, \beta\} \cup \{\psi_i | i \in I\}$ be a universal 2-geometric set of roots such that $M(\Psi) = \bar{M}$, put $s_a := r_\alpha$, $s_b := r_\beta$ and $s_i := r_{\psi_i}$ for $i \in I$. Put also $\Psi_a := \{\alpha\} \cup \{\psi_i | i \in I\}$ and $\Psi_b := \{\beta\} \cup \{\psi_i | i \in I\}$. By (7.2), we know that there is a unique sign $\epsilon_a \in \{+, -\}$ (resp. ϵ_b) such that the set $\epsilon_a \Psi_a$ (resp. $\epsilon_b \Psi_b$) is geometric. Let now $F = \{c, d\} \subset I$ be the fusion edge of the statement, and let $\gamma, \delta \in \Psi$ be the roots corresponding to c and d respectively. Denote by ϕ the unique root associated with $s_c s_d s_c$ which contains $\gamma \cap \delta$. Let $\Psi_a^F := (\Psi_a \backslash \{\gamma, \delta\}) \cup \{\phi\}$, $\Psi_b^F := (\Psi_b \backslash \{\gamma, \delta\}) \cup \{\phi\}$ and $\Psi^F := \{\psi_i | i \in I \backslash F\} \cup \{\phi\}$. It follows from (5.3) that $\epsilon_a \Psi_a^F$ and $\epsilon_b \Psi_b^F$ are geometric sets of roots. On the other hand, we know by assumption that $R(\Psi^F)$ is geometric, and this implies $\epsilon_a = \epsilon_b$ by the converse part of (8.2). Now, using (8.2) again, this time applied to $\epsilon_a \Psi_a$ and $\epsilon_b \Psi_b$, we obtain that $\epsilon_a \Psi = \epsilon_b \Psi$ is geometric. This concludes the proof. □

9 Root bases

In this section, we develop some tools related to spherical Coxeter systems. These tools will be very useful for the proof of Theorem (4.2), which is given in the next section.

(9.1) Let M be a Coxeter matrix over I. A **root basis** associated with M is a triple $(V, b, (e_i)_{i \in I})$ consisting of a real vector space V endowed with a bilinear form b and a basis $(e_i)_{i \in I}$ such that $b(e_i, e_j) = -\cos(\pi/m_{ij})$ for all $i, j \in I$. In that situation, we put $s_i(v) := v - 2b(v, e_i)$ for each $i \in I$ and each $v \in V$. The mapping $s_i : V \to V$ is an isometry of (V, b). We put $S := \{s_i | i \in I\}$, $W := \langle S \rangle$ and $\Phi(V, b) := \{w(e_i) | w \in W, i \in I\}$; the elements of $\Phi(V, b)$ are called **roots**. The coordinates of a root with respect to the basis $(e_i)_{i \in I}$ of V are always greater than or equal to 1 in absolute value. Moreover, they are either all positive or all negative ; we call this root **positive** or **negative** accordingly.

Given $\phi \in \Phi(V,b)$, we define $r_\phi \in W$ by putting $r_\phi(v) := v - 2b(v,\phi)\phi$. Note that $r_{w(\phi)} = wr_\phi w^{-1}$ for all $\phi \in \Phi(V,b)$ and all $w \in W$.

(9.2) Note that the term *root* has been used to designate two distinct notions till now : in (2.5), a root is a subset of the chamber system associated to the Coxeter system (W,S), while in (9.1), a root is a vector in the vector space V of the root basis $(V,b,(e_i)_{i\in I})$ associated with (W,S). However, there is a natural correspondence $\rho : \Phi(V,b) \to \Phi(W,S)$ between the roots of $(V,b,(e_i)_{i\in I})$ and the roots in $\Sigma(W,S)$, defined

$$\rho(e_i) = H(s_i,1) \qquad \text{for each } i \in I$$

and by the requirement that ρ commutes with the action of W on the sets $\Phi(V,b)$ and $\Phi(W,S)$. It is easily checked that the correspondence ρ is well defined. The positive roots in $\Phi(V,b)$ are mapped by ρ onto the roots of $\Sigma(W,S)$ which contain the chamber 1.

(9.3) Proposition. *Let M be a Coxeter matrix over a set I, and let $(V,b,(e_i)_{i\in I})$ be a root basis associated with M. Define S,W and Φ as above. Then*

1) *The pair (W,S) is a Coxeter system of type M ;*
2) *If $J \subset I$ is such that M_J is spherical, then $(U,b|_U)$ is a Euclidean subspace, where $U := \langle e_j | j \in J\rangle$.*

Proof. See [1], in particular §3.2 of Chapter V for 1) and §4.8 of for 2). See also [8]. □

(9.4) Corollary. *Let $\Phi(V,b)$ be the set of roots of the root basis $(V,b,(e_i)_{i\in I})$. Given $\alpha,\beta \in \Phi(V,b)$ such that $|b(\alpha,\beta)| \geq 1$, then $r_\alpha = r_\beta$ or $r_\alpha r_\beta$ has infinite order.*

Proof. Suppose $r_\alpha r_\beta$ has finite order. Then $X := \langle r_\alpha, r_\beta\rangle$ is a finite subgroup of W, and hence it is conjugate to a subgroup of some finite group of the form W_J for $J \subset I$. Thus we can assume that $r_\alpha, r_\beta \in W_J$. It follows that $\alpha,\beta \in U$ where $U := \langle e_j | j \in J\rangle$. As $b|_U$ is positive definite and as $b(\alpha,\alpha) = b(\beta,\beta) = 1$ it follows from the Cauchy-Schwarz inequality that $|b(\alpha,\beta)| \leq 1$ with equality if and only if α and β are linearly dependent, which means that precisely that $r_\alpha = r_\beta$. □

(9.5) Corollary. *Let M be a spherical Coxeter diagram over I, let $\bar{M}$ be a 1-extension of M over $\bar{I} = \{a\} \cup I$, and assume that $\bar{M}$ is a tree. Denote by x the extension vertex. Let $(V,b,(e_i)_{\bar{I}})$ be the corresponding root basis. Let $\phi = \sum_{j\in I} \lambda_j e_j \in \Phi(V,b)$. Then the product $s_a r_\phi$ has finite order if and only if $\lambda_x < 2$ and $\bar{m}_{ax} \neq \infty$. If moreover, $\sqrt{2} \geq \lambda_x < 2$ then the product $s_a r_\phi$ has finite order if and only if $\bar{m}_{ax} = 3$.*

Proof. By hypothesis, we have $|b(e_a,\phi)| = |b(e_a,\lambda_x e_x)| = |\lambda_x|\cos(\pi/\bar{m}_{ax})$. The result follows from (9.4), using $|\lambda_x| \geq 1$. □

(9.6) We know fix a choice of labelling for the irreducible spherical Coxeter diagrams, the list of which is known since their classification by H.S.M. Coxeter. Recall that there are three infinite families, namely A_n, C_n and D_n, some exceptional diagrams, namely E_6, E_7, E_8, F_4, H_3 and H_4, and another family $I_2(k)$ for $k \geq 5$ corresponding to the dihedral groups of order $2k$ (the cases $k = 3$ and $k = 4$ are covered by A_2 and C_2 respectively). Our choice of labelling is as follows. For the diagrams H_3 and H_4, the vertices are numbered in a linear order such that the edge $\{1,2\}$ has label 3 ; the diagrams $I_2(k)$ are indexed by the set $\{1,2\}$. For the other irreducible spherical diagrams, we adopt the labelling from [1] (see Planches I, III, IV, V, VI, VII and VIII at the end of op. cit.) ;

hence the indexing set is $\{1, \ldots, n\}$ where n is the cardinality of the set of vertices of the diagram in question.

(9.7) In the following table, we define one or two roots in the root system associated to each irreducible spherical diagram. These roots are called the **highest roots** of the root system in question. The symbol τ denotes the real number $\tau = \frac{1+\sqrt{5}}{2}$.

A_n	ϕ_h	$=$	$\sum_{i=1}^{n} e_i$
C_n	ϕ_h	$=$	$\sqrt{2}\sum_{i=1}^{n-1} e_i + e_n$
	ϕ_H	$=$	$e_1 + 2\sum_{i=2}^{n-1} e_i + \sqrt{2}e_n$
D_n	ϕ_h	$=$	$e_1 + 2\sum_{i=2}^{n-2} + e_{n-1} + e_n$
E_6	ϕ_h	$=$	$e_1 + 2e_2 + 2e_3 + 3e_4 + 2e_5 + e_6$
E_7	ϕ_h	$=$	$2e_1 + 2e_2 + 3e_3 + 4e_4 + 3e_5 + 2e_6 + e_7$
E_8	ϕ_h	$=$	$2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7 + 2e_8$
F_4	ϕ_h	$=$	$\sqrt{2}e_1 + 2\sqrt{2}e_2 + 3e_3 + 2e_4$
	ϕ_H	$=$	$2e_1 + 3e_2 + 2\sqrt{2}e_3 + \sqrt{2}e_4$
H_3	ϕ_h	$=$	$\tau e_1 + 2\tau e_2 + (\tau+1)e_3$
H_4	ϕ_h	$=$	$(\tau+1)e_1 + (2\tau+1)e_2 + (3\tau+1)e_3 + (2\tau+1)e_4$
$I_2(5)$	ϕ_h	$=$	$\tau e_1 + \tau e_2$
$I_2(6)$	ϕ_h	$=$	$2e_1 + \sqrt{3}e_2$
	ϕ_H	$=$	$\sqrt{3}e_1 + 2e_2$
$I_2(k)$	ϕ_h	$=$	$x_k e_1 + y_k e_2 \qquad (k \geq 7)$

where x_k and y_k are chosen such that $x_k, y_k \geq 2$. In the following, we write r_h and r_H in place of r_{ϕ_h} and r_{ϕ_H} respectively.

(9.8) Lemma. *Let M be an irreducible spherical Coxeter graph over I, and let $\bar{M}$ (over $\bar{I} = \{a\} \cup I$) be a non-spherical 1-extension of M which is not a tree (see (2.3)). Let (W, S) be the Coxeter system of type $\bar{M}$. Then the products $s_a r_h$ and $s_a r_H$ have both infinite order.*

Proof. This follows by considering the root basis associated with $\bar{M}$, and by applying (9.4). □

(9.9) Lemma. *Let M be an irreducible spherical Coxeter graph over I such that there exists a label $m_{ij} \geq 4$. Let $\bar{M}$ (over $\bar{I} = \{a, b\} \cup I$) be a critical extension of M. Suppose that s_a and s_b do not commute. Then $s_a s_b s_a r_h$ (as well as $s_a s_b s_a r_H$ if r_H is defined) has infinite order.*

Proof. Let $(V, b, (e_i)_{i \in \bar{I}})$ be the root basis associated with $\bar{M}$. The reflection associated to $s_a(e_b)$ is precisely $s_a s_b s_a$. Since $\bar{m}_{ab} \geq 3$ we have $s_a(e_b) = x e_a + e_b$ for a real $x \in [1, 2]$. Now a case by case consideration shows that $b(s_a(e_b), \phi_h) \geq 1$ (and $b(s_a(e_b), \phi_H) \geq 1$ if ϕ_H is defined), and the claim follows from (9.4). □

(9.10) Lemma. *Let M be an irreducible spherical Coxeter graph over I, such that either $M = E_8$ or there exists a label $m_{ij} \geq 4$. Let $\bar{M}$ (over $\bar{I} = \{a\} \cup I$) be a non-spherical 1-extension of M which is a tree, and let (W, S) be the Coxeter system of type $\bar{M}$. Then we have the following. If r_H is not defined then $s_a r_h$ has infinite order. If r_h and r_H are both defined, then at least one of the products $s_a r_h$ and $s_a r_H$ has infinite order, except in the following situation : $M = C_n$, $n \geq 4$, the extension vertex is n (with the labelling of M as in (9.6)) and $\bar{m}_{an} = 3$.*

Proof. The claim follows from a case by case consideration, using (9.5) and (9.7). □

We end this section with a more specific lemma.

(9.11) Lemma. *Let (W, S) be a Coxeter system of type $M \in \{C_n, F_4, I_2(6) | n \geq 2\}$ over the set I. Then $\{r_h, r_H\}$ is a geometric set of reflections, and $\{\rho(\phi_h), -\rho(\phi_H)\}$ is a geometric set of roots (see (9.2)).*

Proof. The first assertion follows from the second. Moreover, if $M = I_2(6)$ then the statement is straightforward. Now, we assume $M = C_n$ or $M = F_4$. Let $(V, b, (e_i)_{i \in I})$ be the root basis associated with M. One computes that $b(\phi_h, \phi_H) = \sqrt{2}/2$, whence $o(r_h r_H) = 4$, and that $r_h(\phi_H)$ and $r_H(\phi_h)$ are both negative roots. Now, focusing on a spherical residue of rank 2 of $\Sigma(W, S)$ stabilized by $\langle r_h, r_H \rangle$, it is easy to see that these statements imply that $\{\rho(\phi_h), -\rho(\phi_H)\}$ is a geometric set of roots, as was to be proved. □

10 Proof of Theorem (4.2)

The proof proceeds in three main steps. We first prove that a universal weakly 2-geometric set of reflections is necessarily 2-geometric (see Proposition (10.3)), which implies that Theorem (4.2) is a consequence of the fact (1.6). Then, we prove (1.6) in the special case of critical extensions of spherical diagrams (see Proposition (10.4)). Next, we show that the general situation reduces to that case.

(10.1) Lemma. *Let $\Psi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a universal set and weakly 2-geometric set of roots in a Coxeter system (W, S). For each $i \in \{1, \dots 4\}$, set $t_i := r_{\alpha_i}$. If $\alpha_1 \cap \alpha_2 = \emptyset$ and $(-\alpha_3) \cap (-\alpha_4) = \emptyset$, then for each $i \in \{1, 2\}$ and each $j \in \{3, 4\}$, the product $t_i t_j$ has order 2 (while $t_1 t_2$ and $t_3 t_4$ both have infinite order).*

Proof. The proof consists of a case by case analysis.

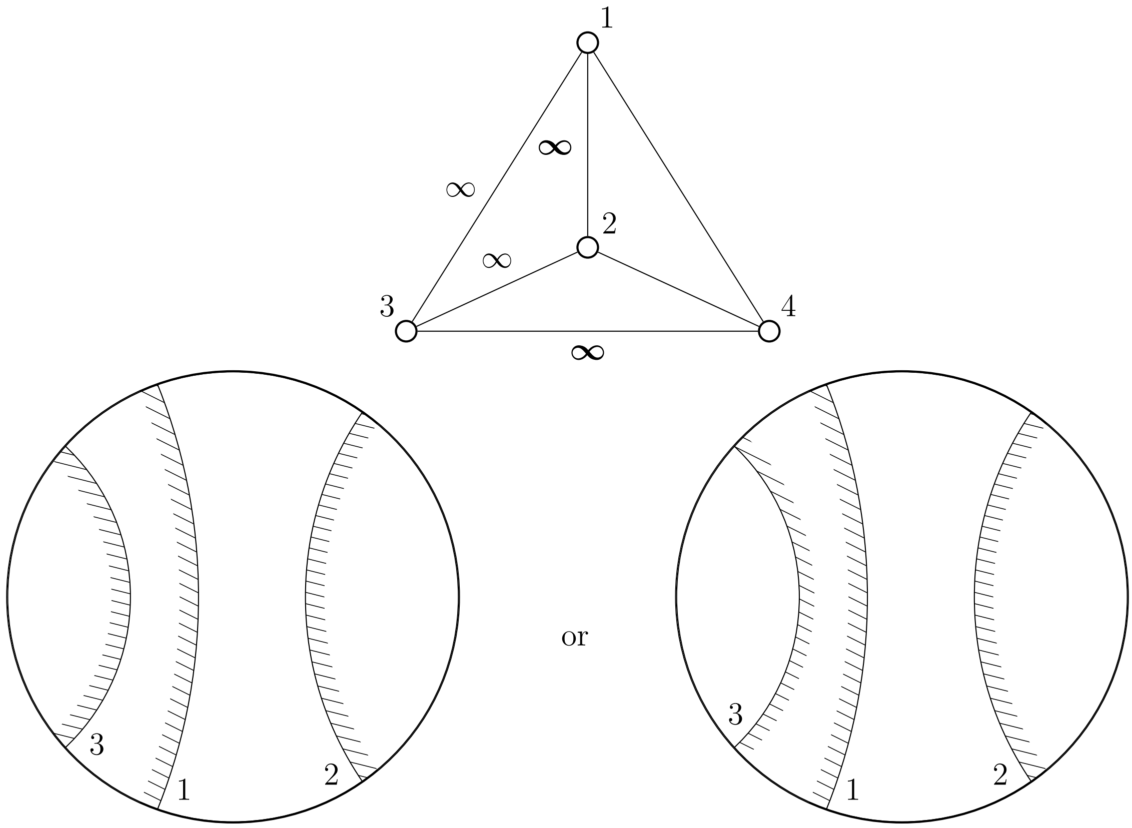
Case 1: *there is an $i \in \{3, 4\}$ such that $t_1 t_i$ and $t_2 t_i$ have both infinite order.* We may assume without restriction that $i = 3$. Assume now that $C(t_3) \subset \alpha_1$. In that case, it is easy to check that $\{\alpha_1, \alpha_3\}$ or $\{\alpha_2, \alpha_3\}$ is not weakly geometric (see Figure 1), which contradicts the hypothesis. Similarly, it cannot happen that $C(t_3) \subset \alpha_2$, and we conclude that $C(t_3) \subset (-\alpha_1) \cap (-\alpha_2)$. This implies that $\alpha_1 \cap \alpha_3 = \alpha_2 \cap \alpha_3 = \emptyset$, whence by hypothesis, we must have $C(t_4) \subset \alpha_3$. Therefore, for each $i \in \{1, 2\}$, the product $t_i t_4$ has infinite order, and furthermore, the pair $\{\alpha_i, \alpha_4\}$ is not weakly geometric (see Figure 2). This is again a contradiction, showing that this case does not occur.

Case 2: *there is an $i \in \{3, 4\}$ such that one of the products $t_1 t_i$ and $t_2 t_i$ has infinite order, and the other has finite order.* We may assume without restriction that $i = 3$, that $t_1 t_3$ has infinite order, and that $t_2 t_3$ has finite order. This implies that $C(t_3) \subset (-\alpha_1)$ because otherwise, the product $t_2 t_3$ would have infinite order. Therefor, we have $\alpha_1 \cap \alpha_3 = \emptyset$ and it is easy to check that this (together with the hypothesis $(-\alpha_3) \cap (-\alpha_4) = \emptyset$) implies that the pair $\{\alpha_1, \alpha_4\}$ is not weakly geometric (see Figure 3). Again, this is impossible.

Case 3: *for each $i \in \{1, 2\}$ and each $j \in \{3, 4\}$, the product $t_i t_j$ has finite order* (see Figure 4). The lemma will be proved if we show that none of these products has an order > 2. Let us thus assume that $t_1 t_3$ is of order $n \geq 3$. Then, for each $i \in \{1, 2\}$ and each $j \in \{3, 4\}$, there exists a spherical residue R_{ij} of rank 2 which is stabilized by $\langle t_i, t_j \rangle$. Let us also define β by $\beta = t_3(\alpha_1)$, and let $t = t_3 t_1 t_3$ (hence $t = r_\beta$). Since the pair $\{\alpha_1, \alpha_3\}$ is geometric, the set $\alpha_1 \cap \alpha_3$ lies entirely in β (indeed, the projection of any chamber in $\alpha_1 \cap \alpha_3$ on R_{13} must belong to $\beta \cap R_{13}$). Therefore, if c_{14} is a chamber lying in $\alpha_1 \cap (-\alpha_4) \cap R_{14}$, then its projection $\mathrm{proj}_{R_{13}}(c_{14})$ onto R_{13} must lie in β, which implies that c_{14} itself lies in β. On the other hand, the set $(-\alpha_1) \cap (-\alpha_3)$ lies entirely in $-\beta$, and one deduces similarly that a chamber $c_{23} \in \alpha_2 \cap (-\alpha_3) \cap R_{23}$ must lie in $-\beta$. Therefore, *any path of* $\Sigma(W, S)$ *joining* c_{14} *to* c_{23} *must cross the wall* $P(t)$.

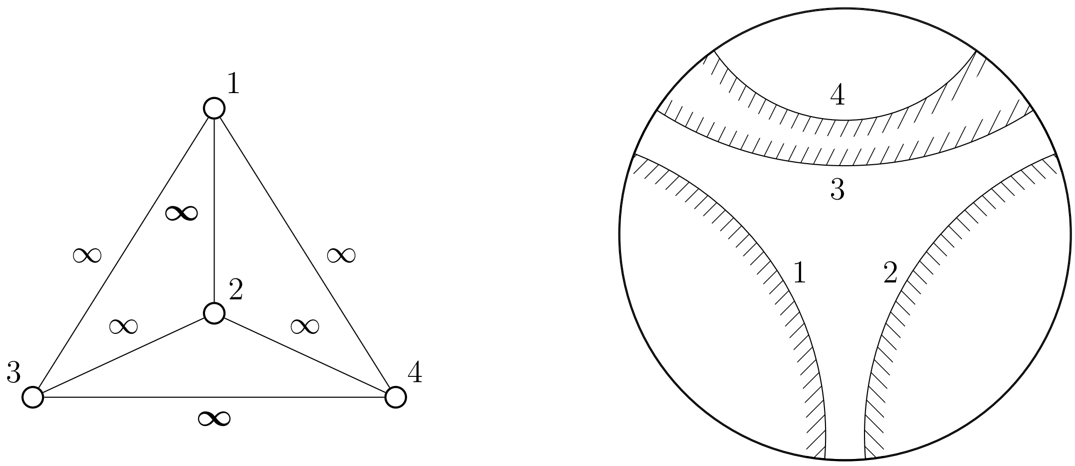
On the other hand, it is easy to compute, using the fact that T is universal, that the products tt_2 and tt_4 have both infinite order. Therefore, we have $C(t) \subset \alpha_4 \cap (-\alpha_2)$. Now, it is easy to see that there exists a path joining c_{14} to c_{23} (passing through R_{24}), and lying entirely in $(-\alpha_4) \cup \alpha_2$. Hence, this path does not cross $P(t)$, which is a contradiction. The proof is complete. □

Figure 1: Proof of Lemma (10.1), Case 1 (1)



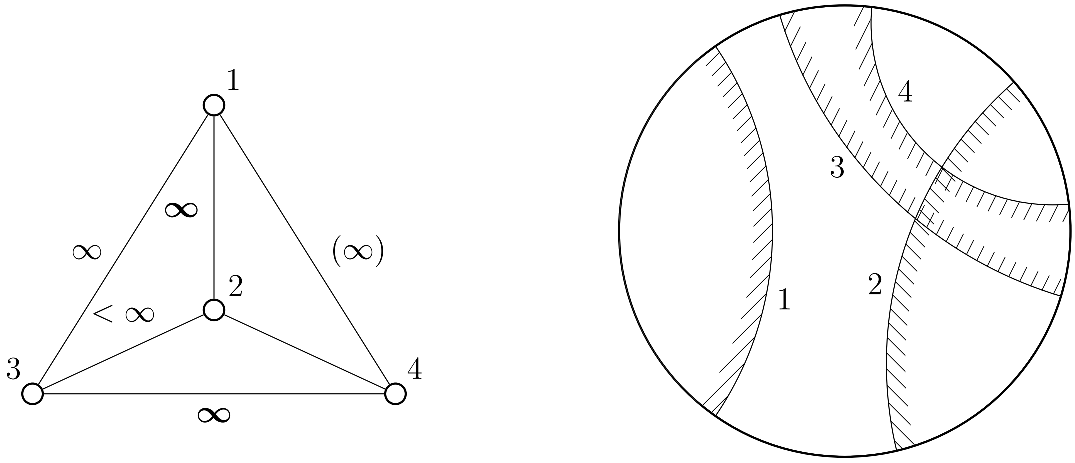
$\{\alpha_1, \alpha_3\}$ is not weakly geometric $\{\alpha_2, \alpha_3\}$ is not weakly geometric

Figure 2: Case 1 (2)



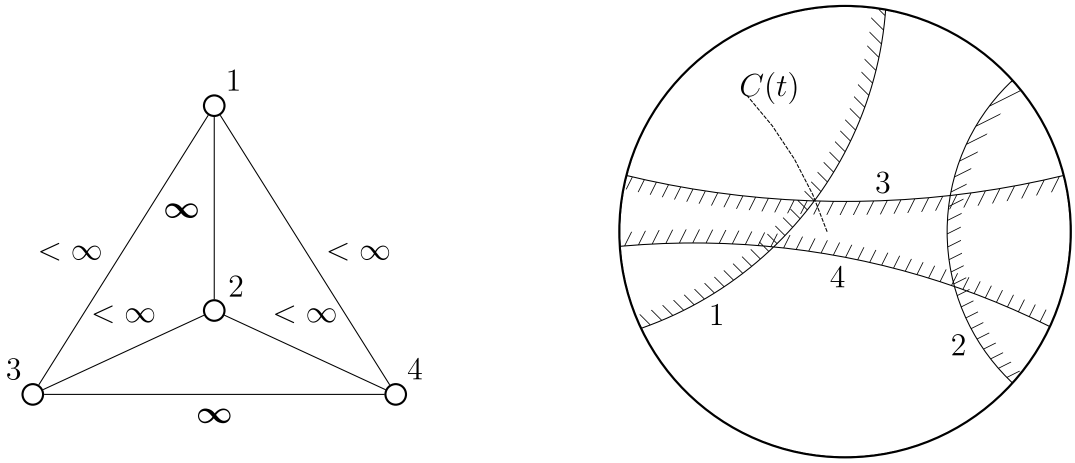
$\{\alpha_1, \alpha_4\}$ and $\{\alpha_2, \alpha_4\}$ are not weakly geometric

Figure 3: Proof of Lemma (10.1), Case 2



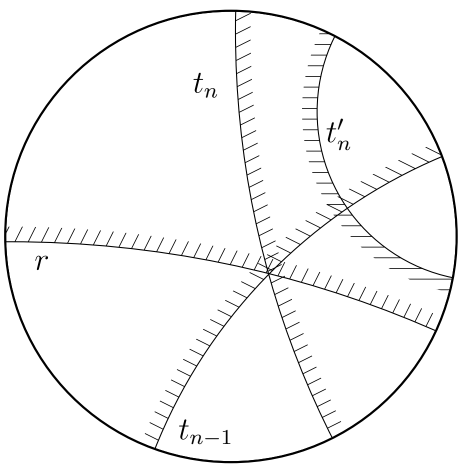
$\{\alpha_1, \alpha_4\}$ is not weakly geometric

Figure 4: Proof of Lemma (10.1), Case 3



$C(t)$ must meet $C(t_2)$ or $C(t_4)$

Figure 5: Proof of Lemma (10.2)



(10.2) Lemma. *Let $\{\psi_0', \psi_0, \psi_1, \ldots, \psi_n, \psi_n'\}$ be a universal weakly 2-geometric set of roots in a Coxeter system (W, S). Assume that the diagram $M(\{\psi_0, \psi_1, \ldots, \psi_n\})$ is linear, irreducible and 2-spherical, and that $M(\{\psi_0, \psi_0'\})$ and $M(\{\psi_n, \psi_n'\})$ have both type $\tilde{A}_1$. Then $\{psi_0, \psi_0'\}$ is geometric if and only if $\{\psi_n, \psi_n'\}$ is geometric.*

Proof. Let us denote by $t_0', t_0, t_1, \ldots, t_n, t_n'$ the reflections associated to $\psi_0', \psi_0, \psi_1, \ldots, \psi_n, \psi_n'$ respectively. By hypothesis, we know that $t_0 t_0'$ and $t_n t_n'$ have both infinite order. By symmetry, it suffices to prove that if $\{\psi_n, \psi_n'\}$ is geometric, then so is $\{\psi_0, \psi_0'\}$. Let us assume that $\{\psi_n, \psi_n'\}$ is geometric and that $\{\psi_0, \psi_0'\}$ is not geometric, and so $\{-\psi_n, -\psi_n'\}$ is geometric. We prove by induction on n that this situation yields a contradiction. The preceding lemma implies that $n \geq 2$.

Let $2 < m < \infty$ be the order of $t_{n-2}t_{n-1}$. Let $r = t_n t_{n-1} t_n$, and set $\psi := H(r, \psi_{n-1} \cap \psi_n)$. Since t_n and t_{n-2} commute, we have $t_n(\psi_{n-2}) = \psi_{n-2}$. On the other hand, it is easy to see that $\psi = t_n(\psi_{n-1})$. Therefore, $\{\psi_{n-2}, \psi\} = t_n(\{\psi_{n-2}, \psi_{n-1}\})$ is a geometric pair of roots. Moreover, $t_{n-2}r = (t_{n-2}t_{n-1})^{t_n}$ has order m.

On the other hand, it is easy to compute that rt_n' has infinite order ; moreover, the definition of ψ implies that $\{\psi, \psi_n'\}$ is a geometric pair of roots (see Figure 5). Therefore, the set $\{\psi_0', \psi_0, \psi_1, \ldots, \psi_{n-2}, \psi, \psi_n'\}$ is weakly 2-geometric.

Now, the set $\{t_0', t_0, t_1, \ldots, t_{n-2}, r\} = \{t_0', t_0, t_1, \ldots, t_{n-2}, r\}^{t_n}$ is obviously a universal set of reflections. Since the product rt_n' is of infinite order, and since t_n' commutes with each reflection of that set distinct from r, we finally conclude that $\{\psi_0', \psi_0, \psi_1, \ldots, \psi_{n-2}, \psi, \psi_n'\}$ constitutes a universal and weakly 2-geometric set of roots which contradicts the induction hypothesis. This concludes the proof of the lemma. □

(10.3) Proposition. *Any universal weakly 2-geometric set of reflections is 2-geometric.*

Proof. Let Ψ be a universal and weakly 2-geometric set of roots in a Coxeter system (W, S). We have to prove that there exists a 2-geometric set of roots Ψ' such that $R(\Psi') = R(\Psi)$. Without loss of generality, we may assume that $M(\Psi)$ is irreducible. If Ψ is 2-geometric, then we are done. Assume that Ψ is not 2-geometric. Then there exists a pair of roots $\{\alpha, \alpha'\} \subset \Psi$ which is not geometric. Hence $\{-\alpha, -\alpha'\}$ is geometric. Since $M(\Psi)$ is irreducible, the preceding lemma implies that for each pair $\{\beta, \beta'\} \subset \Psi$, the pair $\{-\beta, -\beta'\}$ is geometric (notice that if $M(\{\beta, \beta'\})$ is spherical, this assertion is obvious). This means that the set $-\Psi$ is 2-geometric, which concludes the proof. □

(10.4) Proposition. *Let M be an irreducible spherical Coxeter graph over I, and let $\bar{M}$ be a critical extension over $\bar{I} = \{a, b\} \cup I$. Then $\bar{M}$ has property (LG).*

Proof. If $M = E_8, H_3, H_4, I_2(5)$ or if $M = I_2(k)$ with $k \geq 7$ then $s_a r_h$ and $s_b r_h$ have infinite order (see (9.8) and (9.10)), and the result is given by (8.5). In any other case, we may assume that the diagram $\bar{M}_{I \cup \{a\}}$ is a tree by (9.8) ; we denote by $x \in I$ the corresponding extension vertex (see (2.3)).

Let now (W_0, S_0) be a Coxeter system, and let Σ be the corresponding chamber system. Let $\Psi = \{\alpha, \beta\} \cup \{\psi_i | i \in I\}$ be a universal 2-geometric set of roots such that $M(\Psi) = \bar{M}$, put $s_a := r_\alpha$, $s_b := r_\beta$, $s_i := r_{\psi_i}$ for $i \in I$ and $U := \langle s_i | i \in I\rangle$. Let R be a minimal spherical residue of Σ stabilized by U. Put also $\Psi_a := \{\alpha\} \cup \{\psi_i | i \in I\}$ and $\Psi_b := \{\beta\} \cup \{\psi_i | i \in I\}$. By (7.2), we know that there is a unique sign $\epsilon_a \in \{+, -\}$ (resp. ϵ_b) such that the set $\epsilon_a \Psi_a$ (resp. $\epsilon_b \Psi_b$) is geometric. Finally, let $D_a := \bigcap(\epsilon_a \Psi_a)$ and $D_b := \bigcap(\epsilon_b \Psi_b)$.

The proof is divided into the consideration of several cases.

Case 1: *$M = C_n, F_4$ or $I_2(6)$, and if $M = C_n$ with $n \geq 4$, then neither $\bar{M}_{I\cup\{a\}}$ nor $\bar{M}_{I\cup\{b\}}$ contains a subdiagram of type F_4.*

By hypothesis, we see that the reflections r_h and r_H of U are both defined (see (9.7)). Moreover, again by (8.5) together with (9.8) and (9.10), we may assume the following facts:

(i) the 1-extension $\bar{M}_{I\cup\{b\}}$ is also a tree (as is $\bar{M}_{I\cup\{a\}}$) ; we denote by y the corresponding extension vertex ;

(ii) $o(s_a r_h) = \infty = o(s_b r_H)$, while $o(s_a r_H)$ and $o(s_b r_h)$ are both finite (recall that $o(g)$ denotes the order of g) ; in particular, $\bar{m}_{ax}$ and $\bar{m}_{by}$ are both finite.

Note that (ii) implies that x and y are distinct. Moreover, by (9.5), together with (9.7), gives precise information on x, y and $\bar{m}_{ax}, \bar{m}_{by}$, in function of M.

There are three subcases.

Subcase 1 : $\bar{m}_{ab} = \infty$. Since Ψ is 2-geometric, we have $H(s_a, s_b) = \alpha$, $H(s_b, s_a) = \beta$ and $(-H(s_a, s_b)) \cap (-H(s_b, s_a)) = \emptyset$ by (3.3). Since $r_h s_b$ has finite order, we have $H(s_a, s_b) = H(s_a, r_h)$ by the last statement of (3.3). Therefore, we have $\alpha = H(s_a, s_b) = H(s_a, r_h) = H(s_a, R) = \epsilon_a \alpha$ where the last equality follows from (7.2). Similarly, we have $\beta = \epsilon_b \beta$, which proves that $\epsilon_a = \epsilon_b = +$, and the result is given by (8.2).

Subcase 2 : $2 < \bar{m}_{ab} < \infty$. Let R' be a spherical residue of rank 2 of Σ which is stabilized by $\langle s_a, s_b \rangle$. Let $D = \bigcap_{i\in I} \psi_i$ and let $\phi_h := H(r_h, D)$ and $\phi_H := H(r_H, D)$. By (9.11) we know that $\phi_h \cap (-\phi_H)$ is a fundamental domain for the action of $\langle r_h, r_H \rangle$ on Σ.

We claim that, up to switching the names of a and b if necessary, *there exists a reflection $t \in \langle r_h, r_H \rangle$ which commutes with s_a, and such that* $o(s_b t) < \infty$. Let us assume for a while that this claim is proved. Then $o(s_a s_b s_a t) = o(s_b t)$ is finite. On the other hand, we have $o(s_a s_b s_a r_h) = o(s_a s_b s_a r_H) = \infty$ by (9.9). Therefore, if $R' \subset \phi_h \cap (-\phi_H)$, then $C(s_a s_b s_a)$ is completely contained in $\phi_h \cap (-\phi_H)$, and since the latter set is a fundamental domain for the group $\langle r_h, r_H \rangle$ which contains t, we see that there cannot exist a spherical residue of rank 2 which is stabilized by t and $s_a s_b s_a$. But this contradicts the fact that $o(s_a s_b s_a t)$ is finite. Similarly, the case $R' \subset (-\phi_h) \cap \phi_H$ yields a contradiction. This shows that $R' \subset (\epsilon\phi_h) \cap (\epsilon\phi_H)$ for some sign $\epsilon \in \{+, -\}$. Now we have $\epsilon_a \phi_h = H(r_h, D_a) = H(r_h, s_a) = \epsilon\phi_h$, where the second equality follows from **??**, and similarly we obtain $\epsilon_b \phi_H = \epsilon\phi_H$. This shows that $\epsilon_a = \epsilon_b = \epsilon$, and the conclusion follows from (8.2).

It remains to prove the claim. For that purpose, we consider the root basis $(V, b, (e_i)_{i\in\bar{I}})$ associated with $\bar{M}$. Up to switching the names of a and b if necessary, we may assume by (ii) and (9.5) that $y = 1$. Now, if $M = C_n$ or if $M = I_2(6)$, then an easy computation in (V, b) shows that $s_1 \in \langle r_h, r_H \rangle$. In that case the choice $t = s_1$ yields the claim. It remains to treat the case $M = F_4$. By (ii) and (9.5) we have $x = 4$ and $\bar{m}_{ax} = \bar{m}_{by} = 3$. On the other hand, an easy computation in (V, b) yields $r_H r_h r_H = s_1 s_2 s_3 s_2 s_1$, and we define $t = r_H r_h r_H$. Since $\bar{m}_{ai} = 2$ for $i \in \{1, 2, 3\}$, the reflections s_a and t commute. Moreover, since $\bar{M}_{\{b,1,2,3\}} = C_4$, the order of $s_b t$ is finite, and the claim follows.

Subcase 3 : $\bar{m}_{ab} = 2$. In that case, possibly by using repeated applications of (8.6), we are reduced to an almost spherical Coxeter diagram. The conclusion then follows from Lemma (7.2).

This concludes the proof in Case 1.

For the remaining cases, the strategy is to use fusions (namely (8.6)) in order to reduce the problem to cases which have previously been settled. The details are quite tedious, and are not systematically written down. The reader should keep in mind (9.5) as well as the table of (9.7) throughout the discussion.

Case 2: *$M = C_n$, $n \geq 4$, and $\bar{M}_{I\cup\{a\}}$ or $\bar{M}_{I\cup\{b\}}$ contains a subdiagram of type F_4.*

Without loss of generality, we may assume that $x = n$ (with the labelling of M as in (9.6)), and so $\bar{m}_{an} = 3$. Hence we have $\bar{M}_{\{n-2,n-1,n,a\}} = F_4$. By Case 1, the theorem is true for $M = F_4$. Therefore, possibly using repeated applications of (8.6), we are reduced to the situation of (8.4) or (7.2). In any case, the conclusions follows.

Case 3: $M \in \{E_6, E_7\}$.

First assume that $M = E_7$. Still with the labelling of M as in (9.6), we may assume by (9.5) and (8.5) that $x = 7$ and $4 \le \bar{m}_{ax} < \infty$. If $\bar{m}_{a7} = 4$ (whence $\bar{M}_{\{1,3,4,5,6,7,a\}} = C_7$) or if $\bar{m}_{ab} > 2$, then (8.4) gives the conclusion, using the fact that Case 1 and Case 2 are already settled. We may thus assume that $\bar{m}_{ab} = 2$ and $\bar{m}_{a7} \ge 5$. Now if $\bar{M}_{I \cup \{b\}}$ is a tree, with 1 as extension vertex, and if $\bar{m}_{b1} = 3$, then $\bar{M}_{\{a,b\} \cup (I \setminus \{6\})}$ is spherical, and the conclusion follows from (7.2). In any other case, the conclusion follows from (8.4), possibly by using also repeated applications of (8.6).

The case $M = E_6$ can be easily settled by similar arguments.

Case 4: $M = D_n$.

Here again, we use the labelling of M as in (9.6). Thanks to (8.5), we may assume that $x \in \{1, n-1, n\}$ and that $\bar{m}_{ax} < \infty$. As in Case 3, if $\bar{m}_{ax} = 4$ or if $\bar{m}_{ab} > 2$, then (8.4) gives the conclusion. If $\bar{m}_{ax} \ge 5$ and $\bar{m}_{ab} = 2$, then, possibly by using repeated applications of (8.6), we are reduced to the situation of (8.4) or to an almost spherical Coxeter diagram. Again, the conclusion follows.

Case 5: $M = A_n$.

By (8.5), we may assume that $n \ge 2$ and that $m_{ax} < \infty$. We keep the labelling of M as in (9.6). There are several subcases.

Subcase 1 : $n \in \{2, 3\}$. If $\bar{m}_{ab} > 2$, the conclusion follows from (8.4). Otherwise, possibly by applying (8.6) once or twice, we are reduced to the situation of (8.4) or to an almost spherical Coxeter graph. Again, the conclusion follows.

Subcase 2 : $n \ge 4$, and $x \in [2, n-1]$. If $\bar{m}_{ax} \ge 4$, then (8.4) yields the conclusion (again, some applications of (8.6) are possibly necessary), except if $\bar{m}_{ab} = 2$, $\bar{m}_{b1} = \bar{m}_{bn} = 3$ and $\bar{m}_{bi} = 2$ for $i \in [2, n-1]$. But in the latter case, $\bar{M}_{\{a,b\} \cup (I \setminus \{x\})}$ is spherical, and the conclusion then follows from (7.2). Finally, if $\bar{m}_{ax} = 3$, then $n \ge 7$ and $\bar{M}_{I \cup \{a\}}$ contains a subdiagram E_7 or E_8 which is maximal with respect to the property of being irreducible spherical. Therefore, once again by using repeated applications of (8.6) if necessary, we are reduced to the situation of (8.4), and the conclusion follows.

Subcase 3 : $n \ge 4$, and $x \in \{1, n\}$. Since $\bar{M}_{I \cup \{a\}}$ is non-spherical, we have $\bar{m}_{ax} \ge 5$. Here again, the conclusion follows from (8.4), possibly with repeated applications of (8.6).

This concludes the proof of the proposition. □

(10.5) *Proof of (1.6).* Let M be a Coxeter graph over a finite set I. Let (W_0, S_0) be any Coxeter system, let $\Sigma = \Sigma(W_0, S_0)$ and let Ψ be a universal 2-geometric set of roots in Σ such that $M(\Psi) = M$. By (6.1), we may assume that M is irreducible. We prove by induction on $|\Psi|$ that Ψ or $-\Psi$ is geometric, the result being true if M is spherical, thanks to (7.2). The conclusion follows.

Choose a subset Ψ_s of Ψ such that $M(\Psi_s)$ is irreducible and spherical, and assume that Ψ_s is maximal with respect to that property. Hence, for each $\pi \in \Psi \setminus \Psi_s$, the Coxeter diagram $M(\Psi_s \cup \{\pi\})$ is either non-spherical or reducible. Set $S = R(\Psi_s)$. Let

$$\Psi_\perp = \{\psi \in \Psi \setminus \Psi_s \mid r_\psi \text{ centralizes } S\}.$$

Finally, we set $\Psi_1 := \Psi \backslash (\Psi_s \cup \Psi_\perp)$. By (7.2), we may assume that $\Psi_1 \neq \emptyset$.

By (8.4), we may also assume, up to replacing Ψ by $-\Psi$, that $\Psi_s \cup \Psi_1$ is geometric.

Suppose now that $M(\Psi_\perp)$ is reducible, and let $\Psi_\perp^1, \ldots, \Psi_\perp^k$ be the subsets of $\Psi_\perp$ corresponding to the irreducible components of $M(\Psi_\perp)$. Since $M(\Psi)$ is irreducible, we deduce that $M(\Psi_s \cup \Psi_1 \cup \Psi_\perp^i)$ is irreducible for every $i \in [1, k]$. Hence, by induction, the set

$$D_i = \bigcap (\Psi_s \cup \Psi_1 \cup \Psi_\perp^i)$$

is not empty. In particular, the set $\Psi_\perp^i$ is geometric for each $i \in [1, k]$, and so $\Psi_\perp$ itself is geometric by (6.1). Now, the conclusion follows from (8.4).

From now on, we assume that $M(\Psi_\perp)$ is irreducible. There are two cases.

First, suppose that $|\Psi_1| \geq 2$. It is then possible to choose $\psi_1 \in \Psi_1$ such that $M(\Psi \backslash \{\psi_1\})$ is irreducible. Set $\Psi' := \Psi \backslash \{\psi_1\}$. By induction, there is a sign $\epsilon \in \{+, -\}$ such that $\epsilon \Psi'$ is geometric. By (7.2), this sign is unique, and the convention on Ψ taken at the beginning of the proof implies that $\epsilon = +$. In particular, $\bigcap \Psi_\perp$ is not empty, and so $\Psi_\perp$ is geometric. The conclusion follows again from Proposition (8.4).

In a second case, we suppose that $|\Psi_1| = 1$. The symbol ψ_1 now denotes the unique element of Ψ_1. Let ψ_2 be an element of $\Psi_\perp$. Set $\Psi" := \Psi \backslash \{\psi_2\}$. We may choose ψ_2 in such a way that $M(\Psi_2)$ is irreducible. By induction and (7.2), the set $\Psi"$ is geometric (and $-\Psi"$ is not). In particular, the set $\bigcap (\Psi_\perp \backslash \{\psi_2\})$ is not empty.

On the other hand, the induction implies also that $\Psi_\perp$ or $-\Psi_\perp$ is geometric. By (8.4), we may assume that $\Psi_\perp$ is not geometric, whence $-\Psi_\perp$ is geometric, and in particular, the set $\bigcap -(\Psi_\perp \backslash \{\psi_2\})$ is not empty. Now, by (7.1), this implies that $M(\Psi_\perp \backslash \{\psi_2\})$ is spherical. Choose now $\widetilde{\Psi_s} \subset \Psi$ in such a way that $\widetilde{\Psi_s}$ contains $\Psi_\perp \backslash \{\psi_2\}$, that $M(\widetilde{\Psi_s})$ is irreducible and spherical, and that it is maximal with respect to the previous property. If we now put $\widetilde{\Psi_\perp} := \{\psi \in \Psi | r_\psi \text{ centralizes } R(\widetilde{\Psi_s})\}$, then we have $\widetilde{\Psi_\perp} \subset \Psi_s$, whence $M(\widetilde{\Psi_\perp})$ is spherical. In that situation, (8.4) again yields the conclusion.

The proof is complete. □

Notice that Theorem (4.2) is an immediate consequence of (1.6) and (10.3).

11 Rigidity of circuits

In this section we prove that any universal set of reflections whose diagram is a 2-spherical circuit is geometric. An important step is to handle the case of 'large circuits of length 4'. In (11.5), we give a proof of the corresponding result which is due to Niels Mense [16].

(11.1) Lemma. *Let (W_0, S_0) be a Coxeter system and let T be a universal set of reflections, such that $M(T)$ is irreducible and spherical but not of type H_3, H_4 or $I_2(n)$ for $n = 5$ or $n \geq 7$. Then T is geometric.*

Proof. This is a consequence of [10] and the solution of the isomorphism problem for finite irreducible Coxeter groups. □

(11.2) Lemma. *Let (W_0, S_0) be a Coxeter system and let T be a universal 2-spherical set such that $|T| = 3$ and $M(T)$ is irreducible but not of type H_3. Then T is geometric.*

Proof. This is consequence of [10] (or alternatively [6]) and [18]. □

(11.3) Lemma. *Let (W_0, S_0) be a Coxeter system and let T be a universal 2-spherical set of reflections such that $|T| \geq 3$ and $M(T)$ is an extra-large tree (i.e. $o(st) > 2 \Rightarrow o(st) > 3$ for all $s, t \in T$). Then T is geometric.*

Proof. Since $M(T)$ is a tree it suffices to show that any pair of reflections in T is geometric, using the fact (1.6).

Let $s, t \in T$. If $o(st) = 2$ there is nothing to prove. Else, it follows from the hypotheses that there exists $u \in T$ such that $M(\{s, t, u\})$ is compact hyperbolic. By the main result of [6], the set $\{s, t, u\}$ is geometric and, therefore, so is $\{s, t\}$. □

(11.4) Extra-large circuits of length at least five.

In this paragraph we consider the following situation : (W_0, S_0) is a Coxeter system and $S = \{s_{k+1} = s_1, s_2, \ldots, s_k = s_0\}$ is a universal set of reflections of cardinality strictly greater than 4 such that its diagram is a 2-spherical circuit whose labels are at least 4.

Proposition. *The set S is geometric.*

Proof. Given a proper subset S' of S whose diagram is irreducible and whose cardinality is at least 3, then it is geometric by Proposition (11.3). Given any pair (s, S') consisting of a reflection s contained in an S' just described, we denote the unique root associated with s and contained in the geometric set of roots associated with S' by $H(s, S')$. Given two such sets S', S'' having cardinality 4, containing $s \in S$ and such that $S' \cap S''$ has cardinality 3, then $H(s, S') = H(s, S' \cap S'') = H(s, S'')$. It follows by a shifting argument that $H(s, X) = H(s, Y)$ for any two irreducible proper subsets of S of cardinality at least 3 and containing s. Now, for each $1 \leq i \leq k$ we put $H_i := H(s_i, X_i)$ where X_i is an irreducible set of cardinality 3 containing s_i. As one may choose the same subset X_i for s_i and s_{i+1}, it follows that the set $\{H_i \mid 1 \leq i \leq k\}$ is 2-geometric and hence geometric as it is universal. □

(11.5) Extra-large circuits of length four.

In this paragraph we consider the following situation : (W_0, S_0) is a Coxeter system and s, t, u, v are four reflection such that $R = \{s, t, u, v\}$ is a universal set and such that $o(vt) = o(su) = 2$ and the orders of all other products are finite and at least 4.

Proposition. *The set R is geometric.*

Proof. By Lemma (11.2) any 3-subset of R is geometric. Given 3 pairwise distinct reflections x, y, z in R, then $H_x(y, z)$ denotes the unique root associated with x that contains all spherical residues fixed by the group $\langle y, z\rangle$ (see Proposition (7.2)). We know by (11.2) and (7.2), that for any 3-subset $\{x, y, z\}$ of R, the set $\{H_x(y, z), H_y(z, x), H_z(x, y)\}$ is a geometric set of roots. It remains to show for all $x \in R$ that $H_x(y, z) = H_x(y, r) = H_x(z, r)$, where y, z, r denote the three reflections distinct from x in R. Suppose that this is not the case. Then we may assume without loss of generality that $H_v(t, u) \neq H_v(s, u)$. We will show in several steps that this yields a contradiction.

Claim 1: *The orders of $vutu, uvsv$ and $utuvsv$ are infinite.*

As R is supposed to be universal, this is an immediate consequence of the solution of the word problem.

Claim 2: $H(v, utu) = H_v(u, t)$.

Let Q be a spherical residue stabilized by the group $\langle u, t\rangle$. The, by definition, Q is contained in $H_v(u, t)$; at the same time Q is a spherical residue fixed by utu. As $H(v, utu)$ and $H_v(u, t)$ are both roots associated with v the claim follows.

Claim 3: $H(vsv, u) = H(vsv, utu)$.

This follows from the fact that the order of ut is finite.

Claim 4: $v(H_s(u,v)) = H(vsv, utu)$.

Observe first that $v(H_s(u,v))$ is a root associated with vsv. Let Q be a spherical residue fixed by $\langle u, v\rangle$; then Q is fixed by v and therefore contained in $v(H_s(u,v))$ (because Q is by definition in $H_s(u,v)$). Now Q is fixed by u and therefore $v(H_s(u,v)) = H(vsv, u)$; now, as tu has finite order and $vsvutu$ has infinite order it follows that $H(vsv, u) = H(vsv, utu)$ yielding the claim.

Claim 5: *The pair* $\{H(v, utu), H(vsv, utu)\}$ *is geometric.*

Since the pair $\{H_v(s,u), H_s(u,v)\}$ is geometric, so is the pair $\{v(H_v(s,u)), v(H_s(u,v))\}$. Now $v(H_v(s,u)) = -H_v(s,u) = H_v(u,t) = H(v, utu)$ where the second equality is our assumption and the third is Claim 2; by Claim 4 we have $v(H_s(u,v)) = H(vsv, utu)$ which yields the assertion.

The contradiction: $\{H(v, utu), H(vsv, utu)\}$ is a geometric pair of roots whose intersection C contains every finite residue stabilized by utu. On the other hand, as $s \in \langle vsv, v\rangle$ it follows by Lemma **??** that C is contained in the root $H(s, C)$ associated with s which contains C. As $sutu$ has finite order, there is a spherical residue Q stabilized by $\langle s, utu\rangle$. Thus $Q \subseteq H(s, C)$ and $s(Q) = Q$ which yields a contradiction. ☐

(11.6) Circuits.

In this paragraph we consider the following situation: (W_0, S_0) is a Coxeter system, $S \subseteq S_0^{W_0}$ is a universal set of reflections such that $M(S)$ is a circuit of length $k \geq 3$.

Proposition. *S is geometric.*

Proof. The proof goes by induction on k. If $k = 3$, then the assertion follows from Lemma (11.2). Suppose $k > 3$. If all labels of the diagram $M(S)$ are at least 4, then the we are done by the previous two paragraphs. If not, we find two reflections $s, t \in S$ such that the order of st is 3. Assume that S is not geometric. By (1.6) S is not 2-geometric (because it is assumed to be universal). By (5.3) $S \setminus \{s, t\} \cup \{sts\}$ is a universal set of reflections whose diagram is a 2-spherical circuit and which is not 2-geometric. This contradicts our induction hypothesis. ☐

(11.7) Proposition. *Let (W_0, S_0) be a Coxeter system and let $S \subseteq S_0^{W_0}$ be a universal set of reflections such that $M(S)$ is 2-spherical. Suppose that each 2-subset of S is geometric. Then S is geometric.*

Proof. By Proposition (6.1) we may assume that the diagram $M(S)$ is irreducible.

For any triple (s, t, α) consisting of two reflections $s, t \in S$ joined by an edge in the diagram $M(S)$ and a root α associated with s, we let $\alpha(t, s, \alpha)$ denote the unique root associated with t such that $\{\alpha, \alpha(t, s, \alpha)\}$ is a geometric pair of roots. Let $s \in S$ and choose a root α_0 associated with s. For any path $P : s = s_0, s_i, \ldots, s_k = t$ we define recursively the $\alpha_{i+1} := \alpha(s_i, s_{i+1}, \alpha_i)$ and $\alpha_{t,P} := \alpha_k$. As 2-spherical circuits are rigid it follows that $\alpha_{t,P}$ does not depend on the path P. Setting $\alpha_t := \alpha_{t,P}$ for a path P from s to t we find a 2-geometric set of roots associated with S. This shows that S is 2-geometric and therefore geometric by (1.6). ☐

12 Proof of the main result

In this section the term 'Coxeter system' always means 'Coxeter system of finite rank'. In order to prove the main theorem we use the following well known fact about Coxeter systems which can be easily seen from the geometric representation.

(12.1) Lemma. *Let (W,S) be a Coxeter system and $R \subseteq S$. Then $(\langle R\rangle, R)$ is a Coxeter system and $S^W \cap \langle R\rangle = R^{\langle R\rangle}$.*

The following proposition is an exercise in [1]. It can be proved by using the geometric representation (see for instance [15]). It follows also from the fact that the Davis complex has non-positive curvature ([7]).

(12.2) Proposition. *Let $(W,S))$ a Coxeter system and U a finite subgroup of W. Then there exists a spherical subset R of S such that $U^w \leq \langle R\rangle$ for some $w \in W$.*

We have the following consequence.

(12.3) Corollary. *Let (W,S) be a Coxeter system and let U be a subgroup of W. Then the following are equivalent:*
a) U is a maximal finite subgroup of W.
b) There exists a maximal spherical subset R of S such that $U^w = \langle R\rangle$ for some $w \in W$.

For a Coxeter system (W,S) we denote the set of all maximal spherical subsets of S by $\mathrm{Maxsph}(S)$; the set of conjugacy classes of maximal finite subgroups of W is denoted by $Conmaxfin(W)$. In view of the previous corollary and the main result of [3] we have the following:

(12.4) Lemma. *Let $\Sigma = (W,S)$ be a Coxeter system. Then the mapping γ_Σ defined by $R \mapsto \{\langle R\rangle^w \mid w \in W\}$ is a bijection from $\mathrm{Maxsph}(S)$ onto $Conmaxfin(W)$.*

(12.5) Proposition. *Let $\Sigma = (W,S), \Sigma' = (W',S')$ be Coxeter systems and let $\alpha : W' \to W$ be an isomorphism. Then there exists a bijection $\overline{\alpha} : \mathrm{Maxsph}(S') \to \mathrm{Maxsph}(S)$ such that $\langle \alpha(R')\rangle$ is conjugate to $\langle \overline{\alpha}(R')\rangle$ for all $R' \in \mathrm{Maxsph}(S')$. If $\alpha(S') \subseteq S^W$, then $M(R')$ and $M(\overline{\alpha}(R'))$ are isomorphic. In particular, if α is reflection-preserving, then there is a type-preserving bijection between the maximal spherical subsets of $M(S')$ and those of $M(S)$.*

Proof. The isomorphism α induces a canonical bijection α_1 from $Conmaxfin(W')$ onto $Conmaxfin(W)$. The bijection $\overline{\alpha}$ is obtained by setting $\overline{\alpha} := \gamma_\Sigma^{-1} \circ \alpha_1 \circ \gamma_{\Sigma'}$ where γ_Σ and $\gamma_{\Sigma'}$ are as in the previous lemma.

Suppose now that $\alpha(S') \subseteq S^W$. Let $R' \in \mathrm{Maxsph}(S')$ and $w \in W$ be such that $\langle \alpha(R')\rangle^w = \langle \overline{\alpha}(R')\rangle$. It follows that $\alpha(R')^w \subseteq S^W \cap \langle \overline{\alpha}(R')\rangle = \overline{\alpha}(R')^{\langle \overline{\alpha}(R')\rangle}$, where the second equality holds by Lemma (12.1). We define $\beta : \langle R'\rangle \to \langle \overline{\alpha}(R')\rangle$ by $x \mapsto \alpha(x)^w$. It follows that β is an isomorphism from $\langle R'\rangle$ onto $\langle \overline{\alpha}(R')\rangle$ with $\beta(R') \subseteq \overline{\alpha}(R')^{\langle \overline{\alpha}(R')\rangle}$. It follows now from Theorem 3.10 in [2] that $M(R') = M(\overline{\alpha}(R'))$. $\square$

(12.6) Lemma. *Let (W_0,S_0) be a Coxeter system and let $S \subseteq S_0^{W_0}$ be a universal set of reflections such that $M(S)$ is 2-spherical and irreducible. If $|S| \geq 3$ and $\{s,t\} \subseteq S$ is such that $o(st) \neq 5$, then $\{s,t\}$ is geometric.*

Proof. If $o(st) \in \{2,3,4,6\}$ then there is nothing to prove. If $o(st) \geq 7$, then we may choose $u \in S$ such that $M(\{s,t,u\})$ is irreducible. It follows from Lemma (11.2) that the set $\{s,t,u\}$ is geometric and hence $\{s,t\}$ is geometric as well. $\square$

(12.7) Definition. *Let $x \geq 2$ be an integer. We set*

$$Y_x := \begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 5 & 5 \\ 2 & 5 & 1 & x \\ 2 & 5 & x & 1 \end{pmatrix} \qquad \textit{and} \qquad Z_x := \begin{pmatrix} 1 & 5 & 2 & 2 \\ 5 & 1 & 3 & 3 \\ 2 & 3 & 1 & x \\ 2 & 3 & x & 1 \end{pmatrix}.$$

The Coxeter diagram associated with the Coxeter matrix Y_x (resp. Z_x) is also denoted by Y_x (resp. Z_x).

(12.8) Lemma. *Let (W_0, S_0) be a Coxeter system and let $S \subseteq S_0^{W_0}$ be a universal set of reflections such that $M(S)$ is 2-spherical and irreducible. Suppose that $|S| = 4$, that $M(S) \neq Z_x$ for $x \geq 3$ and that $M(S) \neq H_4$. Then S is geometric.*

Proof. In view Proposition (11.7) and the previous lemma it suffices to show that $\{s,t\} \subseteq S$ is geometric for all $s,t \in S$ with $o(st) = 5$. Let $\{s,t\}$ be such a pair. If there is $u \in S$ such that $M(\{s,t,u\})$ is irreducible and different from H_3, then it follows from Lemma (11.2) that $\{s,t,u\}$ is geometric and hence also $\{s,t\}$. Thus we are left with the case where there is no such u.

Let $S = \{s,t,u,v\}$. As $M(S)$ is irreducible we may assume now w.l.o.g. that $o(st) = 5, o(tu) = 3$ and $o(su) = 2$. We know also that $M(\{s,t,v\})$ is either reducible or equal to H_3. Hence we are left with the following possibilities:

1. $o(sv) = o(tv) = 2$
2. $o(sv) = 3$ and $o(tv) = 2$
3. $o(sv) = 2$ and $o(tv) = 3$

In the first case it follows that $o(uv) \geq 4$ because $M(S) \neq H_4$. Using Proposition (5.3) we see that $S' := \{s, utu = tut, v\}$ is a universal set of reflections such that $M(S')$ is irreducible and not equal to H_3. By Lemma (11.2) we see that S' is a geometric set of roots. Hence S is 2-geometric by Proposition (5.3).

In the second case it follows that $M(S)$ is either a circuit of length 4 or compact hyperbolic. In the first case the assertion follows from Proposition (11.6). In the second one uses the main result of [6].

In the third case it follows that $M(S)$ is either compact hyperbolic (and we use again [6]) or it is equal to Z_x for some $x \geq 3$. □

(12.9) Lemma. *Let $(W,S), (W',S')$ be Coxeter systems and let α be an isomorphism from W' onto W such that $\alpha(S') \subseteq S^W$. If $M(S) = Z_x$ for some $x \geq 3$, then $M(S') = M(S)$.*

Proof. By Theorem 3.8 in [2] we have $|S'| = 4$. There are precisely 3 maximal spherical subsets of S: two of type H_3 and one of type $I_2(x) \times A_1$. By Proposition (12.5) there are precisely 3 maximal spherical subsets of S' - two of type H_3 and one of type $I_2(x) \times A_1$. It follows that $M(S') = Y_x$ or $M(S') = Z_x$.

Suppose that $M(S') = Y_x$. Then $M(\alpha(S')) = Y_x$ and by the previous lemma we know that $\alpha(S')$ is geometric. As α is an isomorphism it follows that $\alpha(S')$ generates W. It follows from Lemma (3.5) that $\alpha(S')$ is conjugate to S in W and hence $M(S') = M(S)$ which yields a contradiction. We conclude that $M(S') = Z_x$. □

(12.10) Lemma. *Let (W_0, S_0) be a Coxeter system and let $S \subseteq S_0^{W_0}$ be a universal set of reflections such that $M(S) = Z_x$ for some $x \geq 3$. Then S is geometric.*

Proof. There exists a geometric set of refections $R \subseteq S_0^{W_0}$ such that $\langle R \rangle = \langle S \rangle$ and such that $S^{\langle S \rangle} \subseteq R^{\langle S \rangle}$ (see Proposition 4.2 in [18] and Lemma 3.7 in [2]). We have now two Coxeter systems $(\langle S \rangle, S)$ and $(\langle S \rangle, R)$ and the identity on $\langle S \rangle$ maps R onto a subset of $S^{\langle S \rangle}$. Using the previous lemma it follows that $M(R) = M(S)$. Hence there is an automorphism of $\langle S \rangle$ mapping S onto R. By Theorem 39 in [10] we know that this automorphism is inner by graph. Hence it can be written as a product of an automorphism which stabilizes the set S and an inner automorphism. Since geometric sets are sent to geometric set by conjugation and since R is geometric, we conclude that S is geometric. □

Combining the previous lemma with Lemma (12.8) we obtain the following proposition.

(12.11) Proposition. *Let (W_0, S_0) be a Coxeter system and let $S \subseteq S_0^{W_0}$ be a universal set of reflections such that $M(S)$ is 2-spherical and irreducible. Suppose that $|S| = 4$ and that $M(S) \neq H_4$. Then S is geometric.*

(12.12) Lemma. *Let (W_0, S_0) be a Coxeter system and let $S \subseteq S_0^{W_0}$ be a universal set of reflections such that $M(S)$ is 2-spherical and irreducible. Suppose that $|S| = 5$. Then S is geometric.*

Proof. In view of Proposition (11.7) and Lemma (12.6) it suffices to show that $\{s, t\} \subseteq S$ is geometric for all $s, t \in S$ with $o(st) = 5$. Let $\{s, t\}$ be such a set. If there exists an irreducible subset R of S of cardinality 4 containing $\{s, t\}$ and such that $M(R) \neq H_4$, then this set R is geometric by the previous proposition. Hence in this case we are done because subsets of geometric sets are geometric.

We are left with the case where there is no such subset R of S. In this case there exists (w.l.o.g) $u \in S$ such that $o(ut) = 3, o(us) = 2$ and such that $o(vs) = 2 = o(vt)$ for $v \in S \setminus \{s, t, u\}$. By Proposition (5.3) the set $R := \{s, utu = tut, v, v'\}$ is universal (where v, v' denote the elements in $S \setminus \{s, t, u\}$). If $M(R) = H_4$, then $M(S)$ is a compact hyperbolic Coxeter diagram and we use [6] to see that S is geometric. If $M(R) \neq H_4$, then it follows from the previous proposition that R is geometric. The conclusion then follows from Proposition (5.3). □

The main result (1.1) is an immediate consequence of the following more general statement.

(12.13) Theorem. *Let (W_0, S_0) be a Coxeter system and let $S \subseteq S_0^{W_0}$ be a universal set of reflections whose diagram M is 2-spherical. Suppose furthermore that M has no direct factors of type H_4, H_3 or $I_2(n)$ with $n = 5$ or $n \geq 7$. Then S is a geometric set.*

Proof. Let $\{s, t\} \subseteq S$. If $o(st) \in \{2, 3, 4, 6\}$ then $\{s, t\}$ is geometric.

If $o(st) \geq 7$, then it follows by our assumption that we can find $u \in S$ such that $M(\{s, t, u\})$ is irreducible, which implies that $\{s, t, u\}$ is geometric, by Lemma (11.2). This yields that $\{s, t\}$ is geometric in this case as well.

If $o(st) = 5$, then it follows that there is an irreducible non-spherical subset R of S having cardinality 3,4 or 5, which contains $\{s, t\}$. It follows by Lemma (11.2), Proposition (12.11) and Lemma (12.12) respectively that the set R is geometric in each case. Hence $\{s, t\}$ is geometric.

The claim follows now from Proposition (11.7) □

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