CAN AN ANISOTROPIC REDUCTIVE GROUP ADMIT A TITS SYSTEM?

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Abstract. Seeking for a converse to a well-known theorem by Borel–Tits, we address the question whether the group of rational points $G(k)$ of an anisotropic reductive k-group can admit a split spherical BN-pair. We show that if k is a perfect field or a local field, then such a BN-pair must be virtually trivial. We also consider arbitrary compact groups and show that the only abstract BN-pairs they can admit are spherical, and even virtually trivial provided they are split.

Dedicated to Jacques Tits in honour of his 80th birthday

1. Introduction

In a seminal paper [5], Armand Borel and Jacques Tits established — amongst other things — that the group $G(k)$ of k-rational points of a (connected) reductive linear algebraic k-group G always possesses a canonical BN-pair, where k is an arbitrary ground field. More precisely, they showed that if P is a minimal parabolic k-subgroup of G, and if N is the normalizer in G of some maximal k-split torus contained in P, then $(P(k), N(k))$ is a BN-pair for $G(k)$. This result constitutes a cornerstone in understanding the abstract group structure of the group of k-rational points $G(k)$. As an application, it yields for example the celebrated simplicity result of Tits [19]. Of course, the aforementioned BN-pair is trivial when G is **anisotropic** over k. (Abusing slightly the standard conventions, we shall say that G is anisotropic if it has no proper k-parabolic subgroup, *i.e.* if $P = G$. As is well-known, this definition coincides with the standard one in case G is semi-simple (see [4, 11.21])). In fact, the abstract group structure of $G(k)$ remains intriguing and mysterious to a large extent in the anisotropic case. In this context, we propose the following.

Conjecture (Converse to Borel–Tits)**.** *Let G be a reductive algebraic k-group which is anisotropic over k. Then every split spherical BN-pair for $G(k)$ is virtually trivial.*

Recall that a BN-pair (B, N) for a group G is called **spherical** if the associated Weyl group $W := N/T$ is finite, where $T := B \cap N$. It is said to be **split** if it is saturated (*i.e.* $T = \bigcap_{w\in W} wBw^{-1}$), and if there exists a nilpotent normal subgroup $U \lhd B$ such that $B \cong U \rtimes T$. This implies that the associated building enjoys the Moufang property (see *e.g.* [11]). The BN-pair is called **virtually trivial** if the associated building is finite or, equivalently, if B has finite index in G. The BN-pair $(P(k), N(k))$ for $G(k)$ described above is always split ([4, 14.19]). It is virtually trivial if and only if either k is finite or G is k-anisotropic. In particular, over infinite ground fields the conjecture can really be thought as a converse to the Borel–Tits theorem.

Besides the natural search for a converse to Borel–Tits, a motivation to consider the above conjecture is provided by the recent work of Peter Abramenko and Ken Brown [1], who constructed Weyl transitive actions on trees for certain anisotropic groups over global function fields. We refer to [2, Ch. 6] for more details on the relations and distinctions between BN-pairs, strong transitivity and Weyl transitivity.

Our first contribution concerns the special case when the ground field k is a local field. The k-anisotropy of G is then equivalent to the compactness of $G(k)$ (see [13]). In fact, our first step

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will be to establish the following two results, which concern arbitrary compact topological groups (not necessarily associated with algebraic groups).

Theorem 1. *Let G be a compact group. Then every BN-pair for G is spherical.*

Theorem 2. *Let G be a compact group. Then every split spherical BN-pair for G is virtually trivial.*

We emphasize that the BN-pairs appearing in these statements are *abstract*: The corresponding subgroups B and N are *not* supposed to be closed in G. Specializing to anisotropic groups over local fields, we deduce the following immediate corollary.

Theorem 3. *Let k be a local field and G be a connected semi-simple algebraic k-group which is anisotropic over k. Then:*
(1) Every BN-pair for $G(k)$ is spherical.
(2) Every split spherical BN-pair $G(k)$ is virtually trivial.

Finally, we consider the case of perfect ground fields.

Theorem 4. *Let k be a perfect field and G be a reductive algebraic k-group which is anisotropic over k. Then every split spherical BN-pair for $G(k)$ is virtually trivial.*

Notice that Theorems 3 and 4 are logically independent, since there exist local fields which are not perfect and vice-versa.

It would be very interesting to sharpen the conclusion of Theorems 3 and 4, that is, to show that, under suitable assumptions, the BN-pair must be trivial, and not only virtually trivial. However, we expect this to be quite difficult, since it is closely related to a conjecture due to Andrei Rapinchuk and Gopal Prasad (see [14]), which may be stated as follows: "*Let G be a reductive k-group which is anisotropic over k. Then, every finite quotient of $G(k)$ is solvable.*" As of today, this conjecture was confirmed only when G is the multiplicative group of a finite dimensional division algebra (see [15]). We now sketch informally how these two problems are related.

On one side, if $G(k)$ possesses a BN-pair with finite associated building Δ, and if $K := \ker(G(k) \curvearrowright \Delta)$ is the kernel of the corresponding action, then $G(k)/K$ is a finite group whose action on Δ is faithful, and thus $G(k)/K$ is a finite group which possesses a faithful BN-pair. But these groups have been classified: they are simple Chevalley groups, and in particular are not solvable (up to two exceptions). Thus, if the BN-pair for $G(k)$ were nontrivial, there would exist (modulo these two exceptions) a non-solvable finite quotient of $G(k)$.

Conversely, suppose that $G(k)$ possesses a nontrivial and non-solvable finite quotient $F' := G(k)/K$. Let $R \lneq F'$ be the solvable radical of F', that is, its largest solvable normal subgroup. Going to the quotient $F := F'/R$, we thus know that $G(k)$ surjects onto a nontrivial finite group with trivial solvable radical (namely, F). Let now M be a minimal normal subgroup of F. Then M is a direct product of non-Abelian simple groups which are pairwise isomorphic, say $M \cong S_1 \times \cdots \times S_k$ with $S_i \cong S$ for all $i \in \{1, \ldots, k\}$. By the classification of finite simple groups, S is very likely to be a Chevalley group. Such a group possesses a root datum, and thus also a nontrivial BN-pair whose associated (finite) building is in bijection with S/B. Repeating this construction for each S_i, we then get a finite building $\Delta = \Delta_1 \times \cdots \times \Delta_k$ on which $M = S_1 \times \cdots \times S_k$ acts strongly transitively. Finally, the action of $\mathrm{Aut}(M)$ on the set of p-Sylow subgroups of M (where $p = \mathrm{char}\, k$) induces an action of $\mathrm{Aut}(M)$ on Δ making the diagram

$$\begin{tikzcd} F \arrow[r, "\alpha"] & \mathrm{Aut}(M) \arrow[d] \\ M \arrow[u, "\iota"] \arrow[r, "\text{strongly tr.}"'] & \mathrm{Aut}(\Delta) \end{tikzcd}$$

commute, where $\alpha(f)$ denotes the conjugation by f for all $f \in F$. In particular, we get a strongly transitive action of F, and thus also of $G(k)$, on the finite building Δ. This yields a nontrivial and virtually trivial BN-pair for $G(k)$.

General conventions. All algebraic groups considered here are supposed to be affine, all topological groups are assumed Hausdorff and all BN-pairs have finite rank.

2. Proof of Theorem 1

2.1. **Heuristic sketch.** Let G be a compact group and let (B, N) be a BN-pair for G. Let also Δ be the associated building. We consider the Davis realization of Δ, noted $|\Delta|_{\mathrm{CAT}(0)}$ in this paper, and which is a complete CAT(0) space, as well as a simplicial complex, on which G acts by simplicial isometries. The key step in the proof of Theorem 1 is to establish that this action is elliptic (Theorem 2.5 below). To do so, we use a result of Martin Bridson stating that such an action is always semi-simple, and we then argue by contradiction, assuming that G possesses an element with no fixed point. Such an element would then generate a subgroup Q of G which acts by translations on $|\Delta|_{\mathrm{CAT}(0)}$. Moreover, the structure of simplicial complex of $|\Delta|_{\mathrm{CAT}(0)}$ implies that the set of translation lengths of the elements of Q is discrete at 0. The contradiction now comes from divisibility properties of compact and procyclic groups, which we apply to Q.

2.2. **Procyclic groups.** Let G be a profinite group. Recall that G is said to be **procyclic** if there exists a $g \in G$ such that the subgroup generated by g is dense in G, that is, $G = \overline{\langle g \rangle}$. Moreover G is said to be **pro-p** for some prime p if every finite Hausdorff quotient of G is a p-group.

The following basic properties of procyclic groups can be found in [16, 2.7]. The symbol $\mathbb{P}$ denotes the set of all primes.

Proposition 2.1. *Let G be a procyclic group. Then,*

(i) G is the direct product $G = \prod_{p \in \mathbb{P}} G_p$ of its p-Sylow subgroups, and each G_p is a pro-p procyclic group.

(ii) G is, in a unique way, a quotient of $\hat{\mathbb{Z}} := \prod_{p \in \mathbb{P}} \mathbb{Z}_p$. If G is pro-p for some $p \in \mathbb{P}$, then it is a quotient of $\mathbb{Z}_p$.

2.3. **Divisible groups.** Recall that an element $g \in G$ is said to be **n-divisible** for some $n \in \mathbb{N}$ if there exists an $h \in G$ such that $h^n = g$. We say that g is **divisible** if it is n-divisible for each $n \geq 1$. The group G is called **n-divisible** (respectively **divisible**) when all its elements are.

Now, every prime q different from p is invertible in $\mathbb{Z}_p$ since its p-adic valuation is zero. Hence, the additive group $\mathbb{Z}_p$ is q-divisible for each $q \in \mathbb{P} \setminus \{p\}$. In particular, Proposition 2.1 implies that if a procyclic group G has trivial q-Sylow subgroups, then G is q-divisible.

We conclude this paragraph by stating the following characterization of divisibility for compact groups (see [12, Corollaire 2]).

Proposition 2.2. *Let G be a compact topological group. Then, G is divisible if and only if it is connected.*

2.4. **Semi-simple actions on CAT(0) spaces.** Let G be a group acting on a metric space (X, d). For every $g \in G$, we define the **translation length** of g by $|g| := \inf\{d(x, g \cdot x) \mid x \in X\} \in [0, \infty)$ and the **minimal set** of g by $\mathrm{Min}(g) := \{x \in X \mid d(x, g \cdot x) = |g|\}$. An element $g \in G$ is said to be **semi-simple** when $\mathrm{Min}(g)$ is nonempty. In that case, we say that g is **elliptic** if it fixes some point, that is, if $|g| = 0$; otherwise, if $|g| > 0$, we call g **hyperbolic**.

A **geodesic line** (respectively, **geodesic segment**) in X is an isometry $f \colon \mathbb{R} \to X$ (respectively, $f \colon [0; 1] \to X$); by abuse of language, we will identify f with its image in X.

The following lemma follows from Proposition 2.4 in [6].

Lemma 2.3. *Let (X, d) be a complete CAT(0) metric space, and let C be a closed convex nonempty subset of X. Then:*

(i) For every $x \in X$, there is a unique $y \in C$ such that $d(x, y) = d(x, C)$, where $d(x, C) := \inf_{z \in C} d(x, z)$. We call y the **projection** *of x on C and we write $y = \mathrm{proj}_C\, x$.*

(ii) For all $x_1, x_2 \in X$, we have $d(\mathrm{proj}_C\, x_1, \mathrm{proj}_C\, x_2) \leq d(x_1, x_2)$.

Suppose now that (X, d) is a cell complex. We then say that G acts by **cellular isometries** on X if it preserves the metric, as well as the cell decomposition of X.

The following result is due to Martin Bridson [7].

Proposition 2.4. *Let X be a locally Euclidean CAT(0) cell complex with finitely many isometry types of cells, and G be a group acting on X by cellular isometries. Then every element of G is semi-simple. Moreover,* $\inf\{|g| \neq 0 \mid g \in G\} > 0$.

We now establish the following result, which is the key ingredient for the proof of Theorem 1:

Theorem 2.5. *Let X be a locally Euclidean CAT(0) cell complex with finitely many isometry types of cells, and G be a compact group acting on X by cellular isometries (not necessarily continuously). Then every element of G is elliptic.*

Proof. Suppose for a contradiction there exists a $g \in G$ without fixed point. Proposition 2.4 then implies that g is hyperbolic. Let $Q = \overline{\langle g \rangle}$ be the closure of the subgroup generated by g in G. So, Q is compact.

Claim 1: *Q is Abelian.*

This is clear since it contains a dense Abelian (in fact cyclic) subgroup.

Claim 2: *For every $h \in Q$, the minimal set $\mathrm{Min}(h)$ is a closed convex subset of X which is stabilized by Q.*

This follows from [6, Proposition II.6.2].

Claim 3: *For every $h \in Q$ and every nonempty closed convex subset C of X stabilized by Q, the set $C \cap \mathrm{Min}(h)$ is nonempty.*

Note first that $\mathrm{Min}(h)$ is nonempty by Proposition 2.4. Let $x \in \mathrm{Min}(h)$ and consider the projections $y := \mathrm{proj}_C\, x$ and $z := \mathrm{proj}_C\, hx$ provided by Lemma 2.3. Since $hC = C$, we then obtain

$$d(x,y) = \inf_{c\in C} d(x,c) = \inf_{c\in C} d(hx,hc) = \inf_{c\in C} d(hx,c) = d(hx,z).$$

Hence $d(hx,hy) = d(x,y) = d(hx,z)$, and so $z = hy = \mathrm{proj}_C\, hx$ by uniqueness of projections. Since in addition $d(y,z) \leq d(x,hx) = |h|$ by Lemma 2.3, we finally get $d(y,hy) = |h|$ and therefore $y \in C \cap \mathrm{Min}(h)$.

Claim 4: *For all $h_1, h_2 \in Q$, the set $\mathrm{Min}(h_1) \cap \mathrm{Min}(h_2)$ is nonempty.*

As $\mathrm{Min}(h_1)$ and $\mathrm{Min}(h_2)$ are nonempty by Proposition 2.4, the claim follows from Claims 2 and 3.

Claim 5: *Let $h \in Q$ and let C be a nonempty closed convex subset of X stabilized by Q. We may thus consider the action of h on C. Denote by $|h|_C$ the translation length of h for this action. Then, h is semi-simple in C and $|h| = |h|_C$.*

Claim 3 yields that if $x \in \mathrm{Min}(h)$, then $y := \mathrm{proj}_C\, x \in \mathrm{Min}(h)$. Since $\mathrm{Min}(h)$ is nonempty by Proposition 2.4, the claim follows.

Claim 6: *For every $h \in Q$ and $n \geq 1$, we have $|h^n| = n|h|$.*

By Claim 4, we may choose an $x \in \mathrm{Min}(h) \cap \mathrm{Min}(h^n)$. Note that h is elliptic (respectively hyperbolic) if and only if h^n is so (see [6, II.6.7 and II.6.8]). In particular, if h is hyperbolic, then x belongs to some h-axis, which is also an h^n-axis. In any case, we obtain $d(x,h^n x) = nd(x,hx)$, whence $|h^n| = d(x,h^n x) = nd(x,hx) = n|h|$.

Claim 7: *Every divisible element of Q is elliptic.*

Let $h \in Q$ be divisible and suppose for a contradiction it is not elliptic. Then h is hyperbolic by Proposition 2.4. For each natural number $n \geq 1$, choose an $h_n \in Q$ such that $h_n^n = h$. In particular, all h_n are hyperbolic. Moreover, $|h_n^n| = n|h_n|$ by Claim 6. Therefore, we obtain a sequence (h_n) of elements of Q such that $|h_n| = |h|/n > 0$, contradicting the second part of Proposition 2.4.

We now establish the desired contradiction to the hyperbolicity of g. First note that the component group $P := Q/Q^0$ of Q is a profinite group. In fact, it is even procyclic, since the subgroup generated by the projection of g in P is dense in P, the natural mapping $\pi\colon Q \to Q/Q^0$ being continuous. In particular, it follows from Proposition 2.1 that P is the product of its p-Sylow subgroups P_p. Moreover, each P_p is a pro-p group and is therefore q-divisible for every $q \in \mathbb{P} \setminus \{p\}$. For each $p \in \mathbb{P}$, let Q_p be the subgroup of Q which is the pre-image of P_p under π.

Claim 8: *If $h, a, d \in Q$ with $ha = d^n$ for some $n \geq 1$ and a is elliptic, then $|h| = n|d|$.*

Write $C := \mathrm{Min}(h) \cap \mathrm{Min}(a)$. Then C is nonempty by Claim 4. Since d^n stabilizes C, Claim 5 implies that it is semi-simple in C with translation length $|d^n|_C = |d^n|$. Thus, $|d^n|_C = |d^n| = n|d|$ by Claim 6. Note also that ha is semi-simple in C with translation length $|ha|_C = |h|$. Therefore, $|h| = |ha|_C = |d^n|_C = n|d|$, as desired.

Claim 9: *Let $h \in Q$ be hyperbolic. Suppose that $ha_i = d_i^{n_i}$ for all $i \geq 1$, where $a_i, d_i \in Q$, each a_i is elliptic and where $n_i \geq 1$. Then the set $\{n_i \mid i \geq 1\}$ is bounded.*

Indeed, by Claim 8, the sequence (d_i) of elements of Q is such that $|d_i| = |h|/n_i > 0$. The claim now follows from the second part of Proposition 2.4.

Claim 10: *Let $p \in \mathbb{P}$. Then all elements of Q_p are elliptic.*

Suppose for a contradiction there exists an $h \in Q_p$ which is not elliptic, and is thus hyperbolic by Proposition 2.4. Let $q \in \mathbb{P} \setminus \{p\}$. Since $P_p = \pi(Q_p)$ is q-divisible, there exists an $h_q \in Q$ such that $h_q^q Q^0 = hQ^0$. Let $a \in Q^0$ such that $ha = h_q^q$. By Proposition 2.2, since Q^0 is compact and connected, it is divisible, and so a is elliptic by Claim 7. Since the set of natural prime numbers distinct from p is unbounded, the desired contradiction now comes from Claim 9.

Let now $gQ^0 = (g_p)_{p \in \mathbb{P}}$ be the decomposition of $\pi(g)$ in $P = \prod_{p \in \mathbb{P}} P_p$ (that is, each $g_p \in P_p$). Let $q \in \mathbb{P}$, and choose an $a_q \in Q_p$ such that $\pi(a_q) = g_q^{-1}$. Then $\pi(ga_q)$ has no component in the q-Sylow of P, and is therefore q-divisible in P. Hence, there exist an $h_q \in Q$ and an $a \in Q^0$ such that $ga_q a = h_q^q$. By Claim 10, we know that a_q is elliptic. But so is a, and hence the product $a' := a_q a$ is also elliptic by Claim 4. Since q is an arbitrary prime, Claim 9 again yields the desired contradiction. □

2.5. **The Davis realization of a building.** We recall from [10] that any building Δ admits a metric realization, denoted by $|\Delta|_{\mathrm{CAT}(0)}$, which is a locally Euclidean CAT(0) cell complex with finitely many types of cells. Moreover any group of type-preserving automorphisms of Δ acts in a canonical way by cellular isometries on $|\Delta|_{\mathrm{CAT}(0)}$. Finally, the cell supporting any point of $|\Delta|_{\mathrm{CAT}(0)}$ determines a unique spherical residue of Δ. In particular, an automorphism of Δ which fixes a point in $|\Delta|_{\mathrm{CAT}(0)}$ must stabilize the corresponding spherical residue in Δ.

Here is a reformulation of Theorem 1.

Theorem 2.6. *Let G be a compact group acting strongly transitively by type-preserving automorphisms on a thick building Δ. Then, Δ is spherical.*

Proof. Let (W, S) be the Coxeter system associated to Δ, and let Σ be the fundamental apartment of Δ. Then, the action of the stabilizer in G of Σ can be identified with the action of W on this apartment ([20, 2.8]).

Claim 1: *$|\Sigma|_{\mathrm{CAT}(0)}$ is a closed convex subset of $|\Delta|_{\mathrm{CAT}(0)}$.*

A basic fact about buildings is the existence, for each pair (Σ, C) consisting of an apartment Σ and of a chamber $C \in \Sigma$, of a *retraction of Δ onto Σ centered at C*, that is, of a simplicial map $\rho = \rho_{\Sigma,C} \colon \Delta \to \Sigma$ preserving minimal galleries from C and such that $\rho_{|\Sigma} = \mathrm{id}_{|\Sigma}$. The induced mapping $\overline{\rho} \colon |\Delta|_{\mathrm{CAT}(0)} \to |\Sigma|_{\mathrm{CAT}(0)}$ then maps every geodesic segment of $|\Delta|_{\mathrm{CAT}(0)}$ onto a piecewise geodesic segment of $|\Sigma|_{\mathrm{CAT}(0)}$ of same length. In particular, the mapping $\overline{\rho}$ is distance decreasing (see [10, Lemme 11.2]). Hence, if x and y are two points in $|\Sigma|_{\mathrm{CAT}(0)}$, then the geodesic segment from x to y is entirely contained in $|\Sigma|_{\mathrm{CAT}(0)}$ since its image by $\overline{\rho}$ is also a geodesic from x to y. This proves that $|\Sigma|_{\mathrm{CAT}(0)}$ is convex. To see it is closed, it suffices to note that it is complete as a metric space since it is precisely the Davis realization of the building Σ.

Claim 2: *If $g \in G$ is elliptic in $X = |\Delta|_{\mathrm{CAT}(0)}$ and stabilizes $|\Sigma|_{\mathrm{CAT}(0)}$, then g is also elliptic in $|\Sigma|_{\mathrm{CAT}(0)}$.*

This follows from Claim 5 in the proof of Theorem 2.5.

Theorem 2.5 now implies that the induced action of W on $|\Sigma|_{\mathrm{CAT}(0)}$ is elliptic, that is, every $w \in W$ is elliptic. Notice that the W-action on $|\Sigma|_{\mathrm{CAT}(0)}$ is proper, since by construction, it is cellular and the stabilizer of every point is a spherical (in particular finite) parabolic subgroup of W. Recalling now that every infinite finitely generated Coxeter group contains elements of infinite order (in fact, so do all finitely generated infinite linear groups by a classical result of Schur [17]; in the special case of Coxeter groups, a direct argument may be found in [2, Proposition 2.74]), we deduce that W is finite. In other words Δ is spherical. □

3. Proof of Theorem 2

3.1. **Heuristic sketch.** Let G be a compact group possessing a split spherical BN-pair, and let Δ be the associated building. We first establish Theorem 2 when G acts continuously on Δ. In that case, 2-transitive actions (which are closely related to strongly transitive actions) of G on subspaces X of Δ are easily seen to be possible only for finite X. The second step is then to show that the action of G on Δ has to be continuous. This uses the fact that buildings arising from split spherical BN-pairs are Moufang (see Proposition 3.3 below).

3.2. **Continuous actions on buildings.** Recall that a topological space X is said to satisfy the T_1 separation axiom when all its singletons are closed. The following is probably well-known.

Lemma 3.1. *Let G be a compact group. If G admits a continuous 2-transitive action on a T_1 topological space X, then X is finite.*

Proof. Define $Y := \{(x, y) \in X \times X \mid x \neq y\} \subset X \times X$, and fix $x, y \in X$ with $x \neq y$. Since the orbit map $\alpha_x \colon G \to X : g \mapsto g \cdot x$ is continuous, so is $\alpha_x \times \alpha_y \colon G \to X \times X : g \mapsto (g \cdot x, g \cdot y)$. By 2-transitivity, we get $Y = (\alpha_x \times \alpha_y)(G)$, and so Y is compact.

Note also that the mapping $f \colon X \times X \to X \times X : (a, b) \mapsto (x, b)$ is continuous. Setting $Z := X \setminus \{x\}$, we then get $Z \times \{x\} = f^{-1}(\{(x, x)\}) \cap Y$, so that $Z \times \{x\}$ is closed in Y, and hence compact. It follows that Z is compact, being the image of $Z \times \{x\}$ by the projection on the first factor $X \times X \to X$, which is of course continuous.

In particular, Z is closed, and hence $\{x\}$ is open. It follows that X is discrete, and therefore finite since $X = \alpha_x(G)$ is compact. □

Let Δ be a building of type (W, S), and denote by $\mathrm{Ch}\,\Delta$ the set of its chambers. Consider the chamber system Γ of Δ, which is the labelled graph with vertex set $\mathrm{Ch}\,\Delta$ and with an edge labelled by $s \in S$ for each pair of s-adjacent chambers of Δ (see [8, Ch.I Appendix D]). Let $J \subset S$. A **J-gallery** in Γ between two chambers x and y of Δ is a sequence $(x = x_0, x_1, \dots, x_l = y)$ of chambers of Δ such that for each $i \in \{1, \dots, l\}$, there exists an $s \in J$ such that x_{i-1} is s-adjacent to x_i. The natural number l is called the **length** of the gallery. A **minimal** gallery is a gallery of minimal length. The **distance** in Δ between two chambers $x, y \in \mathrm{Ch}\,\Delta$ is the length of a minimal gallery joining x to y. The **diameter** of Γ is the supremum (in $\mathbb{N} \cup \{\infty\}$) of the distances between its vertices.

Let $J \subset S$. The **J-residue** $R = R_J(x)$ of some chamber $x \in \mathrm{Ch}\,\Delta$ is the set of chambers of Δ which are connected to x by a J-gallery. When J has cardinality 1, we call R a **panel**.

In this paper, we will say that a group G acts **continuously** on Δ if the stabilizers of the residues of Δ are closed in G. Note that we can of course restrict our attention to the maximal proper residues, the others being obtained as intersections of those.

Lemma 3.2. *Let G be a compact group acting continuously and strongly transitively by type-preserving automorphisms on a spherical thick building Δ. Then Δ is finite.*

Proof. The stabilizer H in G of a panel P of Δ is a closed and thus compact subgroup of G.

Claim 1: *H acts 2-transitively on* $\mathrm{Ch}(P)$.

Indeed, let C be a chamber of P and let $B := \mathrm{Stab}_G(C) \subset H$. We first show that B, and thus also H, is transitive on the set $\mathcal{C} = P \setminus \{C\}$. Let $C_1, C_2 \in \mathcal{C}$ and let Σ_1 (respectively, Σ_2) be an apartment containing C and C_1 (respectively, C and C_2). By strong transitivity, B is transitive on the set of apartments containing C, and so there exists a $b \in B$ such that $b\Sigma_1 = \Sigma_2$. Hence

$bC_1 = C_2$. It now remains to show that H is transitive on P. But if $C_1, C_2 \in P$, then since Δ is thick, we may choose a chamber C in P different from C_1, C_2. The stabilizer B' of C in G is then contained in H and is transitive on $P \setminus \{C\}$ by the previous argument.

Now, identifying Δ with $\Delta(G, B)$, so that $H = B \cup BsB$ for some generator s of the corresponding Weyl group, we get a 2-transitive, continuous action by left translation of the compact group H on the topological space H/B. Moreover, this space is T_1 since B is closed in G by hypothesis. Lemma 3.1 then implies that P is finite. In other words, as P was arbitrary, the building Δ is *locally finite*, that is, every panel is finite. The following observation now allows us to conclude:

<u>Claim 2</u>: *Every locally finite spherical building is finite.*

Indeed, let $\Gamma = \mathrm{Ch}\,\Delta$ be the graph whose vertices are the chambers of Δ, and such that two chambers of Δ are adjacent if they share a common panel. Since Δ is locally finite, so is Γ. Hence, fixing a vertex $x \in \Gamma$, each ball in Γ centered at x with radius n ($n \in \mathbb{N}$) possesses a finite number of vertices. Moreover, as Δ is spherical, the diameter of Δ is finite ([8, Ch.IV, 3]), and hence the diameter of Γ is also finite. Thus Γ is contained in such a ball, and is therefore finite. $\square$

3.3. **Moufang buildings.** Let $\Delta = \Delta(G, B)$ be the building associated to a split spherical BN-pair ($B = T \ltimes U, N$) of type (W, S). It is well-known (see the main result of [11]) that the existence of a splitting for the above BN-pair is equivalent to the fact that the building Δ enjoys the Moufang property, as defined in [20, Chapter 11].

Two chambers $x, y \in \mathrm{Ch}\,\Delta$ are called **opposite** if they are at maximal distance in the chamber system of Δ. Similarly, one can define *opposite residues* (see for instance [2, 5.7]). The set of chambers (respectively, residues) of Δ which are opposite to a given chamber C (respectively, residue R) will be denoted by C^{op} (respectively, R^{op}).

Proposition 3.3. *Let $P = BW_JB$ be a proper standard parabolic subgroup of $\Delta = \Delta(G, B)$ for some proper subset J of S, let C be the fundamental chamber (i.e. the unique chamber fixed by B) and let R be the unique J-residue containing C. Define the subgroup $V := \bigcap_{p \in P} pUp^{-1}$ of G. Then V acts simply transitively on R^{op}.*

Proof. Let Σ be an apartment containing C. By [20, 9.11], there exists a minimal galery $\gamma_{R'}$, one for each residue $R' \in R^{\mathrm{op}}$, beginning at C and ending at a chamber C' in R' such that the type of $\gamma_{R'}$ is independent of the choice of R' and $C = \mathrm{proj}_R\, C'$. Let $R' \in R^{\mathrm{op}}$ be the unique residue of Σ opposite R and let C' be the last chamber of $\gamma_{R'}$. Let also α be a root of Σ containing C but not C'. By [20, 8.21], $R \cap \Sigma \subset \alpha$. By [20, 9.7], therefore, R is fixed pointwise by the root group U_α. Since P maps R to itself, we have $C \in R \subset \alpha^p$ and hence $p^{-1}U_\alpha p \subset U$ for all $p \in P$ by the definition of root subgroups (see [20, 11.1]) and the fact that the 'radical' U does not depend on the choice of the apartment Σ (see [20, Proposition 11.11(iii)]). Thus $U_\alpha \subset V$. Now, as in [2, 7.67], one shows that the subgroup of V generated by all U_α's of the latter form acts transitively on the set $\{\gamma_{R''} \mid R'' \in R^{\mathrm{op}}\}$, and hence also on R^{op}.

Suppose $h \in V$ maps $R' \in R^{\mathrm{op}}$ to itself. Then h acts trivially on R. Since the restriction of $\mathrm{proj}_{R'}$ to R is a bijection from R to R' (by [20, 9.11] again), it follows that h acts trivially on R'. By [20, 9.8], therefore, h fixes two opposite chambers of Σ and hence h fixes Σ. By [20, 9.7] again, we conclude that $h = 1$. $\square$

In particular, we have the following (compare [8, Ch.IV, 5]).

Lemma 3.4. *Let C be the fundamental chamber of Δ. Then U acts simply transitively on C^{op}. Equivalently, U acts simply transitively on the set of apartments containing C.*

Lemma 3.5. *Let $P = BW_JB$ be a proper standard parabolic subgroup of $\Delta = \Delta(G, B)$ for some proper subset J of S, let C be the fundamental chamber and let R be the unique J-residue containing C. Then there exist two chambers in C^{op} which are opposite to one another. In particular, $|R^{\mathrm{op}}| \geq 2$.*

Proof. The first assertion holds by [2, Proposition 4.104] and the second follows since no proper residue contains two opposite chambers. $\square$

We are now ready to complete the proof of Theorem 2.

Theorem 3.6. *Let G be a compact topological group possessing a spherical split BN-pair ($B = T \ltimes U, N$). Then the associated building is finite.*

Proof. Let $\Delta = \Delta(G, B)$ be the building associated to (B, N), and let (W, S) be the corresponding Coxeter system.

We start with some basic observations in the case (W, S) is not irreducible. Suppose thus that S decomposes as $S = S_1 \amalg S_2$ with $s_1 s_2 = s_2 s_1$ for all $s_1 \in S_1$ and $s_2 \in S_2$. Then W splits as a direct product $W \cong W_1 \times W_2$, where $W_i = \langle S_i \rangle$, and the building Δ decomposes canonically as a product $\Delta = \Delta_1 \times \Delta_2$ of buildings of type (W_1, S_1) and (W_2, S_2) respectively (see [20, Proposition 7.33]).

In particular, we obtain induced actions of G on both Δ_1 and Δ_2, which are obviously strongly transitive. The corresponding BN-pairs for G may be described as follows. Since each $s \in S$ can be written as a coset $nT \in N/T = W$, we may choose, for $i = 1, 2$, a set $\overline{N}_i$ of representatives in N for the elements of S_i. For each $i = 1, 2$, consider now the subgroup N_i of N generated by $\overline{N}_i$ and T, and set $B_i := \langle B \cup N_{3-i} \rangle = BN_{3-i}B \leq G$. Then (B_i, N_i) is a spherical BN-pair for G, and the associated building is nothing but $\Delta_i = \Delta(G, B_i)$.

We claim that the BN-pair (B_i, N_i) is split. This follows readily from the aforementioned equivalence between splittings of BN-pairs and the Moufang property for the associated buildings. More precisely, consider the group $U_i = \bigcap_{g \in B_i} gUg^{-1}$ which is the kernel of the U-action on Δ_{3-i}. Then U_i acts sharply transitively on the chambers of Δ_i which are opposite the standard chamber C, which by definition is the unique chamber fixed by B_i. Therefore we have $B_i \cong T_i \ltimes U_i$, where $T_i = \bigcap_{w \in W_i} wB_iw^{-1}$, and U_i induces a splitting of the BN-pair (B_i, N_i) as claimed.

This shows that the given split BN-pair for G yields various split BN-pairs for G corresponding to the various irreducible components of Δ. Since $\operatorname{Ch}\Delta$ is naturally in one-to-one correspondence with the Cartesian product $\operatorname{Ch}\Delta_1 \times \cdots \times \operatorname{Ch}\Delta_n$ of the chamber sets of the various irreducible components of Δ, the desired finiteness result readily follows provided we establish it for each irreducible BN-pair (B_i, N_i) as above. In other words, there is no loss of generality in assuming that the building Δ is irreducible. We adopt henceforth this additional assumption.

Let now $\mathcal{P}$ denote the set of maximal proper standard parabolic subgroups of G. Pick any $P \in \mathcal{P}$. Thus P is of the form $P = BW_JB$ for some maximal subset $J \subsetneq S$, where $W_J = \langle J \rangle$. In particular, P is a maximal subgroup of G (see [2, Lemma 6.43(1)]). Define the normal subgroup

$$V := \bigcap_{p \in P} pUp^{-1} \trianglelefteq P$$

of P. As V is contained in U, it is also nilpotent. Moreover, V acts faithfully on Δ. Indeed, the kernel $\ker(G \curvearrowright \Delta)$ of the action of G on Δ is obviously contained in the stabilizer of the chambers of the fundamental apartment Σ, that is, in $\bigcap_{w \in W} wBw^{-1} = T$, and so

$$V \cap \ker(G \curvearrowright \Delta) \subseteq U \cap T = \{1\}.$$

Now, since V is normal in P, we have $P \subseteq \mathscr{N}_G(V)$. Moreover, as the conjugation automorphism $\kappa_g \colon G \to G : x \mapsto gxg^{-1}$ is continuous, we get $\mathscr{N}_G(\overline{V}) \supseteq \mathscr{N}_G(V)$ and so $\mathscr{N}_G(\overline{V}) \supseteq P$. Hence, by maximality of P, we obtain that either $\mathscr{N}_G(\overline{V}) = P$ or $\mathscr{N}_G(\overline{V}) = G$.

<u>Claim</u>: $\mathscr{N}_G(\overline{V}) = P$ *for all* $P \in \mathcal{P}$.

Assume for a contradiction that $\mathscr{N}_G(\overline{V}) = G$ for some $P \in \mathcal{P}$. In other words, $\overline{V} \triangleleft G$. In particular, the center $\mathscr{Z}(\overline{V}) \subseteq \overline{V}$ of $\overline{V}$ is also a normal subgroup of G. Moreover, V is nontrivial since, by Proposition 3.3, it acts transitively on R^{op} and since $|R^{\mathrm{op}}| \geq 2$ by Lemma 3.5. As V is nilpotent, this implies that $\mathscr{Z}(V)$ is also nontrivial.

Now, using again the continuity of the conjugation automorphism κ_h (for $h \in G$), we see that $\mathscr{Z}(V) = \mathscr{Z}_G(V) \cap V$ is contained in $\mathscr{Z}(\overline{V}) = \mathscr{Z}_G(\overline{V}) \cap \overline{V}$. Moreover, as V acts faithfully on Δ, so does $\mathscr{Z}(V)$. This implies in particular that $\mathscr{Z}(V)$, and thus also $\mathscr{Z}(\overline{V})$, act nontrivially on Δ.

Tits' transitivity Lemma (see [8, Lemma 6.61]) then guarantees that the group $\mathscr{Z}(\overline{V})$ is transitive on the chambers of Δ. In fact, this action is even simply transitive. Indeed, the stabilizers in $\mathscr{Z}(\overline{V})$ of the chambers of Δ are all conjugate by transitivity. They are thus all equal since

$\mathscr{Z}(\overline{V})$ is Abelian, and are therefore contained in the kernel $\ker(G \curvearrowright \Delta)$ of the action of G on Δ. Since $\mathscr{Z}(V) \subseteq \mathscr{Z}(\overline{V})$, this implies that the action of $\mathscr{Z}(V)$ on $\mathrm{Ch}\,\Delta$ is free. But since $\mathscr{Z}(V) \subseteq V \subseteq U \subseteq B$, and as B stabilizes the fundamental chamber, it follows that $\mathscr{Z}(V)$ acts trivially on Δ. This contradiction establishes the Claim.

Since the normalizer of a closed subgroup is closed, we deduce from the Claim that every $P \in \mathcal{P}$ is closed. But this means that G acts continuously on Δ, and so Lemma 3.2 ensures that Δ is finite, as desired. □

4. Proof of Theorem 4

Let k be a perfect field and let $K = \overline{k}$ be its algebraic closure. In what follows, we identify an algebraic k-group G with its group of K-rational points.

The main tool for the proof of Theorem 4 is the following result due to Borel and Tits (see [3]).

Proposition 4.1. *Let G be a reductive algebraic k-group and let U be a unipotent k-subgroup of G. If k is perfect, then there exists a parabolic k-subgroup P of G whose unipotent radical $R_u(P)$ contains U.*

In particular, if G is anisotropic over k, then U must be trivial.

Proof of Theorem 4. Suppose for a contradiction that the split spherical BN-pair (B, N) for the reductive k-group G is such that B has infinite index in $G(k)$. Let $\Delta = \Delta(G(k), B)$ be the associated building, and let W be the corresponding (finite) Weyl group. Also, denote by $\overline{B}$ the Zariski closure of B in G.

The Bruhat decomposition for G yields $G = \coprod_{w \in W} BwB$. Since $G(k)$ is Zariski dense in G by [4, 18.3], we have

$$G = \overline{G(k)} = \overline{\coprod_{w \in W} BwB} \subseteq \coprod_{w \in W} \overline{BwB}.$$

As G is connected, it cannot be written as a finite union of closed subsets in a nontrivial way. Therefore, we deduce that BwB is dense in G for some $w \in W$. In particular, so is $\overline{B}w\overline{B}$.

Let now $A := (\overline{B})^0$ be the identity component of $\overline{B}$. Since A has finite index in $\overline{B}$, it follows that $\overline{B}w\overline{B}$ is a finite union of double cosets modulo A. As before, this implies that some double coset of the form AzA is dense in G.

<u>Claim</u>: $\overline{B} \neq G$.

Indeed, let U be the nilpotent normal subgroup of B arising from the splitting of the BN-pair, and suppose for a contradiction that B is dense in G. Then the Zariski closure $\overline{U}$ of U in G is a nilpotent normal subgroup of $\overline{B} = G$ ([4, 2.1]). Its identity component $\overline{U}^0$ is thus contained in the radical of G, which coincides with the connected center $\mathscr{Z}(G)^0$ ([4, 11.21]). Hence, since $\overline{U}^0$ has finite index in $\overline{U}$, we get

$$[U : U \cap \mathscr{Z}(G)] \leq [U : U \cap \overline{U}^0] = [U\overline{U}^0 : \overline{U}^0] \leq [\overline{U} : \overline{U}^0] < \infty.$$

Now, if $u \in U \cap \mathscr{Z}(G)$, then u acts trivially on Δ since for any chamber gB, we have $ugB = guB = gB$. As U acts simply transitively on C^{op} by Lemma 3.4, where $C = 1_G B$ is the fundamental chamber of Δ, this implies that $u = 1$: otherwise, Δ would contain only one apartment, so that $[G(k) : B] < \infty$, a contradiction. So $U \cap \mathscr{Z}(G) = \{1\}$ and therefore U is finite. Using again the sharp transitivity of U on C^{op}, we deduce that Δ is the reunion of finitely many apartments, hence is finite, contradicting once more our initial hypothesis. The claim stands proven.

In particular A is a proper closed connected subgroup of G such that AzA is dense in G for some $z \in G$. The main result of [9] now implies that A is not reductive, *i.e.* the unipotent radical $R_u(A)$ is nontrivial. Moreover, since B is contained in $G(k)$ and is dense in $\overline{B}$, we know that $\overline{B}$ is defined on k ([4, AG.14.4]). Hence, A is also k-defined ([4, 1.2]), and so is $R_u(A)$ since k is perfect ([18, 12.1.7(d)]). Thus $R_u(A)$ is a nontrivial unipotent k-subgroup of G. As remarked after Proposition 4.1 above, this contradicts the assumption that G is anisotropic over k. □

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