Simple groups without lattices

Bader, Caprace, Gelander and Mozes

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Abstract

We show that the group of almost automorphisms of a d-regular tree does not admit lattices. As far as we know, this is the first such example among (compactly generated) simple locally compact groups.

1 Introduction

Let G be a locally compact group. A **lattice** in G is a discrete subgroup Γ such that G/Γ carries a finite G-invariant measure. Important and well known examples are provided by $\Gamma = \mathbf{Z}^n$ in $G = \mathbf{R}^n$, or $\Gamma = \mathrm{SL}_n(\mathbf{Z})$ in $G = \mathrm{SL}_n(\mathbf{R})$. Despite of the basic nature of the latter objects, we emphasise that the existence of a lattice in a given group G should be considered as a very strong condition on that group. It notably requires G to be unimodular, but this condition is however not sufficient for the existence of a lattice. This is well illustrated by nilpotent Lie groups: all of them are indeed unimodular, but many fail to contain any lattice (see Remark II.2.14 in [10]). An example due to I. Kaplansky (see [12, Example 2.4.7]) shows that a non-compact Abelian (hence unimodular) locally compact group can even fail to contain any infinite discrete subgroup; such a group contains *a fortiori* no lattice.

The question of existence of lattices is especially interesting in the case where the ambient group is topologically simple (and, hence, necessarily unimodular). The fundamental case of Lie groups is well understood: according to a well-known theorem due to A. Borel [3], [10, XIV.14.1], every non-compact simple Lie group contains a uniform and a non-uniform lattice. More generally, arithmetic groups provide an important source of lattices in semi-simple algebraic groups over any locally compact field. Beyond the linear world, some non-linear simple locally compact groups are also known to possess lattices. A typical example is the group $\mathrm{Aut}(T)^+$ of type-preserving automorphisms of a regular locally finite tree T, which is of index two in the full automorphism group $\mathrm{Aut}(T)$. The group $\mathrm{Aut}(T)^+$

is compactly generated and simple, and contains both uniform and non-uniform lattices.

The purpose of this note is to provide an example of a compactly generated simple locally compact group which does not contain any lattice. In order to describe it, we let $d \geq 2$ be a fixed integer, T be a (non-rooted) $(d+1)$-regular tree and G the group of **almost automorphisms** (also sometimes called **spheromorphisms**) of T. An element in G is defined by a triple (A, B, φ) where $A, B \subset T$ are finite subtrees with $|\partial A| = |\partial B|$ and $\varphi : T \setminus A \to T \setminus B$ is an isomorphism between the complements, and two such triples define the same element in G if and only if up to enlarging A, B they coincide.

The group G was first introduced by Neretin [8]; it is known to be abstractly simple [6]. For each vertex $v \in T$, the stabilizer $\mathrm{Aut}(T)_v$ is a compact open subgroup of $\mathrm{Aut}(T)$ and it is not difficult to see that G commensurates $\mathrm{Aut}(T)_v$. (In fact, the group G can be identified with the group of all **abstract commensurators** of $\mathrm{Aut}(T)_v$ or, equivalently, with the group of **germs of automorphisms** of $\mathrm{Aut}(T)$, see [11] and [4, Cor. E]. This fact is however not relevant to our present purposes.)

We endow G with the group topology defined by declaring that the conjugates of $\mathrm{Aut}(T)_v$ in G form a sub-basis of identity neighbourhoods. Since G commensurates $\mathrm{Aut}(T)_v$, it follows that the embedding $\mathrm{Aut}(T) \hookrightarrow G$ continuous. In this way, the group G becomes a totally disconnected locally compact group containing $\mathrm{Aut}(T)$ as an open subgroup. In particular elements of G close to the identity can be realised as true automorphisms of T. As a locally compact group, the group G is compactly generated; in fact it contains a dense copy of the Higman–Thompson group $V_{d,2}$, which is finitely generated (see [4, Th. 6.10]).

The main result of this note is the following.

Theorem 1. G does not contain any lattice.

Fix an edge e_0 in T. By $B_n(e_0)$ we denote the open ball of radius n in T around e_0. Thus the boundary sphere $\partial B_n(e_0)$ is a set of vertices, consisting of $k_n = 2d^n$ elements, meeting every connected component of the graph $T \setminus B_n(e_0)$. Consider the subgroup $O \leq G$ consisting of all elements represented by triples $(B_n(e_0), B_n(e_0), \varphi)$. Thus O can be identified with the increasing union

$$O = \bigcup_{n \in \mathbf{N}} O_n \qquad \text{where} \qquad O_n = \mathrm{Aut}(T \setminus B_n(e_0)).$$

The groups O_n are compact and open in G and O is their union. Essential to our argument is that O is open in G. Therefore, if L is a (uniform) lattice in G then its intersection with O is a (uniform) lattice in O. Thus Theorem 1 is a consequence of the following.

Theorem 2. O does not contain any lattice.

Although this fact will not be necessary for our argument, we point out that the group O is itself topologically simple (see [4, Lem. 6.9]), and thus constitutes another example of a simple locally compact group without lattices. However, in contrast to the group G, the group O is not compactly generated.

We will prove Theorem 2 by way of contradiction. We assume henceforth that $\Gamma \leq O$ is a lattice. Our argument can be outlined as follows. By construction O is a union of an ascending chain of compact open subgroups O_n. Moreover, for n large the profinite group O_n maps onto a full symmetric group $\mathrm{Sym}(k_n)$ of very large degree k_n. The image of the intersection $\Gamma \cap O_n$ maps onto a subgroup whose index in $\mathrm{Sym}(k_n)$ is controlled by the covolume of Γ in O. A precise estimate of that index will be established in the first subsection below. This leads us to studying subgroups of finite symmetric groups of 'relatively small index'. In the general case, we shall invoke results due to L. Babai [1, 2] which are relevant to the latter question in order to complete our study. However, in some special cases, it is possible to complete the proof of Theorem 2 using exclusively elementary methods. This is notably the case if one assumes that Γ is cocompact in O, or alternatively if one assumes that T is trivalent (*i.e.* $d = 2$). The latter two special cases are presented in separate sections by way of illustration. The reader who is interested in contemplating a single example of a compactly generated simple locally compact group without lattices can read through Sections 2 to 5 below, avoiding the technical complications arising in the general discussion carried in Section 6.

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2 Some notations and bounds

We fix an edge e_0 in T, and denote by $B_n(e_0)$ the open ball of radius n around e_0. We denote its boundary by $K_n = \partial B_n(e_0)$ and view it as a set of vertices in T. Its cardinality is denoted by

$$k_n = |K_n| = 2d^n.$$

The group O_0 as defined above coincides with the stabilizer of e_0 in $\mathrm{Aut}(T)$. We set $U_0 = O_0$ and, for every $n \in \mathbf{N}$ we denote by $U_n \leq \mathrm{Aut}(T)$ the pointwise stabilizer of $B_n(e_0)$. Then $\{U_n\}$ form a base of identity neighbourhoods for the topology on G.

Furthermore, U_n is a normal subgroup of O_n and $O_n/U_n \cong \mathrm{Sym}(K_n) \cong \mathrm{Sym}(k_n)$. We denote by $\pi_n : O_n \to \mathrm{Sym}(k_n)$ the quotient map, and by

$$A_n = \pi(U_0) = U_0/U_n \cong \mathrm{Aut}(B_n(e_0)).$$

The order of the finite group A_n is denoted by

$$a_n = |A_n| = 2(d!)^{2\cdot\frac{d^n-1}{d-1}}.$$

Assume that Γ is a lattice in O. We let

$$\Gamma_{O_n} = \Gamma \cap O_n \qquad \text{and} \qquad \Gamma_n = \pi_n(\Gamma_{O_n}) \leq \mathrm{Sym}(k_n).$$

Note that since Γ is discrete, there is some $n_0 \in \mathbf{N}$ such that $\Gamma \cap U_n = \{1\}$ and hence $\Gamma_n \cong \Gamma_{O_n}$, for all $n \geq n_0$. Let μ be the Haar measure on O, normalized by $\mu(U_0) = 1$ and let

$$c = \mathrm{vol}(O/\Gamma) \qquad \text{and} \qquad c_n = \mathrm{vol}(O_n/\Gamma_{O_n}).$$

Then the sequence c_n is non-decreasing and tends to c as $n \to \infty$.

For $n \geq n_0$ we have the following volume computation:

$$\begin{aligned} c_n &= \mathrm{vol}(O_n/\Gamma_{O_n}) = \frac{\mu(O_n)}{|\Gamma_{O_n}|} = \frac{[O_n : O_0]}{|\Gamma_n|} = \frac{[O_n : U_n]}{[U_0 : U_n]|\Gamma_n|} = \frac{1}{a_n}\frac{|\mathrm{Sym}(k_n)|}{|\Gamma_n|} \\ &= \frac{[\mathrm{Sym}(k_n) : \Gamma_n]}{a_n}. \end{aligned} \tag{1}$$

In particular

$$[\mathrm{Sym}(k_n) : \Gamma_n] \leq c \cdot a_n. \tag{2}$$

The latter inequality is the crucial estimate that will be confronted with the discreteness of Γ in order to establish a final contradiction. More precisely, in most cases we shall prove that this condition on the index of Γ_n will force Γ to meet the identity neighbourhood U_m for m arbitrarily large.

3 The cocompact case

The purpose of this section is to give a simple proof for the inexistence of *uniform* lattices in O. The proof will make use of the following Lemma.

Lemma 3.1. *A subgroup of the symmetric group* $\mathrm{Sym}(k)$*, generated by two prime cycles* α, β *whose respective supports intersect nontrivially but are not contained in one another, acts doubly transitively on its support.*

Proof. Applying a power of either α or β followed by a power the other one we can map any pair of points in $\mathrm{Supp}(\alpha) \cup \mathrm{Supp}(\beta)$ to a pair $\{x_1, x_2\}$ satisfying

$$x_1 \in \mathrm{Supp}(\alpha) \setminus \mathrm{Supp}(\beta) \qquad \text{and} \qquad x_2 \in \mathrm{Supp}(\beta) \setminus \mathrm{Supp}(\alpha).$$

Observing that the group $\langle \alpha, \beta \rangle$ acts transitively on

$$\big(\mathrm{Supp}(\alpha) \setminus \mathrm{Supp}(\beta)\big) \times \big(\mathrm{Supp}(\beta) \setminus \mathrm{Supp}(\alpha)\big),$$

the desired conclusion follows. □

We now come back to the setting of Theorem 2 and suppose that $\Gamma \leq O$ is a uniform lattice. By fixing a relatively compact fundamental domain Ω for Γ in O, and recalling that $O = \bigcup_n O_n$ and the O_n are compact, open and ascend to O, one sees that $\Omega \subset O_n$ for all large n and hence the sequence $c_n = \mathrm{vol}(O_n/\Gamma_{O_n})$ is eventually constant and equal to c. In particular c is rational. By Equation (1), there is some n_1 such that

$$[\mathrm{Sym}(k_n) : \Gamma_n] = c \cdot a_n = c \cdot 2(d!)^{2 \cdot \frac{d^n - 1}{d-1}},$$

for all $n > n_1$. Therefore, for any prime $p \leq k_n$ which does not divide the right hand side, the group Γ_n must contain some p-Sylow subgroup of $\mathrm{Sym}(k_n)$. Notice that if $p \geq k_n/2$, such a p-Sylow is cyclic and generated by a single p-cycle (the equality case $p = k_n/2$ is excluded since $k_n/2 = d^n$ is not prime). From the Prime Number Theorem, it follows that for a sufficiently large integer k, the interval $[k/2, k]$ contains at least three distinct primes. Therefore, there is some $n > \max\{n_0, n_1\}$ such that the interval $[k_n/2, k_n]$ contains two primes, say p, q, such that $p + 3 \leq q$. Conjugating one p-cycle in Γ_n by some q-cycle one produces two p-cycles satisfying the condition of Lemma 3.1. We can further ensure that the union of the supports of these two p-cycles is a set of cardinality $k \geq p + 3$. We now invoke a theorem of Jordan [5] (see also [13, Theorem 13.9]) ensuring that *a primitive group of degree k containing a p-cycle, with $p + 3 \leq k$, is either the full symmetric or the alternating group.* It follows that Γ_n contains the alternating group on some subset X_n of size $k > k_n/2 + 2$.

By the pigeonhole principle, the set X_n contains two pairs of vertices x_i, y_i, $i = 1, 2$ such that $d_T(x_i, y_i) = 2$, *i.e.* the vertices x_i and y_i admit a common "father" in K_{n-1}. The permutation

$$\overline{\gamma} = (x_1, y_1)(x_2, y_2)$$

is an element of $\mathrm{Alt}(X_n)$ and hence belongs to Γ_n. However its pre-image $\gamma \in \Gamma_{O_n}$ acts trivially on K_{n-1} and is thus contained in U_{n-1}. Since $n > n_0$ we get a contradiction. □

4 The proof of Theorem 2

In this section we will prove Theorem 2, relying on the following finite-group-theoretic proposition, which will be proven in the next sections.

Proposition 4.1. *For all $c, d > 0$, and $0 < \alpha < 1$, there exists an integer n_1 (depending on c, d and α) such that for every finite set K with $|K| \geq n_1$, every subgroup $\Lambda < \mathrm{Sym}(K)$ satisfying the index bound*

$$[\mathrm{Sym}(K) : \Lambda] \leq c \cdot d^{|K|}$$

enjoys the following (non-exclusive) alternative. Either:

(1) there exists a subset $Z \subset K$ with $|Z| > \frac{|K|}{d} + 2$ and $\mathrm{Alt}(Z) < \Lambda$; or

(2) there exist d disjoint subsets $Z_1, Z_2, \ldots, Z_d \subset K$ with

$$|\bigcup_{i=1}^{d} Z_i| > (1-\alpha)|K|, \quad \text{and} \quad \prod_{i=1}^{d} \mathrm{Alt}(Z_i) < \Lambda.$$

Remark 3. For the proof given below for Theorem 2 we will need Proposition 4.1 for some α satisfying $\alpha < 1/d^2$ (in fact, a careful inspection of the proof below will reveal that it is enough to assume $\alpha < \frac{d-1}{d^2}$, but we choose not to obscure the proof with unnecessary detailed arguments). This is important to note, as we will give in the next section an independent proof of Proposition 4.1 for $d = 2$ and $\alpha = 0.24$.

Proof of Theorem 2. We assume by contradiction that Γ is a lattice in O. We use the notations and bounds given in Section 2. By Equation (2), we have

$$[\mathrm{Sym}(k_n) : \Gamma_n] \leq c \cdot a_n = c \cdot 2(d!)^{2 \cdot \frac{d^n - 1}{d-1}} \leq c' \cdot d^{2d^n} = c' \cdot d^{k_n}$$

for some appropriate constant c' and every $n \geq n_0$. We fix $\alpha < 1/d^2$. We apply Proposition 4.1 to the set $K = K_n$ and the group $\Lambda = \Gamma_n$ with the constants c', d and α as above.

Fix $n \geq \max\{n_0 + 2, n_1\}$ (the constant n_0 was defined in Section 2 and n_1 is given by Proposition 4.1). Then $\Gamma \cap U_{n-2} = \{1\}$ and we infer that Γ_n satisfies one of the alternatives (1) or (2) in Proposition 4.1.

Assume first that Γ_n satisfies (1). Then either there is a vertex $u \in K_{n-1}$ with three neighbours $x_1, x_2, x_3 \in Z$, in which case the 3-cycle (x_1, x_2, x_3) belongs to $\mathrm{Alt}(Z) < \Gamma_n$, or there are two vertices $u, v \in K_{n-1}$ with $x_1, x_2 \in Z$ neighbours of u and $y_1, y_2 \in Z$ neighbours of v. In the latter case we have $(x_1, x_2)(y_1, y_2) \in$

$\mathrm{Alt}(Z) < \Gamma_n$. Observe that the preimages in Γ_{O_n} of both of these elements actually belong to U_{n-1}, which gives as a contradiction, as $U_{n-1} < U_{n-2}$ and $\Gamma \cap U_{n-2} = \{1\}$.

Next suppose that Γ_n satisfies the alternative (2). As in the previous case, if we had for some $1 \leq i \leq d$, two vertices $u, v \in K_{n-1}$ with $x_1, x_2 \in Z_i$ neighbours of u and $y_1, y_2 \in Z_i$ neighbours of v, we would get a contradiction. So, for each i we have at most one element in K_{n-1} with two neighbours in Z_i. Call these **flexible** elements of K_{n-1}. There are at most d such. Call an element of K_{n-2} **flexible** if it has a flexible neighbour in K_{n-1}. Clearly, there are at most d flexible elements in K_{n-2} too.

Say that a vertex $u \in B_k(e_0)$ with $k < n$ is **fully-covered** if the following conditions holds: for every vertex $v \in K_n = \partial B_n(e_0)$ so that u belongs to the geodesic connecting v and e_0, we have $v \in \bigcup_{i=1}^{d} Z_i$. Since $|K_n - \bigcup_{i=1}^{d} Z_i| < \alpha k_n$, the number of vertices which are not fully covered in each level K_j is strictly smaller than αk_n. Consider the level K_{n-2}. Since $k_{n-2} = k_n/d^2$ and $\alpha < 1/d^2$, we get that there are at least $(1/d^2 - \alpha)k_n$ fully covered vertices in K_{n-2}. Picking n large enough so $(1/d^2 - \alpha)k_n \geq d + 2$ we get at least two vertices in K_{n-2} which are fully covered but not flexible.

For a given vertex $v \in K_{n-2}$ which is fully covered and not flexible we can construct an element of $\mathrm{Aut}(B_n(e_0))$ which, as a permutation of K_n, belongs to $\prod_{i=1}^{d} \mathrm{Sym}(Z_i)$, by picking two neighbours of v in K_{n-1}, switching them, and switching all their neighbors in K_n, preserving the sets Z_i. Observe that this element is trivial on K_{n-2}. Having two vertices in K_{n-2} which are fully covered and not flexible, we can compose two such automorphisms, and get an element of $\prod_{i=1}^{d} \mathrm{Alt}(Z_i) < \Lambda = \Gamma_n$. It follows that such an element can be realized as an element $\gamma \in \Gamma$. Since by construction, $\gamma \in U_{n-2}$, we get that $\Gamma \cap U_{n-2} \neq \{1\}$. This is a contradiction. □

5 The case $d = 2$

In this section we prove Proposition 4.1 in the special case $d = 2$ and $\alpha = 0.24$, which in this case reads:

Proposition 5.1. *For each $c > 0$, there exists an integer n_1 (depending on c) such that for every finite set K with $|K| \geq n_1$, every subgroup $\Lambda < \mathrm{Sym}(K)$ with*

$$[\mathrm{Sym}(K) : \Lambda] \leq c \cdot 2^{|K|}$$

enjoys the following (non-exclusive) alternative. Either

(1) there exists a subset $Z \subset K$ with $|Z| > \frac{|K|}{2} + 2$ and $\mathrm{Alt}(Z) < \Lambda$, or

(2) there exists two disjoint subsets $Z_1, Z_2 \subset K$ with

$$|Z_1 \cup Z_2| > 0.76|K|, \quad \textit{and} \quad \mathrm{Alt}(Z_1) \times \mathrm{Alt}(Z_2) < \Lambda.$$

Our proof relies on the Prime Number Theorem which can be formulated as

$$\lim_{n\to\infty} \big(\prod_{p \text{ prime } <n} p\big)/e^n = 1.$$

Lemma 5.2. *Fix $c > 0$, and let $\Lambda < \mathrm{Sym}(k)$ be a subgroup with $[\mathrm{Sym}(k) : \Lambda] \leq c \cdot 2^k$. For all sufficiently large k, there are two primes $p, q \in [0.3k, k]$ such that Λ contains a copy of the p-Sylow as well as of the q-Sylow subgroup of $\mathrm{Sym}(k)$. Furthermore we may assume that $q \geq p + 3$ and that $q \neq k/2 + 1$.*

Proof. The Prime Number Theorem ensures that the product of all primes smaller than k is approximately e^k. For a prime p denote by i_p the multiplicity of p in $k!$. Our claim is that for some p, q as above $|\Lambda|$ is divisible by $p^{i_p} q^{i_q}$. Indeed, if this is not the case, then the index of Λ in $\mathrm{Sym}(k_n)$ is divisible by the product of all, except perhaps three or less, primes in the interval $[0.3k, k]$ which is roughly $e^{0.7k}$. However $e^{0.7} > 2$ contradicting the fact that $[\mathrm{Sym}(k_n) : \Lambda] < c \cdot 2^k$. □

We shall make use of the following consequence of Lemma 3.1.

Corollary 5.3. *Let $\alpha_1, \ldots, \alpha_t$ be prime cycles in $\mathrm{Sym}(k)$ such that for every $1 < i \leq t$ there is some $1 \leq j < i$ such that α_i, α_j satisfy the condition on α, β in Lemma 3.1. Then the group $\langle \alpha_1, \ldots, \alpha_t \rangle$ is doubly transitive on its support.*

Proof. Given two subgroups $A, B \leq \mathrm{Sym}(k)$ which are doubly transitive on their respective support, if these supports intersect in a subset of cardinality at least two, then it follows that $\langle A \cup B \rangle$ is doubly transitive on its own support. In view of this observation, the desired statement follows from Lemma 3.1 by induction on t. □

Proof of Proposition 5.1. Let K be a set such that $|K| = k$ and $[\mathrm{Sym}(K) : \Lambda] < c \cdot 2^k$. Suppose that $k \geq 100$ and is large enough so that the conclusion of Lemma 5.2 holds, and let $p, q > 0.3k$ be two primes such that $q \geq p + 3$ and Λ contains a p-Sylow and a q-Sylow subgroups of $\mathrm{Sym}(K)$, as ensured by Lemma 5.2. Note that every p-Sylow subgroup of $\mathrm{Sym}(K)$ contains $\lfloor \frac{k}{p} \rfloor$ cycles of length p with disjoint supports, and the same applies to q. Let $c_p^1, \ldots, c_p^r$, $r = \lfloor \frac{k}{p} \rfloor \leq 3$ (resp. $c_q^1, \ldots, c_q^s$, $s = \lfloor \frac{k}{p} \rfloor$) be disjoint p-cycles (resp. q-cycles) of Λ.

Fix $j \in \{1, \ldots, s\}$. We claim that there is a subgroup $\Lambda_j \leq \Lambda$ generated by p-cycles, which is 2-transitive on its support $K_j = \mathrm{Supp}(\Lambda_j)$ and such that K_j contains $\mathrm{Supp}(c_q^j)$. Since the complement of the union of the supports of the c_p^i's

is of size strictly smaller than $p < q$, the support of c_q^j overlaps with the support of some c_p^i. By conjugating c_p^i by a suitable power of c_q^j we obtain a partner with which c_p^i satisfies the condition of Lemma 3.1. In fact, denoting by $\alpha_1, \ldots, \alpha_q$ the various conjugates of c_p^i under the elements of $\langle c_q^j \rangle$ and upon reordering the α_k's appropriately, we obtain a sequence of p-cycles which satisfies the hypotheses of Corollary 5.3 and such that the union of the supports of the α_k's contains $\mathrm{Supp}(c_q^j)$. This proves the claim.

In view of Corollary 5.3, we may further assume, upon enlarging Λ_j if necessary, that every p-cycle in Λ is either contained in Λ_j or has support disjoint from K_j.

Applying the aforementioned result of C. Jordan (see section 3) to Λ_j and recalling that $|K_j| \geq q \geq p+3$, we deduce that Λ_j contains the alternating group on its support. In fact, since Λ_j is generated by odd cycles, we have $\Lambda_j = \mathrm{Alt}(K_j)$.

From the property that every p-cycle in Λ is either contained in Λ_j or has support disjoint from K_j, it follows that for all $j, j' \in \{1, \ldots, s\}$, either $K_j = K_{j'}$ or $K_j \cap K_{j'} = \varnothing$.

If now some K_j has cardinality at least $k/2 + 2$, then we are done showing the first alternative in Proposition 5.1 (Notice that this happens for instance in case $s = 1$, and for this particular case we made the assumption $q \neq \frac{k}{2} + 1$). Otherwise we have $s > 1$ and the sets K_j are pairwise disjoint, and each of them has cardinality at most $k/2+1$. Again, the property that every p-cycle in Λ is either contained in Λ_j or has support disjoint from K_j implies that the sets K_1, K_2 and $K \setminus (K_1 \cup K_2)$ constitute blocks of imprimitivity for the Λ-action. It follows that Λ_j admits a subgroup of index ≤ 6 which preserves each of the blocks. Denoting the sizes of these blocks by ak, bk and rk respectively with $a + b + r = 1$, and bearing in mind that $a, b \leq 0.51$ and $r \leq 0.4$ one immediately derives that $r \leq 0.24$ since otherwise $|\Lambda|$ would be bounded above by $6(ak)!(bk)!(rk)! \leq 0.49^k k!$ contradicting the fact that the index $[\mathrm{Sym}(K) : \Lambda] \leq c \cdot 2^k$. Thus, the second alternative in Proposition 5.1 holds. □

6 The proof of Proposition 4.1

Our goal in this section is to prove Proposition 4.1 for an arbitrary d (independently of the previous subsection). Consider a set K of size k and a subgroup $\Lambda < \mathrm{Sym}(K)$ such that $[\mathrm{Sym}(K) : \Lambda] \in O(d^k)$. Given $\varepsilon > 0$, an orbit will be called **ε-large** if it is of size at least $\varepsilon|K|$.

Lemma 6.1. *For each $\delta \in (0, 1]$, there is some $\varepsilon = f_1(d, \delta) > 0$ such that whenever $k = |K|$ is sufficiently large, the ε-large orbits cover at least a proportion of $(1 - \delta)$ of the set K.*

Proof. We claim that taking $\varepsilon = f_1(d, \delta) = \frac{\delta}{100(d+1)^{1/\delta}}$ is sufficient. Suppose that

the assertion does not hold. Then there is a subset $Z = \bigcup_{i=1}^{t} Z_i \subset K$ of size $|Z| > \delta|K|$ which is covered by orbits $\{Z_i\}_{i=1}^{t}$ with $|Z_i| < \varepsilon|K|$ for each i. This implies that the restriction of Λ to Z is of index at least

$$\left(\begin{array}{c} |Z| \\ |Z_1|, |Z_2|, \ldots, |Z_t| \end{array} \right).$$

This multinomial coefficient is thus also a lower bound on the index $[\mathrm{Sym}(K) : \Lambda]$. Let us denote $z = |Z|$ and $z_i = |Z_i|$. Thus we have for each $1 \leq i \leq t$ that $\frac{z}{z_i} > 100(d+1)^{1/\delta}$ and hence that $\frac{z}{z_i+1} > 50(d+1)^{1/\delta}$. Recall the estimate:

$$\left(\frac{m}{e}\right)^m \leq m! \leq \left(\frac{m+1}{e}\right)^{m+1} (m+1).$$

Hence we have:

$$\begin{aligned}
&\left(\begin{array}{c} z \\ z_1, z_2, \ldots, z_t \end{array} \right) \geq \frac{\left(\frac{z}{e}\right)^z}{\prod_{i=1}^{t} \left(\left(\frac{z_i+1}{e}\right)^{z_i} (z_i+1)\right)} = \\
&= \frac{z^z}{\prod_{i=1}^{t} \left((z_i+1)^{z_i}(z_i+1)\right)} = \prod_{i=1}^{t} \left((\frac{z}{z_i+1})^{z_i} (z_i+1)^{-1} \right) \geq \\
&\geq \prod_{i=1}^{t} \left((50(d+1)^{1/\delta})^{z_i} (z_i+1)^{-1} \right) = (50(d+1)^{1/\delta})^z \prod_{i=1}^{t} (z_i+1)^{-1} \geq \\
&\geq (50(d+1)^{1/\delta})^z \prod_{i=1}^{t} (2z_i)^{-1} \geq \frac{(50(d+1)^{1/\delta})^z}{(2z/t)^t},
\end{aligned}$$

where the last inequality follows from the inequality $\frac{a_1+\cdots+a_t}{t} \geq \sqrt[t]{a_1 \ldots a_t}$ between the arithmetic and geometric means. The denominator of the last term is maximal when $t = \frac{2z}{e}$, hence we deduce:

$$\begin{aligned}
&\frac{(50(d+1)^{1/\delta})^z}{(2z/t)^t} \geq \frac{(50(d+1)^{1/\delta})^z}{e^{2z/e}} \geq (10(d+1)^{1/\delta})^z \geq \\
&\geq (10(d+1)^{1/\delta})^{\delta|K|} \geq (d+1)^k.
\end{aligned}$$

This lower bound however is too large compared to the bound $[\mathrm{Sym}(K) : \Lambda] \in O(d^k)$. □

Lemma 6.2. *There is some $f_2(d, \varepsilon)$ such that if $k = |K|$ is sufficiently large, for any Λ-orbit $Y \subset K$ whose size is at least $\varepsilon|K|$, any non-trivial Λ-invariant block decomposition of Y contains at most $f_2(d, \varepsilon)$ blocks.*

Proof. A group acting transitively on a set Y and whose action preserves a block decomposition with b blocks is of index at least $|Y|!/(b!((|Y|/b)!)^b)$ in $\mathrm{Sym}(Y)$. This quantity is thus also a lower bound on $[\mathrm{Sym}(K) : \Lambda]$. We remark that since the block decomposition is non-trivial, we have $b \in (1, \frac{y}{2}]$, where $y = |Y| \geq \varepsilon k$.

We shall now estimate the function $f(b) = \frac{y!}{(b!((y/b)!)^b}$ in the range $b \in [1, \frac{y}{2}]$. To this end, we use Stirling's approximation under the following form:

- For every $m \in \mathbf{N}$ we have $m! \geq \sqrt{2\pi m}\left(\frac{m}{e}\right)^m$.
- There is some constant $c > 1$ so that for every $m \in \mathbf{N}$ we have $m! \leq c\sqrt{2\pi m}\left(\frac{m}{e}\right)^m$.

Using these we have:

$$\frac{y!}{b!((y/b)!)^b} \geq \frac{\sqrt{2\pi y}\left(\frac{y}{e}\right)^y}{c\sqrt{2\pi b}\left(\frac{b}{e}\right)^b\left(c\sqrt{2\pi\frac{y}{b}}\left(\frac{y}{be}\right)^{y/b}\right)^b} =$$

$$= \frac{\sqrt{2\pi y}\left(\frac{y}{e}\right)^y}{c\sqrt{2\pi b}\left(\frac{b}{e}\right)^b\left(c\sqrt{2\pi\frac{y}{b}}\right)^b\left(\frac{y}{be}\right)^y} = \frac{\sqrt{y/b}}{c}\cdot\frac{b^y}{((c/e)\sqrt{2\pi y b})^b} >$$

$$> \frac{1}{(100c)^y}\cdot\frac{b^y}{\sqrt{yb}^b} =: g(b).$$

Next we claim that the function $g(b)$ (for any fixed y) is **unimodular** in the range $b \in [1, y/2]$, *i.e.* it increases monotonically till it reaches some maximum and then decreases monotonically. In particular, for all $b_0 \in [1, y/2]$ and $b \in [b_0, y/2]$, we have $g(b) \geq \min\{g(b_0), g(y/2)\}$.

Now, at the rightmost point $b = y/2$ we have

$$g(y/2) = \frac{1}{(100c)^y}\cdot\frac{(y/2)^y}{\sqrt{y^2/2}^{(y/2)}} \geq \left(\frac{\sqrt{y}}{400c}\right)^y.$$

Hence for $y \geq (400c(3(d+1))^{1/\varepsilon})^2$ we have $g(y/2) \geq ((3(d+1))^{1/\varepsilon})^y \geq (3(d+1))^k$. (Note that we may assume $y \geq (400c(3(d+1))^{1/\varepsilon})^2$ is satisfied since the right hand side is a constant.)

Consider next $b_0 = 300c(d+1)^{1/\varepsilon}$. We have

$$g(b_0) = g(300c(d+1)^{1/\varepsilon}) = \frac{1}{(100c)^y}\cdot\frac{(300c(d+1)^{1/\varepsilon})^y}{\sqrt{y\cdot 300c(d+1)^{1/\varepsilon}}^{300c(d+1)^{1/\varepsilon}}} =$$

$$= \frac{(3(d+1)^{1/\varepsilon})^y}{\sqrt{y\cdot 300c(d+1)^{1/\varepsilon}}^{300c(d+1)^{1/\varepsilon}}} = \frac{3^y}{\sqrt{y\cdot 300c(d+1)^{1/\varepsilon}}^{300c(d+1)^{1/\varepsilon}}}(d+1)^{y/\varepsilon}.$$

Hence assuming, as we may, that k (and hence y) is larger than a fixed (computable) constant we have $g(b_0) \geq (d+1)^k$. Thus, for all $b \in [b_0, y/2]$, we have $g(b) \geq \min\{(d+1)^k, (3(d+1))^k\} = (d+1)^k$. This gives a lower bound on the index $[\mathrm{Sym}(K) : \Lambda]$ which is larger than $O(d^k)$. It follows that we must have $b < b_0$. In other words, we may choose the requested constant $f_2(d, \varepsilon)$ to equal $b_0 = 300c(d+1)^{1/\varepsilon}$.

It remains to show the unimodularity of the function $g(x)$ in the interval $[1, y/2]$. As y is fixed we need to consider the function

$$g_1(x) = \frac{x^y}{\sqrt{yx}^x}.$$

It is more convenient to consider its logarithm:

$$h(x) = \log g_1(x) = y \log x - \frac{x}{2} \log(yx) = (y - \frac{x}{2}) \log x - \frac{x}{2} \log y.$$

Computing its derivative we have:

$$h'(x) = \frac{y}{x} - \frac{1}{2} - \frac{1}{2} \log x - \frac{\log y}{2}.$$

This is a monotonly decreasing function of $x > 0$ and hence the function $g(x)$ is unimodular. □

Proposition 6.3. *For all $c, d > 0$ and $\delta > 0$, there is some C such that the following holds. For every large enough finite set K and for every subgroup $\Lambda < \mathrm{Sym}(K)$ with $[\mathrm{Sym}(K) : \Lambda] \leq c \cdot d^{|K|}$ there exists a collection $\mathcal{L}$ of pairwise disjoint subsets of K, satisfying the following properties:*

(1) $|\bigcup_{Z \in \mathcal{L}} Z| \geq (1-\delta)|K|$.

(2) $[\mathrm{Sym}(\bigcup_{Z \in \mathcal{L}} Z) : \prod_{Z \in \mathcal{L}} \mathrm{Alt}(Z)] \leq C \cdot d^{|K|}$.

(3) $\prod_{Z \in \mathcal{L}} \mathrm{Alt}(Z) < \Lambda$.

Furthermore, there exist ε_0 and V_0 (also depending only on c, d and δ) such that $|\mathcal{L}| \leq V_0$ and for each $Z \in \mathcal{L}$, $|Z| \geq \varepsilon_0 |K|$.

The proof of Proposition 6.3 relies on the following result of L. Babai.

Theorem 6.4 (L. Babai [1] and [2])**.** *A primitive subgroup $L < \mathrm{Sym}(n)$ which does not contain the alternating group $\mathrm{Alt}(n)$ satisfies:*

- $|L| < e^{4\sqrt{n} \log^2 n}$ *if L is not 2-transitive, and*

- $|L| < e^{e^{c\sqrt{\log n}}}$ *if* L *is 2-transitive.* □

We point out that these estimates can be strengthened to $|L| < 50e^{\sqrt{n}\log n}$ using the Classification of the Finite Simple Groups, see [7, Cor. 1.1(ii)]. The bounds given by Babai's theorem (which is independent of the Classification of the Finite Simple Groups) will however be sufficient for our purposes. Note also that the better estimate $n^{c(logn)^2}$ for the case of 2-transitive groups has been obtained in [9] (with a simpler argument).

Proof of Proposition 6.3. Denote $k = |K|$ and assume it is large enough so that Lemma 6.1 and Lemma 6.2 hold. Let $\varepsilon = f_1(d, \delta)$ be as in Lemma 6.1. Let Y_j, $j = 1, \ldots, m$ be all the ε-large orbits. Note that $m \leq 1/\varepsilon$. By Lemma 6.1, we have $|\bigcup_{j=1}^m Y_j| \geq (1-\delta)k$. For each $1 \leq j \leq m$ let $\{Y_{j,i} \ : \ 1 \leq i \leq m_j\}$ be a block decomposition of Y_j with the largest possible number of non-trivial blocks. Set

$$\mathcal{L} = \{Y_{j,i} \mid j = 1, \ldots, m \ ; \ i = 1, \ldots, m_j\}.$$

Property (1) of the proposition is clear.

Set $V = f_2(d, \varepsilon)$ as given by Lemma 6.2 and $V_0 = V/\varepsilon$. Then we have $m_j \leq V$ for each j and $|\mathcal{L}| \leq mV \leq V_0$. Also for each j we have $|Y_j| \geq \varepsilon k$ and $Y_{j,i} = |Y_j|/m_j \geq \varepsilon k/V$. Setting $\varepsilon_0 = \varepsilon/V$ we have proven the "furthermore" part of the proposition.

Observe that the index of Λ in $\mathrm{Sym}(K)$ gives an upper bound on the index of the restriction $\bar{\Lambda}$ of Λ to $\bigcup_{j=1}^m Y_j$ in $\mathrm{Sym}(\bigcup_{j=1}^m Y_j)$. Thus

$$[\mathrm{Sym}(\bigcup_{j=1}^{m} Y_j) : \bar{\Lambda}] \leq [\mathrm{Sym}(K) : \Lambda] \leq c \cdot d^k.$$

Assume property (3) holds. Then we have

$$[\bar{\Lambda} : \prod_{j=1}^{m} \prod_{i=1}^{m_j} \mathrm{Alt}(Y_{j,i})] \leq \prod_{j=1}^{m} (m_j! 2^{m_j}) \leq (V! 2^V)^{1/\varepsilon}.$$

Thus property (2) follows for $C = c \cdot (V! 2^V)^{1/\varepsilon}$.

We are left to prove property (3). For each $1 \leq j \leq m$ and $1 \leq i \leq m_j$, we consider the action of Λ on the orbit Y_j. Let $\Lambda_{j,i}$ be the subgroup which preserves the block $Y_{j,i}$. Remark that we have the index bound $[\Lambda : \Lambda_{j,i}] \leq m_j \leq V$. Hence $[\mathrm{Sym}(K) : \Lambda_{j,i}]$ is bounded by $O(d^k)$. By definition the $Y_{j,i}$'s provide the finest Λ-invariant block decomposition of Y_j. This implies that the action of $\Lambda_{j,i}$ on $Y_{j,i}$, $1 \leq i \leq m_j$ is primitive.

Using the fact that each of the sets $Y_{j,i}$ is of size $|Y_{j,i}| > \varepsilon_0 |K|$, it follows from Babai's estimates from Theorem 6.4 that the restriction of $\Lambda_{j,i}$ to $Y_{j,i}$ must contain

the corresponding alternating group $\mathrm{Alt}(Y_{j,i})$, otherwise we would deduce that the index of $\Lambda_{j,i}$ in $\mathrm{Sym}(K)$ is not bounded by $O(d^k)$.

We infer that the intersection of Λ with $\prod_{j=1}^{m}\prod_{i=1}^{m_j}\mathrm{Alt}(Y_{j,i})$ is a subgroup which projects onto each of the non-abelian, simple factors $\mathrm{Alt}(Y_{j,i})$. Such a subgroup is either the full product or is of index which is at least the order of some $\mathrm{Alt}(Y_{j,i})$. The latter possibility is excluded because $|\mathrm{Alt}(Y_{j,i})|$ has a much bigger growth rate than the index $[\mathrm{Sym}(K):\Lambda]\in O(d^k)$. □

Recall the definition of the **entropy function**: For $\alpha_i\geq 0$, $\sum_{i=1}^{s}\alpha_i=1$, this is the function

$$H(\alpha_1,\ldots,\alpha_s)=-\sum_{i=1}^{s}\alpha_i\log_2\alpha_i.$$

For a fixed s, the multinomial coefficient satisfies

$$\lim_{n\to\infty}\frac{\log_2\binom{n}{\alpha_1 n,\alpha_2 n,\ldots,\alpha_s n}}{nH(\alpha_1,\alpha_2,\ldots,\alpha_s)}=1.$$

Hence given $\beta>0$ for sufficiently large n

$$\binom{n}{\alpha_1 n,\alpha_2 n,\ldots,\alpha_s n}\geq\left(2^{nH(\alpha_1,\alpha_2,\ldots,\alpha_s)}\right)^{1-\beta}$$

Proof of Proposition 4.1. Let c,d and α be given. Without loss of generality we may assume $\alpha<\frac{1}{d}$. Our goal is to show that there is a choice of δ (depending on c,d and α) such that for each large enough finite set K and every collection $\mathcal{L}$ of subsets of K afforded by Proposition 6.3 with that choice of δ, either $\mathcal{L}$ contains a set of size greater than $\frac{|K|}{d}+2$ or $\mathcal{L}$ contains d sets whose union is of size greater than $(1-\alpha)|K|$. The conclusion of Proposition 4.1 will thus follow.

Observe that

$$d=2^{H(\frac{1}{d},\frac{1}{d},\ldots,\frac{1}{d})}<2^{H(\frac{1}{d},\frac{1}{d},\ldots,\frac{1}{d},\frac{1}{d}-\frac{\alpha}{2},\frac{\alpha}{2})}.$$

Fix $\tilde{d}$ such that $d<\tilde{d}<2^{H(\frac{1}{d},\frac{1}{d},\ldots,\frac{1}{d},\frac{1}{d}-\frac{\alpha}{2},\frac{\alpha}{2})}$, and choose $\delta>0$ and $\beta>0$ small enough such that

$$\delta<\frac{\alpha}{2}\quad\text{and}\quad\left(\frac{1}{d}-\alpha\right)^{\delta}\left(2^{H(\frac{1}{d},\frac{1}{d},\ldots,\frac{1}{d},\frac{1}{d}-\frac{\alpha}{2},\frac{\alpha}{2})}\right)^{1-\beta}>\tilde{d}.$$

We claim that this choice of δ will do the job. We will prove it by a contradiction. We will assume that there is no set in $\mathcal{L}$ of size greater that $\frac{|K|}{d}+2$ and that the union of the d largest sets in $\mathcal{L}$ covers at most $(1-\alpha)$ of the set K.

Set $k = |K|$. By Property (2) of Proposition 6.3 we have

$$[\mathrm{Sym}(\bigcup_{Z\in\mathcal{L}} Z) : \prod_{Z\in\mathcal{L}} \mathrm{Alt}(Z)] \leq C \cdot d^k.$$

Denoting $\mathcal{L} = \{Z_1, \ldots, Z_t\}$ and recalling that $t \leq V_0$, this index coincides, up to a constant, with a multinomial coefficient

$$\binom{|\bigcup Z_i|}{|Z_1|, |Z_2|, \ldots, |Z_t|}$$

We will derive the contradiction by estimating this multinomial coefficient from below, showing that it is too big.

Let us denote $z_i = |Z_i|$, $1 \leq i \leq t$ and $z = |\bigcup Z_i|$. Assume, as we may, that $z_1 \geq z_2 \geq z_3 \geq \cdots \geq z_t$. Note that by the assumption we have $z_i \leq \frac{k}{d} + 2$ for all $1 \leq i \leq t$. Since $\delta < \frac{\alpha}{2}$ Property (1) of Proposition 6.3 implies that $\sum_{j\geq d+1} z_j \geq \frac{\alpha k}{2}$. In a multinomial coefficient if we keep all the terms fixed except for two terms which we change so that their sum is fixed but their difference increases, then the total value decreases. We claim that applying this repeatedly we can deduce that our multinomial is bounded below by:

$$\binom{z}{a_1, a_2, \ldots, a_{d-1}, b, \frac{\alpha k}{2}} \tag{3}$$

where for $1 \leq i \leq d-1$, $\frac{k}{d} \leq a_i \leq \frac{k}{d} + 2$ and $b = z - \sum a_i - \frac{\alpha k}{2}$. (Note that if some of the fractions above are not integers one has to take nearby integers.) This is done by an inductive process. Suppose we have already shown that:

$$\binom{z}{z_1, z_2, \ldots, z_t} \geq \binom{z}{a_1, a_2, \ldots, a_{i-1}, z_i, z_{i+1}, \ldots, z_s}$$

and $i \leq d-1$. Then we may transfer as much as we can from the last term z_s to the term z_i as long as the latter does not increase beyond $\frac{k}{d}$ and as long as $\sum_{j\geq d+1} z_j \geq \frac{\alpha k}{2}$. If z_s becomes 0, we remove it. Once the entry z_i is increased to $\frac{k}{d}$ call it a_i and repeat until $i = d$ or $\sum_{j\geq d+1} z_j = \frac{\alpha k}{2}$.

If we reached $i = d$ (and $\sum_{j\geq d+1} z_j \geq \frac{\alpha k}{2}$) increase z_d in the same way till it is equal to $z - \sum_{i=1}^{d-1} a_i - \frac{\alpha k}{2}$. Then one may collect all the remaining terms from the $d+1$ place onward together and get one term which will be $\frac{\alpha k}{2}$. If along the process we reached $\sum_{j\geq d+1} z_j = \frac{\alpha k}{2}$ with $i \leq d-1$ then we can collect all the terms from the $d+1$ place onwards together to form one term which is equal to $\frac{\alpha k}{2}$ and then we can "transfer mass" from the term z_d to those entries $z_i, \ldots, z_{d-1}$ which are still smaller then $\frac{k}{d}$. Using this process, we eventually reach the form (3) and we have

$$\binom{z}{z_1, z_2, \ldots, z_t} \geq \binom{z}{a_1, a_2, \ldots, a_{d-1}, b, \frac{\alpha k}{2}}$$

as claimed. Observe now that since for $1 \leq i \leq d-1$ we have $\frac{k}{d} \leq a_i \leq \frac{k}{d} + 2$ it follows that for some fixed polynomial $p(x)$ (say, $p(x) = (x/d+2)^{d-1}$) we have

$$\binom{z}{a_1, a_2, \ldots, a_{d-1}, b, \frac{\alpha k}{2}} \geq \frac{1}{p(k)} \binom{z}{\frac{k}{d}, \frac{k}{d}, \ldots, \frac{k}{d}, (z - (\frac{d-1}{d} + \frac{\alpha}{2})k), \frac{\alpha k}{2}} \tag{4}$$

We claim that the choice of δ guarantees that the last multinomial coefficient grows as $\tilde{d}^k$ (which allows us to absorb (i.e. ignore) the polynomial factor $\frac{1}{p(k)}$). Using that $z > (1-\delta)k$ and $\delta < \frac{\alpha}{2}$ we have for k large enough:

$$\begin{aligned}
&\binom{z}{\frac{k}{d}, \frac{k}{d}, \ldots, \frac{k}{d}, (z - (\frac{d-1}{d} + \frac{\alpha}{2})k), \frac{\alpha k}{2}} \geq \\
&\geq \left(\frac{(z - (\frac{d-1}{d} + \frac{\alpha}{2})k)}{k}\right)^{k-z} \binom{k}{\frac{k}{d}, \frac{k}{d}, \ldots, \frac{k}{d}, (\frac{1}{d} - \frac{\alpha}{2})k, \frac{\alpha k}{2}} \geq \\
&\geq \left((1-\delta) - (\frac{d-1}{d} + \frac{\alpha}{2})\right)^{k-z} \binom{k}{\frac{k}{d}, \frac{k}{d}, \ldots, \frac{k}{d}, (\frac{1}{d} - \frac{\alpha}{2})k, \frac{\alpha k}{2}} \geq \\
&\geq \left(\frac{1}{d} - \delta - \frac{\alpha}{2}\right)^{\delta k} \binom{k}{\frac{k}{d}, \frac{k}{d}, \ldots, \frac{k}{d}, (\frac{1}{d} - \frac{\alpha}{2})k, \frac{\alpha k}{2}} \geq \\
&\geq \left(\frac{1}{d} - \alpha\right)^{\delta k} \binom{k}{\frac{k}{d}, \frac{k}{d}, \ldots, \frac{k}{d}, (\frac{1}{d} - \frac{\alpha}{2})k, \frac{\alpha k}{2}} \geq \\
&\geq \left(\left(\frac{1}{d} - \alpha\right)^{\delta}\right)^{k} \left(2^{H(\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d}, \frac{1}{d} - \frac{\alpha}{2}, \frac{\alpha}{2})}\right)^{(1-\beta)k} = \\
&= \left(\left(\frac{1}{d} - \alpha\right)^{\delta} \left(2^{H(\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d}, \frac{1}{d} - \frac{\alpha}{2}, \frac{\alpha}{2})}\right)^{1-\beta}\right)^{k} > \tilde{d}^k.
\end{aligned}$$

Since $\tilde{d} > d$ this clearly contradicts Property (2) of Proposition 6.3.

Note that we have used that for sufficiently large k

$$\binom{k}{\frac{k}{d}, \frac{k}{d}, \ldots, \frac{k}{d}, (\frac{1}{d} - \frac{\alpha}{2})k, \frac{\alpha k}{2}} \geq \left(2^{H(\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d}, \frac{1}{d} - \frac{\alpha}{2}, \frac{\alpha}{2})}\right)^{(1-\beta)k}.$$

□

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Uri Bader, Technion Haifa, Israel; `bader@tx.technion.ac.il`
Pierre-Emmanuel Caprace, UCLouvain, Belgium; `pe.caprace@uclouvain.be`
Tsachik Gelander, Hebrew University Jerusalem, Israel; `gelander@math.huji.ac.il`
Shahar Mozes, Hebrew University Jerusalem, Israel; `mozes@math.huji.ac.il`