Lectures on proper CAT(0) spaces and their isometry groups

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Introduction

CAT(0) spaces, introduced by Alexandrov in the 1950's, were given prominence by M. Gromov, who showed that a great deal of the theory of manifolds of non-positive sectional curvature could be developed without using much more than the CAT(0) condition (see [BGS85]). Since then, CAT(0) spaces have played a central role in geometric group theory, opening a gateway to a form of generalized differential geometry encompassing non-positively curved manifolds as well as large families of singular spaces such as trees, Euclidean or non-Euclidean buildings, and many other cell complexes of non-positive curvature.

Excellent introductions on CAT(0) spaces may be found in the literature, e.g. in the books [Bal95] and [BH99]. The goal of these lectures is to present some material not covered by those references. While rigidity of (usually discrete) *group actions* on non-positively curved space is a standard theme of study in geometric group theory, the main idea we would like to convey is that, in the locally compact case, the *spaces* themselves turn out to be much more rigid than one might expect as soon as they admit a reasonable amount of isometries. This phenomenon will be highlighted by placing a special emphasis on the *full* isometry group of a proper CAT(0) space. Taking into account the fact the this isometry group is naturally endowed with a locally compact group topology which is possibly non-discrete, many structural (and especially rigidity) properties of the underlying space can be derived by combining results on locally compact groups with (mostly elementary) geometric arguments. A number of results obtained with this approach are presented in this course. In the final lecture, we will come back to discrete groups and present some results whose proof relies heavily on the preceding study of non-discrete group actions.

Although some of the very basics on CAT(0) spaces will be recalled, a familiarity with the aforementioned standard references is recommended. We have chosen to present the results not always in their most general form, but rather in a way that makes their statement simpler and hopefully more enlightening. More general statements, detailed arguments and further results may be found in the papers [CM09a, CM09b, CM12a, CZ12]. All the original results presented here have been obtained in collaboration with Nicolas Monod or with Gašper Zadnik.

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LECTURE I

Leading examples

1. The basics

Let (X, d) be a metric space. A **geodesic map** is an isometric map $\rho\colon I \to X$ of a convex subset $I \subseteq \mathbf{R}$ to X, where the real line $\mathbf{R}$ is endowed with the Euclidean distance. The map ρ is called a **geodesic segment** (resp. **ray**, **line**) if I is a closed interval (resp. I is a half-line, $I = \mathbf{R}$). It should be noted that the notion of geodesic introduced here is a global one, as opposed to the corresponding notion in differential geometry.

A **geodesic metric space** is a metric space (X, d) in which any two points are joined by a geodesic segment.

EXAMPLES I.1.

- The Euclidean space $(\mathbf{R}^n, d_{\mathrm{Eucl}})$ is a geodesic metric space.
- More generally, a Riemannian manifold, viewed as a metric space with its canonical distance function, is a geodesic metric space *provided it is complete.* An incomplete Riemannian manifold need not be a geodesic metric space.
- A metric graph, with all edges of length one, is a geodesic metric space.

Let (X, d) be a geodesic metric space. Given a triple $(x, y, z) \in X^3$, a **Euclidean comparison triangle** for (x, y, z) is a triple $(\hat{x}, \hat{y}, \hat{z})$ of points of the Euclidean plane $\mathbf{R}^2$ such that $d(x, y) = d_{\mathrm{Eucl}}(\hat{x}, \hat{y})$, $d(y, z) = d_{\mathrm{Eucl}}(\hat{y}, \hat{z})$ and $d(z, x) = d_{\mathrm{Eucl}}(\hat{z}, \hat{x})$. Notice that any triple in X admits some Euclidean comparison triangle.

A **CAT(0) space** is a geodesic metric space all of whose triple of points $(x, y, z) \in X^3$ satisfy the following condition: given a Euclidean comparison triangle $(\hat{x}, \hat{y}, \hat{z})$ in $\mathbf{R}^2$, any point $p \in X$ which belongs to some geodesic segment joining y to z in X satisfies the inequality

$$d(x, p) \leq d_{\mathrm{Eucl}}(\hat{x}, \hat{p}),$$

where $\hat{p} \in \mathbf{R}^2$ is the unique point of $\mathbf{R}^2$ such that $d(y, p) = d_{\mathrm{Eucl}}(\hat{y}, \hat{p})$ and $d(p, z) = d_{\mathrm{Eucl}}(\hat{p}, \hat{z})$.

The following fundamental properties of CAT(0) spaces are straightforward to deduce from the definition.

PROPOSITION I.2. *Let (X, d) be a* CAT(0) *space. Then:*

(i) (X, d) is ***uniquely geodesic****, i.e. any two points are joined by a* unique *geodesic segment.*

(ii) X is contractible.

EXAMPLES I.3.

- The Euclidean space $(\mathbf{R}^n, d_{\mathrm{Eucl}})$ is a CAT(0) space. So is any pre-Hilbert space.
- A complete simply connected Riemannian manifold M, endowed with its canonical distance function, is a CAT(0) space if and only if M has non-positive sectional curvature. See [BH99, Theorem 1.A.6]. So is in particular the real hyperbolic space $\mathbf{H}^n$.
- A metric graph X is a CAT(0) space if and only if X is a tree.

This short list of examples already illustrates that the category of CAT(0) spaces encompasses both smooth and singular objects. The singular character expresses itself by the fact that geodesics may **branch**, i.e. two distinct geodesic segments may share a common sub-segment of positive length.

There are several ways to construct new examples of CAT(0) spaces from known ones.

A subset Y of a CAT(0) space (X, d) is called **convex** if the geodesic segment joining any two points of Y is entirely contained in Y. Clearly, a convex subset of a CAT(0) space is itself a CAT(0) space when endowed with the induced metric.

Another key feature of the CAT(0) condition is its stability under Cartesian products. The proof is left as an exercise.

PROPOSITION I.4. *Let (X_1, d_1) and (X_2, d_2) be* CAT(0) *spaces. Then the Cartesian product $X = X_1 \times X_2$, endowed with the metric d defined by $d^2 = d_1^2 + d_2^2$, is a* CAT(0) *space.*

Various more exotic constructions, like gluing two CAT(0) spaces along an isometric convex subset, also preserve the CAT(0) condition. We close this section with the following noteworthy facts, for which we refer to Cor. II.3.10 and II.3.11 in [BH99].

PROPOSITION I.5.

(i) The Cauchy completion of a CAT(0) *space is itself* CAT(0).
(ii) An ultraproduct of CAT(0) *spaces is itself* CAT(0). *In particular, the asymptotic cones of a* CAT(0) *space are* CAT(0).

2. The Cartan–Hadamard theorem

A fundamental feature of the CAT(0) condition is that it is a local condition, as is the condition of being non-positively curved in the realm of Riemannian geometry. This matter of fact is made precise by the following basic result, for which we refer to [Bal95, Theorem I.4.5] and [BH99, Theorem II.4.1].

THEOREM I.6 (Cartan–Hadamard). *Let (X, d) be a complete connected metric space. If every point of X admits some neighbourhood which is* CAT(0) *when endowed with the appropriate restriction of d (we then say that (X, d) is* ***locally* CAT(0)***), then there is a unique distance function $\tilde{d}$ on the universal cover $\widetilde{X}$ such that following two conditions hold:*

- *the covering map $\widetilde{X} \to X$ is a local isometry;*
- *$(\widetilde{X}, \tilde{d})$ is a* CAT(0) *space.*

The metric $\tilde{d}$ coincides with the **length metric** (also called **inner metric**) induced by d on $\widetilde{X}$. We refer to [Bal95, §1.1] and [BH99, §I.3] for detailed treatments of those notions. At this point, let us just observe that a non-convex subset of a CAT(0) space may very well be CAT(0) provided it is endowed with the induced *length* metric.

The Cartan–Hadamard theorem yields a wealth of further examples of CAT(0) spaces constructed as universal covers of compact metric spaces that are locally CAT(0). A typical situation is that of a finite piecewise Euclidean cell complex X, endowed with the length metric d induced by the Euclidean metric on each cell. The Cartan–Hadamard theorem ensures that the universal covering cell complex $\widetilde{X}$ is naturally a CAT(0) space provided (X, d) is locally CAT(0). Verifying that a given finite piecewise Euclidean cell complex is locally CAT(0) is usually highly non-trivial (although, in theory, it can be done algorithmically, see [EM04]). There are only two special cases where this question can be decided by means of an easy combinatorial criterion, as described in the following (see [BH99, §II.5] and the lectures by M. Sageev).

THEOREM I.7. *Let X be a connected piecewise Euclidean cell complex endowed with the length metric d induced by the Euclidean metric on each cell. If any of the following conditions holds, then (X, d) is locally* CAT(0)*, and hence $(\widetilde{X}, \tilde{d})$ is a* CAT(0) *space:*

(i) X is two-dimensional, and for each vertex $v \in X^{(0)}$ and each sequence $(\sigma_1, \sigma_1, \dots, \sigma_n)$ of pairwise distinct 2-faces such that $\sigma_i \cap \sigma_{i+1}$ is an edge containing v for all $i \in \mathbf{Z}/n\mathbf{Z}$, the sum over all i of the interior angles of the faces σ_i at the vertex v is at least 2π.
(ii) Each cell in X is a Euclidean cube with edge length one, and the link of every vertex is a flag complex.

The CAT(0) spaces constructed as in Theorem I.7(ii), which are called **CAT(0) cube complexes**, are endowed with a rich combinatorial structure which provides an important addition tool in their study. This explains why results known about CAT(0) cube complexes are usually much finer than those describing more general classes of CAT(0) spaces. Nevertheless, it turns out that CAT(0) cube complexes are much more ubiquitous that one might think at a first sight. We refer to the lectures by M. Sageev for more information.

From now on, a metric space (X, d) will simply be denoted by its underlying set of points X, the distance function being by default denoted by the letter d, unless explicitly mentioned otherwise.

3. Proper cocompact spaces

The class of all CAT(0) spaces is vast and wild; it is not a realistic goal to understand it exhaustively. In the rest of the course, we shall frequently impose that the spaces under consideration satisfy (some of) the following conditions:

- **Properness**. A metric space is called **proper** if all of its closed balls are compact. In particular such a space is locally compact.
- **Cocompactness**. A metric space X is called **cocompact** if its full isometry group $\mathrm{Is}(X)$ acts cocompactly, i.e. if the orbit space $\mathrm{Is}(X)\backslash X$ is compact.
- **Geodesic completeness**. A geodesic metric space X is called **geodesically complete** (one also says that X has **extendible geodesics**) if every geodesic segment can be prolonged to a (potentially non-unique) bi-infinite geodesic line.

As in Riemannian geometry, the notions of properness, completeness and geodesic completness are related in the case of locally compact spaces:

THEOREM I.8 (Hopf–Rinow). *Let X be a locally compact* CAT(0) *space.*

(i) X is proper if and only if it is complete.
(ii) If X is geodesically complete, then it is proper.

PROOF. See [Bal95, Theorem I.2.4] and [BH99, Proposition I.3.7]. □

Among all proper cocompact CAT(0) spaces, there are two leading families of examples, namely *symmetric spaces* and *Euclidean buildings*. Those are the spaces naturally associated with semi-simple Lie groups or semi-simple linear algebraic groups over local fields. We shall now briefly recall the basic definitions.

4. Symmetric spaces

A **symmetric space** is a Riemannian manifold M such that the geodesic symmetric σ_x centered at each point $x \in M$ is a global isometry. Equivalently, for each $x \in M$ there is an isometry $\sigma_x \in \mathrm{Is}(M)$ fixing x, whose differential is the central symmetry of T_xM.

Basic examples are provided by the sphere $\mathbf{S}^n$, the Euclidean space $\mathbf{R}^n$ and the real hyperbolic space $\mathbf{H}^n$. As the example of the sphere shows, a symmetric space can be positively curved. A symmetric space is said to be of **non-compact type** if it has non-positive sectional curvature and no non-trivial Euclidean factor.

Any such space M is thus a CAT(0) space which is proper, cocompact (in fact homogeneous!) and geodesically complete. It can be constructed as a coset space $M = G/K$, where G is a non-compact, connected semi-simple Lie group and $K < G$ is a maximal compact subgroup. The metric on G/K comes from the Killing form of the Lie algebra of G.

A prominent example is provided by the case $G = \mathrm{SL}_n(\mathbf{R})$ and $K = \mathrm{SO}(n)$. The coset space $M = G/K$ can be identified with the collection of scalar products on $\mathbf{R}^n$ for which the unit ball has the same volume as the unit ball with respect to the standard Euclidean metric d_{Eucl}. A description of the symmetric space M may be consulted in [BH99, §II.10]; the discussion below is an alternative approach.

The distance function on M can be defined as follows. Given two scalar products $x_1 = (\cdot,\cdot)_1$ and $x_2 = (\cdot,\cdot)_2$ on $\mathbf{R}^n$, it is a standard fact (see e.g. [HJ90, Th. 7.6.4]) that there exists some basis of $\mathbf{R}^n$ with respect to which both products are represented by a diagonal Gram matrix, say $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $\mathrm{diag}(\mu_1, \dots, \mu_n)$. The distance from x_1 to x_2 is then defined by

$$d(x_1, x_2) = \sqrt{\sum_{i=1}^{n} \left(\log \frac{\lambda_i}{\mu_i} \right)^2}. \tag{4.1}$$

It then turns out that (M, d) is a CAT(0) space (this is however non-trivial to verify, see Exercise I.4). The following key feature of that space is easy to deduce from the definition given above.

PROPOSITION I.9. *Let (M, d) be the symmetric space associated with $\mathrm{SL}_n(\mathbf{R})$. Then any two points of M are contained in a common flat of dimension $n-1$.*

A **flat** of dimension k in a CAT(0) space is a subset isometric to the Euclidean space $\mathbf{R}^k$. The **rank** of a symmetric space is the maximal dimension of a flat. The above property is a special instance of a general property: *in a symmetric space of rank r, any two points are contained in a common r-flat.*

5. Euclidean buildings

Let $W \leq \mathrm{Is}(\mathbf{R}^n)$ be a **discrete reflection group**, i.e. a discrete subgroup generated by orthogonal reflections through hyperplanes.

The discreteness of W implies that the collection $\mathcal{H}$ of all hyperplanes associated with reflections in W is **locally finite**, i.e. every ball meets only finitely many hyperplanes in $\mathcal{H}$. In fact, the pattern determined by $\mathcal{H}$ defines a cellular decomposition of $\mathbf{R}^n$, which is called a **Euclidean Coxeter complex**. A **chamber** in that complex is defined as a connected component of the space $\mathbf{R}^n - \bigcup_{H \in \mathcal{H}} H$. The group W acts sharply transitively on the set of chambers. The top-dimensional cells in a Coxeter complex coincide with the closures of the chambers, which may be non-compact. Any lower dimension cell is the intersection of a closed chambers with a set of hyperplanes in $\mathcal{H}$.

A **Euclidean building** is a cell complex Δ satisfying the following two conditions:

(1) Any two cells are contained in a common subcomplex, called an **apartment**, which is (combinatorially) isomorphic to a Euclidean Coxeter complex.
(2) Given any two apartments A_1 and A_2 in Δ, there is an isomorphism $\varphi\colon A_1 \to A_2$ fixing the intersection $A_1 \cap A_2$ pointwise.

A Euclidean building is thus primarily a combinatorial object. It always possesses a CAT(0) metric realization:

Proposition I.10. *Let Δ be a Euclidean building. Then Δ has a metric realisation $(|\Delta|, d)$ such that for each apartment $A \subset \Delta$, the restriction of d to $|A|$ is the Euclidean metric. The metric space $(|\Delta|, d)$ is a complete* CAT(0) *space.*

Proof. Axiom (2) implies that all apartments are combinatorially isomorphic. Fix a Euclidean metric on one of them, and transport this metric to all the others via the isomorphisms provided by (2). The axioms imply that this yields a geometric realisation $|\Delta|$ endowed with a well defined map $d\colon |\Delta| \times |\Delta| \to \mathbf{R}_+$ whose restriction to each apartment is the Euclidean metric. One may then verify that $(|\Delta|, d)$ is a metric space which satisfies the CAT(0) condition. The fact that it is geodesic is immediate from (1). See [AB08, Theorem 1.16] for details. □

The simplest example of a building is when W is the infinite dihedral group acting properly on the real line. In that case, the corresponding Coxeter complex is the simplicial line, and a Euclidean building having that Coxeter complex as type of apartments is a simplicial tree without vertex of valency one. Conversely any simplicial tree without vertex of valency one is a Euclidean building. Likewise, if W is a product of n copies of the infinite dihedral groups acting properly on $\mathbf{R}^n$, the corresponding buildings are products of n trees.

Euclidean buildings are the natural 'discrete' analogues of symmetric spaces. In fact, to any semi-simple linear algebraic group over a local field (e.g. $\mathrm{SL}_n(\mathbf{Q}_p)$), one may associate a Euclidean building on which the group acts isometrically, transitively on the chambers. This is part of the Buhat–Tits theory [BT72]. Let us merely mention here that the key feature of symmetric spaces pointed out in Proposition I.9 is shared by Euclidean buildings:

Proposition I.11. *In the* CAT(0) *realization of a Euclidean building of dimension n, any two points are contained in a common n-flat.* □

The fact that the rank coincides with the dimension is of course peculiar to buildings; the symmetric space associated with $\mathrm{SL}_n(\mathbf{R})$ has rank $n-1$ and dimension $\frac{(n-1)(n+2)}{2}$. In fact, one has the following characterization of Euclidean buildings among locally compact CAT(0) spaces:

Theorem I.12 (Kleiner). *Let X be a locally compact* CAT(0) *space of geometric dimension n. If any two points are contained in a common n-flat, then X is the metric realization of a Euclidean building.*

The notion of **geometric dimension** was introduced by B. Kleiner [Kle99]. It can be defined as the supremum over all compact subsets $K \subset X$ of the topological dimension of K. If X is a piecewise Euclidean cell complex, the geometric dimension coincides with the maximal dimension of a cell. For further information and alternative characterizations, see [Kle99].

A more detailed introduction on Euclidean buildings can be found in [Bro89]. See also [AB08] for a comprehensive account. The Euclidean buildings defined above are sometimes called **discrete** Euclidean buildings, in order to distinguish them within a more general class of objects, called **R-buildings** (or **non-discrete Euclidean buildings**). Those generalize discrete buildings in the same way as **R**-trees generalize simplicial trees; they appear naturally in the Bruhat–Tits theory of reductive groups over fields with a non-discrete valuation. They also pop up as asymptotic cones of symmetric spaces of non-compact type, as proved by Kleiner and Leeb (see [KL97], as well as the lectures by M. Kapovich).

6. Rigidity

Symmetric spaces and Euclidean buildings should be considered as leading examples of CAT(0) spaces. This is not only justified by the fact that their features serve as a basis for the intuition in the study of more general CAT(0) spaces, but also because these spaces (especially in rank > 1) seem to be the most rigid among all proper CAT(0) spaces. We finish this first lecture by mentioning some instances of this matter of fact.

Products of symmetric spaces and Euclidean buildings arise naturally in the study of arithmetic groups (see the lectures by T. Gelander and by D. Morris). The prototypical example is $\Gamma = \mathrm{SL}_n(\mathbf{Z}[\frac{1}{p_1 \cdots p_r}])$, with $p_1, \dots, p_r$ distinct primes. Indeed the diagonal embedding of Γ in $G = \mathrm{SL}_n(\mathbf{R}) \times \mathrm{SL}_n(\mathbf{Q}_{p_1}) \times \cdots \times \mathrm{SL}_n(\mathbf{Q}_{p_r})$ is a lattice embedding. In particular the discrete group Γ acts properly on the model CAT(0) space of G, which is the product

$$X = M \times \Delta_{p_1} \times \cdots \times \Delta_{p_r}$$

of the symmetric space M of $\mathrm{SL}_n(\mathbf{R})$ with the Euclidean buildings Δ_{p_i} of $\mathrm{SL}_n(\mathbf{Q}_{p_i})$.

The following result highlights a strong rigidity property of the arithmetic group Γ.

Theorem I.13 ([CM09a, Th. 1.14 and 1.15]). *Let $\Gamma = \mathrm{SL}_n(\mathbf{Z}[\frac{1}{p_1 \cdots p_r}])$, with $n \geq 3$ and $r \geq 0$, act by isometries on a proper cocompact* CAT(0) *space Y. Assume that Γ acts* ***minimally*** *in the sense that it does not preserve any non-empty closed convex subset $Z \subsetneq Y$.*

Then Y is a product of symmetric spaces and Euclidean buildings, which is a subproduct of the model space X.

Notice that no properness assumption is made on the action of Γ on Y. The theorem shows that Γ admits only few minimal actions on proper cocompact CAT(0) spaces: all of them occur as projections of the Γ-action on the model space X on a subproduct. In particular, when the model space has only one factor, i.e. when $\Gamma = \mathrm{SL}_n(\mathbf{Z})$, it follows that any minimal action of Γ on a proper cocompact CAT(0) space is either trivial, or proper and coincides with the standard Γ-action on the symmetric space $\mathrm{SL}_n(\mathbf{R})/\mathrm{SO}(n)$.

Theorem I.13 can be viewed as a rigidity property of the arithmetic group $\mathrm{SL}_n(\mathbf{Z}[\frac{1}{p_1 \cdots p_r}])$. The following result should rather be interpreted as a rigidity property of its model space X. We recall the isometries of a CAT(0) fall into three families, called **elliptic**, **hyperbolic** and **parabolic** respectively. Elliptic isometries are those which fix points. Hyperbolic isometries are those which preserve some geodesic line and act non-trivially along it. Parabolic isometries are all the others; they can have translation length zero or not, and should be viewed as the wilder type of isometries, especially when the ambient space is not locally compact. See [Bal95, §II.3] and [BH99, §II.6].

Theorem I.14 ([CM09b, Th. 1.5]). *Let X be a locally compact, geodesically complete, cocompact* CAT(0) *space. Assume that the full isometry group* $\mathrm{Is}(X)$ *contains a lattice Γ which is finitely generated, residually finite, and* ***indecomposable*** *in the sense that it does not split non-trivially as a direct product, even virtually.*

If X admits some parabolic isometry, then X is a product of symmetric spaces and Euclidean buildings.

The hypothesis that $\Gamma < \mathrm{Is}(X)$ be a finitely generated lattice is automatically satisfied if Γ is a discrete group acting properly and cocompactly on X.

The content of this course includes some of the main ingredients coming into the proofs of Theorems I.13 and I.14.

We finish this section by mentioning a conjecture geometric characterization of symmetric spaces and Euclidean buildings, independent of any discrete group action. To this end, let us denote by $(\mathcal{P}_n)$ the property that any two points are contained in a common n-flat. Clearly, a CAT(0) space satisfies $(\mathcal{P}_1)$ if and only if it is geodesically complete. We have seen that for all n, symmetric spaces of rank n and Euclidean buildings of dimension n satisfy $(\mathcal{P}_n)$. Moreover, one has the following easy observation:

Lemma I.15. *Let $X = X_1 \times X_2$ be a* CAT(0) *product space. If X_1 and X_2 satisfy $(\mathcal{P}_{n_1})$ and $(\mathcal{P}_{n_2})$ respectively, then X satisfies $(\mathcal{P}_{n_1+n_2})$.*

We have thus three sources of CAT(0) spaces satisfying $(\mathcal{P}_n)$ with $n > 1$: symmetric spaces, Euclidean buildings, and products of geodesically complete spaces. It is an important question to determine to what extent these are the only sources:

Conjecture I.16 (Ballmann–Buyalo [BB08]). *Let X be a proper cocompact* CAT(0) *space. If X satisfies $(\mathcal{P}_n)$ for some $n \geq 2$, then X is a symmetric space, or a Euclidean building, or a (non-trivial)* CAT(0) *product space.*

This conjecture is closely related to the the phenomenon called **Rank Rigidity**. It is known in case X is a manifold of non-positive curvature, see [Bal95] and references therein. It has also been verified when X is a CAT(0) cell complex of dimension 2 or 3 by Ballmann and Brin [BB95, BB00]. It is true if X has dimension n by Kleiner's theorem (Theorem I.12 above). It is moreover true when X is a CAT(0) cube complex of arbitrary dimension, see [CS11] as well as M. Sageev's lecture notes in this volume.

7. Exercises

Exercise I.1. Let X be a proper metric space and let $G \leq \mathrm{Is}(X)$.
Show that the orbit space $G\backslash X$ is compact if and only if there is a ball in X which meets every G-orbit.

Exercise I.2. Let (X_1, d_1) and (X_2, d_2) be CAT(0) spaces. Given $p \in [1, \infty)$, let d_p be the metric on the cartesian product $X = X_1 \times X_2$ defined by $d^p = d_1^p + d_2^p$. Show that (X, d_p) is a CAT(0) space if and only if $p = 2$.

Exercise I.3. (i) Let $X = X_1 \times X_2$ be a CAT(0) product space. Show that X is geodesically complete if and only if X_1 and X_2 are both so.
(ii) Show that every CAT(0) space embeds as a convex subset in some geodesically complete CAT(0) space.
(iii) Show that Theorem I.8(ii) fails for spaces that are not locally compact.

Exercise I.4. Let $M = \mathrm{SL}_n(\mathbf{R})/\mathrm{SO}(n)$ and $d : M \times M \to \mathbf{R}$ be the map defined by (4.1).

(i) Show that d is well defined, i.e. it does not depend on the choice of a diagonalizing basis.
(ii) Given a positive definite $n \times n$ matrix A, we denote by $\lambda(A)$ the vector formed by its eigenvalues put in non-increasing order. A result by Lidskii (see [Bha97]) asserts that for A, B positive definite, one has $\log(\lambda(AB)) \prec \log \lambda(A) + \log \lambda(B)$. The expression $(x_1, \ldots, x_n) \prec (y_1, \ldots, y_n)$ for two non-increasing sequences means that the latter sequence **majorizes** the former, i.e. $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ for all k, and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$.

Use Lidskii's result to show that (M, d) is a metric space.

(iii) (Open problem) Find a direct proof that (M, d) is a CAT(0) space, without using differential geometry.

LECTURE II

Geometric density

1. A geometric relative of Zariski density

Let X be a CAT(0) space and $G < \mathrm{Is}(X)$ be a group of isometries. The G-action is called **minimal** if G does not preserve any non-empty closed convex subset $X' \subsetneq X$. The group G is called **geometrically dense** if G acts minimally and without a fixed point at infinity on X. (For a brief recap on points at infinity, see §2 below.)

This notion can be viewed as coarsely related to *Zariski density* in the case of linear groups. Indeed, if X is a symmetric space of non-compact type, then any geometrically dense subgroup $G < \mathrm{Is}(X)$ is Zariski dense. This can be deduced from the Karpelevic–Mostow theorem. If in addition X is irreducible of rank ≥ 2, the converse holds by a theorem of Kleiner and Leeb [KL06]. In rank one symmetric spaces, there exist Zariski dense subgroups which do not act minimally (see Exercise II.10).

A CAT(0) space is called **irreducible** if it does not split as a CAT(0) product space in a non-trivial way. The symmetric space associated with a *simple* Lie group is always irreducible, as is the Euclidean building associated with a *simple* algebraic group over a local field. The following property of the full isometry group of a proper CAT(0) space could be viewed as some very weak form of 'simplicity'.

Theorem II.1 ([CM09a, Th. 1.10]). *Let X be a proper* CAT(0) *space which is irreducible, not isometric to the real line, and has finite-dimensional visual boundary ∂X.*

Given a geometrically dense subgroup $G < \mathrm{Is}(X)$, any normal subgroup $N \lhd G$ is either trivial or geometrically dense.

If $X = \mathbf{R}$ is the real line, a non-trivial normal subgroup $N \lhd G$ still acts minimally on X, but may obviously fix the two elements of ∂X.

The notion of *dimension* referred to in the theorem and appearing frequently in the rest of these notes, is Kleiner's *geometric dimension* defined in the previous lecture. The condition that X has finite-dimension visual boundary ∂X is automatic if X is cocompact (see [Kle99, Th. C]), or if X itself is finite-dimensional (see [CL10, Prop. 2.1]).

2. The visual boundary

The **visual boundary** of X is the set of asymptotic classes of geodesic rays. It is denoted by ∂X. The visual boundary ∂X comes equipped with two different natural topologies, which are both preserved by $\mathrm{Is}(X)$:

- The **cone topology**, which is defined by viewing $X \cup \partial X$ as the space of all geodesic segments and rays issuing from some fixed base point, endowed with the topology of uniform convergence on bounded subsets. That topology is independent of the choice of a base point. Moreover, when X is proper, the space $X \cup \partial X$ is compact by the Arzela–Ascoli theorem, the subset X is open and dense, and ∂X is closed, hence compact. In particular $X \cup \partial X$ with the cone topology is a compactification of X, usually called the **visual compactification**.

- The topology induced by the **angular metric**. The **angle** between two points $\xi, \eta \in \partial X$ is defined by
$$\angle(\xi, \eta) = \sup_{x \in X} \angle_x(\xi, \eta),$$
where $\angle_x(\xi, \eta)$ denotes the Alexandrov angle at x between the unique geodesic rays issuing from x and pointing to ξ and η respectively. The angular metric is indeed a metric, and the associated topology is finer (and often strictly finer) than the cone topology.

For a detailed treatment of the visual boundary, see [BGS85, §3–4], [Bal95, § II.1–II.4] and [BH99, §II.8–II.9]. A fundamental fact is that if X is complete, then the metric space $(\partial X, \angle)$ is a complete CAT(1) space, see [BH99, Th. II.9.20]. Here we shall content ourselves with mentioning a few facts needed in the sequel.

It is a well known fact that a bounded subset Z of a complete CAT(0) space X admits a unique **circumcenter**, i.e. a unique point c such that Z is contained in the closed ball $B(c, R)$ of radius R around c, where c is defined as $c = \inf_{x \in X}\{r \in \mathbf{R} \mid Z \subset B(x, r)\}$. The number r is called the **circumradius** of Z. In CAT(1) geometry, a similar statement holds *provided the subset Z is assumed to have circumradius $< \pi/2$*, see [BH99, Prop. II.2.7]. The following important result, due to Balser and Lytchak, shows that this can be extended to sets of circumradius $\leq \pi/2$ provided the set Z is convex and finite-dimensional:

THEOREM II.2 ([BL05, Prop. 1.4]). *Let Z be a finite-dimensional complete CAT(1) space. If Z has circumradius $\pi/2$, then the set of circumcenters of Z has circumradius $< \pi/2$. In particular the full isometry group $\mathrm{Is}(Z)$ fixes a point in Z.* □

We emphasize that Theorem II.2 fails without the finite-dimensionality assumption, see Exercise II.3. The typical situation in which we shall apply Theorem II.2 is the following: the space Z will be a closed convex subset of the visual boundary ∂X of a proper CAT(0) space X. As mentioned above, the finite-dimensionality hypothesis is automatically satisfied if X is cocompact, since the full visual boundary ∂X is then finite-dimensional. It should be noted that the circumradius of Z as a subset of X may be smaller than the **intrinsic** circumradius of Z, defined by $\inf_{z \in Z}\{r \in \mathbf{R} \mid Z \subset B(z, r)\}$. It is of course the intrinsic circumradius that has to be used when applying Theorem II.2 to a closed convex subset $Z \subseteq \partial X$. In the situations we shall encounter, the upper bound of $\pi/2$ on the circumradius will be deduced from the following observation.

PROPOSITION II.3. *Let X be a proper* CAT(0) *space, and $(Y_i)_{i \in I}$ be a descending chain of closed convex subsets.*

If $\bigcap_{i \in I} Y_i$ is empty, then $\bigcap_{i \in I} \partial Y_i$ is a non-empty closed convex subset of ∂X, whose circumradius is at most $\pi/2$.

PROOF. Pick $x \in X$ and let y_i be its orthogonal projection to Y_i. If the set $(y_i)_{i \in I}$ is bounded, then $\bigcap_{i \in I} Y_i$ is non-empty. Assume that this is not the case. We can then extract a countable chain $(Y_{i(n)})_{n \geq 0}$ such that the sequence $(y_{i(n)})$ converges to some boundary point $\xi \in \partial X$ with respect to the cone topology. In particular $\bigcap_n Y_{i(n)}$ is empty and $Z = \bigcap_n \partial Y_{i(n)} = \bigcap_i Y_i$. Notice moreover that ξ belongs to Z.

It remains to show that for each $\eta \in Z$, we have $\angle(\xi, \eta) \leq \pi/2$. To this end, observe that there is a sequence $y'_n \in Y_{i(n)}$ converging to η in the cone topology. We have $\pi/2 \leq \angle_{y_{i(n)}}(x, y'_n)$ by the properties of the projection [BH99, Prop. II.2.4], and $\angle_{y_{i(n)}}(x, y'_n) \leq \overline{\angle}_{y_{i(n)}}(x, y'_n)$ by the CAT(0) condition, where $\overline{\angle}$ denotes the angle in a Euclidean comparison triangle. It follows that $\overline{\angle}_x(y_{i(n)}, y'_n) \leq \pi/2$. By [BH99, Prop. II.9.16], this implies that $\angle(\xi, \eta) \leq \pi/2$, as desired. □

3. Convexity

A map $f\colon X \to \mathbf{R}$ is called **convex** if for each geodesic $\rho\colon I \to X$, the composed map $f \circ \rho\colon I \to \mathbf{R}$ is convex. In that case, sublevel sets of f are convex subsets of X. Here are a few examples:

- Given a point $p \in X$, the distance to p, namely
$$d_p\colon X \to \mathbf{R} : x \mapsto d(x,p)$$
is convex: this follows right away from the CAT(0) condition. Its sublevel sets are nothing but balls around p.
- Given a complete convex subset $Y \subset X$, the distance to Y, namely
$$d_Y\colon X \to \mathbf{R} : x \mapsto d(x,Y) = \inf_{y\in Y} d(x,y)$$
is convex, see [BH99, Cor. II.2.5]. Its sublevel sets are called **tubular neighbourhoods** of Y and denote by $\mathcal{N}_r(Y) = f^{-1}([0,r])$.
- Given a geodesic ray $\rho\colon [0,\infty) \to X$, the function $b_\rho\colon X \to \mathbf{R}$ defined by
$$b_\rho(x) = \lim_{t\to\infty} d(x,\rho(t)) - t$$
is well defined, convex and 1-Lipschitz, see Exercise II.7. It is called the **Busemann function** associated with ρ. Its sublevel sets are called **horoballs** centered at the endpoint $\xi = \rho(\infty)$. If ρ' is another geodesic ray having ξ as endpoint, then the Busemann functions b_ρ and $b_{\rho'}$ differ by a constant, so that the collection of horoballs centered at ξ does not depend on the choice of a geodesic ray pointing to ξ.
- Given an isometry $g \in \mathrm{Is}(X)$, its **displacement function** $d_g\colon X \to \mathbf{R}$ defined by $d_g(x) = d(x, g.x)$ is convex and 2-Lipschitz, see Exercise II.5. The infimum of the displacement function is called the **translation length**, and is denoted by $|g|$. The sublevel set $f^{-1}([0,|g|]) = f^{-1}(|g|)$, which is thus closed and convex, is denoted by $\mathrm{Min}(g)$. It is non-empty if and only if g is not parabolic.

The existence of isometries with a constant displacement function witnesses the presence of a Euclidean factor:

PROPOSITION II.4 ([BH99, Th. II.6.5]). *A* CAT(0) *space X admits a non-trivial isometry with constant displacement function if and only if X splits as a product $X \cong \mathbf{R} \times X'$.* □

We record the following consequence:

COROLLARY II.5. *Let X be a* CAT(0) *space without non-trivial Euclidean factor.*
For any group $G < \mathrm{Is}(X)$ acting minimally on X, the centraliser $\mathscr{Z}_{\mathrm{Is}(X)}(G)$ is trivial. In particular so is the center $\mathscr{Z}(G)$.

PROOF. Let $g \in \mathscr{Z}_{\mathrm{Is}(X)}(G)$. Then the displacement function d_g is G**-invariant** in the sense that d_g is constant on each G-orbit. In particular G preserves all sublevel sets of d_g, which are closed and convex. By minimality, it follows that d_g has no non-trivial sublevel set; in other words d_g is constant, and Proposition II.4 concludes the proof. □

Here is another straightforward application of convexity.

LEMMA II.6. *Let X be a complete* CAT(0) *space. Given $G < \mathrm{Is}(X)$ and two points $y, z \in X$, we have $\partial\overline{\mathrm{Conv}(G.y)} = \partial\overline{\mathrm{Conv}(G.z)}$, where $\mathrm{Conv}(Y)$ denotes the convex hull of Y, and $G.y$ the G-orbit of y.*

Proof. Set $Y = \overline{\mathrm{Conv}(G.y)}$ and $Z = \overline{\mathrm{Conv}(G.z)}$. Then Y and Z are both G-invariant. Setting $r = d(Y, z)$, we obtain $G.z \subseteq \mathcal{N}_r(Y)$. Since $\mathcal{N}_r(Y)$ is closed and convex, this yields $Z \subseteq \mathcal{N}_r(Y)$. Similarly $Y \subseteq \mathcal{N}_r(Z)$, and hence Y and Z are a bounded Hausdorff distance apart. Therefore $\partial Y = \partial Z$, see Exercise II.1. □

Lemma II.6 allows one to associate a canonical subset $\Delta G \subseteq \partial X$ to the group G, defined as the visual boundary of the closed convex hull of some (arbitrarily chosen) orbit. We call ΔG the **convex limit set** of G. It contains (generally as a proper subset) the usual **limit set** ΛG, which is defined as the intersection with the visual boundary ∂X of the closure of some G-orbit in the union $X \cup \partial X$, endowed with the cone topology. Notice that the convex limit set is defined as the visual boundary of a complete CAT(0) subspace of X, and is thus a CAT(1) space. In other words, it is closed and convex in ∂X.

4. A product decomposition theorem

Let X be a complete CAT(0) space and ℓ be a geodesic line in X. It is then a standard fact (see [BH99, Th. II.2.14]) that the union $\mathcal{P}(\ell)$ of all geodesic lines having the same endpoints as ℓ in the visual boundary, is a closed convex subset of X, which splits as a CAT(0) product $P(\ell) \cong \mathbf{R} \times C$. This fact is actually the key point in the proof of Proposition II.4. Our next task is to extend that statement to more general subspaces than lines. To this end, we need an additional piece of terminology.

A closed convex subset $Y \subseteq X$ is called **boundary-minimal** if for every closed convex subset $Z \subsetneq Y$, we have $\partial Z \subsetneq Y$. For instance, a geodesic line is boundary-minimal while a geodesic ray is not.

Theorem II.7 ([CM09a, Prop. 3.6]). *Let X be a proper* CAT(0) *space and let $\Delta \subseteq \partial X$. Set*

$$\mathscr{C}_\Delta = \{Y \subseteq X \mid Y \text{ is boundary-minimal and } \partial Y = \Delta\}.$$

Then the union $\bigcup \mathscr{C}_\Delta$ is a closed convex subset which splits as a CAT(0) *product $\bigcup \mathscr{C}_\Delta \cong Y \times C$. Moreover $\mathscr{C}_\Delta$ coincides with the set of fibers $\big\{Y \times \{c\} \mid c \in C\big\}$.*

Proof. We shall only prove the key point, namely the fact that for any two sets $Z_1, Z_2 \in \mathscr{C}_\Delta$, the distance function to Z_1, denoted by d_{Z_1}, is constant on Z_2.

Let $Z_2' \subseteq Z_2$ be a non-empty sublevel set of the restriction of d_{Z_1} to Z_2. Thus there is some $r > 0$ such that $Z_2' = \{z \in Z_2 \mid d_{Z_1}(z) \leq r\}$, and Z_2' is closed and convex.

Let then $\xi \in \Delta$ and pick any $p \in Z_2'$. Let also $\rho : [0, \infty) \to X$ be the geodesic ray issuing from p and pointing to ξ. Since $\xi \in \Delta = \partial Z_2$ and since Z_2 is closed and convex, it follows that the point $\rho(t)$ belongs to Z_2 for all t. Since ξ also belongs to ∂Z_1, the ray $\rho([0, \infty))$ is entirely contained in a tubular neighbourhood of Z_1. It follows that the map $t \mapsto d_{Z_1}(\rho(t))$ is bounded convex function. It must therefore be non-increasing. Since $\rho(0) = p \in Z_2'$, it follows that $\rho(t) \in Z_2'$ for all t. In particular ξ belongs to $\partial Z_2'$. This proves that $\partial Z_2' = \partial Z_2$. Since Z_2 is boundary-minimal, we deduce that $Z_2' = Z_2$ which proves that the function d_{Z_1} is constant on Z_2, as claimed.

The rest of the proof of the theorem uses the Sandwich Lemma [BH99, Ex. II.2.12], and is similar to the special case of the parallel set of a geodesic line mentioned above. Further details are provided in [CM09a, Prop. 3.6]. □

Remark that the set $\mathscr{C}_\Delta$ is potentially empty.

5. Geometric density of normal subgroups

We are now in a position to complete the proof of geometric density for normal subgroups.

Proof of Theorem II.1. Let $N \lhd G$ be a non-trivial normal subgroup and $\Delta = \Delta N$ be its convex limit set.

If Δ is empty, then the N-orbits are bounded and the fixed point set X^N of N is thus a non-empty closed convex G-invariant subset. By minimality, we must have $X^N = X$, whence N is trivial.

We assume henceforth that Δ is non-empty and consider the set $\mathscr{C}_\Delta$.

Suppose now that $\mathscr{C}_\Delta$ is empty. Then by Zorn's lemma, there exists a chain of closed convex subspaces $(Y_i)_{i \in I}$ such that $\partial Y_i = \Delta$ for all i, and $\bigcap_i Y_i = \varnothing$. By Proposition II.3, it follows that Δ has intrinsic circumradius at most $\pi/2$. Since Δ is G-invariant, Theorem II.2 implies that G fixes a point in ∂X, a contradiction.

Thus $\mathscr{C}_\Delta$ is non-empty. By Theorem II.7, the union $\bigcup \mathscr{C}_\Delta$ is then a non-empty closed convex subset splitting as a product of the form $Y \times C$ with all fibers $Y \times \{c\}$ belonging to $\mathscr{C}_\Delta$. Since $\mathscr{C}_\Delta$ is G-invariant and since the G-action on X is minimal, it follows that $X = \bigcup \mathscr{C}_\Delta$. Since X is irreducible, the product decomposition $X \cong Y \times C$ must be trivial. Thus either Y or C is reduced to a singleton. The former case is impossible, since it would mean that the elements of $\mathscr{C}_\Delta$ are singletons, which is absurd since they have a non-empty visual boundary. Thus $X \cong Y \times \{c\}$, which implies that X belongs to $\mathscr{C}_\Delta$. Thus X is boundary-minimal. It follows that N-acts minimally on X. Indeed, given a non-empty closed convex N-invariant subset $Z \subsetneq X$, we have $\Delta N \subseteq \partial Z \subseteq \partial X$. Since $\Delta N = \partial X$, we have $\partial Z = \partial X$, whence $Z = X$ since X is boundary-minimal.

This proves that any non-trivial normal subgroup $N \lhd G$ acts minimally on X. It remains to show that N does not fix any point at infinity. Suppose on the contrary that N fixes some $\xi \in \partial X$. Then the commutator subgroup $[N, N]$ annihilates the Busemann character centered at ξ (see Exercise II.7) and therefore stabilises each horoball around ξ. In particular it does not act minimally on X. But N being normal in G, its commutator subgroup $[N, N]$ is also normal in G, and is thus trivial by the first part of the proof. Thus N is abelian. This is absurd, since a group acting minimally on CAT(0) space without Euclidean factor must be center-free by Corollary II.5. □

Remark that the finite-dimensionality of ∂X was only used through the application of Theorem II.2. It is an interesting question to determine whether Theorem II.1 holds if X is a proper CAT(0) space with infinite-dimensional visual boundary.

Clearly Theorem II.1 can be bootstrapped, thereby giving information on subnormal subgroups:

Corollary II.8. *Let X be a proper cocompact* CAT(0) *space which is irreducible, and not isometric to the real line. Let $G < \mathrm{Is}(X)$ be a geometrically dense subgroup and $H < G$ be a non-trivial subnormal subgroup. Then H is still geometrically dense; in particular:*

(i) $\mathscr{Z}_G(H) = 1$,
(ii) H does not split non-trivially as a direct product,
(iii) H is not soluble,
(iv) H does not have fixed points in X.

Proof. That H is geometrically dense is immediate from an iterated application of Theorem II.1, and (iv) follows right away. Part (i) is a consequence of Corollary II.5, Part (ii) follows from (i). Part (i) also implies that a subnormal subgroup cannot be abelian, which implies (iii). □

6. Exercises

Exercise II.1. Let X be a CAT(0) space and $Y, Z \subseteq X$ be two convex subsets. Show that if Y and Z are a bounded Hausdorff distance apart, then $\partial Y = \partial Z$. The converse does not hold in general.

EXERCISE II.2. Construct an example of a proper cocompact CAT(0) space X whose full isometry group is minimal, but not geometrically dense.

EXERCISE II.3. Show that Theorem II.2 fails if Z is infinite-dimensional. (Hint: a counterexample may be constructed as a closed convex subset of the unit sphere in a Hilbert space).

EXERCISE II.4. Let X be a proper CAT(0) space and let $G < \mathrm{Is}(X)$.

(i) Show that if G does not fix any point in ∂X, then G stabilises a non-empty closed convex subset $X' \subseteq X$ on which its action is minimal. This minimal G-invariant subspace X' need not be unique, even if G acts without fixed point in X.
(ii) Show that if G acts cocompactly on X, then the same conclusions hold.
(iii) Show that if X is geodesically complete and G acts cocompactly, then G acts minimally.

EXERCISE II.5. Show that the displacement function of an isometry of a CAT(0) space is convex and 2-Lipschitz.

EXERCISE II.6. Let X be a metric space and $G < \mathrm{Is}(X)$. A function $f \colon X \to \mathbf{R}$ is called **G-invariant** if f is constant on G-orbits, namely $f(g.x) = f(x)$ for all $x \in X$ and $g \in G$. A function $f \colon X \to \mathbf{R}$ is called **G-quasi-invariant** if for all $g \in G$, the map $X \to \mathbf{R} : x \mapsto f(g.x) - f(x)$ is constant. Assuming this is the case, we denote the difference by $c(g)$. Show that the map

$$G \to \mathbf{R} : g \mapsto c(g)$$

is a homomorphism.

EXERCISE II.7. Let X be a CAT(0) space.

(i) Show that Busemann functions associated with geodesic rays in X are well defined, convex and 1-Lipschitz.
(ii) Show that any Busemann function associated with a geodesic ray pointing to $\xi \in X$ is quasi-invariant under the stabiliser G_ξ of ξ in the full isometry group $G = \mathrm{Is}(X)$.
(iii) Show that the corresponding homomorphism $G_\xi \to \mathbf{R}$ defined as in Exercise II.6 depends only on ξ. This homomorphism is called the **Busemann character** at ξ.

EXERCISE II.8. Let X be a complete CAT(0) space and $G < \mathrm{Is}(X)$.

(i) Show that if X is geodesically complete, then every bounded convex function is constant.
(ii) Show that if X is boundary-minimal, then every bounded convex function is constant.
(iii) Show that G acts minimally on X if and only if every continuous G-invariant convex function is constant.
(iv) Show that G is geometrically dense if and only if every continuous G-quasi-invariant convex function is constant.

EXERCISE II.9. An action of a group G on a topological space Z by homeomorphism is called **(topologically) minimal**[1] if G does not preserve any non-empty closed subset $Z' \subsetneq Z$. Equivalently, the G-action is minimal if and only if every G-orbit is dense in Z.

Let M denote the symmetric space of $G = \mathrm{SL}_n(\mathbf{R})$. Show that the G-action on the visual boundary ∂M is minimal if and only if $n = 2$.

[1]This standard notion of minimality in topological dynamics should not be confused with the notion of minimality introduced above in the realm of CAT(0) geometry.

Exercise II.10. Let $G = \mathrm{SL}_2(\mathbf{R})$.

(i) Show that a subgroup $\Gamma < G$ is Zariski dense if and only if Γ is not virtually soluble.
(ii) Show that G contains Zariski dense subgroups that are not geometrically dense as isometry groups of the hyperbolic plane $\mathbf{H}^2$.

Exercise II.11. Let X be a proper CAT(0) space.

(i) Show that if X is boundary-minimal, then ∂X has circumradius $> \pi/2$.
(ii) Show that if X has finite-dimensional boundary and if $\mathrm{Is}(X)$ acts minimally, then X is boundary-minimal.

LECTURE III

The full isometry group

1. Locally compact groups

Our strategy in studying the full isometry group of a proper CAT(0) space is to combine geometric arguments with information arising from the structure theory of locally compact groups. The following classical fact shows that locally compact groups pop up naturally in our setting:

THEOREM III.1. *Let X be a proper metric sapce. Then the full isometry group $\mathrm{Is}(X)$, endowed with the compact open topology, is a locally compact (Hausdorff) topological group, and the natural action of $\mathrm{Is}(X)$ on X is continuous and proper.*

PROOF. See Exercise III.1. $\square$

The **continuity** of the action of $G = \mathrm{Is}(X)$ on X means that the map $G \times X \to X$ is continuous. The **properness** of the action means that for each ball B in X, the set $\{g \in G \mid g.B \cap B \neq \varnothing\}$ has compact closure in G.

A deep result in the theory of locally compact groups which we shall invoke is the following:

THEOREM III.2 (Gleason; Montgomery–Zippin [MZ55, Th. IV.4.6]). *Let G be a connected locally compact group. Then any identity neighbourhood in G contains a compact normal subgroup $K \lhd G$ such that G/K is a Lie group.*

2. The isometry group of an irreducible space

Combining the results obtained thus far yields the following.

COROLLARY III.3. *Let X be a proper $\mathrm{CAT}(0)$ space with finite-dimensional boundary, such that X is irreducible and $\mathrm{Is}(X)$ is geometrically dense.*

Then $\mathrm{Is}(X)$ is either a virtually connected simple Lie group, or $\mathrm{Is}(X)$ is totally disconnected (potentially discrete).

PROOF. Let $G = \mathrm{Is}(X)$. Thus G is a locally compact group by Theorem III.1. The connected component of the identity G° is a closed normal subgroup of G. By Theorem III.1, any compact subgroup of G has a bounded orbit, hence a fixed point in X. Corollary II.8(iv) thus ensures that G the only compact subnormal subgroup of G is trivial. In particular G° has no non-trivial compact normal subgroup, and must thus be a connected Lie group by Theorem III.2.

Corollary II.8(iii) implies that the solvable radical, as well as the center, of G° is trivial, hence G° is a center-free semi-simple Lie group. It is thus a product of simple groups, which can have at most one non-trivial factor by Corollary II.8(ii). This shows that G° is a centerfree simple Lie group.

A consequence of the classification of simple Lie groups is that the outer automorphism group $\mathrm{Out}(G^\circ)$ is finite. The conjuation action of G on G° yields a continuous map $\varphi \colon G \to \mathrm{Out}(G^\circ)$, whose kernel is thus a closed normal subgroup of G of finite index. Notice that $\mathrm{Ker}(\varphi) = G^\circ \cdot \mathscr{Z}_G(G^\circ)$. By Corollary II.8(i), either G° or its centraliser must be trivial. In the former case, the group G is totally disconnected. In the latter case, the identity component G° has finite index in G, so that G is virtually a connected simple Lie group. $\square$

It is not surprising that, in the Lie group case of Corollary III.3, much finer information on X can be extracted from the structure theory of simple Lie groups. Each maximal compact subgroup $K < G = \mathrm{Is}(X)$ fixes a point in X, and we thus get an equivariant embedding of the symmetric space $M = G/K$ into X. Notice however that *this embedding need not be isometric*, even up to scaling. Explicit examples of this phenomenon have recently been constructed by Monod and Py [MP12] with $G = \mathrm{SO}(n, 1)$ acting cocompactly on a proper CAT(0) space X, containing no isometric (and even homothetic) copy of the hyperbolic space $\mathbf{H}^n$. Of course, the cocompactness of the action implies that X is *quasi-isometric* to the symmetric space of G. That X is genuinely *isometric* to the symmetric space is however true if one imposes in addition that X be geodesically complete:

Theorem III.4 ([CM09a, Th. 7.4]). *Let X be a locally compact geodesically complete* CAT(0) *space and G be a virtually connected semi-simple Lie group acting continuously, properly and cocompactly on X by isometries.*

Then X is equivariantly isometric to the symmetric space of G (up to an appropriate scaling of each irreducible factor).

The same conclusion holds under the slightly weaker hypotheses that the action is minimal with full limit set, and that the boundary of X is finite-dimensional.

One should next analyze the totally disconnected case of Corollary III.3. Since that case includes the situation that $\mathrm{Is}(X)$ be discrete, conclusions in the same vein as those of Theorem III.4 cannot be expected. The following useful facts can however be derived under the hypothesis of geodesic completeness:

Theorem III.5 ([CM09a, §6]). *Let X be a locally compact geodesically complete* CAT(0) *space and G be a totally disconnected locally compact group acting continuously and properly on X by isometries. Then:*

*(i) The action is **smooth** in the sense that the pointwise stabiliser of every open set is open in G.*
(ii) Every G-orbit is discrete.
(iii) If the G-action is cocompact, then G does not contain parabolic isometries.
(iv) If the G-action is cocompact, then X admits a locally finite G-equivariant decomposition into convex pieces, such that the piece $\sigma(x)$ supporting a point $x \in X$ is defined as the fixed-point-set of the stabiliser G_x.

Remark that if X is geodesically complete, any group acting cocompactly automatically acts minimally (see Exercise II.4). Therefore, the following dichotomy follows immediately by combining the previous three results.

Corollary III.6. *Let X be a locally compact, geodesically complete, irreducible* CAT(0) *space such that* $\mathrm{Is}(X)$ *acts cocompactly without a fixed point at infinity. Then either* $\mathrm{Is}(X)$ *acts transitively on X, or* $\mathrm{Is}(X)$ *has discrete orbits.*

It should be emphasized that this dichotomy no longer holds without the hypothesis that $\mathrm{Is}(X)$ has no fixed point at infinity. Concrete examples illustrating this matter of fact are provided by the **millefeuille spaces** constructed in [CCMT12, §7].

3. de Rham decomposition

In the previous section, we focused on irreducible CAT(0) spaces. One should next show that the general case reduces to the irreducible one. This will require to impose suitable assumptions, since a 'de Rham decomposition theorem' cannot be expected in full generality for CAT(0) spaces, due to the possible presence of infinite-dimensional pieces. This happens even for locally compact spaces: a CAT(0) space can even be *compact* and infinite-dimensional, as is easily seen by considering compact convex subsets of a Hilbert space.

The following remarkable result, due to Foertsch and Lytchak, shows that infinite-dimensionality is the only obstruction to a 'de Rham decomposition' at a very broad level of generality:

THEOREM III.7 (Foertsch–Lytchak [FL08]). *Let X be a finite-dimensional geodesic metric space. Then X admits a canonical product decomposition*

$$X \cong \mathbf{R}^n \times X_1 \times \cdots \times X_p,$$

where $n, p \geq 0$, and where each factor X_i is irreducible, and neither reduced to a singleton, nor isometric to the real line (the right-hand side is given the ℓ^2-metric). Every isometry of X preserves the decomposition, up to a permutation of possibly isometric factors among the X_i. In particular $\mathrm{Is}(\mathbf{R}^n) \times \mathrm{Is}(X_1) \times \cdots \times \mathrm{Is}(X_p)$ *is a finite-index normal subgroup of* $\mathrm{Is}(X)$.

In the case of CAT(0) spaces, we have the following analogue:

THEOREM III.8 ([CM09a, Cor. 5.3]). *Let X be a proper* CAT(0) *space with finite-dimensional visual boundary ∂X, and such that* $\mathrm{Is}(X)$ *acts minimally.*

Then X admits a canonical CAT(0) *product decomposition, with the same properties as in Theorem III.7.*

The latter statement cannot be deduced directly from Theorem III.7, since the hypotheses do not imply in general that X itself be finite-dimensional. A detailed proof of Theorem may be found in [CM09a, §5.A]. An alternative approach in case X is cocompact can be taken using the following.

PROPOSITION III.9. *Let Z be a finite-dimensional, complete CAT(1) space. Then Z admits a canonical decomposion as a join*

$$Z \cong \mathbf{S}^n \circ Z_1 \circ \cdots \circ Z_p,$$

where $\mathbf{S}^n$ is the Euclidean n-sphere and each Z_i is not a sphere and does not decompose non-trivially as a join for all i. Every isometry of Z preserves the decomposition, up to a permutation of possibly isometric factors among the Z_i.

PROOF. Let X be the Euclidean cone over Z, defined as in [BH99, Def. I.5.6]. By Berestovskii's theorem [BH99, Th. II.3.14], the space X is a CAT(0) space, which is finite-dimensional since Z is so. (However X is not locally compact in general.) Every isometry of Z extends to an isometry of the cone X. The conclusion now follows by applying Theorem III.7 to X. □

One may now conclude the proof of Theorem III.8 as follows.

PROOF OF THEOREM III.8. By Exercise II.11, the space X is boundary-minimal. It follows that for every product decomposition $X \cong Y_1 \times \cdots \times Y_q$, each factor Y_i is unbounded (see Exercise II.1) and thus has a non-empty visual boundary ∂Y_i. In other words, every product decomposition of X determines a join decomposition of the visual boundary ∂X, the factors in both decompositions being canonically in one-to-one correspondence. From Proposition III.9, it follows that X admits at least one product decomposition $X \cong \mathbf{R}^n \times X_1 \times \cdots \times X_q$ with a maximal Euclidean factor $\mathbf{R}^n$ and finitely many irreducible non flat factors.

At this point, we know that X admits at least one decomposition as a product of flat and irreducible factors, and that each such decomposition corresponds to some regrouping of factors in the canonical join decomposition of ∂X afforded by Proposition III.9. The desired result now follows from Proposition III.10 below, which implies that the visual boundary of an irreducible factor of X does not admit any non-trivial join decomposition. Indeed, this shows that the various factors in the maximal decompositions of X and ∂X are canonically in one-to-one correspondence, so that the canonicity of the decomposition of X follows from that of ∂X. □

Proposition III.10. *Let X be a proper* CAT(0) *space such that* Is(X) *acts cocompactly and minimally.*

Then X admits a non-trivial product decomposition $X = X_1 \times X_2$ if and only if ∂X admits a join decomposition $\partial X = \Delta_1 \circ \Delta_2$ with $\Delta_i = \partial X_i$.

Proof. (See [BH99, Th. II.9.24] for the case when X is geodesically complete.) The 'only if' part is clear. We assume henceforth that $\partial X = \Delta_1 \circ \Delta_2$. Since X is cocompact, it follows from [GO07] that every point ξ in ∂X admits some **opposite** ξ', i.e. ξ' is such that ξ and ξ' are the endpoints of some geodesic line. Moreover, given $\xi \in \Delta_i$, any point ξ' opposite ξ also belongs to Δ_i. By intersecting two horoballs respectively centered at ξ and ξ', one constructs closed convex subsets of X whose visual boundary is exactly Δ_{3-i} (see [CM09a, Lem. 3.5]). From Exercise II.11, we infer that X is boundary-minimal, and hence that each factor Δ_i must have radius $> \pi/2$. Therefore, the set $\mathscr{C}_{\Delta_i}$ from Theorem II.7 is non-empty by Proposition II.3. Theorem II.7 then provides a product decomposition $X = X_1 \times X_2$ such that $\partial X_i = \Delta_i$, as desired. □

The possibility that Is(X) may fix a point at infinity is not excluded in Theorem III.8, and does indeed occur sometimes (see Exercise II.2). However, assuming that the full isometry group is geometrically dense, the results obtained thus far assemble to yield the following, which already sheds some light on the conclusions of Theorem I.14.

Corollary III.11. *Let X be a locally compact geodesically complete* CAT(0) *space. Assume that* Is(X) *acts cocompactly without a fixed point at infinity. Then X admits a canonical product decomposition*

$$X \cong M_1 \times \cdots \times M_p \times \mathbf{R}^n \times Y_1 \times \cdots \times Y_q,$$

which is preserved by all isometries upon permutations of isomorphic factors, where M_i is an irreducible symmetric space of non-compact type, and Y_j has a totally disconnected isometry group, which acts smoothly and does not contain any parabolic isometry.

Proof. Since X is geodesically complete, any cocompact group action is minimal (see Exercise II.4). Theorem III.8 provides a canonical product decomposition for X, and the various properties of the irreducible non Euclidean factors were established in Corollary III.3 and Theorems III.4 and III.5. □

4. Exercises

Exercise III.1. Let X be a proper metric space and let $\mathrm{Is}(X)_{\mathrm{p.o.}}$ denote the full isometry group of X endowed with the point-open topology. Let also $\varphi\colon \mathrm{Is}(X) \to X^X$ be the natural embedding of Is(X) in the space X^X of all maps from X to X, endowed with the product topology.

(i) Show that $\varphi(\mathrm{Is}(X))$ is closed in X^X.
(ii) Show that $\varphi\colon \mathrm{Is}(X)_{\mathrm{p.o.}} \to X^X$ is a homeomorphism onto its image.
(iii) Deduce from (i) and (ii) that $\mathrm{Is}(X)_{\mathrm{p.o.}}$ is locally compact.

(iv) Show that the point-open and the compact-open topology on $\mathrm{Is}(X)$ coincide.
(v) Conclude the proof of Theorem III.1.

Exercise III.2. Let G be a locally compact group acting by isometries on a proper metric space X.

(i) Show that the following conditions are equivalent:
 (a) the G-action is continuous,
 (b) the orbit maps $G \to X : g \mapsto g.x$ are continuous for all $x \in X$,
 (c) the homomorphism $\alpha\colon G \to \mathrm{Is}(X)$ induced by the action is continuous.
(ii) Assuming that the G-action is continuous, show that the following conditions are equivalent:
 (a) the G-action is proper,
 (b) the homomorphism $\alpha\colon G \to \mathrm{Is}(X)$ induced by the action is continuous is proper.

 If in addition G is separable, then those conditions are also equivalent to:
 (c) $\mathrm{Ker}(\alpha)$ is compact and $\alpha(G)$ is closed in $\mathrm{Is}(X)$.

Exercise III.3. Let X be a locally compact geodesically complete CAT(0) space.

(i) Prove point (ii) in Theorem III.5 using point (i).
(ii) Show that if a non-discrete totally disconnected locally compact group G acts continuously and properly on X, then some geodesics in X must branch.
(iii) Show that if $\mathrm{Is}(X)$ is geometrically dense and every geodesic can be prolonged into a *unique* bi-infinite geodesic line, then $\mathrm{Is}(X)$ is a Lie group.

LECTURE IV

Lattices

1. Geometric Borel density

The phenomenon of geometric density of normal subgroups has been discussed in Theorem II.1. We shall now present a related statement for lattices. In the light of the analogy between geometric density and Zariski density, this could be interpreted as a geometric version of the Borel density theorem (in fact, the classical statement can indeed be deduced from the geometric version, see [CM09b, Prop. 2.8]).

THEOREM IV.1 ([CM09b, Th. 2.4]). *Let X be a proper* CAT(0) *space without non-trivial Euclidean factor. Let G be a locally compact group and $\varphi\colon G \to \mathrm{Is}(X)$ be a continuous homormorphism.*

If $\varphi(G)$ is geometrically dense, then so is $\varphi(\Gamma)$ for each lattice $\Gamma < G$ (and, more generally, for each closed subgroup of finite covolume).

The proof consists in two parts: the first is to show the absence of Γ-fixed points at infinity, which is established by adapting an argument of Adams and Ballmann [AB98]; the second is to show that the Γ-action is minimal. Since some technicalities can be avoided when Γ is assumed cocompact, we will content ourselves with a discussion of the second part of the proof in that special case.

PROOF OF THEOREM IV.1. For simplicity, we assume that $\Gamma < G$ is cocompact and only discuss the proof of Γ-minimality; for a complete proof in the general case, the reader should consult [CM09b].

Hence we admit that the first part of the proof has already been accomplished, namely that $\varphi(\Gamma)$ does not fix any point in ∂X. It follows that there is a non-empty Γ-invariant closed convex subsets $Y \subseteq X$ on which Γ acts minimally (see Exercise II.4).

We need to show that $Y = X$. To this end, consider the distance function d_Y to Y. Since Γ is cocompact in G, it follows that for each $x \in X$, the map $G \to \mathbf{R} : g \mapsto d_{gY}(x)$ is continuous and bounded. In particular the function

$$f\colon X \to \mathbf{R} : x \mapsto \int_{G/\Gamma} d_{gY}(x)\mathrm{d}g$$

is well defined. Moreover it is convex and 2-Lispschitz since d_Y is so. By construction, it is G-invariant. Since G acts minimally on X, the map f must be constant. It follows that for almost all $g\Gamma \in G/\Gamma$, the map d_{gY} is **affine**, i.e. it is both convex and concave (see Exercises IV.1 and IV.2).

We have seen that there exists $g \in G$ such that d_{gY} is affine. Since $d_{hY} = d_Y \circ h^{-1}$ for all $h \in G$, we infer that d_{hY} is affine for all h; in particular so is d_Y. It follows from Exercise IV.1(ii) that the level sets of d_Y are all convex.

Let Y' be a level set of d_Y. Thus Y' is closed, convex and Γ-invariant. Since Y is Γ-invariant and Γ-minimal, the restriction of $d_{Y'}$ to Y is also constant. Using the Sandwich Lemma [BH99, Ex. II.2.12], one may conclude that Y' is equivariantly isometric to Y via the orthogonal projection. This implies that Y' is Γ-minimal, and that X decomposes as a product $X \cong Y \times C$, so that the fibers $Y \times \{c\}$ are precisely the minimal Γ-invariant closed convex subsets.

If C contains two distinct points, we define m as their midpoint and consider the fiber $Y'' = Y \times \{m\}$. Since it is a minimal Γ-invariant subspace, the arguments above show that $d_{Y''}$ is affine. This is impossible by Exercise IV.1(iii). Thus C is reduced to a singleton, and hence Γ acts minimally. $\square$

As in the case of normal subgroups, Theorem IV.1 yields algebraic restrictions on lattices:

Corollary IV.2. *Let X be a proper* CAT(0) *space without non-trivial Euclidean factor, $G < \mathrm{Is}(X)$ be a closed subgroup which is geometrically dense and Γ be a lattice in G. Then:*

(i) $\mathscr{Z}_G(\Gamma) = 1$.

(ii) If Γ is finitely generated, then $\mathscr{N}_G(\Gamma)$ is a lattice (containing Γ as a finite index subgroup).

Proof. (i) follows from Theorem IV.1 and Corollary II.5. For (ii), observe that the finite generation assumption implies that $\mathrm{Aut}(\Gamma)$ is countable. Hence so is $\mathscr{N}_G(\Gamma)$ by (i). Since the normaliser of a closed subgroup is closed, it follows that $\mathscr{N}_G(\Gamma)$ is a countable locally compact group, and must thus be discrete by Baire's category theorem. A discrete subgroup containing a lattice is itself a lattice, whence the conclusion. $\square$

2. Fixed points at infinity

Most results obtained so far used the condition that $\mathrm{Is}(X)$ be geometrically dense as a hypothesis. Our next task is to discuss how severe this restriction is.

If a group G acts cocompactly, or without a fixed at infinity, on X, then there always exists some non-empty G-invariant closed convex subset $Y \subseteq X$ on which G acts minimally (see Exercise II.4). On the other hand, one cannot expect that the full isometry group $\mathrm{Is}(X)$ of a proper CAT(0) space be always geometrically dense on some minimal invariant subspace $Y \subseteq X$ (see Exercise II.2). The next result shows that this is however indeed the case provided the full isometry group contains a lattice.

Theorem IV.3 ([CM12a, Th. L]). *Let X be a proper cocompact* CAT(0) *space and assume that $\mathrm{Is}(X)$ acts minimally. Let $\Gamma < \mathrm{Is}(X)$ be a lattice (e.g. a discrete group acting properly cocompactly on X).*

Then the only points in the visual boundary fixed by Γ lie in the boundary of the maximal Euclidean factor of X.

Since a locally compact group containing a lattice is unimodular, Theorem IV.3 follows by combining the following result with the geometric Borel density from the previous section:

Theorem IV.4 ([CM12a, Th. M]). *Let X be a proper cocompact* CAT(0) *space and assume that $\mathrm{Is}(X)$ acts minimally.*

If $\mathrm{Is}(X)$ is unimodular, then $\mathrm{Is}(X)$ has no fixed point at infinity.

Notice that the minimality assumption in both theorems is harmless: indeed, since the action is assumed cocompact, we may simply replace X by some minimal $\mathrm{Is}(X)$-invariant subspace $Y \subseteq X$. One should however be aware that Y may admit isometries that do not extend to X. It is thus conceivable (and it indeed happens, see Exercise IV.3) that $\mathrm{Is}(X)$ fixes points at infinity, while $\mathrm{Is}(Y)$ never does by Theorem IV.4.

The proof of Theorem IV.4 requires further geometric preliminaries and is thus postponed to the next section. A weaker version of Theorem IV.3 was first proved in [CM09b, Th. 3.14] under the additional hypothesis that Γ be finitely generated. At this point, let us merely present the simplest version of the argument, due to Burger–Schroeder [BS87], under the stronger assumption that Γ is cocompact; it can be proved directly, without invoking Theorem IV.4:

LEMMA IV.5. *Let X be a proper* CAT(0) *space and Γ be a discrete group acting properly cocompactly on X. If a finitely generated subgroup $\Lambda < \Gamma$ fixes some $\xi \in \partial X$, then Λ fixes some ξ' which is* ***opposite*** *ξ in the sense that $\{\xi, \xi'\}$ are the endpoints of a geodesic line.*

Applying the lemma to the whole group $\Lambda = \Gamma$, which is finitely generated since it is cocompact, we find an opposite pair of Γ-fixed points. Assuming in addition that the Γ-action is minimal, the Product Decomposition Theorem (see Theorem II.7) then yields a Γ-invariant splitting $X \cong \mathbf{R} \times X'$ such that ξ and ξ' are the endpoints of the line factor. Thus the conclusion of Theorem IV.3 holds in case Γ is cocompact.

3. Boundary points with a cocompact stabiliser

Let X be a proper CAT(0) space and $\xi \in \partial X$ be a boundary point. We define the following subgroup of the stabiliser $\mathrm{Is}(X)_\xi$ of ξ:

$$\mathrm{Is}(X)_\xi^{\mathrm{u}} = \Big\{ g \in \mathrm{Is}(X)_\xi \ : \ \lim_{t\to\infty} d\big(g\cdot r(t), r(t)\big) = 0 \ \ \forall\, r \text{ with } r(\infty) = \xi \Big\},$$

where r is a geodesic ray. One verifies that $\mathrm{Is}(X)_\xi^{\mathrm{u}}$ is a closed normal subgroup of $\mathrm{Is}(X)_\xi$ (see Exercise IV.4). Notice that $\mathrm{Is}(X)_\xi^{\mathrm{u}}$ is contained in the kernel of the Busemann character β_ξ centered at ξ (see Exercise II.7). In fact, the subgroup $\mathrm{Is}(X)_\xi^{\mathrm{u}}$ may be view as the intersection of $\mathrm{Ker}(\beta_\xi)$ with the kernel of the action of $\mathrm{Is}(X)_\xi$ on some other CAT(0) space denoted X_ξ and called the **transverse space** at ξ. It is defined as a completed quotient of the space of all geodesic rays pointing to ξ. We refer to [CM12a] for details. The subgroup $\mathrm{Is}(X)_\xi^{\mathrm{u}}$ can be interpreted as a *unipotent radical* of the stabiliser $\mathrm{Is}(X)_\xi$, which justifies the choice of notation. This interpretation is motivated by a version of the *Levi decomposition theorem* for parabolic subgroups of semi-simple Lie or algebraic groups, which can be established for CAT(0) spaces as soon as the stabiliser $\mathrm{Is}(X)_\xi$ acts cocompactly on X (see [CM12a, Th. J and Th. 3.12]).

In the present notes, we will only use the following fact, which can be deduced from the aforementioned Levi decomposition:

PROPOSITION IV.6. *Let X be a proper* CAT(0) *space and $G < \mathrm{Is}(X)$ be a closed subgroup. Let $\xi \in \partial X$ be such that the stabiliser G_ξ acts cocompactly on X.*

Then the group $G_\xi^{\mathrm{u}} = G \cap \mathrm{Is}(X)_\xi^{\mathrm{u}}$ acts transitively on the set $\mathrm{Opp}(\xi)$*, which is non-empty.*

It is not difficult to show that if G_ξ acts cocompactly on X, then the set $\mathrm{Opp}(\xi)$ is non-empty, and the group G_ξ acts transitively on it. Proposition IV.6 ensures that the smaller subgroup G_ξ^{u} remains transitive on $\mathrm{Opp}(\xi)$. This fact plays a crucial role in excluding fixed points at infinity for cocompact actions of unimodular groups, as we shall now see it.

PROOF OF THEOREM IV.4. The isometry group $\mathrm{Is}(\mathbf{R}^n)$ of the Euclidean space is unimodular and acts without a fixed point at infinity. By Theorem III.8, there is thus no loss of generality in assuming that X has no non-trivial Euclidean factor.

Assume for a contradiction that $G = \mathrm{Is}(X)$ fixes some point $\xi \in \partial X$. Since G acts cocompactly on X by hypothesis, the set of opposites $\mathrm{Opp}(\xi)$ is non-empty (see Exercise IV.4), and the subgroup G_ξ^{u} acts transitively on it by Proposition IV.6.

We claim that G_ξ^{u} is compact. Since $G = G_\xi$ acts minimally, this implies that the G_ξ^{u}, which is normal in G_ξ, must be trivial. Therefore the set $\mathrm{Opp}(\xi)$ is reduced to a singleton, say $\{\xi'\}$. In particular ξ' is fixed by G, and Theorem II.7 applied to the pair $\{\xi, \xi'\}$ yields a product decomposition of X with a line factor, contradicting that the maximal Euclidean factor of X is trivial.

In order to prove the claim, we proceed as follows. Let $\ell : \mathbf{R} \to X$ be a geodesic line such that $\ell(-\infty) = \xi$ and $\ell(+\infty) = \xi'$. By cocompactness, there is a sequence (g_n) in G such that $d(g_n.\ell(0), \ell(n))$ is bounded.

Since G fixes ξ it follows that for each individual element $g \in G$, the sequence of conjugates $(g_n g g_n^{-1})$ is bounded (i.e. relatively compact) in G. By an application of the Baire category theorem, one deduces that for each compact subset $U \subset G$, the union $\bigcup_n g_n U g_n^{-1}$ has compact closure.

Consider now an element $g \in G_\xi^{\mathrm{u}}$. This implies that any limit point of the sequence of conjugate $(g_n g g_n^{-1})$ fixes pointwise the line ℓ. Choosing some compact neighbourhood Q of the pointwise stabiliser of ℓ in G, we infer that $g_n g g_n^{-1}$ belongs to Q for all sufficiently large n. This holds for any individual element $g \in G_\xi^{\mathrm{u}}$, and another application of the Baire category theorem implies that for each compact subset $V \subset G_\xi^{\mathrm{u}}$, one has $g_n V g_n^{-1} \subset Q$ for all sufficiently large n.

We now fix some compact identity niehgourhood U in X. Thus $0 < \mathrm{vol}(U) < \infty$, where vol denotes a left Haar measure on G. We have seen that the set $P = \overline{\bigcup_n g_n U g_n^{-1}}$ is compact, and thus has finite volume. Now, for each compact subset $V \subset G_\xi^{\mathrm{u}}$, we find

$$g_n U V g_n^{-1} = g_n U g_n^{-1} g_n V g_n^{-1} \subset PQ$$

for all sufficiently large n. Since PQ is compact, it has finite volume. The unimodularity of G implies that the Haar measure is conjugacy invariant. Thus $\mathrm{vol}(UV) < \mathrm{vol}(PQ) < \infty$. This holds for every compact subset $V \subset G_\xi^{\mathrm{u}}$. Thus $\mathrm{vol}\left(UG_\xi^{\mathrm{u}}\right) < \infty$, from which it follows that G_ξ^{u} is compact, as claimed. $\square$

4. Back to rigidity

We finally come back to Theorem I.14 and describe the main steps of its proof:

(1) Since $\mathrm{Is}(X)$ is cocompact and X geodesically complete, the $\mathrm{Is}(X)$-action is minimal (Exercise II.4).
(2) The existence of a lattice in $\mathrm{Is}(X)$ implies that $\mathrm{Is}(X)$ is geometrically dense by Theorem IV.4.
(3) We are then in a position to invoke Corollary III.11, which yields a canonical decomposition $X \cong M_1 \times \cdots \times M_p \times \mathbf{R}^n \times Y_1 \times \cdots \times Y_q$, where M_i is an irreducible symmetric space of non-compact type, and Y_j has a totally disconnected isometry group, which acts smoothly and does not contain any parabolic isometry. The hypothesis that X has some parabolic isometry can now be re-interpreted: it simply means that X has at least one non-trivial symmetric space factor.
(4) At this point, if the space X is irreducible, we are done. Otherwise we may assume that X has several non-trivial factors. This implies that Γ may be viewed as a lattice in a product of locally compact groups; this gives access to superrigidity results, that are available for lattices in product groups at a high level of generality, notably through works by Burger [Bur95], Shalom [Sha00], Monod [Mon06], Gelander–Karlsson–Margulis [GKM08].
(5) The residual finiteness assumption, combined with the indecomposability of Γ is then used is an essential way: it is shown to imply that the Γ-action on each irreducible factor of X. The connection between residual finiteness of the lattice and the faithfulness of its action on the factors was first discovered by Burger and Mozes in their work on lattices in products of trees [BM00]. It was extended to lattices in products of CAT(0) spaces in [CM09b, Th. 4.10] (see also [CM12b, Prop. 2.4]).

Here, we deduce that the Γ-action on the non-trivial symmetric space factor yields in particular a faithful representation of Γ.

(6) The rest of the proof consists in using this linear representation combined with superrigidity tools to establish that Γ is an S-arithmetic group; this step closely follows the way in which Margulis deduced his arithmeticity theorems from superrigidity (see [Mar91] and the lectures by T. Gelander).

(7) Finally, once Γ has been identified as an S-arithmetic group, further applications of superrigidity imply that the closure of the image of Γ in the isometry group of each irreducible factor Y_j of X is a semi-simple algebraic group. That Y_j must be the model space (symmetric space or Euclidean building) for the semi-simple group in question is finally established, using the geodesic completeness hypothesis.

5. Flats and free abelian subgroups

Some key problems on proper CAT(0) spaces, groups and lattices, remain open; the most famous among them are perhaps the Rank Rigidity Conjecture, the Tits alternative, the existence of infinite torsion subgroups in CAT(0) groups, or the Flat Closing conjecture. It goes beyond the scope of these lectures to discuss all those problems, or to present the state of the art in each case. We shall content ourselves with a brief discussion of the latter. We start with the following well known open (and notoriously difficult) problem.

Question IV.7. Let X be a proper CAT(0) space and Γ be a discrete group acting properly and cocompactly on X.

Is it true that Γ is Gromov hyperbolic if and only if Γ does not contain any subgroup isomorphic to $\mathbf{Z}^2$?

The 'only if' direction is true by a well known property of hyperbolic groups (independent of CAT(0) geometry). Evidence for the reverse implication is provided by the following result.

Theorem IV.8 ([BH99, Th. III.H.1.5]). *Let X be a proper* CAT(0) *space and Γ be a discrete group acting properly and cocompactly on X.*

Then Γ is Gromov hyperbolic if and only if X does not contain any 2*-flat.*

In view of that theorem, answering Question IV.7 amounts to deciding whether the existence of a 2-flat in X implies the existence of a $\mathbf{Z}^2$ subgroup in Γ. More generally, one can ask the following.

Question IV.9 (Flat Closing conjecture, see [Gro93, 6.B_3]). Let X be a proper CAT(0) space and Γ be a discrete group acting properly and cocompactly on X.

Given $n > 0$, does the existence of an n-flat in X imply the existence of a $\mathbf{Z}^n$ subgroup in Γ?

For $n = 1$, the answer is known to be positive, due to E. Swenson [Swe99, Th. 11]. It is generally believed that the answer for $n = 2$ should be negative, and that examples should be found among CAT(0) square complexes. For general n, the answer is positive when X is a symmetric space, see the lectures by T. Gelander in this volume. When X is a Hadamard manifold (not necessarily symmetric), the answer is also positive for all n, due to Bangert–Schroeder [BS91], but much more delicate to establish. We finish by mentioning a general result for CAT(0) groups in this direction.

Theorem IV.10 ([CZ12, Cor. 1]). *Let X be a locally compact geodesically complete* CAT(0) *space and Γ a discrete group acting properly and cocompactly on X.*

If X splits non-trivially as a CAT(0) *product of n factors, then Γ contains a copy of $\mathbf{Z}^n$.*

Remark that if X is a non-trivial product of n factors, then each factor is unbounded and X contains some n-flat (because X is geodesically complete). Thus the hypotheses in Theorem IV.10 are (strictly) stronger than the existence of some n-flat in X. The proof of Theorem IV.10 is indirect: it relies on the fundamental decomposition provided by Corollary III.11, and then treats separately the case of symmetric spaces (where the result is well known, as mentioned above) and the case of spaces with a totally disconnected isometry group.

6. Exercises

Exercise IV.1. Let X be a CAT(0) space. A map $f\colon X \to \mathbf{R}$ is called **affine** if for each geodesic $\rho\colon I \to X$, the composed map $f \circ \rho\colon I \to \mathbf{R}$ is affine.

(i) Show that f is affine if and only if f and $-f$ are both convex.
(ii) Show that the level sets of an affine map are convex.
(iii) Suppose that X splits as a CAT(0) product $X = Y \times [0,1]$. Show that the distance function to the fiber $Y \times \{0\}$ is affine, while the distance to $Y \times \{\frac{1}{2}\}$ is not.

Exercise IV.2. Let X be a complete CAT(0) space. Let also (Ω, μ) be a measure space and $(f_\omega)_{\omega\in\Omega}$ be a family of convex functions on X such that the map $\omega \mapsto f_\omega(x)$ is integrable for all $x \in X$. Show that if the map $f\colon x \mapsto \int_\Omega f_\omega(x)\mathrm{d}\mu(\omega)$ is constant, then f_ω is affine for μ-almost all ω.

Exercise IV.3. Show that Theorem IV.4 can fail if $\mathrm{Is}(X)$ does not act minimally.

Exercise IV.4. Let X be a proper CAT(0) space and $\xi \in \partial X$ be a boundary point.

(i) Show that $\mathrm{Is}(X)^{\mathrm{u}}_\xi$ is a closed normal subgroup of $\mathrm{Is}(X)_\xi$.
(ii) Assume that $\mathrm{Is}(X)_\xi$ is cocompact on X. Show that $\mathrm{Opp}(\xi)$ is non-empty, and that $\mathrm{Is}(X)_\xi$ acts transitively on it.

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