ON GEOMETRIC FLATS IN THE CAT(0) REALIZATION OF COXETER GROUPS AND TITS BUILDINGS

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Abstract. Given a complete CAT(0) space X endowed with a geometric action of a group Γ, it is known that if Γ contains a free abelian group of rank n, then X contains a geometric flat of dimension n. We prove a converse of this statement in the special case where X is a convex subcomplex of the CAT(0) realization of a Coxeter group W, and Γ is a subgroup of W. In particular a convex cocompact subgroup of a Coxeter group is Gromov-hyperbolic if and only if it does not contain a free abelian group of rank 2. Our result also provides an explicit control on geometric flats in the CAT(0) realization of arbitrary Tits buildings.

Introduction

Let X be a complete CAT(0) space and Γ be a group acting properly discontinuously and cocompactly on X. It is a well known consequence of the so called flat torus theorem (see [BH99, Corollary II.7.2]) that:

($\mathbb{Z}^n \Rightarrow \mathbb{E}^n$): *if Γ contains a free abelian group of rank n, then X contains a geometric flat of dimension n.*

Recall that a **(geometric) flat** of dimension n, also called **(geometric) n–flat**, is a closed convex subset of X which is isometric to the Euclidean n-space. One may wonder whether a converse of this statement does hold, that is to say, whether the presence of a geometric n–flat in X is reflected in Γ by the existence of a free abelian group of rank n. This question goes back at least to Gromov [Gro93, §6.B_3].

In the case $n = 2$, in view of the flat plane theorem (see [BH99, Corollary III.H.1.5]), this question can be stated as follows:

If X is not hyperbolic, does Γ contains a copy of $\mathbb{Z} \times \mathbb{Z}$?

The answer is known to be positive in the following cases:

- Γ is the fundamental group of a closed aspherical 3-manifold, see [KK04].
- X is a square complex satisfying certain technical conditions, see [Wis05].

A **combinatorially convex subcomplex** of the Davis complex $|W|_0$ of a Coxeter group W is an intersection of closed half-spaces of $|W|_0$. The following result shows that, if X is a such a combinatorially convex subcomplex of $|W|_0$, and if $\Gamma \subset W$ acts cellularly, then the converse of the property ($\mathbb{Z}^n \Rightarrow \mathbb{E}^n$) above holds for all n:

Theorem A. *Let X be a combinatorially convex subcomplex of the Davis complex $|W|_0$ of a Coxeter group W. Let Γ be a subgroup of W which preserves X and whose induced action on X is cocompact. If X contains a geometric n–flat, then Γ contains a free abelian group of rank n.*

Since half-spaces are CAT(0)-convex, combinatorially convex subcomplexes are CAT(0)-convex as well. We do not know if the theorem above is still true when X is only assumed to be a CAT(0)-convex subset of $|W|_0$. We note that in general the intersection $\bar{X}$ of the

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closed half-spaces of $|W|_0$ containing X is not cocompact under Γ. Yet Γ is still cofinite on the set of walls separating X (or $\bar{X}$), and perhaps this is enough.

Corollary B. *Let X be a CAT(0) convex subcomplex of the Davis complex $|W|_0$ of a Coxeter group W. Let Γ be a subgroup of W which preserves X and whose induced action on X is cocompact. If X contains a geometric n–flat, then Γ contains a free abelian group of rank n.*

Proof. The Corollary follows by Theorem A because, since X is a subcomplex, the intersection of the closed half-spaces of $|W|_0$ containing X is a combinatorially convex Γ-cocompact subcomplex $\bar{X}$.

We sketch the argument. The key-point is that X^0 is convex for the combinatorial distance. First, any two vertices x, y of X may be joined by a combinatorial geodesic $(x_0 = x, \dots, x_n = y)$ all of whose vertices belong to the smallest subcomplex of $|W|_0$ containing the CAT(0) geodesic between x ad y (see [HP98, Lemme 4.9]). Since X is a CAT(0) convex subcomplex, it follows that $x_0, \dots, x_n$ belong to X^0. Now any combinatorial geodesic between x, y may be joined to $(x_0, \dots, x_n)$ by a sequence of geodesics, two consecutive of which differ by replacing half the boundary of some polygon of $|W|_0$ by the other half. Since X is a CAT(0) convex subcomplex it contains a polygonal face of $|W|_0$ as soon as it contains two consecutive edges of the boundary. It follows that X^0 contains the vertices of any combinatorial geodesic joining two of its points.

For any edge e with endpoints $x \in X, y \notin X$ we claim that the geometric wall m separating x from y does not separate x from any other vertex z of X. Indeed any vertex separated from x by m can be joined to x by a combinatorial geodesic through y. So by combinatorial convexity X would contain y, contradiction. This shows that X is contained in the intersection $\tilde{X}$ of closed half-spaces whose boundary wall separates an edge with one endpoint in X and the other one outside.

We claim that $\tilde{X}$ contains no vertex outside X. Indeed let $v \notin X^0$ denote some vertex. Choose a vertex $w \in X^0$ such that the combinatorial distance $d(v, w)$ is minimal. Consider any geodesic from w to v. Then the first edge e of this geodesic ends at a vertex $y \notin X$, and the wall separating w from y does not separate y from v. Thus $v \notin \tilde{X}$. Since $X^0 \subset \bar{X} \subset \tilde{X}$ and $\tilde{X}$ is the union of chambers with center in X^0, it follows that $\bar{X} = \tilde{X}$. Since Γ is cofinite on X^0 by assumption it follows that Γ is cocompact on $\bar{X}$, and we may apply Theorem A. □

The **algebraic flat rank** of a group Γ, denoted alg-rk(Γ), is the maximal $\mathbb{Z}$-rank of abelian subgroups of Γ. The **geometric flat rank** of a CAT(0) space X, denoted rk(X), is the maximal dimension of isometrically embedded flats in X. As an immediate consequence of Theorem A combined with the flat torus theorem, one obtains:

Corollary C. *Let X and Γ be as in Theorem A. Then* $\mathrm{rk}(X) = \text{alg-rk}(\Gamma)$. *In particular, one has* $\mathrm{rk}(|W|_0) = \text{alg-rk}(W)$.

It is an important result of Daan Krammer [Kra94, Theorem 6.8.3] that the algebraic flat rank of W can be easily computed in the Coxeter diagram of (W, S).

The equality between the algebraic flat rank of W and the geometric flat rank of $|W|_0$ was conjectured in [BRW05]. Actually, it is shown in loc. cit. that this equality allows to compute very efficiently the so called *(topological) flat rank* of certain automorphism groups of locally finite buildings whose Weyl group is W. The groups in question carry a canonical structure of locally compact totally discontinuous topological groups; furthermore they are topologically simple [Rém04]. The topological flat rank mentioned above is a natural invariant of the structure of topological group (see [BRW05] for more details).

The class of pairs (X, Γ) satisfying the assumptions of Theorem A is larger than one might expect. Assume for example that Γ acts geometrically by cellular isometries on a CAT(0) cubical complex X, and that Γ acts in a 'special' way on hyperplanes:

(1) for any hyperplane H of X and any element $g \in \Gamma$, either $gH = H$, or H and gH have disjoint neighbourhoods
(2) for any two distinct, intersecting hyperplanes H, H' of X and any element $g \in \Gamma$, either gH' intersects H, or H and gH' have disjoint neighbourhoods

Such 'special' actions are studied in [HW06], where it is proved that in the above situation there exists a right-angled Coxeter group W, an embedding $\Gamma \to W$ and an equivariant cellular isometric embedding $X \to |W|_0$. Thus Corollary B applies to groups acting geometrically and specially on CAT(0) cubical complexes. When the action is free we obtain:

Corollary D. *The fundamental group of a compact non positively curved special cube complex is hyperbolic iff it does not contain $\mathbb{Z} \times \mathbb{Z}$.*

The fundamental groups of the "clean" (VH-)square complexes studied in [Wis05] are examples of virtually special groups (by Theorem 5.7 of [HW06]). Thus our Theorem A provides in this case a new proof of the equivalence between hyperbolicity and absence of $\mathbb{Z} \times \mathbb{Z}$. Note that Wise's result applies to malnormal or cyclonormal VH-complexes, which are a priori more general than the virtually clean ones. But in [Wis05] Wise asks explicitly wether malnormal or cyclonormal implies virtually clean; and he proved already this converse implication for many classes of VH-complexes.

Not surprisingly, Theorem A also provides a control on geometric flats isometrically embedded in the CAT(0) realization of arbitrary Tits buildings. More precisely, we have:

Theorem E. *Let (W, S) be a Coxeter system and $\mathscr{B}$ be a building of type (W, S). Every geometric flat of the* CAT(0) *realization $|B|_0$ of B is contained in an apartment. In particular, one has* $\mathrm{rk}(|B|_0) = \text{alg-rk}(W)$.

Note that in [BRW05] the authors had established the equality $\mathrm{rk}(|B|_0) = \mathrm{rk}(|W|_0)$.

Finally, we recall from [Kle99, Theorem B] that if X is a locally compact complete CAT(0) space on which $\mathrm{Isom}(X)$ acts cocompactly, then the geometric flat rank of X coincides with five other quantities, among which the following ones:

- The maximal dimension of a quasi–flat of X.
- $\sup\{k \mid H_{k-1}(\partial_T X) \neq \{0\}\}$, where $\partial_T X$ denotes the Tits boundary of X.
- The geometric dimension of any asymptotic cone of X.

This applies of course to the Davis complex $|W|_0$, but also to many locally finite buildings of arbitrary type, including all locally finite Kac-Moody buildings. In particular, Corollary C and Theorem E above, combined with Daan Krammer's computation of alg-rk(W), provide a very efficient way to compute all these quantities for these examples.

In Section 1, we first recall basic facts on the Davis–Moussong geometric realization of Coxeter groups. In particular we introduce the walls, the half-spaces and the chambers.

In Section 2 we define combinatorial convex subsets of the Davis–Moussong geometric realization, and we establish an important Lemma.

In Section 3 we present the main technical tools of this article. If a family of walls behaves as if it was contained in a Euclidean triangle subgroup, then in fact it generates a Euclidean triangle subgroup (see Lemmas 3.1 and 3.4 for precise statements).

In Section 4 we describe completely the combinatorial structure of the set of walls separating a given flat. The reflections along these walls generate a subgroup that we also describe.

In Section 5 we explain how to get a rank n free abelian group out of a rank n flat.

And in Section 6 we explain how to deduce the statement on buildings from the statement on Coxeter complexes.

1. Preliminaries

Let (W, S) be a Coxeter system with S finite. The Davis complex associated with (W, S), denoted $|W|_0$, is a CAT(0) cellular complex equipped with a faithful, properly discontinuous, cocompact action of W (see [Dav98]).

Recall that a **reflection** of W is, by definition, any conjugate of an element of S. The fixed point set of a reflection in $|W|_0$ is called a **wall**. Note that a wall is a closed convex subset of $|W|_0$. A fundamental property is that every wall separates $|W|_0$ into two open convex subsets, whose respective closures are called **half-spaces**. If a is a half-space, its boundary is a wall which is denoted by ∂a. If $x \in |W|_0$ is a point which is not contained in any wall, then the intersection of all half-spaces containing x is compact; this compact set is called a **chamber** of $|W|_0$. The W-action on the chambers of $|W|_0$ is free and transitive.

Let x, y denote two non-empty convex subsets of $|W|_0$. We say that a wall m separates x from y whenever x is contained in one of the half-spaces delimited by m, y is contained in the other half-space, and neither x nor y are contained in m.

We will use the following notation. Given a wall m of $|W|_0$, the unique reflection fixing m pointwise is denoted by r_m. For any set M of walls, we set $W(M) := \langle r_m |\ m \in M\rangle$. Recall that $W(M)$ is itself a Coxeter system on a certain set of reflections $(r_\nu)_{\nu \in N}$, where each wall $\nu \in N$ is of the form $\nu = w\mu$ for some $w \in W(M)$ and some $\mu \in M$ (see [Deo89]). Such a subgroup will be called a **reflection subgroup**.

Finally, given two points (resp. two convex subsets) x, y of $|W|_0$, we denote by $\mathscr{M}(x, y)$ the set of all walls which separate x from y. Two chambers c, c are said to be **adjacent** whenever $\mathscr{M}(c, c')$ is empty, or consists in a single wall m (in which case $r_m(c) = c'$). A gallery (of length n) is a sequence $(c_0, c_1, \dots, c_n)$ of chambers such that c_i and c_{i+1} are adjacent chambers for $i = 0, \dots, n-1$. The gallery defines a unique sequence of walls it crosses (this sequence might be empty if the gallery is a constant sequence).

We get a (discrete) distance on the set of chambers by considering the infimum of the length of all galleries from the first chamber to the second. Using the simple transitive action of W on the chambers, this gallery distance is identified with the word metric on (W, S).

It is well known that for two chambers c, c' the gallery distance $d_{\mathrm{gal}}(c, c')$ is the cardinality of $\mathscr{M}(c, c')$, and that a gallery from c to c' has length $d_{\mathrm{gal}}(c, c')$ if and only if the sequence of walls it crosses has no repetition. Furthermore for any gallery from c to c' the set of walls separating c from c' is the set of walls appearing an odd number of times in the sequence of walls that the gallery crosses.

The following basic lemmas are well known; their proofs are easy exercises.

Lemma 1.1. *Let x, y be two points of $|W|_0$. There are two chambers c_x, c_y such that $x \in c_x$, $y \in c_y$ and $\mathscr{M}(x, y) = \mathscr{M}(c_x, c_y)$.* □

Lemma 1.2. *Let $x, y \in |W|_0$. There exists $\gamma \in W(\mathscr{M}(x, y))$ such that x and $\gamma.y$ are contained in a common chamber.* □

2. Combinatorial convexity

A subset $F \subset |W|_0$ is called **combinatorially convex** if either $F = |W|_0$ or F coincides with the intersection of all half-spaces containing it. The **combinatorial convex closure** of a subset $F \subset |W|_0$ will be denoted by $\mathrm{Conv}(F)$. Hence $\mathrm{Conv}(F)$ is either the whole $|W|_0$ (if F is not contained in any half-space) or the intersection of all half-spaces of $|W|_0$

containing F. Since half-spaces are subcomplexes of the first barycentric subdivision of $|W|_0$ we note that combinatorially convex subsets are subcomplexes as well.

Since half-spaces are CAT(0) convex, combinatorially convex subcomplexes are CAT(0) convex, but we will rather use the following elementary combinatorial convexity property: all chambers of a geodesic gallery from a chamber c to a chamber c' belong to $\mathrm{Conv}(c \cup c')$.

Lemma 2.1. *Let $x, y \in |W|_0$ and assume that the set $\mathscr{M}(x,y)$ possesses a subset M such that for all $m \in M$ and $\mu \in \overline{M} = \mathscr{M}(x,y) \backslash M$, the reflections r_m and r_μ commute. Then the combinatorial convex closure of $\{x,y\}$ contains a point z such that $\mathscr{M}(y,z) = M$ and $\mathscr{M}(x,z) = \overline{M}$.*

Proof. Let c_x, c_y be chambers such that $x \in c_x$, $y \in c_y$ and $\mathscr{M}(x,y) = \mathscr{M}(c_x, c_y)$ (see Lemma 1.1). We prove that there exists a chamber c_z such that $\mathscr{M}(c_y, c_z) = M$ and $\mathscr{M}(c_x, c_z) = \overline{M}$ (note that such a chamber necessarily lies in the combinatorial convex closure of $c_x \cup c_y$).

This implies the desired result. Indeed since $\mathscr{M}(x,y) = \mathscr{M}(c_x, c_y)$ we have $\mathrm{Conv}(\{x,y\}) = \mathrm{Conv}(c_x \cup c_y)$. Furthermore since $\mathscr{M}(y, c_y) = \emptyset$ we have $\mathscr{M}(c_z, y) \subset \mathscr{M}(c_z, c_y)$. Conversely if $m \in \mathscr{M}(c_z, c_y)$ then m does not separate c_z from c_x – otherwise c_z would not be inside $\mathrm{Conv}(c_x \cup c_y)$. Thus m separates c_y from c_x, and so $m \in \mathscr{M}(x,y)$. In particular $y \notin m$. Thus in fact $m \in \mathscr{M}(c_z, y)$. Consequently $\mathscr{M}(c_z, y) = \mathscr{M}(c_z, c_y)$ $(= M)$, and similarly $\mathscr{M}(c_z, x) = \mathscr{M}(c_z, c_x)$ $(= \overline{M})$. We then define z to be any point in the interior of the chamber c_z.

It remains to prove the statement for chambers. To this end, we argue by induction on the cardinality n of $\mathscr{M}(c_x, c_y)$. We may assume $n > 0$.

Consider some geodesic gallery $(c_0 = c_x, \ldots, c_{n-1}, c_n = c_y)$. Let μ denote the unique wall separating c_{n-1} from c_n. By induction there is a chamber d such that $\mathscr{M}(c_x, d) = \overline{M} \setminus \{\mu\}, \mathscr{M}(d, c_{n-1}) = M \setminus \{\mu\}$. We then have $\mathscr{M}(d, c_y) = \mathscr{M}(d, c_{n-1}) \cup \{\mu\}$.

If $\mu \in M$, then the chamber d satisfies $\mathscr{M}(c_x, d) = \overline{M}$ and $\mathscr{M}(d, c_y) = M$, so we are done.

Assume now that $\mu \in \overline{M}$, so $M = \mathscr{M}(d, c_{n-1})$. Consider a gallery from d to c_{n-1} of minimal length. If this gallery has length 0 then $M = \emptyset$ and we take $c_z = c_y$. Otherwise let $m \in M$ denote the last wall that the gallery crosses. Let d' denote the chamber $r_m r_\mu(c_{n-1})$. Then d' is adjacent to c_{n-2}, and d' is also adjacent to c_n because $r_m r_\mu = r_\mu r_m$. It follows that there exists a gallery of minimal length from c_x to c_y whose last crossed wall is m. So in fact we are back to the first case, and thus we are done. $\square$

Note that the corresponding statement (for vertices) is true in an arbitrary CAT(0) cubical complex X. Indeed for any two vertices x, y of X such that the set $\mathscr{M}(x,y)$ of hyperplanes of X separating x from y may be written $\mathscr{M}(x,y) = M \sqcup \overline{M}$ so that every hyperplane of M is perpendicular to every hyperplane of $\overline{M}$, there exists a vertex z such that $\mathscr{M}(z,y) = M$ and $\mathscr{M}(z,x) = \overline{M}$. Clearly z is on some combinatorial geodesic from x to y, thus z is in the convex hull of $\{x,y\}$.

3. The Euclidean triangle lemmas

In what follows, a **Euclidean triangle subgroup** of the Coxeter group W is a reflection subgroup which is isomorphic to one of the three possible irreducible Coxeter groups containing $\mathbb{Z} \times \mathbb{Z}$ as a finite index subgroup. We say that a set P of walls is **Euclidean** whenever there exists a wall m such that $P \cup \{m\}$ generates a Euclidean triangle subgroup of W. We will be mainly interested in the case when P is a set of pairwise disjoint walls.

The following lemma relates the combinatorial configuration of a certain set of walls M of $|W|_0$ with the algebraic structure of $W(M)$. This provides the key ingredient which allows to understand the walls of a geometric flat of $|W|_0$, see Proposition 4.9 below.

Lemma 3.1. *There exists a constant L, depending only on the Coxeter system (W, S), such that the following property holds. Let $a, b, h_0, h_1, \dots, h_n$ be a collection of half-spaces of $|W|_0$ such that:*

1. $\emptyset \neq a \cap b \subsetneq h_0 \subsetneq h_1 \subsetneq \cdots \subsetneq h_n$,
2. $\emptyset \neq \partial a \cap \partial b \subset \partial h_0$,
3. *∂a and ∂b both meet ∂h_i for each $i = 1, \dots, n$.*

If $n \geq L$, then the group generated by the reflections through the walls ∂a, ∂b, ∂h_0, $\partial h_1, \dots, \partial h_n$ is a Euclidean triangle subgroup.

Proof. See [Cap06, Theorem A]. □

A set P of walls of $|W|_0$ is called a **chain of walls** if there exists a set A of half-spaces of $|W|_0$ such that A is totally ordered by inclusion and $P = \{\partial a, a \in A\}$ (for short we write $P = \partial A$). There are three kinds of chains of walls. We say that P is a **segment of walls** if it is a finite chain of walls. We say that P is a **line of walls** if $P = \partial A$, with A a set of half-spaces such that the ordered set $(A, \subset)$ is isomorphic to $(\mathbb{Z}, \leq)$. And we say that P is a **ray of walls** if $P = \partial A$, with A a set of half-spaces such that the ordered set $(A, \subset)$ is isomorphic to $(\mathbb{N}, \leq)$.

Lemma 3.2. *Let P denote a nonempty set of walls which are all disjoint from a given wall μ. Assume that $P \cup \{\mu\}$ is Euclidean. Then $P \cup \{\mu\}$ is a chain and $W(P \cup \{\mu\})$ is infinite dihedral.*

Proof. Let μ' denote some wall such that $W(P \cup \{\mu, \mu'\})$ is a Euclidean triangle subgroup. Represent $W(P \cup \{\mu, \mu'\})$ as a group of isometries of the Euclidean plane (in such a way that the abstract reflections act as geometric reflections).

Let m, m' denote two walls of $P \cup \{\mu\}$. Note that $m \cap m' = \emptyset$ if and only if the order of $r_m r_{m'}$ is infinite. In the geometric representation we have $m \cap m' = \emptyset$ if and only if the Euclidean lines $L(m), L(m')$ fixed pointwise by m and m' are parallel. Since we assume $m \cap \mu = \emptyset$ or $m = \mu$, we deduce that $L(m)$ is parallel to $L(\mu)$. Similarly $L(m')$ is parallel to $L(\mu)$. Thus $L(m)$ and $L(m')$ are parallel, which implies that $m = m'$ or $m \cap m' = \emptyset$.

Thus $P \cup \{\mu\}$ is a set of pairwise disjoint walls (of cardinality ≥ 2). By looking at the geometric representation we deduce that $W(P \cup \{\mu\})$ is infinite dihedral. Note that the set of walls associated with all the reflections of any infinite dihedral reflection subgroup is a line of walls (this can be seen by considering a generating set consisting of two reflections; the associated walls cut $|W|_0$ into three pieces, one of which is a fundamental domain for the reflection subgroup that we consider). It follows that $P \cup \{\mu\}$ is a chain. □

Let T denote any subset of the generating set S. Then any conjugate of the subgroup $W(T)$ is called a **parabolic subgroup**. The **parabolic closure** of any subgroup $\Gamma \subset W$ is the intersection of all parabolic subgroups of W containing Γ; we denote it by $\widetilde{\Gamma}$. With this terminology, we have:

Lemma 3.3. *Let P be a set of pairwise disjoint walls of $|W|_0$. Assume that there exists a wall m such that $W(P \cup \{m\})$ is a Euclidean triangle subgroup. Then the parabolic closure $\widetilde{W(P)}$ satisfies the following conditions:*

1. *$\widetilde{W(P)}$ is isomorphic to an irreducible affine Coxeter group.*
2. *For all walls μ, μ', μ'', if μ separates μ' from μ'' and if $r_{\mu'}$ and $r_{\mu''}$ both belong to $\widetilde{W(P)}$, then r_μ also belongs to $\widetilde{W(P)}$.*
3. *For any line of walls P' and any wall μ, if $W(P') \leq \widetilde{W(P)}$ and if $W(P' \cup \{\mu\})$ is a Euclidean triangle subgroup, then r_μ belongs to $\widetilde{W(P)}$.*

Proof. Point (1) follows from a theorem of D. Krammer which appears in [CM05, Theorem 1.2] (see also Theorem 3.3 in loc. cit.); (2) and (3) follow from (1) using convexity arguments, see [Cap06, Lemma 8] for details. □

We may now deduce an other useful result of the same kind as Lemma 3.1:

Corollary 3.4. *Let P be a set of pairwise disjoint walls of $|W|_0$ and let m be a wall such that $W(P \cup \{m\})$ is a Euclidean triangle subgroup. Then W possesses a Euclidean triangle subgroup, denoted by $\overline{W(P \cup \{m\})}$, containing $W(P \cup \{m\})$ and such that $r_\mu \in \overline{W(P \cup \{m\})}$ for each wall μ satisfying either of the following conditions:*

1. *There exist $\mu', \mu'' \in P$ such that μ separates μ' from μ''.*
2. *μ is disjoint from m and moreover $W(P \cup \{\mu\})$ is a Euclidean triangle subgroup.*

Proof. Let $\widetilde{W(P)} \leq W$ be the irreducible affine Coxeter group provided by Lemma 3.3. By Lemma 3.3(3) we have $r_m \in \widetilde{W(P)}$. Let P' be the set consisting of all those walls p' such that $r_{p'} \in \widetilde{W(P)}$ and that there exists $p \in P \cup \{m\}$ which does not meet p'. Define $\overline{W(P \cup \{m\})} := W(P' \cup P \cup \{m\})$. The group $\overline{W(P \cup \{m\})}$ is a Euclidean triangle subgroup, because it is a subgroup of an affine Coxeter group generated by reflections corresponding to two directions of hyperplanes. Given a wall μ satisfying (1) or (2), we obtain successively $r_\mu \in \widetilde{W(P)}$ by Lemma 3.3 and then $\mu \in P'$ by the definition of P'. □

4. The walls of a geometric flat

Let F be a geometric flat which is isometrically embedded in the Davis complex $|W|_0$ of W. Let $\mathscr{M}(F)$ denote the set of all walls which separate points of F:

$$\mathscr{M}(F) := \bigcup_{x,y \in F} \mathscr{M}(x, y).$$

Lemma 4.1. *For all $\mu \in \mathscr{M}(F)$, the set $\mu \cap F$ is a Euclidean hyperplane of F.*

Proof. Let x, y be points of F which are separated by μ. We know that $\mu \cap F$ is a closed convex subset of F which separates F into two open convex subsets. Thus the result will follow if we prove that the geodesic segment $[x, y]$ joining x to y meets μ in a single point. This is a local property, which can easily be checked in a single (Euclidean) cell of $|W|_0$ (see [NV02, Lemma 3.4] for details). □

Lemma 4.2. *Let μ be a wall which meets F. Assume that F contains a Euclidean half-space F^+ such that $F^+ \cap \mu \neq \emptyset$ and F^+ is contained in a ε–neighborhood of μ for some $\varepsilon > 0$. Then $F \subset \mu$.*

Proof. Let d be the distance function of the Davis complex $|W|_0$. Since μ is a closed convex subset, the function $d_\mu : |W|_0 \to \mathbf{R}^+ : x \mapsto \inf\{d(x, y) |\ y \in \mu\}$ is convex (see [BH99, §II.2]). By assumption, the restriction $d_\mu|_{F^+}$ of d_μ to F^+ is bounded. Therefore $d_\mu|_{F^+}$ must be constant, as it is the case for any bounded convex function on an unbounded convex domain. Since μ meets F^+ by hypothesis, we have $d_\mu|_{F^+} = 0$, that is to say, $F^+ \subset \mu$. By Lemma 4.1, this implies $F \subset \mu$. □

Two elements μ, μ' of $\mathscr{M}(F)$ will be called F–**parallel** if their respective traces on F are parallel in the Euclidean sense. In symbols, this writes:

$$\mu \|_F\, \mu' \quad \Leftrightarrow \quad \mu \cap F = \mu' \cap F \text{ or } \mu \cap F \cap \mu' = \emptyset.$$

The relation of F–parallelism is an equivalence relation on $\mathscr{M}(F)$.

Besides the relation of F–parallelism, there is an other relation of **global parallelism** on the walls of F defined by

$$\mu \| \mu' \quad \Leftrightarrow \quad \mu = \mu' \text{ or } \mu \cap \mu' = \emptyset.$$

Clearly $\mu \| \mu' \Rightarrow \mu \|_F \mu'$. Given $\mu \in \mathscr{M}(F)$, we set $P_F(\mu) := \{m \in \mathscr{M}(F) \mid m \| \mu\}$. Thus $P_F(\mu)$ is contained in the F–parallel class of μ. Note that, in contrast with the F–parallelism, the relation of global parallelism is not transitive in general: two distinct walls of $P_F(\mu)$ may have non trivial intersection.

Any large set of walls contains two non-intersecting ones (see [NR03, Lemma 3]). Consequently, the set of F–parallel classes is finite. Since chambers are compact and F is unbounded, it follows that some F–parallel class must be infinite. Actually, all of them are, as follows from the following:

Lemma 4.3. *Given any $\mu \in \mathscr{M}(F)$, there exist two rays of walls $M^+(\mu), M^-(\mu) \subset \mathscr{M}(F)$ such that μ separates any element of $M^+(\mu)$ from any element of $M^-(\mu)$. In particular, μ does not meet any element of $M^+(\mu) \cup M^-(\mu)$, and $P_F(\mu)$ contains a line of walls (passing through μ).*

Proof. Consider a line of F which meets orthogonally the F-hyperplane $\mu \cap F$. Using Lemma 4.2 we see that when a point p goes at infinity on the line, its distance to μ must tend to infinity. Now by the so called *parallel wall theorem* (see [BH93, Theorem 2.8]) any point at large distance from a given wall in $|W|_0$ is separated from that wall by some other wall of $|W|_0$. The Lemma follows. □

Remarks 4.4. For $\mu \in \mathscr{M}(F)$, any subset $P \subset P_F(\mu)$ of pairwise disjoint walls is a chain of walls. Indeed for three distinct walls $p_1, p_2, p_3 \in P$ we have $p_i \|_F \mu$, thus p_1, p_2, p_3 are mutually F–parallel. The Euclidean hyperplanes $p_i \cap F$ are pairwise disjoint, so we may assume that $p_2 \cap F$ separates $p_1 \cap F$ from $p_3 \cap F$. It follows that p_2 separates p_1 from p_3. Hence $\{p_1, p_2, p_3\}$ is a segment of walls. Since any 3–subset of P is a chain, it follows that P itself is a chain.

We will see in Proposition 4.7 below that the restriction of the relation of global parallelism to a certain subset $\mathscr{M}_{\mathrm{Eucl}}(F)$ of $\mathscr{M}(F)$ is an equivalence.

By definition, the subset $\mathscr{M}_{\mathrm{Eucl}}(F) \subset \mathscr{M}(F)$ consists of all those walls $\mu \in \mathscr{M}(F)$ which satisfy the following property:

There exists a wall $\mu' \in \mathscr{M}(F)$ such that $W(P_F(\mu) \cup \{\mu'\})$ is a Euclidean triangle subgroup.

Applying Lemmas 3.2 and 4.3, we get the following:

Lemma 4.5. *Assume $\mu \in \mathscr{M}_{\mathrm{Eucl}}(F)$, and more precisely that $W(P_F(\mu) \cup \{\mu'\})$ is a Euclidean triangle subgroup for some $\mu' \in \mathscr{M}(F)$. Then:*

(i) *$P_F(\mu)$ is a line of walls.*
(ii) *For all $m \in P_F(\mu)$, one has $P_F(\mu) \subset P_F(m)$. In particular $P_F(\mu) = P_F(m)$ provided $m \in \mathscr{M}_{\mathrm{Eucl}}(F)$.*
(iii) *$W(P_F(\mu))$ is an infinite dihedral subgroup of W, and is a maximal one.*
(iv) *$r_{\mu'}$ does not centralize $W(P_F(\mu))$.* □

The following lemma outlines the main combinatorial properties of the set $\mathscr{M}_{\mathrm{Eucl}}(F)$.

Lemma 4.6. *We have the following:*

(i) *Let $P \subset \mathscr{M}(F)$ be a line of walls. If there exists $m \in \mathscr{M}(F)$ such that the group $W(P \cup \{m\})$ is a Euclidean triangle subgroup, then $P \subset \mathscr{M}_{\mathrm{Eucl}}(F)$.*
(ii) *Let $m \in \mathscr{M}(F)$. If $m \notin \mathscr{M}_{\mathrm{Eucl}}(F)$, then m meets every element of $\mathscr{M}_{\mathrm{Eucl}}(F)$.*
(iii) *Let $m, m' \in \mathscr{M}(F)$. If the reflections r_m and $r_{m'}$ do not commute and if m and m' are not F–parallel, then $m \in \mathscr{M}_{\mathrm{Eucl}}(F)$.*
(iv) *Let $m, m' \in \mathscr{M}(F)$. If the reflections r_m and $r_{m'}$ do not commute and if $m' \in \mathscr{M}_{\mathrm{Eucl}}(F)$, then $m \in \mathscr{M}_{\mathrm{Eucl}}(F)$.*

Before proving the lemma, it is convenient to introduce the following additional terminology. A set P of walls of $|W|_0$ is said to be **convex** whenever the following holds: for each wall m of $|W|_0$ separating two walls of P, we have $m \in P$. For example, for all $x, y \in |W|_0$ the set $\mathscr{M}(x, y)$ is convex; moreover, the set $\mathscr{M}(F)$ is convex as well.

Proof of Lemma 4.6(i). Let $\mu \in P$. Since $P \subset \mathscr{M}(F)$ is a line of walls we have $P \subset P_F(\mu)$. There are finitely many walls separating two disjoint walls of $|W|_0$. The line of walls P may be written as a union of segments of walls $\{\mu_n, \mu_{n+1}\}$ $(n \in \mathbb{Z})$ so that no $m \in P$ separates μ_n from μ_{n+1}. Choose then a segment of walls $P_n \subset P_F(\mu)$ such that $P_n \cap P = \{\mu_n, \mu_{n+1}\}$ and any wall $m \in P_n \setminus \{\mu_n, \mu_{n+1}\}$ separates μ_n from μ_{n+1} and moreover P_n is maximal with respect to these properties. Set $\bar{P} = \cup_k P_k$. Then $P \subset \bar{P} \subset P_F(\mu)$, $\bar{P}$ is a line and for every wall m' of $P_F(\mu) \setminus \bar{P}$ the set $\bar{P} \cup \{m'\}$ is not a line anymore.

By construction for every $p \in \bar{P}$ there exist $p', p'' \in P$ such that p separates p' from p''. Therefore, since $W(P \cup \{m\})$ is a Euclidean triangle subgroup, we have $W(\bar{P} \cup \{m\}) \subset \overline{W(P \cup \{m\})}$ by Corollary 3.4. In particular, $W(\bar{P} \cup \{m\})$ is a Euclidean triangle subgroup. Hence we are done if we show that $\bar{P} = P_F(\mu)$. This is what we do now.

Let m' denote a wall separating two walls p', p'' of $\bar{P}$. Then $m' \in \mathscr{M}(F)$ and by Corollary 3.4 the subset $\bar{P} \cup \{m'\}$ is still Euclidean. By Lemma 3.2 $\bar{P} \cup \{m'\}$ is a line, and by the maximality of $\bar{P}$ we have $m' \in \bar{P}$. Thus $\bar{P}$ is a convex set of walls.

Assume by contradiction that there exists $m' \in P_F(\mu) \setminus \bar{P}$. By the maximality of $\bar{P}$, the set $\bar{P} \cup \{m'\}$ is not a line anymore. By Remark 4.4 this implies that m' meets at least one element of $\bar{P}$. Let $\bar{P}'$ denotes the (nonempty) subset of $\bar{P}$ consisting of all those walls which meet m'. Note that by the definition of $\bar{P}'$, for all $p \in \bar{P}$, if there exist $p', p'' \in \bar{P}'$ such that p separates p' from p'', then $p \in \bar{P}'$. Since $\bar{P}$ is convex, this shows in particular that $\bar{P}'$ is convex.

If $\bar{P}'$ is finite, it is a segment of the line $\bar{P}$ and there exist $p', p'' \in \bar{P}$ such that m' separates p' from p''. Since $\bar{P}$ is convex, this implies that $m' \in \bar{P}$, a contradiction.

Hence $\bar{P}'$ is infinite. Since $\mu \notin \bar{P}'$ and $\bar{P}'$ is convex, we see that $\bar{P}'$ is a ray of walls (contained in $\bar{P}$, and not containing μ).

By Lemma 3.2 the group $W(\bar{P})$ is infinite dihedral. Since $\bar{P}$ is a line of walls, the wall π of any reflection r_π of $W(\bar{P})$ separates two walls p', p'' of $\bar{P}$. By convexity we then have $\pi \in \bar{P}$: the reflections of $W(\bar{P})$ are precisely the reflections along walls of $\bar{P}$. We note two consequences of that. Firstly $\bar{P}$ is invariant under $W(\bar{P})$. Secondly we have $W(\bar{P}) = W(\bar{P}_0)$ for any convex subset $\bar{P}_0 \subset \bar{P}$ of cardinality at least 2. In particular we have $W(\bar{P}) = W(\bar{P}')$.

The reflection $r_{m'}$ does not centralize $W(\bar{P}')$, otherwise it would centralize $W(\bar{P})$ and, hence, m' would meet μ. Consequently $r_{m'}$ does not centralize $W(\bar{P}'_0)$ for all convex subset $\bar{P}'_0 \subset \bar{P}'$ of cardinality at least 2. Hence there are infinitely many walls $\bar{p}'$ in the ray $\bar{P}'$ such that the reflections $r_{m'}$ and r_p do not commute. Let $\bar{p}' \in \bar{P}'$ denote some wall such that the reflections $r_{m'}$ and $r_{\bar{p}'}$ do not commute, and that the collection of all walls of $\bar{P}'$ which separate $\bar{p}'$ from μ is of cardinality greater than the constant $L (\geq 1)$ of Lemma 3.1.

Let $m'' := r_{\bar{p}'}(m')$. Let $\{\bar{p}_1, \dots, \bar{p}_k\}$ denote the segment of walls of $\bar{P}'$ which separate μ from $\bar{p}'$ (we have $k \geq L$). Then the walls $\bar{m}_i = r_{\bar{p}'}(\bar{p}_i)$ belong to the ray $\bar{P}'$ by convexity (remember that $r_{\bar{p}'}(\mu) \in \bar{P}$). Hence each of them meets m'. By construction each of them also meets m''. By Lemma 3.1 we deduce that $W(m', m'', \bar{m}_1, \dots, \bar{m}_k, \bar{p}') = W(m', \bar{m}_1, \dots, \bar{m}_k, \bar{p}')$ is a Euclidean triangle subgroup. Since $\{\bar{m}_1, \dots, \bar{m}_k, \bar{p}'\}$ is a convex subsegment of $\bar{P}$ containing at least two walls we see that $W(\bar{P} \cup \{m'\})$ is a Euclidean triangle subgroup. Since $\mu \in \bar{P}$ and $m' \cap \mu = \emptyset$, this contradicts Lemma 3.2, thereby completing the proof of the desired assertion. $\square$

Proof of Lemma 4.6(ii). Let $m \in \mathscr{M}(F)$. Assume that there exists $\mu \in \mathscr{M}_{\mathrm{Eucl}}(F)$ which does not meet m. In other words $m \in P_F(\mu)$. By Lemma 4.6(i), $\mu \in \mathscr{M}_{\mathrm{Eucl}}(F)$ implies $P_F(\mu) \subset \mathscr{M}_{\mathrm{Eucl}}(F)$. Thus $m \in \mathscr{M}_{\mathrm{Eucl}}(F)$.

□

Proof of Lemma 4.6(iii). Let M be the F–parallel class of m and let $m'' := r_m(m')$. Since m and m' are not F–parallel, there are points x, y on $m \cap F$ which are separated by m'. Thus m'' separates x from y as well. It follows that $m'' \in \mathscr{M}(F)$.

We now show that m'' is not F–parallel to m. To this end, first note that m'' contains $m \cap m' \cap F$ which is nonempty. Hence, if m'' were F-parallel to m, then we would have $m \cap F = m'' \cap F$. This yields successively $m \cap F = r_m(m') \cap F$ and then $m \cap r_m(F) = m' \cap r_m(F)$. Since $m \cap F$ is pointwise fixed by r_m, we have $m \cap F \subset m \cap r_m(F)$, whence finally $m \cap F \subset m'$, which contradicts the fact that m and m' are not F-parallel. This shows that m'' is not F–parallel to m and it follows that m' and m'' both meet every element of the F–parallel class M.

By Lemma 4.3, M contains a line P containing m. In particular P is infinite. By Lemma 3.1, the group $W(P \cup \{m'\})$ is a Euclidean triangle subgroup. Therefore, we deduce from Lemma 4.6(i) that $m \in \mathscr{M}_{\mathrm{Eucl}}(F)$. □

Proof of Lemma 4.6(iv). Let $m \in \mathscr{M}(F)$ and $m' \in \mathscr{M}_{\mathrm{Eucl}}(F)$ be such that the reflections r_m and $r_{m'}$ do not commute. By Lemma 4.6(i), we have $P_F(m') \subset \mathscr{M}_{\mathrm{Eucl}}(F)$. Let $P' := P_F(m')$. Hence P' is a line of walls and for all $\mu' \in P'$, we have $P_F(\mu') = P'$.

By Lemma 4.6(ii) we may assume that m meets every element of P', and in fact that every element of $P_F(m)$ meets every element of P', otherwise $m \in \mathscr{M}_{\mathrm{Eucl}}(F)$ and we are done. By Lemma 4.3, $P_F(m)$ contains a line of walls P which contains m.

Let C (resp. C') denote the set of walls of P (resp. P') which meet $m'' := r_m(m')$.

Assume that C' is finite. Then there exists a (convex) segment of walls $(p_+, p_1, \dots, p_n, p_-)$ contained in P' such that $C' = \{p_1, \dots, p_n\}$ and m'' is disjoint from p_+ and p_-. We let x_+, x_- denote points lying on $m \cap p_+, m \cap p_-$ respectively. Since m' separates p_+ from p_- and $m \backslash m'' = m \backslash m'$ we deduce that m'' separates x_+ from x_-. Thus m'' separates p_+ from p_-. It follows that $m'' \in \mathscr{M}(F)$, and in fact $m'' \in P_F(p^+)$. By hypothesis $m' \in \mathscr{M}_{\mathrm{Eucl}}(F)$, whence $P_F(p^+) = P_F(m')$. Since m'' meets m', this implies $m' = m''$ from which it follows that the reflections r_m and $r_{m'}$ commute, a contradiction. Thus C' is infinite.

By Lemma 3.1, it follows that $W(C' \cup \{m\})$ is a Euclidean triangle subgroup. Since P' is Euclidean, we have $W(P') = W(P'_0)$ for any convex chain $P'_0 \subset P'$ of cardinality at least 2 (see Lemma 3.2). Since C' is infinite and convex, we deduce $W(P') \subset W(C' \cup \{m\})$. Since $r_{m''}$ belongs to the Euclidean triangle subgroup $W(C' \cup \{m\})$ and $r_{m''} r_{\mu'}$ has finite order for every $\mu' \in C'$, we see that $r_{m''} r_{\mu'}$ has finite order for every $\mu' \in P'$. Thus $C' = P'$. Moreover for all $\mu' \in P'$, the reflections $r_{\mu'}$ does not commute with r_m.

Let μ be any element of P different from m. Let a denote the half-space bounded by m such that $\mu \cap a = \emptyset$. Let h_0 denote the half-space bounded by m' such that $a \cap h_0 \subset r_m(h_0)$. Extend h_0 to a chain of half-spaces $(h_i)_{i \in \mathbb{Z}}$ such that $h_i \subset h_{i+1}$ for all $i \in \mathbb{Z}$ and that $\{\partial h_i |\ i \in \mathbb{Z}\} = P'$. Since $W(P' \cup \{m\})$ is a Euclidean triangle subgroup it follows that the relation $a \cap h_i \subset r_m(h_i)$ holds for every $i \in \mathbb{Z}$. For each $i \in \mathbb{Z}$, choose a point $y_i \in \mu \cap \partial h_i$ and a point $y'_i \in \partial h_i$ in the interior of a. Then $y_i \in \partial h_i$ and $y'_i \in \partial h_i$ are separated by m. Since r_m and $r_{\partial h_i}$ do not commute it follows that y_i and y'_i are separated by $r_m(\partial h_i)$. Since $y'_i \in a \cap h_i$ we deduce that $y_i \notin r_m(h_i)$ for all $i \in \mathbb{Z}$. Now choose a point $x_i \in m \cap \partial h_i$ for each $i \in \mathbb{Z}$. We have

$$x_0 \in m \cap \partial h_0 \subset r_m(\partial h_0) \subset r_m(h_0) \subset r_m(h_1) \subset r_m(h_2) \subset \dots$$

Since $\mathscr{M}(x_0, y_0)$ is finite and since $y_0 \notin r_m(h_0)$, there exists $j > 0$ such that $y_0 \in r_m(h_j)$. Thus the wall $r_m(\partial h_i)$ separates y_0 from y_i for all $i \geq j$. Since y_0 and y_i both lie on the wall μ, it follows that ∂h_i meets μ for all $i \geq j$.

This argument holds for any $\mu \in P \backslash \{m\}$. In particular, if we choose μ such that m and μ are separated by at least L elements of P, where L is the constant of Lemma 3.1, we deduce from this lemma that $W(\{m, \mu, \partial h_j\})$ is a Euclidean triangle subgroup. By Corollary 3.4, we obtain $r_{\partial h_i} \in \overline{W(\{m, \mu, \partial h_j\})}$ for all $i \geq j$. As before, this implies that $W(P') < \overline{W(\{m, \mu, \partial h_j\})}$ and, in particular, that $m'' = r_m(m') = r_m(\partial h_0)$ meets μ. Thus we have $\mu \in C$.

Since this holds for all walls $\mu \in P$ which are sufficiently far apart from m, and since C is convex, we finally deduce that $C = P$. By Lemma 3.1 this implies that $W(P \cup \{m'\})$ is a Euclidean triangle subgroup. By Lemma 4.6(i), we have $P \subset \mathscr{M}_{\text{Eucl}}(F)$ whence $m \in \mathscr{M}_{\text{Eucl}}(F)$. □

The main results of this section are the following two propositions.

Proposition 4.7. *The group $W(\mathscr{M}_{\text{Eucl}}(F))$ is isomorphic to a direct product of finitely many irreducible affine Coxeter groups.*

Proof. We claim that for all $m, m' \in \mathscr{M}_{\text{Eucl}}(F)$, either $P_F(m) = P_F(m')$ or the groups $W(P_F(m))$ and $W(P_F(m'))$ centralize each other or $W(P_F(m) \cup P_F(m'))$ is a Euclidean triangle subgroup.

We first deduce the desired result from the claim. We know that $W(\mathscr{M}_{\text{Eucl}}(F))$ is isomorphic to a Coxeter group. Let $W(\mathscr{M}_{\text{Eucl}}(F)) = W_1 \times \cdots \times W_k$ be the decomposition of $W(\mathscr{M}_{\text{Eucl}}(F))$ in its direct components. Hence W_i is an irreducible Coxeter group for each $i = 1, \ldots, k$. Let M_i denote the set of walls $m \in \mathscr{M}_{\text{Eucl}}(F)$ such that $r_m \in W_i$. We note that $\mathscr{M}_{\text{Eucl}}(F) = M_1 \sqcup \cdots \sqcup M_k$ and $W_i = W(M_i)$.

We must prove that W_i is affine. We record the following easy observations which follow from the fact that the W_i's are the irreducible components of $W(\mathscr{M}_{\text{Eucl}}(F))$:

(1) If $m \in \mathscr{M}_{\text{Eucl}}(F)$ is a wall such that $r_m \in W_i$, then $W(P_F(m)) \leq W_i$.
(2) If $m, m' \in \mathscr{M}_{\text{Eucl}}(F)$ are two walls such that $P_F(m) \neq P_F(m')$ and that r_m and $r_{m'}$ both belong to W_i, then there exists a sequence of walls $m = m_0, m_1, \ldots, m_\ell = m'$ such that for each j, one has $m_j \in M_i$, $r_{m_j} \in W_i$ and r_{m_j} does not commute with $r_{m_{j-1}}$ (a priori the order of $r_{m_j} r_{m_{j-1}}$ might be infinite).

We show that, in view of the claim above, these two observations imply that for any wall $m \in \mathscr{M}_{\text{Eucl}}(F)$ such that $r_m \in W_i$, one has

$$W(P_F(m)) \leq W_i \leq W(\widetilde{P_F(m)}),$$

where $W(\widetilde{P_F(m)})$ is the irreducible affine Coxeter group provided by Lemma 3.3.

By the first observation we just have to check that $W_i \leq W(\widetilde{P_F(m)})$. Since $W_i = W(M_i)$ it is enough to show that $r_{m'} \in W(\widetilde{P_F(m)})$ for any $m' \in M_i$. For such an m' we have a sequence of walls $m = m_0, m_1, \ldots, m_\ell = m'$ such that for each j, one has $m_j \in M_i$ and r_{m_j} does not commute with $r_{m_{j-1}}$. We are going to show by induction that for each $\mu \in P_F(m_i)$ we have $r_\mu \in W(\widetilde{P_F(m)})$, which implies in particular $r_{m'} \in W(\widetilde{P_F(m)})$.

This is clearly true for $i = 0$. Assume this is true for $P_F(m_{i-1})$, with $i > 0$. Either $m_i \in P_F(m_{i-1})$, thus $P_F(m_i) = P_F(m_{i-1})$ and we have nothing to prove. Or, by the initial claim, $r_{m_i} r_{m_{i-1}}$ has finite order > 2 and $W(P_F(m_{i-1}) \cup P_F(m_i))$ is a Euclidean triangle subgroup. Since r_{m_i} and $r_{m_{i-1}}$ do not commute it follows that $W(P_F(m_{i-1}) \cup \{m_i\})$ is a Euclidean triangle subgroup. Thus by Lemma 3.3 we have $r_{m_i} \in W(\widetilde{P_F(m)})$. In fact the same argument applies to any wall $\mu \in P_F(m_i)$, which ends the proof.

The inclusion $W_i \leq W(\widetilde{P_F(m)})$, is now established. In particular W_i is an infinite reflection subgroup of an irreducible affine Coxeter group; hence it must be itself an affine Coxeter group, as desired.

It remains to prove the claim. Let $m, m' \in \mathscr{M}_{\mathrm{Eucl}}(F)$.

Suppose that $P_F(m) \neq P_F(m')$. Then by Lemma 4.5 m meets m'.

If there exists $m'' \in P_F(m) \cap P_F(m')$ then, by Lemma 4.6(i), we have $m'' \in \mathscr{M}_{\mathrm{Eucl}}(F)$ which implies that the elements of $P_F(m'')$ are pairwise disjoint (see Lemma 4.5). Since $m'' \in P_F(m) \cap P_F(m')$, we have $\{m, m'\} \subset P_F(m'')$ and, hence, $m = m'$ because m meets m'. This contradicts the fact that $P_F(m) \neq P_F(m')$, thereby showing that $P_F(m) \cap P_F(m')$ is empty. In other words, m meets every element of $P_F(m')$ and m' meets every element of $P_F(m)$.

For all $\mu \in P_F(m)$ we have $\mu \in \mathscr{M}_{\mathrm{Eucl}}(F)$ by Lemma 4.6(i) and, hence, $P_F(m) = P_F(\mu)$ by Lemma 4.5. Similarly, for all $\mu' \in P_F(m')$, we have $P_F(m') = P_F(\mu')$. Therefore, we deduce from the previous paragraph that every element of $P_F(m)$ meets every element of $P_F(m')$.

Suppose moreover that $W(P_F(m))$ does not centralize $W(P_F(m'))$. Then there exist $p \in P_F(m)$ and $p' \in P_F(m')$ such that r_p and $r_{p'}$ do not commute. Let $p'' := r_p(p')$.

Suppose p'' meets only finitely elements of the line of walls $P_F(m)$. Then there is a segment of walls $(p_-, p_1, p_2, \dots, p_n, p^+)$ inside $P_F(m)$ such that $\{p_1, p_2, \dots, p_n\}$ is the set of walls of $P_F(m)$ which meet p'', and p'' is disjoint from p_- and p_+. We let x_-, x_+ denote points in $p' \cap p_-, p' \cap p_+$ respectively. Since p separates p_- from p_+ and $p' \setminus p = p' \setminus p''$ we deduce that p'' separates x_- from x_+. Thus p'' separates p_- from p_+. In particular since p_- and p_+ meet F we have $p'' \in \mathscr{M}(F)$ and clearly $p'' \in P_F(p_-)$. As we have already observed we have $P_F(p_-) = P_F(m) = P_F(p)$. Thus $p'' \in P_F(p)$, contradiction.

Thus in fact p'' meets infinitely many elements of $P_F(m)$. By Lemma 3.1, this shows that $W(P_F(m) \cup \{p'\})$ is a Euclidean triangle subgroup. Similarly $W(P_F(m') \cup \{p\})$ is a Euclidean triangle subgroup. The order of the product $r_p r_{n'}$ is thus independent of the wall n' chosen in the line of walls $P_F(m')$. It follows that for each $n' \in P_F(m')$ the reflections r_p and $r_{n'}$ do not commute. Then by Lemma 3.1 the subgroup $W(P_F(m) \cup \{n'\})$ is also a Euclidean triangle subgroup. By Corollary 3.4 we now deduce that $r_{n'} \in \overline{W(P_F(m) \cup \{p'\})}$. Thus $W(P_F(m')) \subset \overline{W(P_F(m) \cup \{p'\})}$, and in particular the group $W(P_F(m) \cup P_F(m'))$ is a Euclidean triangle subgroup, which proves the claim. □

Corollary 4.8. *For all $m \in \mathscr{M}_{\mathrm{Eucl}}(F)$ and $\gamma \in W(\mathscr{M}_{\mathrm{Eucl}}(F))$, if $\gamma.m \cap m = \emptyset$ then $\gamma.m \in \mathscr{M}_{\mathrm{Eucl}}(F)$.*

Proof. By assumption, the group $\langle r_m, r_{\gamma.m}\rangle$ is an infinite dihedral group which is contained in $W(\mathscr{M}_{\mathrm{Eucl}}(F))$. Therefore, since $W(\mathscr{M}_{\mathrm{Eucl}}(F))$ is an affine Coxeter group by Proposition 4.7, the group $W(P_F(m) \cup \{\gamma.m\})$ is an infinite dihedral group and, by Lemma 4.5(iii), we have $r_{\gamma.m} \in W(P_F(m))$. Since $P_F(m)$ is a convex line of walls, we deduce finally that $\gamma.m \in P_F(m) \subset \mathscr{M}_{\mathrm{Eucl}}(F)$. □

Proposition 4.9. *One the following assertions holds:*

(i) *There exists an infinite subset $M \subset \mathscr{M}(F)$ which satisfies the following conditions:*
- *For all $m, m' \in M$, either $m \cap F = m' \cap F$ or $m \cap F \cap m' = \emptyset$;*
- *The groups $W(M)$ and $W(\mathscr{M}(F) \backslash M)$ centralize each other.*

(ii) *The group $W(\mathscr{M}(F))$ is isomorphic to an affine Coxeter group.*

Proof. Assume first that $\mathscr{M}_{\mathrm{Eucl}}(F) = \mathscr{M}(F)$. Then by Proposition 4.7 property (ii) holds.

Assume now there exists $m \in \mathscr{M}(F) \backslash \mathscr{M}_{\mathrm{Eucl}}(F)$. Let M be the set of all those elements of $\mathscr{M}(F)$ which do not belong to $\mathscr{M}_{\mathrm{Eucl}}(F)$ and which are F–parallel to m. By Lemma 4.6(ii), we have $P_F(m) \subset M$; in particular M is infinite. Let $m' \in \mathscr{M}(F) \backslash M$. If m' is not F–parallel to m then $r_{m'}$ centralizes $W(M)$ by Lemma 4.6(iii). If m' is F–parallel to m, then $m' \in \mathscr{M}_{\mathrm{Eucl}}(F)$ since $m' \notin M$. In view of Lemma 4.6(iv), this implies that $r_{m'}$ centralizes $W(M)$. This shows that the groups $W(M)$ and $W(\mathscr{M}(F) \backslash M)$ centralize each other. Thus property (i) holds. □

5. From geometric flats to free abelian groups

Let X be a combinatorially convex subcomplex of the Davis complex $|W|_0$, and Γ be a subgroup of W which stabilizes X and whose induced action on X is cocompact. The distance function on $|W|_0$ is denoted by d.

Lemma 5.1. *Let $\rho \subset X$ be any unbounded subset through a given point x, and let $\mathscr{M}(\rho) := \bigcup_{y,z \in \rho} \mathscr{M}(y,z)$ be the set of walls which separate points of ρ. There exists a constant K (depending on ρ and Γ) with the following property: given any positive real number r, there exists a chamber c at distance at most K from x and an element $\gamma \in \Gamma \cap W(\mathscr{M}(\rho))$ such that c and $\gamma.c$ both meet ρ, and that $d(c, \gamma.c) > r$.*

Proof. Recall that a combinatorially convex subcomplex is a (CAT(0) convex) union of chambers.

Let $\mathcal{C}(\rho)$ denote the set of chambers of X meeting ρ: thus ρ is covered by the chambers of $\mathcal{C}(\rho)$. Recall that Γ has finitely many orbits on the set of all chambers of X. Since ρ is unbounded, the set $\mathcal{C}(\rho)$ is infinite and it follows that there exists a chamber $c \in C(\rho)$ such that $\Gamma.c \cap \mathcal{C}(\rho)$ is infinite.

We write $\Gamma.c \cap \mathcal{C}(\rho) = \{\gamma_0.c, \gamma_1.c, \dots, \gamma_i.c, \dots\}$ (with $\gamma_0 = 1$). We pick a point x_i in each intersection $\rho \cap \gamma_i.c$. By Lemma 1.2 there exists $g_i \in W(\mathscr{M}(x_0, x_i))$ such that $g_i x_0$ and x_i lie in a common chamber. Thus ${g_i}^{-1}\gamma_i$ is an element of W sending c to a chamber meeting c. There are finitely many such elements.

Thus up to extracting a subsequence we may suppose that the sequence $({g_i}^{-1}\gamma_i)_{i \geq 1}$ is constant. Then for each i the element $\gamma'_i = \gamma_i {\gamma_1}^{-1}$ belongs to $\Gamma \cap W(\mathscr{M}(\rho))$. And also γ'_i sends the chamber $\gamma_1 c$ to the chamber $\gamma_i c$. The Lemma follows because the set of chambers $(\gamma_i c)_{i \geq 1}$ is infinite. □

As before, let $\mathscr{M}(F)$ denote the set of all walls which separate points of F. Theorem A of the introduction is a straightforward consequence of the following:

Theorem 5.2. *Let F be a geometric flat which is isometrically embedded in X; let n denote its dimension. Then the intersection $\Gamma \cap W(\mathscr{M}(F))$ contains a free abelian group of rank n.*

Proof. By Selberg's lemma, the group Γ has a finite index subgroup which is torsion free. Since Γ is cocompact on X, any finite index subgroup of Γ is cocompact as well, hence we may assume without loss of generality that Γ is torsion free.

The proof works by induction on the dimension n of the flat F. We may assume that $n > 0$.

Suppose first that $\mathscr{M}(F)$ possesses a subset M which satisfies the conditions (i) of Proposition 4.9. Let then m be any element of M and set $F' := F \cap m$. By Lemma 4.1, F' is a geometric flat of dimension $n-1$.

Let ρ denote any geodesic ray of F meeting transversally infinitely many walls of M. Let x denote the origin of ρ, and let x_n denote the unique point of ρ with $d(x, x_n) = n$. By Lemma 2.1 there exists a point $z_n \in X$ such that $\mathscr{M}(x, x_n) = \mathscr{M}(x, z_n) \sqcup \mathscr{M}(z_n, x_n)$, with $\mathscr{M}(x, z_n) = \mathscr{M}(x, x_n) \cap M$. Observe that the cardinality of $\mathscr{M}(x, z_n)$ tends to infinity with n, and thus $d(x, z_n) \to +\infty$. There is a subsequence $(z_{n_k})_{k \geq 0}$ such that the geodesic segment $[x, z_{n_k}] \subset X$ converges to a geodesic ray $\rho' \subset X$ (with origin x). Note that for every $y \in \rho'$ we have $\mathscr{M}(x, y) \subset \mathscr{M}(x, z_{n_k})$ for k large enough. In particular $\mathscr{M}(x, y) \subset M$. Thus $\mathscr{M}(\rho') \subset M$.

We now apply Lemma 5.1 to the ray ρ' for some (large) positive real number $r > 0$. We then get a nontrivial element $\gamma \in \Gamma \cap W(M)$. Observe that γ must be of infinite order since Γ is torsion free.

It follows from the definition of M that γ centralizes $W(\mathscr{M}(F'))$. Furthermore, since $W(\mathscr{M}(F'))$ is isomorphic to a Coxeter group and since the center of any Coxeter group is a torsion group (this is well known and is a straightforward consequence of [Hum90, Exercise 1, p.132]), the intersection $W(\mathscr{M}(F')) \cap \langle\gamma\rangle$ is trivial. We deduce that the group generated by $W(\mathscr{M}(F'))$ together with γ is isomorphic to the direct product $W(\mathscr{M}(F')) \times \langle\gamma\rangle$. The desired result follows by induction.

Suppose now that assertion (ii) of Proposition 4.9 holds. Let μ_1 be any element of $\mathscr{M}(F)$. Again by Lemma 4.1 the intersection $\mu_1 \cap F$ is a geometric flat of dimension $n-1$. Note that any flat Φ of dimension ≥ 1 is unbounded and thus has $\mathscr{M}(\Phi) \neq \emptyset$. Thus for each $i = 2, \dots, n$ we may choose successively

$$\mu_i \in \mathscr{M}\big(\big(\bigcap_{j=1}^{i-1} \mu_j\big) \cap F\big).$$

In view of Lemma 4.1, the set $\big(\bigcap_{i=1}^{n} \mu_i\big) \cap F$ consists of a single point x of F and for each $i \in \{1, \dots, n\}$, the set

$$\lambda_i := \big(\bigcap_{j \in \{1,\dots,n\} \setminus \{i\}} \mu_j\big) \cap F$$

is a geodesic line of F.

We need the following auxiliary result:

Lemma 5.3. *Γ has a finite index subgroup Γ' such that for any wall m and any chamber c meeting m, if $\gamma \in \Gamma'$ sends c to a chamber meeting m, then $\gamma m = m$.*

Proof. It is enough to prove the Lemma when $\Gamma = W$. Recall that the stabilizer of a wall m is the centralizer of the involution r_m. Since W is residually finite the centralizer $Z(r_m)$ is a separable subgroup, that is to say $Z(r_m)$ is an intersection of finite index subgroups. (In any residually finite group W the centralizer of any element g is separable. Indeed, for $x \notin C = Z_W(g)$, we have $[x, g] \neq 1$, thus there is a finite quotient $\phi : W \to \bar{G}$ such that $[\phi(x), \phi(g)] \neq 1$. Then $\phi(x) \notin Z_{\bar{G}}(\phi(g))$ and the finite index subgroup $\phi^{-1} Z_{\bar{G}}(\phi(g))$ separates x from C.)

We fix some wall m and claim that there is a finite index subgroup $W_m \subset W$ such that for any chamber c meeting m, if $\gamma \in W_m$ sends c to a chamber meeting m, then $\gamma m = m$. The lemma will follow since we may assume that W_m is normal, and there are only finitely many orbits of walls under W.

Let B_m be the subset of W consisting of all those elements $\gamma \in W$ such that there exists a chamber c such that m and $\gamma.c$ both meet m. Note that B_m is invariant by left- and right-multiplication under $Z(r_m)$. In fact it is a finite union of double classes: $B_m = Z(r_m) \sqcup Z(r_m)\gamma_1 Z(r_m) \sqcup \cdots \sqcup Z(r_m)\gamma_k Z(r_m)$, where $\gamma_1, \dots, \gamma_k$ do not belong to $Z(r_m)$ (the finiteness follows from the fact that $Z(r_m)$ acts co-finitely on the set of chambers meeting m, and from the local compactness of the Davis complex). The claim follows if we take for W_m any finite index subgroup of W containing the separable subgroup $Z(r_m)$ but none of the elements $\gamma_1, \dots, \gamma_k$. □

By Lemma 5.3 we may assume that for any wall m and any chamber c meeting m, if $\gamma \in \Gamma$ sends c to a chamber meeting m, then $\gamma m = m$. Note that this implies in particular that if γm intersects m, then $\gamma m = m$.

Let r be any positive real number. For each i we choose one of the two rays contained in λ_i with origin x, and denote it by ρ_i. For each $i \in \{1, \dots, n\}$, Lemma 5.1 provides a chamber c_i at distance at most K_i of x, and an element $\gamma_i(r) \in W(\mathscr{M}(\lambda_i)) \cap \Gamma$

$(\subset W(\mathscr{M}(F)) \cap \Gamma)$, such that $c_i \cap \rho_i$ and $\gamma_i(r).c_i \cap \rho_i$ are both nonempty, and that $d(c_i, \gamma_i(r).c_i) > r$. Here c_i and $\gamma_i(r)$ depend on r, but K_i depends only on ρ_i. Note that $\gamma_i(r)$ is of infinite order because Γ is torsion free.

It immediately follows from the fact that $\rho_i \subset \mu_j$ that each $\gamma_i(r)$ preserves μ_j $(j \neq i)$.

Since $x \in \rho_i \cap \mu_i$ but $\rho_i \not\subset \mu_i$, it follows from Lemma 4.2 that there is a constant r_i such that, given any point y of ρ_i, if y is at distance at least r_i from x, then y is at distance larger than $K_i + D$ from μ_i, where D is the diameter of a chamber. Therefore, for each $r \geq r_i$, we have $d(x, \gamma_i(r).c_i) \geq d(c_i, \gamma_i(r).c_i) > r$ and hence any point on $\gamma_i(r).c_i \cap \rho_i$ is at distance larger than $K_i + D$ from μ_i. Thus $\gamma_i(r).c_i$ is at distance larger than K_i from μ_i. On the other hand $\gamma_i(r).c_i$ is at distance at most K_i from $\gamma_i(r)\mu_i$, from which it follows that $\gamma_i(r)\mu_i \neq \mu_i$ for all $r \geq r_i$. By the above, this yields $\gamma_i(r)\mu_i \cap \mu_i = \emptyset$ for all $r \geq r_i$.

Let a_i be the half-space bounded by μ_i and containing ρ_i. We define an element γ_i as follows.

If $\gamma_i(r_i)a_i \subset a_i$ we set $\gamma_i = \gamma_i(r_i)$.

If not, then we choose $r > r_i$ as follows. Note that $\gamma_i(r)\mu_i \in \mathscr{M}_{\mathrm{Eucl}}(F)$ for all r by Corollary 4.8. In particular $\gamma_i(r_i)\mu_i$ meets ρ_i, but $\rho_i \not\subset \gamma_i(r_i)\mu_i$ because $x \in \rho_i \cap \mu_i$ and $\mu_i \cap \gamma_i(r_i)\mu_i = \emptyset$. Thus, by Lemma 4.2, every point of ρ_i sufficiently far away from x is also far way from $\gamma_i(r_i)\mu_i$. Repeating the arguments used to define the constant r_i, we obtain a constant $r > r_i$ such that $\gamma_i(r)\mu_i \neq \gamma_i(r_i)\mu_i$.

Now, if $\gamma_i(r)a_i \subset a_i$ we set $\gamma_i = \gamma_i(r)$. Otherwise we set $\gamma_i = \gamma_i(r)^{-1}\gamma_i(r_i)$. Let us check that, in the latter case, we have also $\gamma_i a_i \subset a_i$. The walls μ_i, $\gamma_i(r_i)\mu_i$ and $\gamma_i(r)\mu_i$ belong to $\mathscr{M}_{\mathrm{Eucl}}(F)$ by Corollary 4.8 and are pairwise disjoint by construction. Thus they form a chain and it follows that $\gamma_i(r_i)a_i \subset \gamma_i(r)a_i$ whence $\gamma_i a_i \subset a_i$. Therefore, for all $m > 0$, we have $\gamma_i^m a_i \subset a_i$ and hence $\gamma_i^m \mu_i \cap \mu_i = \emptyset$ while $\gamma_i^m \mu_j = \mu_j$ for $j \neq i$.

Choose integers $m_1, \dots, m_n$ divisible enough so that each $\gamma_i' := \gamma_i{}^{m_i}$ belongs to the translation subgroup of the affine Coxeter group $W(\mathscr{M}(F))$. Thus the γ_i''s generate an abelian group. In view of the action of each γ_i' on the walls $\mu_1, \dots, \mu_n$, the intersection $\langle \gamma_i' \rangle \cap \langle \gamma_j' |\ j \neq i \rangle$ is trivial for all i. This implies that the γ_i''s generate a free abelian group of rank n. $\square$

We note that the complete proof of Theorem 5.2 is much shorter when (W, S) is assumed to be right-angled (in this case $\mathscr{M}_{\mathrm{Eucl}}(F)$ is empty).

6. Geometric flats in Tits buildings

The purpose of this section is to prove Theorem E of the introduction.

As before, let (W, S) be a Coxeter system of finite rank. Let $\mathscr{B} = (\mathcal{C}(\mathscr{B}), \delta)$ be a building of type (W, S). Recall that $\mathcal{C}(\mathscr{B})$ is a set whose elements are called **chambers**, and that $\delta : \mathcal{C}(\mathscr{B}) \times \mathcal{C}(\mathscr{B}) \to W$ is a mapping, called W**-distance**, which satisfies the following conditions, where $x, y \in \mathcal{C}(\mathscr{B})$ and $w = \delta(x, y)$:

- **Bu1:** $w = 1$ if and only if $x = y$;
- **Bu2:** if $z \in \mathcal{C}(\mathscr{B})$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) = w$ or ws, and if, furthermore, $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$;
- **Bu3:** if $s \in S$, there exists $z \in \mathcal{C}(\mathscr{B})$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

For example the map $W \times W \to W$ sending (x, y) to $x^{-1}y$ satisfies the above. An **apartment** of the building B is a subset $\mathcal{C}(\mathscr{A}) \subset \mathcal{C}(\mathscr{B})$ such that there exists a bijection $f : \mathcal{C}(\mathscr{A}) \to W$ satisfying $\delta(x, y) = f(x)^{-1} f(y)$.

The composed map $\ell \circ \delta : \mathcal{C}(\mathscr{B}) \times \mathcal{C}(\mathscr{B}) \to \mathbb{N}$, where ℓ is the word metric on W with respect to S, is called the **numerical distance** of $\mathscr{B}$. It is a discrete metric on $\mathcal{C}(\mathscr{B})$.

The following lemma is well known:

Lemma 6.1. *Let $\mathcal{C}(\mathscr{A})$ be an apartment and C be a subset of $\mathcal{C}(\mathscr{B})$. Suppose that there exists a map $f : C \to \mathcal{C}(\mathscr{A})$ such that $\delta(f(c), f(d)) = \delta(c, d)$ for all $c, d \in C$. Then there exists an apartment $\mathcal{C}(\mathscr{A}')$ such that $C \subset \mathcal{C}(\mathscr{A}')$.*

Proof. Follows from [Tit81, §3.7.4]. □

Let $T \subset S$ and let c be a chamber of the building B. The **residue of type T of** c is the set $\rho_T(c)$ of those chambers c' for which $\delta(c, c') \in W(T)$. The residue is called **spherical** whenever $W(T)$ is finite. Given any residue ρ of B and any chamber x, there exists a unique chamber c in ρ at minimal numerical distance from x. This chamber has the property that $\delta(x, d) = \delta(x, c)\delta(c, d)$ for each chamber d of ρ. The chamber c is called the **projection of x onto** ρ and is denoted by $\mathrm{proj}_\rho(c)$ (see [Ron89, §Corollary 3.9]).

Lemma 6.2. *Let $\mathcal{C}(\mathscr{A})$ be an apartment of $\mathscr{B}$ and $C \subset \mathcal{C}(\mathscr{A})$ be a set of chambers. Suppose that there exists a residue ρ and a chamber $c \in C$ such that $c \in \mathcal{C}(\rho)$ and $\mathrm{proj}_\rho(c') = c$ for all $c' \in C$. Then, for any chamber $d \in \mathcal{C}(\rho)\backslash\{c\}$, there exists an apartment $\mathcal{C}(\mathscr{A}_d)$ such that $C \cup \{d\}$ is contained in $\mathcal{C}(\mathscr{A}_d)$.*

Proof. Let $d \in \mathcal{C}(\rho)\backslash\{c\}$ and let $w_d := \delta(c, d)$. Let d' be the unique chamber of $\mathcal{C}(\mathscr{A})$ such that $\delta(c, d') = w_d$. For any $c' \in C$, we have $\delta(c', d) = \delta(c', c).w_d = \delta(c', d')$ because $\mathrm{proj}_\rho(c') = c$. It follows that the function $f : C \cup \{d\} \to C \cup \{d'\}$, which maps d to d' and induces the identity on C, preserves the W-distance δ. Therefore, the existence of an apartment $\mathcal{C}(\mathscr{A}_d)$ such that $\mathcal{C}(\mathscr{A}_d)$ contains $C \cup \{d\}$ follows from Lemma 6.1. □

Before stating the main result of this section, we need to introduce some additional terminology and notation:

- $|\mathscr{B}|_0$ denotes the CAT(0)-realization of the building $\mathscr{B}$, as defined in [Dav98]; it is a piecewise Euclidean simplicial complex. For each chamber $c \in B$ there is an associated CAT(0)-convex subcomplex $|c|_0 \subset |\mathscr{B}|_0$, which we call **the associated geometric chamber**. For every subset $C \subset \mathcal{C}(\mathscr{B})$ we denote by $|C|_0$ the union of geometric chambers $|c|_0$ associated to chambers $c \in C$. We say that a subcomplex $X \subset |\mathscr{B}|_0$ is **combinatorial** whenever it is a union of geometric chambers. If $\mathscr{A}$ is any apartment of $\mathscr{B}$ the subcomplex $|\mathscr{A}|_0$ is isometric to $|W|_0$. As a simplicial complex, $|\mathscr{A}|_0$ is isomorphic to the first barycentric subdivision of the Davis complex $|W|_0$.
- Given $x \in |\mathscr{B}|_0$, we set
$$\rho(x) := \{c \in \mathcal{C}(\mathscr{B}) |\ x \in |c|_0\}$$
and
$$\sigma(x) := \bigcap_{c \in \rho(x)} |c|_0.$$
The set $\rho(x)$ is a (spherical) residue. The subcomplex $|\rho(x)|_0$ is a neighbourhood $N(x)$ of x in $|\mathscr{B}|_0$. For every chamber $c \in \mathcal{C}(\mathscr{B})$, the set $\mathrm{Int}(c)$ of points $x \in |\mathscr{B}|_0$ such that $\rho(x) = \{c\}$ is an open subset of $|\mathscr{B}|_0$. It is the interior of $|c|_0$ and its closure is $|c|_0$.
- Given $E \subset |\mathscr{B}|_0$, we set
$$\mathcal{C}(E) := \{c \in \mathcal{C}(\mathscr{B}) |\ |c|_0 \subset E\}.$$
For example given any $x \in |\mathscr{B}|_0$ we have $\mathcal{C}(N(x)) = \rho(x)$. We say that a subcomplex $\mathscr{A} \subset |\mathscr{B}|_0$ is a **geometric apartment** provided $\mathscr{A}$ is combinatorial and $\mathcal{C}(\mathscr{A})$ is an apartment of $\mathscr{B}$.
- Given a geometric flat $F \subset |\mathscr{B}|_0$ and any subset $E \subset |\mathscr{B}|_0$, we denote by $\dim(F \cap E)$ the dimension of the Euclidean subspace of F generated by $E \cap F$; by convention, the empty set is a Euclidean subspace of dimension -1.

Let now $F \subset |\mathscr{B}|_0$ be a geometric flat of dimension n. Since the combinatorial subcomplexes $N(x)$ are neighborhoods of x, we have:

$$\forall x \in F,\ \exists\ c \in \mathcal{C}(\mathscr{B}) \text{ such that } x \in |c|_0 \text{ and } \dim(F \cap |c|_0) = n.$$

And since every geometric chamber is the closure of its interior, we deduce:

$$\forall x \in F,\ \exists\ y \in F \text{ such that } x \in \sigma(y) \text{ and } \dim(F \cap \sigma(y)) = n.$$

These two basic facts will be used repeatedly in the following.

Theorem 6.3. *Let $F \subset |\mathscr{B}|_0$ be a geometric flat of dimension n and let c_0 be a chamber such that* $\dim(F \cap c_0) = n$ *(the geometric chamber associated to c_0 is also denoted by c_0). Define*

$$C(F, c_0) := \{\mathrm{proj}_{\rho(x)}(c_0) \mid x \in F\}.$$

Then there exists a geometric apartment $\mathscr{A}$ such that $C(F, c_0) \subset \mathcal{C}(\mathscr{A})$. In particular, we have $F \subset \mathscr{A}$.

Proof. The proof is by induction on n, the case $n = 0$ being trivial. We assume now that $n > 0$.

Let $F_0 \subset F$ be a Euclidean hyperplane such that $\dim(F_0 \cap c_0) = n-1$. By induction, the set $C(F_0, c_0)$ is contained in the set of chambers of some apartment. In view of Lemma 6.1, it follows from Zorn's lemma that the collection of all those subsets of $C(F, c_0)$ which contain $C(F_0, c_0)$ and which are contained in the set of chambers of some apartment, has a maximal element.

Let C_1 be such a maximal element and choose a geometric apartment $\mathscr{A}_1$ such that $C_1 \subset \mathcal{C}(\mathscr{A}_1)$. Set $X := \mathscr{A}_1 \cap F$. Note that X is closed and convex.

Suppose by contradiction that C_1 is properly contained in $C(F, c_0)$. The rest of the proof is divided into several steps. The final claim below contradicts the maximality of C_1, thereby proving the theorem.

Claim 1. *For all $x \in X$, we have* $\mathrm{proj}_{\rho(x)}(c_0) \in C_1$.

Since $\mathscr{A}_1$ is a combinatorial subcomplex, we have $\sigma(x) \subset \mathscr{A}_1$. Since $c_0 \in \mathcal{C}(\mathscr{A}_1)$, we have $\mathrm{proj}_{\rho(x)}(c_0) \in \mathcal{C}(\mathscr{A}_1)$. Therefore, the claim follows from the maximality of C_1.

Claim 2. *For all $c \in C_1$, there exists $x \in X$ such that* $\mathrm{proj}_{\rho(x)}(c_0) = c$.

Given $c \in C(F, c_0)$, there exists $x \in F$ such that $\mathrm{proj}_{\rho(x)}(c_0) = c$. If now $c \in C_1$, then $\sigma(x) \subset |c|_0 \subset \mathscr{A}_1$. Thus $x \in F \cap \mathscr{A}_1 = X$.

Claim 3. $\dim(F \cap X) = n$.

Clear since $c_0 \cap F \subset X$ and $\dim(F \cap c_0) = n$.

Claim 4. *There exists a Euclidean hyperplane $F_1 \subset F$ which is contained in $\mathscr{A}_1$ and which bounds an open half-space of F, none of whose points is contained in $\mathscr{A}_1$. In other words, the hyperplance F_1 is contained in the Euclidean boundary ∂X of X.*

Let $c \in C(F, c_0) \backslash C_1$ and let $x \in F$ be such that $\mathrm{proj}_{\rho(x)}(c_0) = c$. By Claim 1, x does not belong to X. Given $x_0 \in c_0 \cap F$, we have $[x_0, x] \cap X = [x_0, y]$ for some $y \in X$ because X is closed and convex. Let $F_1 \subset F$ be the Euclidean hyperplane parallel to F_0 and containing y. We have $F_1 \subset X$ by convexity. Furthermore, it is clear from the definition of y and F_1 that any point $z \in F \backslash F_1$ on the same side of F_1 as x does not belong to $\mathscr{A}_1$.

Claim 5. *Let $x_1 \in F_1$ be such that* $\dim(F_1 \cap \sigma(x_1)) = n - 1$. *For all $c \in C_1$, we have* $\mathrm{proj}_{\rho(x_1)}(c) = \mathrm{proj}_{\rho(x_1)}(c_0)$.

Let $c_1 := \mathrm{proj}_{\rho(x_1)}(c_0)$. Suppose by contradiction that there exists $c \in C_1$ such that $\mathrm{proj}_{\rho(x_1)}(c) \neq c_1$. Let h be a (Coxeter) half-space of the apartment $\mathcal{C}(\mathscr{A}_1)$ containing c_1 but not $c_2 := \mathrm{proj}_{\rho(x_1)}(c)$. Thus h contains c_0 but not c.

Since $\sigma(x_1) \subset |c_1|_0 \cap |c_2|_0$, we have $\sigma(x_1) \subset \partial|h|_0$. Therefore, since $F_1 \subset \mathscr{A}_1$ (see Claim 4) and since $\dim(F_1 \cap \sigma(x_1)) = n-1$, we deduce from Lemma 4.1 that $F_1 \subset \partial|h|_0$. By Claim 4, the set X, as a subset of F, is entirely contained in one of the Euclidean half-spaces of F determined by F_1. Since $F_1 \subset \partial|h|_0$, we deduce that X, as a subset of $\mathscr{A}_1$, is entirely contained in one of the Coxeter half-spaces of $\mathscr{A}_1$ determined by $\partial|h|_0$. Since $c_0 \subset X \cap |h|_0$, we obtain $X \subset |h|_0$.

Since $c \in C_1$, there exists $x \in X$ such that $\mathrm{proj}_{\rho(x)}(c_0) = c$ by Claim 2. Since $X \subset |h|_0$ and since $|h|_0$ is a combinatorial subcomplex, we have $\sigma(x) \subset |h|_0$ and hence $\mathrm{proj}_{\rho(x)}(c_0) \in h$ by the combinatorial convexity of Coxeter half-spaces. This contradicts the fact that h does not contain c.

Claim 6. *There exists $d \in C(F, c_0)$ and an apartment $\mathscr{A}_d$ such that $C_1 \cup \{d\} \subset \mathcal{C}(\mathscr{A}_d)$.*

Let $x_1 \in F_1$ be as in Claim 5. By Claim 1 we have $c_1 := \mathrm{proj}_{\rho(x_1)}(c_0) \in C_1$. Let $y \in F \backslash X$ be such that $x_1 \in \sigma(y)$. Let $d := \mathrm{proj}_{\sigma(y)}(c_0)$. Clearly $d \in C(F, c_0)$. Furthermore $d \notin C_1$, otherwise we would have $y \in \sigma(y) \subset d \subset \mathscr{A}_1$, whence $y \in X$, which is absurd. Since $\sigma(x_1) \subset \sigma(y) \subset d$, the claim follows from Lemma 6.2 together with Claim 5. □

Clearly, Theorem E of the introduction is an immediate consequence of Theorem 6.3, combined with Corollary C.

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