AT INFINITY OF FINITE-DIMENSIONAL CAT(0) SPACES

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Abstract. We show that any filtering family of closed convex subsets of a finite-dimensional CAT(0) space X has a non-empty intersection in the visual bordification $\overline{X} = X \cup \partial X$. Using this fact, several results known for proper CAT(0) spaces may be extended to finite-dimensional spaces, including the existence of canonical fixed points at infinity for parabolic isometries, algebraic and geometric restrictions on amenable group actions, and geometric superrigidity for non-elementary actions of irreducible uniform lattices in products of locally compact groups.

1. Introduction

Several families of finite-dimensional CAT(0) spaces naturally include specimens which are not locally compact; *e.g.* buildings of finite rank (Euclidean or not), finite-dimensional CAT(0) cube complexes, or asymptotic cones of Hadamard manifolds or of CAT(0) groups.

A major difficulty one encounters when dealing with non-proper spaces is that the visual boundary may have a very pathological behaviour. For example, an unbounded CAT(0) space may well have an *empty* visual boundary. The purpose of this paper is to show that for finite-dimensional spaces, the visual boundary nevertheless enjoys similarly nice properties as in the case of proper spaces.

Following B. Kleiner [Kle99], we define the **(geometric) dimension** of a CAT(0) space X to be the supremum over all compact subsets $K \subset X$ of the topological dimension of K. We refer to *loc. cit.* for more details and several characterizations of this notion. A 0-dimensional CAT(0) space is reduced to a singleton, while 1-dimensional CAT(0) spaces coincide with $\mathbf{R}$-trees. We emphasize that the notion of geometric dimension is *local*. It turns out that, for our purposes, it will be sufficient to demand that the spaces have finite dimension *at large scale*. In order to define this condition precisely, we shall say that a CAT(0) space X has **telescopic dimension** $\leq n$ if every asymptotic cone $\lim_\omega(\varepsilon_n X, \star_n)$ has geometric dimension $\leq n$. A space has telescopic dimension 0 if and only if it is bounded. It has telescopic dimension ≤ 1 if and only if it is Gromov hyperbolic. A CAT(0) space of finite geometric dimension has finite telescopic dimension. We refer to §2.1 below for more details and some examples.

Theorem 1.1. *Let X be a complete* CAT(0) *space of finite telescopic dimension and $\{X_\alpha\}_{\alpha\in A}$ be a filtering family of closed convex subspaces. Then either the intersection $\bigcap_{\alpha\in A} X_\alpha$ is non-empty, or the intersection of the visual boundaries $\bigcap_{\alpha\in A} \partial X_\alpha$ is a non-empty subset of ∂X of intrinsic radius at most $\pi/2$.*

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Recall that a family $\mathcal{F}$ of subsets of a given set is called **filtering** if for all $E, F \in \mathcal{F}$ there exists $D \in \mathcal{F}$ such that $D \subseteq E \cap F$. In particular the preceding applies to nested families of closed convex subsets, and provides a criterion ensuring that the visual boundary ∂X is non-empty. In the course of the proof, we shall establish a result similar to Theorem 1.1 for finite-dimensional CAT(1) spaces (see Proposition 5.3 below). We remark however that Theorem 1.1 fails for complete CAT(0) spaces with finite-dimensional Tits boundary, see Example 5.6 below.

Remark 1.2. Theorem 1.1 may be reformulated using the topology $\mathscr{T}_c$ introduced by Nicolas Monod [Mon06, §3.7] on the set $\overline{X} = X \cup \partial X$. It is defined as the coarsest topology such that for any convex subset $Y \subseteq X$, the (usual) closure $\overline{Y}$ in $\overline{X}$ is $\mathscr{T}_c$-closed. It is known that any bounded closed subset of X is $\mathscr{T}_c$-quasi-compact (see [Mon06, Theorem 14]) and that, if X is Gromov hyperbolic, then $\overline{X}$ is $\mathscr{T}_c$-quasi-compact (see Proposition 23 in *loc. cit.*). However, if X is infinite-dimensional then $\overline{X}$ is generally not $\mathscr{T}_c$-quasi-compact. Theorem 1.1 just means that, *given a complete* CAT(0) *space of finite telescopic dimension, the set* $\overline{X}$ *is quasi-compact for the topology* $\mathscr{T}_c$. This compactness property is thus shared by proper CAT(0) spaces, Gromov hyperbolic CAT(0) spaces and finite-dimensional CAT(0) spaces.

A key idea in the proof of Theorem 1.1 is to obtain points at infinity by applying (a very special case of) a result of A. Karlsson and G. Margulis [KM99] to the gradient flow of a convex function that is associated in a canonical way to the given filtering family. This strategy requires to show that the velocity of escape of the gradient flow in question is strictly positive. This is where the assumption on the telescopic dimension of the ambient space is used; the main point in estimating that velocity is the following natural generalisation to non-positively curved spaces of H. Jung's classical theorem [Jun01]. Another closely related generalisation was established in [LS97].

Theorem 1.3. *Let X be a* CAT(0) *space and n be a positive integer.*

Then X has geometric dimension $\leq n$ if and only if for each subset Y of X of finite diameter we have $\mathrm{rad}_X(Y) \leq \sqrt{\frac{n}{2(n+1)}}\mathrm{diam}(Y)$.

Similarly X has telescopic dimension $\leq n$ if and only if for any $\delta > 0$ there exists some constant $D > 0$ such that for any bounded subset $Y \subset X$ of diameter $> D$, we have $\mathrm{rad}_X(Y) \leq \left(\delta + \sqrt{\frac{n}{2(n+1)}}\right)\mathrm{diam}(Y)$.

Recall that the **circumradius** $\mathrm{rad}_X(Y)$ of a subset $Y \subseteq X$ is defined as the infimum of all positive real numbers r such that Y is contained in some closed ball of radius r of X.

Remark 1.4. In the case of an n-dimensional regular Euclidean simplex one has equality in the theorem above. For a short discussion of the case of equality as well as analogous statements in other curvature bounds we refer to Section 3.

It turns out that Theorem 1.1 provides a key property that allows one to extend to finite-dimensional CAT(0) spaces several results which are known to hold for proper spaces. We now proceed to describe a few of these applications.

Parabolic isometries. A first elementary consequence of Theorem 1.1 is the existence of canonical fixed points at infinity for parabolic isometries. This extends the results obtained in [FNS06, Theorem 1.1] and [CM08, Corollary 2.3] in the locally compact setting.

Corollary 1.5. *Let g be a parabolic isometry of a* CAT(0) *space X of finite telescopic dimension. Then the centraliser $\mathscr{Z}_{\mathrm{Is}(X)}(g)$ possesses a canonical fixed point in ∂X.*

Amenable group actions. The next application provides obstructions to isometric actions of amenable groups; in the locally compact case the corresponding statement is due to S. Adams and W. Ballmann [AB98], and generalizes earlier results by M. Burger and V. Schroeder [BS87].

Theorem 1.6. *Let X be a complete* CAT(0) *space of finite telescopic dimension. Let G be an amenable locally compact group acting continuously on X by isometries. Then either G stabilises a flat subspace (possibly reduced to a point) or G fixes a point in the ideal boundary ∂X.*

Combining this with the arguments of [Cap07], one obtains the following description of the algebraic structure of amenable groups acting on CAT(0) cell complexes.

Theorem 1.7. *Let X be a* CAT(0) *cell complex with finitely many types of cells and G be a locally compact group admitting an isometric action on X which is continuous, cellular and metrically proper. Then a closed subgroup $H < G$ is amenable if and only if it is (topologically locally finite)-by-(virtually Abelian).*

By definition, a subgroup H of a topological group G is **topologically locally finite** if the closure of every finitely generated subgroup of H is compact. We refer to [Cap07] for more details. The proof of Theorem 1.7 proceeds as in *loc. cit.* One introduces the **refined boundary** $\partial_{\mathrm{fine}}X$ of the CAT(0) space and shows, using Theorem 1.6, that any amenable subgroup of G virtually fixes a point in $X \cup \partial_{\mathrm{fine}}X$; conversely any point of $X \cup \partial_{\mathrm{fine}}X$ has an amenable stabilizer in G.

Minimal and reduced actions. A basic property of CAT(0) spaces with finite telescopic dimension is that their Tits boundary has finite geometric dimension (see Proposition 2.1 below). Given this observation, Theorem 1.1 may be used to extend several results of [CM08, Part I] to the finite-dimensional case. The following collects a few of these statements.

Proposition 1.8. *Let X be a complete* CAT(0) *space of finite telescopic dimension and let $G < \mathrm{Is}(X)$ be any group of isometries.*

(i) *If the G-action is evanescent, then G fixes a point in $X \cup \partial X$.*
(ii) *If G does not fix a point in the ideal boundary, then there is a non-empty closed convex G-invariant subset $Y \subseteq X$ on which G acts minimally.*
(iii) *Suppose that X is irreducible. If G acts minimally without fixed point at infinity on X, then so does every non-trivial normal subgroup of G; furthermore, the G-action is reduced.*
(iv) *If $\mathrm{Is}(X)$ acts minimally on X, then for each closed convex subset $Y \subsetneq X$ we have $\partial Y \subsetneq \partial X$.*

Following Nicolas Monod [Mon06], we say that the action of a group G on a CAT(0) space X is **evanescent** if there is an unbounded subset $T \subseteq X$ such that for every compact set $Q \subset G$ the set $\{d(gx, x) : g \in Q, x \in T\}$ is bounded. Recall further that the G-action is said to be **minimal** if there is no non-empty closed convex G-invariant subset $Y \subsetneq X$. Finally, it is called **reduced** if there is no non-empty closed convex subset $Y \subsetneq X$ such that for each $g \in G$, the sets Y and $g.Y$ are at bounded Hausdorff distance from one another. The relevance of the notions of evanescent and reduced actions was first highlighted by Nicolas Monod [Mon06] in the context of geometric superrigidity. In particular, the combination of [Mon06, Theorem 6] with Proposition 1.8(iii) yields the following (see [CM08, Theorem 9.4] for the corresponding statement in the locally compact case).

Corollary 1.9. *Let Γ be an irreducible uniform (or square-integrable weakly cocompact) lattice in a product $G = G_1 \times \cdots \times G_n$ of $n \geq 2$ locally compact σ-compact groups. Let X be a complete* CAT(0) *space of finite telescopic dimension without Euclidean factor. Then any minimal isometric Γ-action on X without fixed point at infinity extends to a continuous G-action by isometries.*

On the other hand, combining Proposition 1.8 with Theorem 1.6 yields the following extension of [CM08, Corollary 4.8].

Corollary 1.10. *Let G be a locally compact group acting continuously and minimally on a* CAT(0) *space X of finite telescopic dimension, without fixing any point at infinity. Then the amenable radical R of G stabilizes the maximal Euclidean factor of X. In particular, if X has no non-trivial Euclidean factor, then R acts trivially.*

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2. Preliminaries

2.1. **Geometric and telescopic dimension.** We recall some facts about dimensions of spaces with upper curvature bounds. The **geometric dimension** (sometimes simply called *dimension*) of such spaces was defined inductively in [Kle99], by setting the dimension of a discrete space to be 0 and be defining $\dim(X) = \sup\{\dim(S_x X) + 1 | x \in X\}$, where $S_x X$ denotes the space of directions at the point x. It turns out that this notion of dimension is closely related to more topological notions. Namely $\dim(X) \leq n$ if and only if for all open subsets $V \subset U$ of X the relative singular homology $H_k(U, V)$ vanishes for all $k > n$. Moreover, this is equivalent to the fact that the topological dimension of all compact subsets of X is bounded above by n, see *loc. cit.*

By definition, a CAT(0) space X is said to have **telescopic dimension** $\leq n$ if every asymptotic cone $\lim_\omega(\varepsilon_n X, \star_n)$ has geometric dimension $\leq n$. Although this will not play any role in the sequel, we remark that the telescopic dimension is a quasi-isometry invariant. Moreover, it follows from [Kle99, Th. C] that a locally compact CAT(0) space with a cocompact isometry group has finite telescopic dimension.

Convex subsets inherit the geometric dimension bound from the ambient space. Moreover, if (X_i, x_i) is a sequence of pointed $\mathrm{CAT}(\kappa)$ spaces of geometric dimension $\leq n$, then their ultralimit $\lim_\omega(X_i, x_i)$ with respect to some ultrafilter is a $\mathrm{CAT}(\kappa)$ space of dimension at most n, see [Lyt05b, Lemma 11.1]. In particular, it follows that a $\mathrm{CAT}(0)$ space of geometric dimension $\leq n$ has telescopic dimension $\leq n$. Furthermore, we have the following.

Proposition 2.1. *Let X be a $\mathrm{CAT}(0)$ space. If X has telescopic dimension $\leq n$, then the visual boundary ∂X endowed with the Tits metric has geometric dimension at most $n-1$.*

Proof. Let $o \in X$ be a base point and $C_\omega X$ be the asymptotic cone $\lim_\omega(\frac{1}{n}X, o)$. The Euclidean cone $C(\partial X)$ embeds isometrically into $C_\omega X$, see [Kle99, Lemma 10.6]. Thus $\dim(\partial X) = \dim(C(\partial X)) - 1 \leq \dim(C_\omega X) \leq n$. $\square$

We emphasize that a $\mathrm{CAT}(0)$ space X may have finite-dimensional Tits boundary without being of finite telescopic dimension, even if X is proper. Indeed, consider for instance the positive real half-line and glue at each point $n \in \mathbb{N}$ an n-dimensional Euclidean ball of radius n. The resulting space is proper and $\mathrm{CAT}(0)$, its ideal boundary consists of a single point, but each of its asymptotic cones contains an infinite-dimensional Hilbert space.

We shall use a topological version Helly's classical theorem that holds in much greater generality (see [Dug67] as well as [Far08, §3] for a related discussion). The following statement is an immediate consequence of [Kle99, Proposition 5.3] since intersections of convex sets are either empty or contractible.

Lemma 2.2. *Let X be a $\mathrm{CAT}(0)$ space of geometric dimension $\leq n$. Let $\{U_\alpha\}_{\alpha\in A}$ be a finite family of open convex subsets of X. If for each subset $B \subset A$ with at most $n+1$ elements the intersection $\bigcap_{\alpha\in B} U_\alpha$ is non-empty, then $\bigcap_{\alpha\in A} U_\alpha$ is non-empty.* $\square$

2.2. **Inner points.** Following [LS07], we shall say that a point o of a $\mathrm{CAT}(0)$ space X is a **topologically inner** point if $X \setminus \{o\}$ is not contractible. For each topologically inner point there is some $\varepsilon > 0$ and a compact subset K of X with $d(o, K) \geq \varepsilon$ with the following property: For each $x \in X$ there is some $\bar{x} \in K$ such that $xo\bar{x}$ is a geodesic. Thus every geodesic segment which terminates at o may be locally prolonged beyond o; in loose terms, the space X is *geodesically complete at the point o*. In a $\mathrm{CAT}(0)$ space which is locally of finite geometric dimension, the set of topologically inner points is dense, see [LS07, Theorem 1.5]. In particular it is non-empty.

3. Jung's theorem

3.1. **CAT(0) case.** Throughout the paper, we shall adopt the following notational convention. Given a subset $Y \subseteq X$ we denote the distance to Y by d_Y, namely $d_Y : X \to \mathbf{R} : x \mapsto \inf_{y\in Y} d(x, y)$. We further recall that the **intrinsic radius** of a subset Z of a metric space X is defined as

$$\mathrm{rad}(Z) = \inf_{z\in Z}\{r \in \mathbf{R}_{>0} \mid Z \subseteq B(z, r)\}.$$

This notion should not be confused with the **circumradius** (or **relative radius**), defined as

$$\mathrm{rad}_X(Z) = \inf_{x\in X}\{r \in \mathbf{R}_{>0} \mid Z \subseteq B(x, r)\}.$$

Bounded closed convex subsets of non-positively curved spaces have the **finite intersection property** (see [LS97, Proof of Theorem B] or [Mon06, Theorem 14]). This means that for any family $\{X_\alpha\}_{\alpha\in A}$ of *bounded* closed convex subsets of a CAT(0) space X the intersection $\bigcap_{\alpha\in A} X_\alpha$ is non-empty whenever the intersection of each finite sub-family is non-empty.

Lemma 3.1. *Let X be a* CAT(0) *space of geometric dimension $\leq n$ and $Y \subseteq X$ be a subset of finite diameter. If for all subsets $Y' \subseteq Y$ of cardinality $|Y'| \leq n+1$ we have $\mathrm{rad}_X(Y') \leq r$ then $\mathrm{rad}_X(Y) \leq r$.*

Proof. Fix an arbitrary $r' > r$. For $y \in Y$, denote by O_y the open ball of radius r' around y. These balls are convex and, by assumption, the intersection of any collection of at most $(n+1)$ such balls is non-empty. By Lemma 2.2 the intersection of any finite collection of such balls is non-empty. Since $r' > r$ is arbitrary, this implies that each finite subset Y' of Y has radius at most r. For $y \in Y$, denote now by B_y the closed ball of radius r around y. Then the intersection of each finite collection of B_y is non-empty, hence the intersection of all B_y is non-empty. For any point x in this intersection, we get $d(x, y) \leq r$ for all $y \in Y$. Hence $\mathrm{rad}_X(Y) \leq r$. □

Proof of Theorem 1.3. Theorem A from [LS97] ensures that for any CAT(0) space X and each subset $Y \subset X$ of cardinality at most $n+1$, the inequality $\mathrm{rad}_X(Y) \leq \sqrt{\frac{n}{2(n+1)}}\mathrm{diam}(Y)$ holds. In view of this, it follows from Lemma 3.1 that the inequality $\mathrm{rad}_X(Y) \leq \sqrt{\frac{n}{2(n+1)}}\mathrm{diam}(Y)$ holds for any subset Y of a CAT(0) space X of geometric dimension $\leq n$.

Assume conversely that X has geometric dimension $> n$. By [Kle99, Theorem 7.1], there exist a sequence (λ_k) of positive real numbers such that $\lim_k \lambda_k = \infty$, and a sequence $(Y_k, \star_k)_{k\geq 0}$ of pointed subsets of X such that

$$\lim_\omega (\lambda_k Y_k, \star_k) = \mathbf{R}^{n+1}$$

for any non-principal ultrafilter ω. We may then find $n + 2$ sequences $(y_k^i)_{k\geq 0}$ of points of Y_k indexed by $i \in \{0, 1, \dots, n+1\}$ such that the set $\Delta = \{\lim_\omega (y_k^i) \mid i = 0, \dots, n + 1\}$ coincides with the vertex set of a regular simplex of diameter 1 in $\mathbf{R}^{n+1}$. Since the equality case of the $(n + 1)$-dimensional Jung inequality is achieved in the case of Δ, we deduce that there exists some $k \geq 0$ such that the n-dimensional Jung inequality fails for the subset $\Delta_k = \{y_k^i \mid i = 0, \dots, n+1\} \subset X$.

Assume now that X has telescopic dimension $\leq n$ and suppose for a contradiction that for some fixed $\delta > 0$ and for each integer $k > 0$ there is a subset $Y_k \subset X$ such that $\mathrm{diam}(Y_k) > k$ and $\mathrm{rad}_X(Y_k) \geq (\sqrt{\frac{n}{2(n+1)}} + \delta)\mathrm{diam}(Y_k)$. Let $\star_k$ be the circumcentre of Y_k. Setting $\lambda_k = \mathrm{rad}_X(Y_k)$, it then follows that the asymptotic cone $\lim_\omega(\frac{1}{\lambda_k} X, \star_k)$ possesses a subset $\lim_\omega(Y_k)$ which fails to satisfy the n-dimensional Jung inequality. The contradicts the first part of the statement which has already been established.

Assume conversely that X has telescopic dimension $> n$. Then, by [Kle99, Theorem 7.1] there exists a sequence $(Y_k, \star_k)_{k\geq 0}$ of pointed subsets of X such that $\lim_\omega(Y_k, \star_k) = \mathbf{R}^{n+1}$. In particular $\mathrm{diam}(Y_k)$ tends to ∞ with k and we conclude by the same argument as before. □

3.2. **Rigidity and other curvature bounds.** In this subsection, we briefly sketch the analogues of Theorem 1.3 in the case of non-zero curvature bounds and address the equality case. Since the results are not used in the sequel, we do not provide complete proofs.

Following word by word the proof of Theorem 1.3 and using the results of [LS97] for other curvature bounds, one obtains the following.

Proposition 3.2. *Let X be a* $\mathrm{CAT}(-1)$ *space of geometric dimension at most n. Let Y be a subset of X of diameter at most D. Then the radius of Y in X is bounded above by $r_n(D)$, where $r_n(D)$ denote the radius of the regular n-dimensional simplex Δ_D in the n-dimensional real hyperbolic space $\mathbb{H}^n$ of diameter D.* □

In the positively curved case one needs to assume a bound on the radius in order for the balls in question to be convex. An additional technical difficulty arises from the fact the the whole space may be non-contractible in this case, and the statement of Lemma 2.2 has therefore to be slightly modified in that case. The resulting radius–diameter estimate is the following.

Proposition 3.3. *Let X be a* $\mathrm{CAT}(1)$ *space of dimension $\leq n$. Let Y be a subset of X of circumradius $r < \frac{\pi}{2}$. Then the diameter of Y is at least $s_n(r)$, where $s_n(r)$ is the diameter of the regular simplex of radius r in the round $\mathbb{S}^n$.*

Remark 3.4. In a similar way it can be shown, that the assumption $r = \mathrm{rad}_X(Y) < \frac{\pi}{2}$ is fulfilled as soon as $\mathrm{diam}(Y) < k_n = \arccos(-1/(n+1))$.

It is shown in [LS97] that for a subset Y of cardinality $\leq n+1$, the equality in Theorem 1.3 holds if and only if the convex hull of these points is isometric to a regular Euclidean simplex. Arguing as in the proof of Theorem 1.3 one obtains that if X is locally compact, the inequality becomes an equality if and only if the convex hull of Y contains a regular n-dimensional Euclidean simplex of diameter equal to the diameter of Y. If X is not locally compact the same statement holds for the convex hull of the ultraproduct $Y^\omega \subset X^\omega$. Similarly, the analogous rigidity statements hold for spaces with other curvature bounds for the same reasons.

4. Convex functions and their gradient flow

4.1. **Gradient flow.** We recall some basics about gradient flows associated to convex functions. We refer to [May98] for the general case and to [Lyt05a] for the simpler case of Lipschitz continuous functions; only the latter is relevant to our purposes.

Given a CAT(0) space X, a map $f : X \to \mathbf{R}$ is called **convex** if its restriction $f \circ \gamma$ to each geodesic γ is convex. Basic examples of convex functions on CAT(0) spaces are distance functions to points or to convex subsets, and Busemann functions, see [BH99, II.2 and II.8].

Let f be a continuous convex function on a CAT(0) space X. For a point $p \in X$, the **absolute gradient** of the concave function $(-f)$ at p is defined by the formula

$$|\nabla_p(-f)| = \max\left\{0, \limsup_{x\to p} \frac{f(p) - f(x)}{d(p,x)}\right\}.$$

The absolute gradient is a non-negative, possibly infinite function. It is bounded above by the Lipschitz constant if f is Lipschitz continuous. A fundamental object attached to the function f is the **gradient flow** which consists of a map $\phi : [0, \infty) \times X \to X$ which, loosely speaking, has the property that $\phi_0 = \mathrm{Id}$ and $\phi_t(x)$ follows for each x the path of steepest descent of f from x. The gradient flow is indeed a flow in the sense that it satisfies $\phi_{s+t}(x) = \phi_s \circ \phi_t(x)$ for all $x \in X$. The most important property of gradient flows, originally observed by Vladimir Sharafutdinov [Šar77] in the Riemannian context, is that the flow ϕ_t is **semi-contracting**. In other words, for each $t \geq 0$, the map $\phi_t : X \to X$ is 1-Lipschitz (see [Lyt05a, Theorem 1.7]). We refer to [May98] or [Lyt05a, §9] for more details and historical comments.

Remark 4.1. Originally, the gradient lines and flows were defined for concave functions by Sharafutdinov [Šar77] in the case of manifolds; they are also commonly used for semi-concave (but not semi-convex) functions. Moreover the gradient usually represents the direction of the maximal *growth* of the function rather than its maximal *decay*. This explains the slightly cumbersome notation $|\nabla_x(-f)|$ that we use here.

For each $x \in X$ the **gradient curve** $t \mapsto \phi_t(x)$ of f has the following properties (and is uniquely characterised by them).

(1) The curve $t \mapsto \phi_t(x)$ has velocity $|\phi_t(x)'| = |\nabla_{\phi_t(x)}(-f)|$ for almost all $t \geq 0$.

(2) The restriction $t \mapsto f(\phi_t(x))$ of f to the gradient curve is convex. Furthermore it satisfies $(f \circ \phi_t(x))' = -|\nabla_{\phi_t(x)}(-f)|^2$.

We define the **velocity of escape** of the flow ϕ_t at the point $x \in X$ by

$$\limsup_{t \to \infty} \frac{d(x, \phi_t(x))}{t}.$$

Since the flow ϕ_t is semi-contracting, the lim sup in the above definition may be replaced by a usual limit. Moreover, it does not depend on the starting point x. The following statement is an application of the main result of [KM99] (to a deterministic setting).

Proposition 4.2. *Let f be a convex Lipschitz function on a* CAT(0) *space X. If $\varepsilon = \inf_{x \in X} |\nabla_x(-f)| > 0$ then there is a unique point $\xi_f \in \partial X$ such that for all $x \in X$ the gradient curve $\phi_t(x)$ defined by f converges to ξ_f for $t \to \infty$.*

Remark 4.3. In particular, the existence of a function f as in Proposition 4.2 implies that the ideal boundary of X is non-empty.

The following construction due to Anton Petrunin shows that the conclusion of Proposition 4.2 fails without a uniform lower bound on the absolute gradient.

Example 4.4. Choose an acute angle in $\mathbf{R}^2$ enclosed by two rays $\gamma^{\pm}(t) = t \cdot v^{\pm}$ emanating from the origin. Let $f_n(w) = \langle w, x_n \rangle$ be linear maps on $\mathbf{R}^2$ such that the following conditions hold. First, for all n, we require that $\langle x_n, v^{\pm} \rangle$ be positive. For odd (resp. even) n, the direction v^+ (resp. v^-) is between x_n and v^- (resp. x_n and v^+). Moreover, the sequence (x_n) satisfies the recursive condition $\langle x_n, v^- \rangle = \langle x_{n-1}, v^+ \rangle$ for even n and $\langle x_n, v^+ \rangle = \langle x_{n-1}, v^- \rangle$ for odd n. Finally, we require that the length $\|x_n\|$ tends to 0 as n tends to infinity. It is easy to see that such a sequence (x_n) exists.

Now let $p_1 = v^-$ and define inductively p_n on γ^+ (resp. γ^-) for n even (resp. odd) to be the point such that $p_n - p_{n-1}$ is parallel to x_n. This just means that p_n arises from p_{n-1} by following the gradient flow of the affine (and hence concave) function f_n.

Define the numbers C_n by $C_0 = 0$ and $f_n(p_{n+1}) - f_{n+1}(p_{n+1}) = C_{n+1} - C_n$. Consider the concave function $f(x) = \inf(f_n(x) + C_n)$. One verifies that on the geodesic segment (p_n, p_{n+1}) the function f coincides with f_n (in fact on a neighbourhood of all points except p_{n+1}). Hence the segment joining p_n to p_{n+1} is part of a gradient curve of f. Therefore the appropriately parametrised piecewise infinite geodesic γ running through all p_i is a gradient curve of f. It is clear that both v^- and v^+ (and all unit vectors between them) considered as points in the ideal boundary are accumulation points of γ at infinity.

Proof of Proposition 4.2. From the assumption that $\varepsilon = \inf_{x\in X} |\nabla_x(-f)| > 0$, we deduce that $f(\phi_t(x)) - f(x) \leq -\varepsilon^2 t$ for all $x \in X$. In view of Property (2) of the gradient curve recalled above and the fact that f is Lipschitz, we deduce that the velocity of escape of the gradient curve is strictly positive.

An important consequence of [KM99, Theorem 2.1] is that any semi-contracting map $F : X \to X$ of a complete CAT(0) space X with strictly positive velocity of escape $\limsup_{n\to\infty} \frac{d(p,F^n(p))}{n}$ has the following convergence property: There is a unique point ξ_F in the ideal boundary ∂X of X, such that for all $p \in X$ the sequence $p_n = F^n(p)$ converges to ξ_F in the cone topology. In view of the above discussion, we are in a position to apply this result to $F = \phi_1$, from which the desired conclusion follows. □

4.2. **Asymptotic slope and a radius estimate.** Finally we recall an observation of Eberlein ([Ebe96], Section 4.1) about the size of the set of points in the ideal boundary with negative asymptotic slopes.

Let $f : X \to \mathbf{R}$ be a continuous convex function. For each geodesic ray $\gamma : [0,\infty) \to X$ one defines the **asymptotic slope** of f on γ by $\lim_{t\to\infty}(f \circ \gamma'(t))$. This defines a number in $(-\infty, +\infty]$ which depends only on the point at infinity $\gamma(\infty) \in \partial X$. Thus one obtains a function $slope_f : \partial X \to (-\infty, +\infty]$. One says that a point $\xi \in \partial X$ is f**-monotone** if $slope_f(\xi) \leq 0$. This is equivalent to saying that the restriction of f to any ray asymptotic to ξ is non-increasing. One denotes the set of all f-monotone points by $X_f(\infty)$.

Lemma 4.5. *Let f be a convex Lipschitz function on a complete* CAT(0) *space X such that $\inf_{x\in X} |\nabla_x(-f)| > 0$. Then for each point $\xi \in X_f(\infty)$, we have $d_{\mathrm{Tits}}(\xi, \xi_f) \leq \frac{\pi}{2}$, where ξ_f is the canonical point provided by Proposition 4.2.*

Proof. Eberlein's argument for the proof of [Ebe96, Proposition 4.1.1] (which is also reproduced in the proof of [FNS06, Theorem 1.1]) shows, that for any $p \in X$ and any sequence t_i, such that $\phi_{t_i}(p)$ converges to some point $\xi \in \partial X$, the Tits-distance between ξ and any other point $\psi \in X$ is at most $\frac{\pi}{2}$. □

4.3. **The space of convex functions.** Pick a base point $o \in X$. Denote by $\mathcal{C}_0$ the set of all 1-Lipschitz convex functions f on X with $f(o) = 0$. We view it as subset of the locally convex topological vector space $\mathcal{B}$ of all functions f on X with $f(o) = 0$, where the latter is considered with the topology of pointwise convergence. The subset $\mathcal{C}_0$ may thus be considered as a closed subset of the infinite product $\prod_{x\in X} I_x$, where I_x is the interval $I_x = [-d(o,x), d(o,x)]$. Since a convex combination of convex 1-Lipschitz functions is convex and 1-Lipschitz, the set $\mathcal{C}_0$ is a convex compact subset of $\mathcal{B}$.

The isometry group $G = \mathrm{Is}(X)$ of X acts continuously on $\mathcal{B}$ by $g \cdot f : x \mapsto f(gx) - f(go)$ and preserves the subset $\mathcal{C}_0$. Consider the map $i : X \to \mathcal{C}_0$ given by $i(x) := \bar{d}_x$, where $\bar{d}_x$ is the normalized distance function $\bar{d}_x(y) = d(x, y) - d(x, o)$. Note that the map i is G-equivariant. In particular, the subset $\mathcal{C} = i(X) \subset \mathcal{C}_0$ as well as its closure and closed convex hull are G-invariant. If X is locally compact, then $\overline{\mathcal{C}}$ consists precisely of normalized distance and Busemann functions on X, and is thus nothing but the visual compactification of X. However, if X is not locally compact, then $\bar{\mathcal{C}}$ may be much larger, and the convergence in $\mathcal{C}_0$ may be rather strange.

Example 4.6. Let X' be a separable Hilbert space with origin o and an orthonormal base $\{e_n\}_{n\geq 0}$. Then the sequence $\bar{d}_{ne_n}$ converges in $\mathcal{C}_0$ to the constant function.

Example 4.7. Let X'' be a metric tree consisting of a single vertex o from which emanate countably many infinite rays η_n. In other words X'' is the Euclidean cône over a discrete countably infinite set. Let b_n denote the Busemann function associated with η_n. Then b_n converge in $\mathcal{C}_0$ to the distance function d_o.

We emphasize that the choice of the base point o does not play any role: any change of base point amounts to adding an additive constant.

In some sense, the set $\mathcal{C}$ may serve in the non-locally compact case as a generalized ideal boundary. It is therefore important to understand how "large" it really is. This will be the purpose of the next subsection.

4.4. **Affine functions on spaces of finite telescopic dimension.** Recall that a function $f : X \to \mathbf{R}$ is called **affine** if its restriction to any geodesic is affine. Equivalently, for all pairs $x^+, x^- \in X$ with midpoint m we have $f(x^+) + f(x^-) = 2f(m)$. A simple-minded but noteworthy observation is that affine functions are precisely those convex functions f whose opposite $(-f)$ is also convex. Clearly, constant functions are affine; thus any CAT(0) space admit affine functions. However, the very existence of *non-constant* affine functions imposes very strong restrictions on the underlying space, see [LS07]. The following result also provides an illustration of this phenomenon, which will be relevant to the proof of Theorem 1.6.

Proposition 4.8. *Let X be a* CAT(0) *space of finite telescopic dimension which is not reduced to a single point and such that* Is(X) *acts minimally. If $\overline{\mathcal{C}}$ contains an affine function, then there is a splitting $X = \mathbf{R} \times X'$.*

Recall from [BH99, II.6.15(6)] that any complete CAT(0) space X admits a canonical splitting $X = \mathbb{E} \times X'$ preserved by all isometries, where $\mathbb{E}$ is a (maximal) Hilbert space called the **Euclidean factor** of X. It is shown in [LS07, Corollary 4.8] that if X is locally finite-dimensional and if Is(X) acts minimally, then X' does not admit *any* non-constant affine function. The main technical point in the proof of the latter fact is the existence of inner points (see §2.2).

In order to deal with the case of *asymptotic* dimension bounds, we need to substitute this by some coarse equivalent. This substitute is provided by Lemma 4.9, which is of technical nature. In the special case of spaces of finite *geometric* dimension, it follows quite easily from the existence of inner points ; therefore, the reader who is only interested in those spaces may wish to skip it.

Lemma 4.9. *Let X be an unbounded space of finite telescopic dimension. Then there are sequences of positive numbers $D_j \to \infty$, $\delta_j \to 0$ and sequences of points $p_j \in X$ and of finite subsets $Q_j \subset X$ with the following two properties.*

1. *Q_j is contained in the ball of radius $D_j(1+\delta_j)$ around p_j.*
2. *For all $s \in X$, there is some $q_j \in Q_j$ with $d(s, q_j) - d(s, p_j) \geq D_j$.*

Proof. Consider $\tilde{X} = \lim_\omega(\frac{1}{n}X, o)$ and let $\tilde{p} = (p_n)$ be an inner point of $\tilde{X}$. Let $\varepsilon > 0$ and the compact subset $K \subset \tilde{X}$ be chosen as in §2.2. Moving points of K towards $\tilde{p}$ we may assume that all point of K have distance ε to $\tilde{p}$. Furthermore, there is no loss of generality in assuming $\varepsilon < 1$.

Since K is compact, there exist finite subsets $Q_n \in \frac{1}{n}X$ with $\lim_\omega Q_n = K$ and $d(p_n, q) \leq \varepsilon n$ for all $q \in Q_n$.

In view of the defining property of K, we deduce that for all $\delta \in (0, \varepsilon)$ and all n_0, there is some $n = n(\delta, n_0) > n_0$ such that for any $s \in X$ with $d(s, p_n) \leq n$, there is some $q \in Q_n$ with $d(s, q) \geq d(s, p_n) + n(\varepsilon - \delta)$.

Assume now $\delta \in (0, \frac{\varepsilon}{2})$. Given $\tilde{s} \in X$ with $d(\tilde{s}, p_n) \geq n$ and choose the point s between p_n and $\tilde{s}$ with $d(p_n, s) = n$. Let $q \in Q_n$ be such that $d(s, q) \geq d(s, p_n) + n(\varepsilon - \delta)$. Using the *law of cosines* in a comparison triangle for $\Delta(\tilde{s}, p_n, q)$, we deduce from that CAT(0) inequality that

$$\frac{d(\tilde{s}, q)^2 - d(\tilde{s}, p_n)^2 - d(p_n, q)^2}{d(\tilde{s}, p_n)} \geq \frac{n^2(\varepsilon - \delta)(2 + \varepsilon - \delta) - d(p_n, q)^2}{n}.$$

Since $d(p_n, q) \leq \varepsilon n$ and $d(\tilde{s}, p_n) = n + d(s, \tilde{s})$, we deduce

$$\begin{aligned} d(\tilde{s}, q)^2 - d(\tilde{s}, p_n)^2 &\geq n^2(\varepsilon - \delta)(2 + \varepsilon - \delta) + 2n(\varepsilon - \delta(1+\varepsilon) + \tfrac{\delta^2}{2})d(s, \tilde{s}) \\ &\geq n^2(\varepsilon - 2\delta)(2 + \varepsilon - 2\delta) + 2n(\varepsilon - 2\delta)d(s, \tilde{s}) \\ &= n^2(\varepsilon - 2\delta)^2 + 2n(\varepsilon - 2\delta)d(\tilde{s}, p_n). \end{aligned}$$

This implies that $d(\tilde{s}, q) \geq d(\tilde{s}, p_n) + n(\varepsilon - 2\delta)$.

It remains to define the desired sequences by making the appropriate choices of indices. This may be done as follows. For each integer $j > 0$, we now set $\delta_j = \frac{\varepsilon}{2j}$ and $n_j = n(\frac{\varepsilon\delta_j}{4}, j)$, where $n : (0, \varepsilon) \times \mathbf{N} \to \mathbf{N} : (\delta, n_0) \mapsto n(\delta, n_0)$ is the map considered above. Finally we set $p_j = p_{n_j}$, $Q_j = Q_{n_j}$ and $D_j = \varepsilon n_j(1 - \frac{\delta_j}{2})$. $\square$

Proof of Proposition 4.8. Assume that the set $\mathcal{A}$ of affine functions contained in $\overline{\mathcal{C}}$ is nonempty. For each integer $j > 0$ we set

$$C_j = \{p \in X \mid \forall f \in \mathcal{A}\ \exists z \in X, f(z) - f(p) = D_j \text{ and } d(p, z) \leq (1 + \delta_j)D_j\},$$

where (D_j) and (δ_j) are the sequences provided by Lemma 4.9.

We claim that C_j is non-empty.

Let us fix the index j and consider the point $p_j \in X$ provided by Lemma 4.9. We shall show that p_j belongs to C_j. To this end, let $f \in \mathcal{A}$. By definition, there is a sequence (s_n) of points of X such that the sequence $(\bar{d}_{s_n})$ converges pointwise to f, where $\bar{d}_s$ denotes the normalised distance function $\bar{d}_s(\cdot) = d(s, \cdot) - d(o, s)$. Now for each n, Lemma 4.9 provides a point $q_n \in Q_j$ such that $d(p_j, q_n) \leq D_j(1 + \delta_j)$ and $\bar{d}_{s_n}(q_n) - \bar{d}_{s_n}(p_j) = d_{s_n}(q_n) - d_{s_n}(p_j) \geq D_j$. Upon extracting, we may assume that (q_n) is constant and is equal to $q \in Q_j$, since Q_j is finite. Now, passing to the limit as $n \to \infty$, we obtain $f(q) - f(p_j) \geq D_j$ and $d(p, q) \leq (1 + \delta_j)D_j$. Finally, since f is affine, there exists a unique point $z \in [p_j, q]$ such that $f(z) - f(p_j) = D_j$. This confirms that $p_j \in C_j$ and the claim stands proven.

We claim that C_j is convex.

Indeed, let $p_1, p_2 \in C_j$, $f \in \mathcal{A}$ and z_1, z_2 such that for $i = 1, 2$ we have $f(z_i) - f(p_i) = D_j$ and $d(p_i, z_i) \leq (1 + \delta_j)D_j$. Given $p \in [p_1, p_2]$ at distance $\lambda d(p_1, p_2)$ from p_1, where $\lambda \in (0, 1)$, let $z \in [z_1, z_2]$ be the unique point at distance $\lambda d(z_1, z_2)$ from z_1. Since the distance function is convex, we have $d(p, z) \leq (1 + \delta_j)D_j$. Furthermore, since f is affine we have $f(z) - f(p) = D_j$. Hence $p \in C_j$.

We claim that $C_j = X$.

Since $\mathcal{A}$ is $\mathrm{Is}(X)$-invariant, so is C_j. In view of the assumption of minimality on the $\mathrm{Is}(X)$-action, it follows that C_j is dense in X for all $D_j, \delta_j > 0$. Now the claim follows from a routine continuity argument using the fact that f is 1-Lipschitz.

For each $f \in \mathcal{A}$ and $p \in C_j$, we have $|\nabla_p(f)| \geq \frac{1}{1+\delta_j}$ since f is affine. By the previous claim, the latter inequality holds for all $p \in X$ and all $\delta_j > 0$. Since f is 1-Lipschitz it follows that $|\nabla_p(f)| = 1$ for all $p \in X$. Now Proposition 4.2 yields a point $\xi \in \partial X$ to which the gradient curve $t \mapsto \phi_t(p)$ converges as t tends to infinity. Since the gradient curve as velocity 1 (see §4.1) we deduce that it is a geodesic ray pointing to ξ. It follows that $-f$ is a Busemann function associated with ξ. In particular $-f = \lim_n \bar{d}(\phi_n(p), \cdot)$ belongs to $\overline{C}$, hence to $\mathcal{A}$ since f is affine. This yields another point $\xi' \in \partial X$ and a geodesic ray $\phi'_t(p)$ pointing to ξ'. The concatenation of both rays is a geodesic line γ joining ξ' to ξ such that $(f \circ \gamma)' = 1$. At this point, [LS07, Lemma 4.1] yields the desired splitting. $\square$

We conclude this section with a technical property of the space of functions $\mathcal{C}_0$ valid for arbitrary CAT(0) spaces.

Lemma 4.10. *Let X be any* CAT(0) *space and $\mathcal{C}_0$ be as above. Given a compact subset $\mathcal{A} \subset \mathcal{C}_0$, if $\mathcal{A}$ does not contain any affine function, then the closed convex hull $Conv(\mathcal{A})$ does not contain any affine function either.*

Proof. For any $f \in \mathcal{A}$ we find some pair of points $x_f^+, x_f^- \in X$ with midpoint m_f, such that $\varepsilon_f = f(x_f^+) + f(x_f^-) - 2f(m_f) > 0$. For each $f \in \mathcal{A}$, let U_f be the open subset of $\mathcal{A}$ consisting of all h with $h(x_f^+) + h(x_f^-) - 2h(m_f) > \varepsilon_f/2$. Since $\mathcal{A}$ is compact, finitely many U_{f_i} cover $\mathcal{A}$. Therefore, using the convexity of h, we deduce that

$$r(h) := \sum_i \big(h(x_{f_i}^+) + h(x_{f_i}^-) - 2h(m_{f_i})\big) \geq \inf\{\varepsilon_{f_i}\} > 0$$

for all $h \in \mathcal{A}$. Thus the continuous functional $h \mapsto r(h)$ is strictly positive on $\mathcal{A}$, hence it is positive on the compact convex hull of $\mathcal{A}$. Therefore each $f \in Conv(\mathcal{A})$ is non-affine on at least one of the geodesics $[x_{f_i}^+, x_{f_i}^-]$. $\square$

5. Filtering families of convex sets

The purpose of this section is to prove Theorem 1.1. We start by considering an analogous property for finite-dimensional CAT(1) spaces.

5.1. CAT(1) case.

We start with the following analogue of the finite intersection property for bounded convex sets in CAT(0) spaces.

Lemma 5.1. *Let X be a complete* CAT(1) *space of radius $< \frac{\pi}{2}$. Then any filtering family $\{X_\alpha\}_{\alpha \in A}$ of closed convex subspaces has a non-empty intersection.*

Proof. Given [BH99, II.2.6(1) and II.2.7], the proof is identical to that in [Mon06, Theorem 14]. $\square$

Lemma 5.2. *Let X be a finite-dimensional* $\mathrm{CAT}(1)$ *space and $\{X_i\}_{i\geq 0}$ be a decreasing sequence of closed convex subsets such that* $\mathrm{rad}(X_i) \leq \frac{\pi}{2}$. *Then the intersection $\bigcap_i X_i$ is a non-empty subset of intrinsic radius $\leq \pi/2$.*

Proof. Let z_i be a centre of X_i and $Z = \{z_i \mid i \geq 0\}$. By assumption $d(z_i, z_j) \leq \frac{\pi}{2}$ for all i, j. Since any ball of radius $\leq \frac{\pi}{2}$ is convex, it follows that the closed convex hull C of Z has intrinsic radius $\leq \frac{\pi}{2}$.

We claim that $\mathrm{rad}(C) < \frac{\pi}{2}$. Otherwise we have $\mathrm{rad}(C) = \frac{\pi}{2}$ and every $z \in Z$ is a centre of C. Since the set of all centres is *convex*, it follows that every point of C is a centre. This implies $\mathrm{diam}(C) \leq \mathrm{rad}(C)$, which contradicts [BL05, Proposition 1.2] and thereby establishes the claim.

Let C_i be the convex hull of $\{z_j \mid j \geq i\}$. Then $(C_i)_{i\geq 0}$ is a decreasing sequence of closed convex subsets in a $\mathrm{CAT}(1)$ space of radius $< \pi/2$. By Lemma 5.2, the intersection $Q = \bigcap_i C_i$ is non-empty. Notice that $C_i \subseteq X_i$ whence $Q \subseteq \bigcap_i X_i$. The latter intersection is thus non-empty.

For each $x \in \bigcap_i X_i$ we have $d(x, z_j) \leq \frac{\pi}{2}$ for all j. Thus C_j is contained in the ball of radius $\frac{\pi}{2}$ around x. Therefore $d(x, q) \leq \pi/2$ for all $x \in \bigcap_i X_i$ and $q \in Q$. This shows that $\bigcap_i X_i$ has radius at most $\frac{\pi}{2}$. $\square$

Proposition 5.3. *Let X be a finite-dimensional* $\mathrm{CAT}(1)$ *space and $\{X_\alpha\}_{\alpha\in A}$ be a filtering family of closed convex subsets such that* $\mathrm{rad}(X_\alpha) \leq \frac{\pi}{2}$ *for each $\alpha \in A$. Then the intersection $\bigcap_{\alpha\in A} X_\alpha$ is a non-empty subset of intrinsic radius $\leq \pi/2$.*

Proof. We proceed by induction on $n = \dim X$. There is nothing to prove in dimension 0, hence the induction can start.

If $\dim(X_0) < n$ for some index $0 \in A$, then the induction hypothesis applied to the filtering family $\{X_0 \cap X_\alpha\}_{\alpha\in A}$ yields the desired conclusion. We assume henceforth that $\dim(X_\alpha) = n$ for each $\alpha \in A$.

For $\beta \in A$, let z_β be a centre of X_β. If $d_{X_\alpha}(z_\beta) = \frac{\pi}{2}$ for some $\beta \in A$, then the closed convex hull of z_β and X_α coincides with the spherical suspension of z_β and X_α (see [Lyt05b, Lemma 4.1]) and hence has dimension $1 + \dim(X_\alpha)$. This is absurd since $\dim(X_\alpha) = \dim(X)$. We deduce $d_{X_\alpha}(z_\beta) < \frac{\pi}{2}$ for all $\alpha, \beta \in A$.

Assume now that $\sup_{\alpha\in A} d_{X_\alpha}(z_\beta) = \frac{\pi}{2}$. Then there is a countable sequence $(X_{\alpha_i})_{i\geq 0}$ with $\alpha_i \in A$ such that $\lim_i d_{X_{\alpha_i}}(z_\beta) = \frac{\pi}{2}$. Upon replacing X_{α_j} by $\bigcap_{i=0}^{j} X_{\alpha_i}$ we may and shall assume that the sequence $(X_{\alpha_i})_{i\geq 0}$ is decreasing. By Lemma 5.2 the intersection $Y = \bigcap_{i\geq 0} X_{\alpha_i}$ is a non-empty closed convex subset of X. Furthermore by definition we have $d_Y(z_\beta) = \frac{\pi}{2}$. In particular, we deduce by the same argument as above that $\dim(Y) < n$.

Now for each $\alpha \in A$, we apply Lemma 5.2 to the nested family $(X_\alpha \cap X_{\alpha_i})_{i\geq 0}$, which shows that $Y_\alpha = \bigcap_{i\geq 0}(X_\alpha \cap X_{\alpha_i})$ is a closed convex non-empty subset of Y with intrinsic radius at most $\frac{\pi}{2}$. Moreover, the family $\{Y_\alpha\}_{\alpha\in A}$ is filtering and we have $\bigcap_\alpha Y_\alpha = \bigcap_\alpha (X_\alpha)$. It follows by induction that $\bigcap_\alpha X_\alpha$ is non-empty and of intrinsic radius at most $\frac{\pi}{2}$, as desired.

It remains to consider the case when $r = \sup_\alpha d_{X_\alpha}(z_\beta) < \pi/2$. We are then in a position to apply Lemma 5.1 to the filtering family $\{B(z_\beta, r) \cap X_\alpha\}_{\alpha\in A}$. We deduce that $Y = \bigcap_\alpha X_\alpha$ is non-empty. Moreover, since $d_Y(z_\beta) \leq r < \frac{\pi}{2}$, we deduce by considering the nearest point projection of z_β to Y (see [BH99, II.2.6(1)]) that $\mathrm{rad}(Y) < \frac{\pi}{2}$. $\square$

5.2. **CAT(0) case.** We start with the special case of nested sequences of convex sets.

As pointed out to us by the referee, the use of the gradient flow in the argument below is reminiscent of the proof of Lemma 5 on p. 217 in [BGS85].

Lemma 5.4. *Let X be a complete* CAT(0) *space of telescopic dimension $n < \infty$ and $(X_i)_{i\geq 0}$ be a nested sequence of closed convex subsets such that $\bigcap_{i\geq 0} X_i$ is empty. Let $o \in X$ be a base point and set $f_i : x \mapsto d_{X_i}(x) - d_{X_i}(o)$. Then the sequence $(f_i)_{i\geq 0}$ sub-converges to a 1-Lipschitz convex function f which satisfies $\inf_{p\in X} |\nabla_p(-f)| \geq \frac{1}{2}\big(1 - \sqrt{\frac{n}{n+1}}\big)$.*

Proof. The functions f_i are 1-Lipschitz and convex ([BH99, II.2.5(1)]), hence they are elements of the space $\mathcal{C}_0$ defined in Section 4.3. Since $\mathcal{C}_0$ is compact, the sequence (f_i) indeed sub-converges to a function $f \in \mathcal{C}_0$. It remains to estimate the absolute gradient of f.

Pick a point $p \in X$. By assumption the intersection $\bigcap_i X_i$ is empty. Since bounded closed convex sets enjoy the finite intersection property (see Section 3.1), it follows that $d_{X_i}(p)$ tends to infinity with i. Thus for each $t > 0$ there is some N_t such that $d_{X_i}(p) > t$ for all $i \geq N_t$. We may and shall assume without loss of generality that $N_1 = 0$.

Let x_i denote the nearest point projection of p to X_i (see [BH99, Proposition II.2.4]) and $\rho_i : [0, d(p, x_i)] \to X$ be the geodesic path joining p to x_i. Set

$$D_t = \mathrm{diam}\{\rho_i(t) \mid i \geq N_t\}.$$

We distinguish two cases.

Assume first that $\sup_t D_t < \infty$. It then follows that for all $t > 0$, the sequence $(\rho_i(t))_{i\geq N_t}$ is Cauchy. Denoting by $\rho(t)$ its limit, the map $\rho : t \mapsto \rho(t)$ is a geodesic ray emanating from p. Therefore $f = \lim_i f_i$ is a Busemann function and we have $|\nabla_p(-f)| = 1$. Thus we are done in this case.

Assume now that $\sup_t D_t = \infty$. Then D_t tends to infinity with t. Choose $\delta > 0$ small enough so that $\sqrt{2}\delta < 1 - \sqrt{\frac{n}{n+1}}$ and let $D > 0$ be the constant provided by Theorem 1.3. We now pick t large enough so that $D_t > D$ and set $y_i = \rho_i(t)$ for all $i \geq N_t$. For $j > i$ we have $\angle_{x_i}(p, x_j) \geq \frac{\pi}{2}$ and considering a comparison triangle for $\Delta(p, x_i, x_j)$ we deduce $d(y_i, y_j) \leq t\sqrt{2}$. Set $Y = \{y_i \mid i \geq 0\}$. By Theorem 1.3 we have $\mathrm{rad}(Y) \leq t(\sqrt{2}\delta + \sqrt{\frac{n}{n+1}})$.

Let z be the circumcentre of Y. We have $d_{X_i}(z) \leq d_{X_i}(y_i) + d(y_i, z)$ and $d(y_i, z) \leq \mathrm{rad}(Y)$. Since moreover $d_{X_i}(p) = d_{X_i}(y_i) + t$, we deduce

$$\begin{aligned} f_i(z) - f_i(p) &= d_{X_i}(z) - d_{X_i}(p) \\ &\leq d(y_i, z) - t \\ &\leq -t\Big(1 - \sqrt{\tfrac{n}{n+1}} - \sqrt{2}\delta\Big) \end{aligned}$$

for each $i \geq 0$. Therefore $f(p) - f(z) \geq \delta' t$, where $\delta' = 1 - \sqrt{\frac{n}{n+1}} - \sqrt{2}\delta$. On the other hand, we have $d(z, p) \leq d(p, y_i) + d(y_i, z) \leq 2t$, thus $\frac{f(p)-f(z)}{d(p,z)} \geq \delta'/2$.

Since the restriction of $-f$ to the geodesic segment $[p, z]$ is concave by assumption, we deduce

$$|\nabla_p(-f)| \geq \delta'/2.$$

Finally, recalling that $\delta' = 1 - \sqrt{\frac{n}{n+1}} - \sqrt{2}\delta$ and that $\delta > 0$ may be chosen arbitrary small, the desired estimate follows. $\square$

Lemma 5.5. *Let X be a complete* CAT(0) *space of finite telescopic dimension and $(X_i)_{i\geq 0}$ be a nested sequence of closed convex subsets. If $\bigcap_{i\geq 0} X_i$ is empty, then $\bigcap_{i\geq 0} \partial X_i$ is a non-empty subset of the visual boundary ∂X of intrinsic radius $\leq \frac{\pi}{2}$.*

Proof. Let $\phi_t : X \to X$ denote the gradient flow associated to the convex function f defined as in Lemma 5.4. Proposition 4.2 provides some point ξ in the ideal boundary ∂X such that the gradient line $t \mapsto \phi_t(p)$ converges to ξ for any starting point $p \in X$.

We claim that ξ is contained in ∂X_i for each i. To this end, we fix an index i and consider the restriction h of f to X_i. This is a convex function on X_i and it is sufficient to prove that the gradient flow of h coincides with the gradient flow of f starting at any point of X_i. Hence it is enough to prove that for all $p \in X_i$ the equality $|\nabla_p(-f)| = |\nabla_p(-h)|$ holds.

Pick a point $x \in X$ and let x_i denote the nearest point projection of x to X_i. We have $d_{X_j}(x) \geq d_{X_j}(x_i)$ and $d(p,x) \geq d(p,x_i)$ for all $p \in X_i$. Hence for $p \in X_i$ and all $j \geq i$ we get the inequality

$$\frac{f_j(p) - f_j(x)}{d(p,x)} \leq \frac{f_j(p) - f_j(x_i)}{d(p,x_i)}.$$

Hence the same is true for the limiting function f, which implies the desired equality $|\nabla_p(-f)| = |\nabla_p(-h)|$. This shows that ξ is contained in the intersection $\bigcap_i \partial X_i$, which is thus non-empty.

For any geodesic ray η in X with endpoint in $\bigcap_i \partial X_i$, the restriction of f_i to η is bounded from above, hence non-increasing. Therefore the same holds true for the restriction of the limiting function f to the ray η. In other words the endpoint of η is f-monotone. From Lemma 4.5 we deduce that $d(\xi, \psi) \leq \pi/2$ for all $\psi \in \cap \partial X_i$. $\square$

Proof of Theorem 1.1. Pick a base point $o \in X$. If the set $\{d_{X_\alpha}(o)\}_{\alpha\in A}$ is bounded, then $\bigcap_\alpha X_\alpha$ has a non-empty intersection by the finite intersection property (see Section 3.1). We assume henceforth that this is not the case. In particular there exists a sequence of indices $(\alpha_n)_{n\geq 0}$ such that $\lim_n d_{X_{\alpha_n}}(o) = \infty$. Now for each $\alpha \in A$, we may apply Lemma 5.5 to the nested sequence $(X_\alpha \cap X_{\alpha_n})_{n\geq 0}$. This shows that $Y_\alpha = \bigcap_{n\geq 0} \partial(X_\alpha \cap X_{\alpha_n})$ is a non-empty subset of intrinsic radius $\leq \frac{\pi}{2}$ of ∂X. Notice that $\{Y_\alpha\}_{\alpha\in A}$ is a filtering family. Proposition 2.1 then allows one to appeal to Proposition 5.3, which shows that $\bigcap_\alpha Y_\alpha$ is a non-empty subset of intrinsic radius $\leq \frac{\pi}{2}$. This provides the desired statement since $\bigcap_\alpha \partial X_\alpha = \bigcap_\alpha Y_\alpha$. $\square$

We end this section by an example illustrating that Theorem 1.1 fails if one assumes only that the Tits boundary ∂X be finite-dimensional.

Example 5.6. Let $\mathcal{H}$ be a separable (real) Hilbert space with orthonormal basis $\{e_i\}$ and $X \subset \mathcal{H}$ be the subset consisting of all points $\sum_i a_i e_i$ with $|a_i| \leq i$ for all i. Thus X is a closed convex subset of $\mathcal{H}$ with *empty* (hence finite-dimensional) ideal boundary. Let now $X_n = \{\sum_i a_i e_i \in X \mid a_i \geq 1 \text{ for all } i \leq n\}$. Then $\{X_n\}$ is a nested family of closed convex subsets with empty intersection.

6. Applications

6.1. Parabolic isometries.

Proof of Corollary 1.5. By Proposition 2.1, the boundary ∂X is finite-dimensional. The sublevel sets of the displacement function of g define a $\mathscr{Z}_{\mathrm{Is}(X)}(g)$-invariant nested sequence of closed convex subspace. The intersection of their boundaries is nonempty by Theorem 1.1 and possesses a barycentre by [BL05, Prop. 1.4], which is the desired fixed point. □

6.2. **Minimal and reduced actions.** We begin with a de Rham type decomposition property. It was shown by Foertsch–Lytchak [FL08] that any finite-dimensional CAT(0) space (and more generally any geodesic metric space of finite affine rank) admits a canonical isometric splitting into a flat factor and finitely many non-flat irreducible factors. Building upon [FL08], it was then shown by Caprace–Monod [CM08, Corollary 4.3(ii)] that the same conclusion holds for proper CAT(0) spaces whose isometry group acts minimally, assuming that the Tits boundary is finite-dimensional. We shall need the following 'improper' variation of this result.

Proposition 6.1. *Let X be a complete* CAT(0) *space of finite telescopic dimension, such that* $\mathrm{Is}(X)$ *acts minimally. Then there is a canonical maximal isometric splitting*

$$\mathbf{R}^n \times X_1 \times \cdots \times X_m$$

where each X_i is irreducible, unbounded and $\not\cong \mathbf{R}$. Every isometry preserves this decomposition upon permuting possibly isometric factors X_i.

Proof. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_i\}_{i>0}$ and denote by C_k the convex hull of the set $\{0\} \cup \{2^i e_i \mid 0 < i \leq k\}$. Let now X be a CAT(0) space such that for every isometric splitting $X = X_1 \times \cdots \times X_p$ with each X_i unbounded, some factor X_i admits an isometric splitting $X_i = X_i' \times X_i''$ with unbounded factors. Then there is a point $o \in X$ and for each $k > 0$ an isometric embedding $\varphi_k : C_k \to X$ with $\varphi(0) = o$. Since for all $k > 0$ the set $2.C_k$ embeds isometrically in C_{k+1}, it follows that C_k embeds isometrically in the asymptotic cone $\lim_\omega(\frac{1}{n}X, o)$. In particular X does not have finite telescopic dimension. This shows that any CAT(0) space of finite telescopic dimension admits a *maximal* isometric splitting into a product of finitely many unbounded (necessarily irreducible) subspaces.

In view of the latter observation and given Proposition 2.1, the proof of [CM08, Corollary 4.3(ii)] applies *verbatim* and yields the desired conclusion. □

Proof of Proposition 1.8. (i) We claim that the statement of (i) follows from (ii) and (iii). Indeed, if G has no fixed point at infinity, then there is a minimal non-empty G-invariant subspace $Y \subseteq X$ by (ii). Upon replacing G by a finite index subgroup, this subspace Y admits a G-equivariant decomposition as in Proposition 6.1. The induced action of G on each of these spaces is minimal without fixed point at infinity. Therefore, it is non-evanescent by (iii), unless Y is bounded, in which case it is reduced to a single point by G-minimality. This means that G fixes a point in X.

(ii) Assume that G has no minimal invariant subspace. By Zorn's lemma this implies that there is a chain of G-invariant subspaces with empty intersection.

By Theorem 1.1 the intersection of the boundaries at infinity of the subspaces in this chain provide a closed convex G-invariant set $Y \subseteq \partial X$ of radius $\leq \frac{\pi}{2}$. By Proposition 2.1, the set Y is finite-dimensional. Hence it possesses a unique barycentre by [BL05, Prop. 1.4], which is thus fixed by G.

By Proposition 2.1 the boundary ∂X is finite-dimensional. Therefore, for (iii) and (iv), Theorem 1.1 (in fact, Lemma 5.5 is sufficient) allows one to repeat *verbatim* the proofs of the corresponding statements that are given in [CM08], namely Theorem 1.6 in *loc. cit.* for the fact that normal subgroups act minimally without fixed point at infinity, Corollary 2.8 in *loc. cit.* for the fact that the G-action is reduced and Proposition 1.3(i) in *loc. cit.* for the fact that X is boundary-minimal provided $\mathrm{Is}(X)$ acts minimally. □

Proof of Corollary 1.9. By Proposition 6.1 the space X admits a canonical decomposition as a product of finitely many irreducible factors. The lattice Γ admits a finite index normal subgroup Γ^* which acts componentwise on this decomposition (the finite quotient Γ/Γ^* acts by permuting possibly isometric irreducible factors). Let G_i^* be the closure of the projection of Γ^* to G_i and set $G^* = G_1^* \times \cdots \times G_n^*$. Thus G^* is a closed normal subgroup of finite index of G and we have $G = \Gamma \cdot G^*$. In particular it is sufficient to show that the Γ^*-action extends to a continuous G^*-action. To this end, we work one irreducible factor at a time. Given Proposition 1.8(iii), the desired continuous extension is provided by [Mon06, Theorem 6]. □

6.3. **Isometric actions of amenable groups.**

Proof of Theorem 1.6. Assume that G has no fixed point at infinity. Thus there is a minimal closed convex invariant subset by Proposition 1.8(ii) and we may assume that this subset coincides with X. In other words G acts minimally on X. Let $X = \mathbb{E} \times X'$ be the maximal Euclidean decomposition (see [BH99, II.6.15(6)]). Thus G preserves the splitting $X = \mathbb{E} \times X'$ and the induced G-action on both $\mathbb{E}$ and X' is minimal and does not fix any point at infinity. We need to show that X' is reduced to a single point. To this end, it is thus sufficient to establish the following claim.

If an amenable locally compact group G acts continuously, minimally and without fixed points at infinity on a CAT(0) *space X of finite telescopic dimension without Euclidean factor, then X is reduced to a single point.*

Assume that this is not the case. Pick a base point $o \in X$ and consider the spaces $\mathcal{C} \subset \mathcal{C}_0$ defined in Subsection 4.3. Let $\mathcal{A}$ denote the closed convex hull of $\overline{\mathcal{C}}$ in the locally convex topological vector space $\mathcal{B}$ of all functions vanishing at o. By Proposition 6.1 the subset $\overline{\mathcal{C}}$ does not contain any affine function. It follows from Lemma 4.10 that $\mathcal{A}$ does not contain any affine function either. The induced action of G on $\mathcal{B}$ is continuous and preserves the compact convex set $\mathcal{A}$. By the definition of amenability G has a fixed point in $\mathcal{A}$. Thus we have found some *non-constant* 1-Lipschitz convex function f which is **quasi-invariant** with respect to G in the sense that, for each $g \in G$, one has $f(gx) = f(x) + f(go)$. (In other words, this means that for each g, the map $x \mapsto f(gx) - f(x)$ is constant.) The following lemma, analogous to [AB98, Lemma 2.4], implies that G has a fixed point at infinity, which is absurd. □

Lemma 6.2. *Let a group G act minimally by isometries on a complete* CAT(0) *space X of finite telescopic dimension. There is a G-quasi-invariant continuous non-constant convex function f on X if and only if G fixes a point in ∂X.*

Proof. If G fixes a point in ∂X, then the Busemann function of this point (that is uniquely defined up to a positive constant) is quasi-invariant.

Assume that f is quasi-invariant and define $a : G \to \mathbb{R}$ by $a(g) = f(gx) - f(x)$. By assumption a does not depend on x; furthermore a is a homomorphism. If a were constant, then f would be G-invariant and, hence, so would be any sub-level set of f. This contradicts the minimality assumption on the G-action. Therefore a is non-constant; more precisely the image of a is unbounded and $\inf f = -\infty$. For each $r \in \mathbf{R}$ set $X_r := \phi^{-1}(-\infty, -r]$. Then $(X_r)_{r \in R}$ is a chain of closed convex subspaces with empty intersection; furthermore every element of G permutes the sets X_r. It follows that $C = \bigcap_{r \in \mathbf{R}} \partial X_r$ is G-invariant. Theorem 1.1 now shows that C is nonempty of radius $\leq \frac{\pi}{2}$, and [BL05, Prop. 1.4] implies that G fixes a point in $C \subset \partial X$. □

Proof of Theorem 1.7. The proof mimicks the arguments given in [Cap07]; we do not reproduce all the details. As in *loc. cit.* the key point is to establish that every point of the **refined boundary** $\partial_{\mathrm{fine}} X$ (defined in *loc. cit.*, §4.2) has an amenable stabiliser in G and that, conversely, any amenable subgroup of G possesses a finite index subgroup which fixes a point in $X \cup \partial_{\mathrm{fine}} X$. The proof that amenable groups stabilise point in $X \cup \partial_{\mathrm{fine}} X$ uses Theorem 1.6 together with an induction on the geometric dimension (see the remark following Corollary 4.4 in *loc. cit.* showing that there is a uniform upper-bound on the level of a point in the refined boundary). For the converse, one shows directly that the G-stabiliser of a point in $\partial_{\mathrm{fine}} X$ is (topologically locally finite)-by-(virtually Abelian); the cocompactness argument used in Proposition 4.5 of *loc. cit.* is replaced by a compactness argument relying on the hypothesis that X has finitely many types of cells, all of which are compact. □

Proof of Corollary 1.10. By Proposition 6.1, the space X admits a canonical decomposition as a product of a maximal Euclidean factor and a finite number of irreducible non-Euclidean factors. The Euclidean factor is G-invariant and G possesses a closed normal subgroup of finite index G^* that acts componentwise on the above product. By hypothesis, the G^*-action on each non-Euclidean factor is minimal and does not fix any point at infinity. Theorem 1.6 and Proposition 1.8(iii) therefore imply that the amenable radical of G^* acts trivially. This implies that the amenable radical of G acts as a finite group on the product of all non-Euclidean factors of X. Thus this action is trivial since G acts minimally. □

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