Isomorphisms of Kac-Moody groups

Pierre-Emmanuel CAPRACE* and Bernhard MÜHLHERR

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Département de Mathématiques,
Université libre de Bruxelles, CP216
Bd du Triomphe, B-1050 Bruxelles, Belgium
e-mail: *pcaprace@ulb.ac.be* and *bmuhlherr@ulb.ac.be*

ABSTRACT. We give the solution of the isomorphism problem for Kac-Moody groups over algebraically closed fields of any characteristic. In particular, we prove a conjecture of Kac and Peterson and compute the automorphism group of a Kac-Moody group over an algebraically closed field of characteristic zero.

1 Introduction

Kac-Moody groups are infinite-dimensional generalizations of Chevalley groups. It is known that each automorphism of a Chevalley group (of irreducible type and over a perfect field) can be written as a product of an inner, a diagonal, a graph and a field automorphism (see Theorem 30 in [22]). In [14] it was conjectured that the same statement holds for Kac-Moody groups over algebraically closed fields of characteristic 0 up to the addition of a so called sign automorphism. In [7] this conjecture is shown to be true for Kac-Moody groups of affine type. In [15] it is proved that each automorphism which preserves the set of all $\mathrm{Ad}_{\mathfrak{g}'}$-finite elements can be written in the way described above; this holds for Kac-Moody groups over (not necessarily algebraically closed) fields of characteristic 0 under the additional assumption that the underlying generalized Cartan matrix is symmetrizable. In this paper we prove the conjecture above to be true for Kac-Moody groups over algebraically closed fields without any restrictions neither on the type nor on the characteristic.

Throughout the paper we use Tits' definition for Kac-Moody groups over fields [25]. This definition does not only provide the abstract Kac-Moody group G but also a canonical system $(U_\alpha)_{\alpha\in\Phi}$ of root subgroups. The pair $(G, (U_\alpha)_{\alpha\in\Phi})$ is an example of a so called

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twin root datum. Twin root data have been introduced by Tits in order to give suitable axioms for these pairs arising from his definition of Kac-Moody groups.

It turns out (Theorem 2.2 below) that the proof of the conjecture above is essentially equivalent to showing that each automorphism of the group G maps the system $(U_\alpha)_{\alpha\in\Phi}$ onto one of its G-conjugates. This is certainly achieved if one can show that there is - up to conjugation - only one system of subgroups $(U_\alpha)_{\alpha\in\Phi}$ in G such that the corresponding pair is a twin root datum. In this way we are naturally led to the isomorphism problem for groups with a twin root datum.

The main result of the present paper (Theorem 2.3 below) is the solution of the isomorphism problem for (locally split) twin root data over algebraically closed fields. As Kac-Moody groups over algebraically closed fields are special cases of them, the validity of the conjecture above turns out to be a corollary of it (see Theorem 2.4). We also mention that our proof of the main theorem does not appeal to the corresponding well known result in the spherical case. In particular, it yields a new approach to Steinberg's theorem for Chevalley groups over algebraically closed fields.

Before explaining our results in detail we remark that twin root data over algebraically closed fields are not classified. There are essentially two known constructions how to obtain examples of twin root data which are not Kac-Moody groups in the sense of Tits (see also the remark preceding the statement of the main theorem below). One can use the techniques of [16] in order to construct examples which are in a sense still of algebraic nature. There is also a construction in [20] where one has more freedom (non-isomorphic fields might be used) in one sense, but also the restriction that the Weyl group has to be right-angled.

Finally, we would like to remark that the isomorphism problem for Kac-Moody groups over finite fields has been considered in [19] for a special case. In [4], a complete solution has been found recently by the authors for finite fields of cardinality at least 4.

2 Notation and Results

2.1 Twin root data

Let (W,S) be a Coxeter system and let ℓ denote the corresponding length function. For each $s\in S$ we set $\alpha_s := \{w\in W \mid \ell(sw)=\ell(w)+1\}$. A **root** of (W,S) is a subset of W of the form $w\alpha_s$ for some $w\in W$ and $s\in S$. The root α_s is called a **simple root**. The set of all roots is denoted by $\Phi(W,S)$, or simply by Φ. The roots containing (resp. not containing) 1 are called **positive** (resp. **negative**); the set of positive (resp. negative) roots is denoted by Φ_+ (resp. Φ_-). The complement $W\backslash\alpha$ of a root α in W is again a root which is denoted by $-\alpha$. A pair of roots $\{\alpha,\beta\}$ is said to be **prenilpotent** if $\alpha\cap\beta\neq\emptyset$ and $(-\alpha)\cap(-\beta)\neq\emptyset$. For a prenilpotent pair of roots $\{\alpha,\beta\}$ we set $[\alpha,\beta] := \{\gamma\in\Phi \mid \alpha\cap\beta\subset\gamma, (-\alpha)\cap(-\beta)\subset-\gamma\}$ and $(\alpha,\beta) := [\alpha,\beta]\backslash\{\alpha,\beta\}$. For each $\alpha\in\Phi$ there is a unique element of W conjugate to an element of S and permuting α and $-\alpha$ by left translation; we denote it by s_α: it is the **reflection** associated to the root α. For instance, we have $s_\alpha = s_{-\alpha}$ and if $\alpha=\alpha_s$ then $s_\alpha = s$.

Let (W,S) be a Coxeter system and let Φ be the set of its roots. A **twin root datum** (abbreviated by **TRD**) of type (W,S) is a system consisting of a group G and a family of subgroups U_α indexed by Φ such that the following axioms are satisfied, where H and

U_+ denotes respectively the intersection of the normalizers of all U_α's and the subgroup of G generated by the U_α's such that α is positive:

(TRD0) $U_\alpha \neq \{1\}$ for all $\alpha \in \Phi$;
(TRD1) if $\{\alpha, \beta\}$ is a prenilpotent pair of distinct roots, the commutator $[U_\alpha, U_\beta]$ is contained in the group $U_{(\alpha,\beta)}$ generated by all U_γ's with $\gamma \in (\alpha, \beta)$;
(TRD2) if $s \in S$ and $u \in U_{\alpha_s} \backslash \{1\}$, there exist elements u', u'' of $U_{-\alpha_s}$ such that the product $\mu(u) = u'uu''$ conjugates U_β onto $U_{s_\alpha(\beta)}$ for each $\beta \in \Phi$;
(TRD3) if $s \in S$, then $U_{-\alpha_s}$ is not contained in U_+;
(TRD4) the group G is generated by H and the U_α's.

The group generated by H and all $\mu(u)$ for $u \in U_{\alpha_s} \backslash \{1\}$ $(s \in S)$ normalizes H and is denoted by N. It is a fact that the quotient N/H is isomorphic to W (see Theorem 1.5.4 in [18]).

If $\mathcal{Z} := (G, (U_\alpha)_{\alpha \in \Phi})$ then we set $G^{\mathcal{Z}} := G$.

Example 2.1. Let $\mathbb{K}$ be a field. Then the group $G = SL_2(\mathbb{K})$ is naturally involved in a twin root datum $(G, \{U_+, U_-\})$, where U_+ (resp. U_-) is the group of upper (resp. lower) unipotent matrices in G. The type of this TRD is the Coxeter system of type A_1 whose Coxeter group has order 2. Similarly, the group $PSL_2(\mathbb{K})$ is naturally involved in a twin root datum of the same type.

2.2 Isomorphisms of twin root data

Let $\mathcal{Z} := (G, (U_\alpha)_{\alpha \in \Phi(W,S)})$ and $\mathcal{Z}' := (G', (U'_\alpha)_{\alpha \in \Phi(W',S')})$ be twin root data. Let $S = S_1 \cup \cdots \cup S_n$ be the finest partition of S such that $[S_i, S_j] = 1$ whenever $1 \leq i < j \leq n$. Then $\mathcal{Z}$ and $\mathcal{Z}'$ are called **isomorphic** if there exist an isomorphism $\varphi : G \to G'$, an isomorphism $\pi : W \to W'$ with $\pi(S) = S'$, an element $x \in G'$ and a sign ϵ_i for each $1 \leq i \leq n$ such that

$$\varphi(U_{\alpha_s}) = xU'_{\epsilon_i \alpha_{\pi(s)}} x^{-1}$$

for every $s \in S$. If φ is as above, then we say that the isomorphism φ **induces an isomorphism** of $\mathcal{Z}$ to $\mathcal{Z}'$. The following result shows that the type of a twin root datum is uniquely determined by the system of root subgroups $(U_\alpha)_{\alpha \in \Phi}$.

Theorem 2.2. *Let $\mathcal{Z} := (G, (U_\alpha)_{\alpha \in \Phi})$ (resp. $\mathcal{Z}' := (G', (U'_{\alpha'})_{\alpha' \in \Phi'})$) be a twin root datum of type (W, S) (resp. (W', S')), where S (resp. S') is finite. If $\xi : G \to G'$ is an isomorphism and if there exists $x \in G'$ such that*

$$\{\xi(U_\alpha) | \alpha \in \Phi\} = \{xU'_\alpha x^{-1} | \alpha \in \Phi'\},$$

then ξ induces an isomorphism of $\mathcal{Z}$ to $\mathcal{Z}'$. In particular, the Coxeter systems (W, S) and (W', S') are isomorphic.

2.3 The main result

A twin root datum $\mathcal{Z} := (G, (U_\alpha)_{\alpha \in \Phi})$ is called **locally split** if the group $H := \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$ is abelian and if, moreover, for each $\alpha \in \Phi$ there is a field $\mathbb{K}_\alpha$ such that the twin root datum $(\langle U_\alpha \cup U_{-\alpha} \rangle, \{U_\alpha, U_{-\alpha}\})$ of type A_1 is isomorphic to the natural twin root datum involving $SL_2(\mathbb{K}_\alpha)$ or $PSL_2(\mathbb{K}_\alpha)$ as described in Example 2.1. We also say that $\mathcal{Z}$ is

locally split over $(\mathbb{K}_\alpha)_{\alpha\in\Phi}$. Notice that $\mathbb{K}_\alpha = \mathbb{K}_{-\alpha}$. In that case, the group H is called the **fundamental torus** of G (with respect to the given TRD). Moreover, if S is finite, then there are only a finite number of non-isomorphic fields in $(\mathbb{K}_\alpha)_{\alpha\in\Phi}$ because every pair $\{U_\alpha, U_{-\alpha}\}$ is conjugate to a pair $\{U_{\alpha_s}, U_{-\alpha_s}\}$ for some $s \in S$. If each field $\mathbb{K}_\alpha$ is isomorphic to a fixed field $\mathbb{K}$, then $\mathcal{Z}$ is called **locally split over the field** $\mathbb{K}$ or simply $\mathbb{K}$**-locally split**. If $\mathcal{Z}$ is the natural twin root datum arising from a Kac-Moody group over $\mathbb{K}$, then it is said to be **split over** $\mathbb{K}$ or simply $\mathbb{K}$**-split**. It follows from the definition of a Kac-Moody group over $\mathbb{K}$ (see Section 8) that '$\mathbb{K}$-split' implies '$\mathbb{K}$-locally split'.

One may wonder whether 'locally split over a field $\mathbb{K}$' implies 'split over $\mathbb{K}$' as it is the case for algebraic groups. In the context of twin root data, this is no longer true – even if the field in question is algebraically closed. Given a field $\mathbb{K}$ and an automorphism σ of $\mathbb{K}$, then we can twist the multiplication in $\mathbb{K}[t, t^{-1}]$ by putting $\lambda t = t\sigma(\lambda)$. The group $GL_n(\mathbb{K}[t, t^{-1}])$ has a twin root datum of type $\tilde{A}_n$ which is locally split over $\mathbb{K}$; if σ is non-trivial, this group is however not a Kac-Moody group over $\mathbb{K}$ as defined in [25].

Our main result is the following.

Theorem 2.3. *Let $\mathcal{Z}$ and $\mathcal{Z}'$ be twin root data of type (W, S) and (W', S') respectively. Assume that S and S' are finite, that $\mathcal{Z}$ and $\mathcal{Z}'$ are locally split over algebraically closed fields $(\mathbb{K}_\alpha)_{\alpha\in\Phi(W,S)}$ and $(\mathbb{K}'_\alpha)_{\alpha\in\Phi(W',S')}$ respectively. Then any isomorphism of $G^{\mathcal{Z}}$ to $G^{\mathcal{Z}'}$ induces an isomorphism of $\mathcal{Z}$ to $\mathcal{Z}'$. In this situation we have in particular an induced bijection $\eta : \Phi(W, S) \to \Phi(W', S')$ and for each $\alpha \in \Phi(W, S)$, the field $\mathbb{K}_\alpha$ is isomorphic to $\mathbb{K}'_{\eta(\alpha)}$.*

The main application of the theorem above we have in mind is the solution of the isomorphism problem of Kac-Moody groups over algebraically closed fields. The specific nature of these groups makes it possible to prove the existence of certain classes of automorphisms which enables us to make the main result more precise in that case. In the following statement we use the terminology of [18] concerning Kac-Moody groups. Moreover, we use some specific terminology regarding automorphisms of these groups which is directly inspired from [7] and [22]. All of these notions are described in detail in Section 8 below.

Theorem 2.4. *Let $\mathbb{K}$ be an algebraically closed field. Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum such that the generalized Cartan matrix A is indecomposable and that the lattice $\Lambda^\vee$ is generated by the h_i's. Let $G := \mathcal{G}_{\mathcal{D}}(\mathbb{K})$ be the corresponding Kac-Moody group. Then every automorphism of G is a product of an inner automorphism, a sign automorphism, a diagonal automorphism, a graph automorphism and a field automorphism. If moreover, either* $\mathrm{char}(\mathbb{K}) = 0$ *or every off-diagonal entry of the generalized Cartan matrix A is prime to* $\mathrm{char}(\mathbb{K})$ *then the term 'graph automorphism' may be replaced by 'diagram automorphism' in the previous statement.*

In the special case when $\mathbb{K}$ has characteristic 0 and $\mathcal{D}$ is the simply connected Kac-Moody root datum this result was conjectured by Kac and Peterson in [14] (see Remark (g) on p. 136 in *loc. cit.*).

The previous theorem gives only an "upper-bound" for the automorphism group of a Kac-Moody group G over an algebraically closed field $\mathbb{K}$. However, the definitions of sign automorphisms, diagonal automorphisms, diagram automorphisms and field automorphisms are constructive (see Section 8.2 below). Therefore, it follows from the previous theorem that $\mathrm{Aut}(G)$ reaches this upper-bound in the case where $\mathrm{char}(\mathbb{K}) = 0$ or where $\mathrm{char}(\mathbb{K})$ is prime to every off-diagonal entry of the generalized Cartan matrix which defines A. Using a theorem about the existence of graph automorphisms of Kac-Moody groups due to A. Chosson [8] (see Theorem 8.5 below), one can verify that this upper-bound is also reached for other classes of Kac-Moody groups. Those include the affine Kac-Moody groups over algebraically closed fields of any characteristic.

2.4 An outline of the proof of Theorem 2.3

Let us write $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$, where $\Phi = \Phi(W, S)$. We have to show that the abstract structure of the group G determines the twin root datum $\mathcal{Z}$. The main tool is the theory of twin buildings which had been introduced by M. Ronan and J. Tits in the late 1980's. The twin root datum $(G, (U_\alpha)_{\alpha\in\Phi})$ can be used to construct a so called twin building on which the group G acts naturally; the action of the root groups U_α has very special combinatorial properties. We show that any action of G on an abstract twin building forces the groups U_α to act in exactly this way. This enables us to establish the desired uniqueness of the system $(U_\alpha)_{\alpha\in\Phi}$.

The following two propositions constitute the main intermediate steps of the proof of Theorem 2.3. They both relate the structure of certain small subgroups of G to their behavior with respect to the given root datum $\mathcal{Z}$. The small subgroups in question are finite abelian subgroups for the first proposition and rank 1 subgroups (namely, subgroups isomorphic to $(P)SL_2(\mathbb{K})$) for the second.

Before stating those propositions we make one more definition. Let p be a prime and X be a group. The supremum of the ranks of elementary abelian p-subgroups of X is called the **p-rank** of X and denoted by $m_p(X)$.

Proposition 2.5. *Any finite abelian subgroup of G whose order is prime to the characteristic of $\mathbb{K}_\alpha$ for each $\alpha \in \Phi$ is conjugate to a subgroup of the fundamental torus H of G. Assume moreover that G is center-free. Then for each prime p as above the p-rank of G is finite and bounded by the cardinality of S. If A is an elementary abelian p-subgroup of rank $m_p(G)$, then $gN_G(A)g^{-1} = N_G(H)$ for every $g \in G$ such that $gAg^{-1} \leq H$.*

In the special case when W is finite, this result is essentially equivalent to the classical fact that a finite abelian subgroup of a linear algebraic group over $\mathbb{K}$ is diagonalizable whenever its order is prime to the characteristic of $\mathbb{K}$. The previous proposition is obtained by combining the latter fact with the fixed point theorem for finite groups acting on buildings.

Proposition 2.6. *Assume that G is center-free. Let L be a subgroup of G which is isomorphic to $SL_2(\mathbb{K})$ or $PSL_2(\mathbb{K})$ for some algebraically closed field $\mathbb{K}$. Then the following two statements are equivalent:*

(i) there exists a root $\alpha \in \Phi$ such that L is conjugate to a subgroup of $\langle U_\alpha, U_{-\alpha}\rangle$;

(ii) there exists an elementary abelian p-subgroup A^1 of p-rank $m_p(G) - 1$ centralizing L, where p is a prime such that $p > 3$, $p \neq \mathrm{char}(\mathbb{K})$ and $p \notin \{\mathrm{char}(\mathbb{K}_\alpha) | \alpha \in \Phi\}$.

Let Δ be the building associated to $\mathcal{D}$. The important point is that the centralizer of the group $C_{A^1}(L)$ in G is naturally involved in a TRD, which is canonically related to the subbuilding of Δ fixed by $C_{A^1}(L)$. The building $\bar{\Delta}$ associated to the latter TRD turns out to be of universal type which essentially means that it is a tree. Since the group L

centralizes $C_{A^1}(L)$, it acts on $\bar{\Delta}$ and this action can be easily described thanks to a result of [24]. The conclusive step is to use the relation between $\bar{\Delta}$ and Δ in order to interpret the previous analysis in terms of the action of L on Δ.

Once those two propositions are established it will be easy to check that the hypotheses of Theorem 2.2 are satisfied, whence the conclusion.

The paper is organized as follows.

In the next section we introduce some terminology, fix notation and recall some results from the theory of (twin) buildings which are needed later. In Section 4.2 we give the proof of Theorem 2.2. In the following three sections the respective proofs of Proposition 2.5, Proposition 2.6 and the main theorem are given. The final Section 8 is devoted to Kac-Moody groups in the strict sense. There we recall some definitions and introduce the five types of automorphisms which appear in the statement of Theorem 2.4. Moreover, necessary conditions for the existence of nontrivial graph automorphisms are given (Proposition 8.3). Finally, the proof of the latter is given.

3 Twin structures

3.1 Twin buildings

In this subsection we recall some notions and fix the notation used throughout. The main references, beyond the specific ones mentioned in the text, are [1], [18], [21], [26] and [28].

We view buildings as chamber systems and refer the reader to [21] and [30] for the basic theory.

Let (W, S) be a Coxeter system. A **twinned pair of buildings** or **twin building** of type (W, S) is a pair $((\Delta_+, \delta_+), (\Delta_-, \delta_-))$ of buildings of that type, endowed with a **W-codistance**

$$\delta^* : (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \to W$$

satisfying the following axioms, where $\epsilon \in \{+, -\}$, $x \in \Delta_\epsilon$, $y \in \Delta_{-\epsilon}$ and $w = \delta^*(x, y)$:

- **(Tw1)** $\delta^*(y, x) = w^{-1}$;
- **(Tw2)** if $z \in \Delta_\epsilon$ is such that $\delta_{-\epsilon}(y, z) = s \in S$ and $\ell(ws) < \ell(w)$, then $\delta^*(x, z) = ws$;
- **(Tw3)** if $s \in S$, there exists $z \in \Delta_{-\epsilon}$ such that $\delta_{-\epsilon}(y, z) = s$ and $\delta^*(x, z) = ws$.

Two chambers $x \in \Delta_+$ and $y \in \Delta_-$ are called **opposite** if $\delta^*(x, y) = 1$. It can be proved that the W-codistance δ^* is completely determined by the opposition relation together with the W-distances δ_+ and δ_- [26]. Two residues are called **opposite** if they are of the same type and contain opposite chambers. A pair of opposite residues of type S' endowed with the W-codistance induced by δ^* is itself a twin building of type $(\langle S' \rangle, S')$.

An ordered pair of apartments $\Sigma = (\Sigma_+, \Sigma_-)$ with $\Sigma_\epsilon \subset \Delta_\epsilon$ for $\epsilon = +, -$ is called a **twin apartment** if the restriction op_Σ of the opposition relation to Σ defines a bijection between Σ_+ and Σ_-. In this case δ^* induces a W-codistance on Σ. This endows Σ with the structure of a thin twin building (namely a twinned pair of thin buildings). If $(c_+, c_-) \in \Delta_+ \times \Delta_-$ is a pair of opposite chambers, then there exists a unique twin apartment (Σ_+, Σ_-) such that $(c_+, c_-) \in \Sigma_+ \times \Sigma_-$. By abuse of notation, we often write $x \in \Sigma$ in place of $x \in \Sigma_+ \cup \Sigma_-$ for a chamber x. An apartment of Δ_ϵ is called **admissible** if it is involved in some twin apartment. Any two chambers of Δ_ϵ are contained in some admissible apartment.

If $\Sigma = (\Sigma_+, \Sigma_-)$ is a twin apartment in a twin building $\Delta = (\Delta_+, \Delta_-, \delta^*)$ of type (W, S), then the group W acts naturally on both Σ_+ and Σ_-. This action preserves the restriction of the W-codistance of Δ to Σ. Therefore, if $t \in W$ is a reflection of Σ_+, then it is also a reflection of Σ_- and vice versa. We call it a reflection of Σ. The symbol $P(t)$ is used to denote the union of the walls associated to t in both Σ_+ and Σ_- (these walls are viewed as sets of panels). We set $C(t) := \bigcup_{\pi \in P(t)} \pi$. Conversely, if $\alpha = (\alpha_+, \alpha_-)$ is a twin root of Σ, we equally write s_α, s_{α_+} of s_{α_-} for the corresponding reflection. Let $\epsilon \in \{+, -\}$. Given $c \in \Sigma_\epsilon$, we denote by $H(t, c)$ the root of Σ_ϵ associated with t and containing c. A pair of roots $(\alpha, \tilde{\alpha})$ in the twin apartment Σ is called a **twin root** if we have $\tilde{\alpha} = -\mathrm{op}_\Sigma \alpha$, where op_Σ denotes the restriction of the opposition relation to Σ.

An important feature of twin buildings is that their structure is rather rigid. The following result makes this statement more precise. It is an extension to twin buildings of Theorem 4.1.1. of [23]; for a proof in this more general context, see Theorem 1 in [27]. The symbol $E_1(c)$ denotes the set of all chambers adjacent to the chamber c.

Proposition 3.1. *Let $\Delta = (\Delta_+, \Delta_-, \delta^*)$ be a thick twin building, let $c_+ \in \Delta_+$ and $c_- \in \Delta_-$ be opposite chambers and let $\epsilon \in \{+, -\}$. Let ϕ_1 and ϕ_2 be automorphisms of Δ. Then $\phi_1 = \phi_2$ if and only if, they coincide on the set $E_1(c_\epsilon) \cup \{c_{-\epsilon}\}$.*

3.2 Twin BN-pairs

Twin buildings naturally arise from twin root data. In order to describe the link between these notions in more detail we need an auxiliary concept which is the twin analogue of a BN-pair.

Let G be a group and let B_+, B_- and N be subgroups of G such that $B_+ \cap N = B_- \cap N$. Set H to be the common value and set $W := N/H$. Suppose that (G, B_+, N, S) and (G, B_-, N, S) are both BN-pairs in G. The system (G, B_+, B_-, N, S) is called a **twin BN-pair** if the following conditions are satisfied:

- **(TBN1)** for $w \in W$, $s \in S$ with $\ell(ws) < \ell(w)$ and $\epsilon \in \{+, -\}$, we have $B_\epsilon w B_{-\epsilon} s B_{-\epsilon} = B_\epsilon ws B_{-\epsilon}$;
- **(TBN2)** for $s \in S$, we have $B_+ s \cap B_- = \emptyset$.

For $\epsilon \in \{+, -\}$, the map $\beta_\epsilon : W \to B_\epsilon \backslash G / B_{-\epsilon}, w \mapsto B_\epsilon w B_{-\epsilon}$ is bijective. This is the **Birkhoff decomposition** of G. For each $\epsilon \in \{+, -\}$ let $(\Delta_\epsilon, \delta_\epsilon)$ be the building associated to the BN-pair (G, B_ϵ, N, S). Define a mapping δ^* by

$$\delta^*(gB_\epsilon, hB_{-\epsilon}) = w \Leftrightarrow g^{-1}h \in B_\epsilon w B_{-\epsilon},$$

which makes sense in view of the Birkhoff decomposition of G. Then $\Delta(G) := ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ is a twin building of type (W, S); the group G acts on $\Delta(G)$ by automorphisms and it is transitive on pairs of opposite chambers. Moreover, the group B_ϵ fixes a unique chamber $c_\epsilon \in \Delta_\epsilon$. The chambers c_+ and c_- are called the **fundamental chambers**; they opposite and the unique twin apartment they determine is said to be **fundamental** with respect to the given twin BN-pair. This twin apartment is stabilized by the group N and chamberwise fixed by $B_+ \cap N = B_- \cap N$.

Remark 3.2. The notation $\Delta(G)$ is abusive, because the building it denotes depends on the twin BN-pair (G, B_+, B_-, N, S) and (in general) not only on the group G. We shall however make this abuse when there is no ambiguity on which twin BN-pair we have in mind.

3.3 Twin root data and the associated twin buildings

Let now $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum of type (W, S). Set $B_\epsilon = HU_\epsilon$ for $\epsilon \in \{+, -\}$ and $N = \langle H, \mu(u) | u \in U_{\alpha_s}\backslash\{1\}, s \in S\rangle$. Then (G, B_+, B_-, N, S) is a twin BN-pair. We have $B_\epsilon \cap U_{-\epsilon} = \{1\}$ for $\epsilon \in \{+, -\}$. In particular N is the full stabilizer in G of the fundamental twin apartment Σ associated with the twin BN-pair, H is the chamberwise stabilizer of it. The group U_ϵ acts regularly on chambers opposite c_ϵ in the twin building $\Delta(G)$, where c_+ and c_- are the fundamental chambers with respect to $\mathcal{Z}$. Moreover, if $\mathcal{Z}$ is locally split over fields $(\mathbb{K}_\alpha)_{\alpha\in\Phi}$ each of which has at least 4 elements, then H fixes no chamber outside the fundamental twin apartment Σ (this is easily seen thanks to arguments as in the proof of Lemma 5.1 below), and so we have $N = N_G(H)$.

Assume that H is abelian. Then the center Z of G coincides with the kernel of the action of G on the twin building $\Delta(G)$. Let $\bar{G} = G/Z$ and $p : G \to \bar{G}$ be the canonical projection. Then the restriction of p to each U_α is injective and $(\bar{G}, (\bar{U}_\alpha)_{\alpha\in\Phi})$ is a TRD of type (W, S) where $\bar{U}_\alpha = p(U_\alpha)$. This TRD is called the **reduction** of $(G, (U_\alpha)_{\alpha\in\Phi})$. We have $\Delta(\bar{G}) = \Delta(G)$ and $\bar{G}$ acts faithfully on that building. Finally, if $(G, (U_\alpha)_{\alpha\in\Phi})$ is locally split over $(\mathbb{K}_\alpha)_{\alpha\in\Phi}$ then so is $(\bar{G}, (\bar{U}_\alpha)_{\alpha\in\Phi})$. (These assertions follow essentially from Section 3.3 of [28]. See also Section 4.1 in [2] for a more detailed exposition.)

Let $\Sigma = (\Sigma_+, \Sigma_-)$ be the fundamental apartment of $\Delta(G)$ and for $\epsilon \in \{+, -\}$ let ι_ϵ be the isometry of $\Sigma(W, S)$ to Σ_ϵ such that $\iota_\epsilon(1) = c_\epsilon$. For each $\alpha \in \Phi$ the ordered pair $(\alpha_+, \alpha_-) := (\iota_+(\alpha), -\iota_-(\alpha))$ is a twin root of Σ which we denote abusively by α as well. The group U_α fixes any chamber of $\Delta(G)$ which is contained in a panel π such that $\pi \cap \alpha_+$ or $\pi \cap \alpha_-$ has cardinality 2. We say that U_α **fixes** α **thickly**. Moreover, U_α acts sharply transitively on the set of twin apartments which contain the twin root (α_+, α_-). Equivalently, it acts sharply transitively on the set $\pi\backslash\alpha$ for each panel belonging to the wall associated to α. Assume now that there exists an s-panel in this wall such that $s \in S$ is not a central element of W. Then there exists a chamber x of Σ such that no panel containing x is in the wall $P(s_\alpha)$. Proposition 3.1 then implies that U_α is the group of all automorphisms of $\Delta(G)$ which fix α thickly. In this case U_α is called a **root group** of the building $\Delta(G)$. If no element of S is central in W, then every U_β is a root group and the building $\Delta(G)$ is said to have the **Moufang property** or to be **Moufang**.

For example, any rank 2 residue of $\Delta(G)$ of irreducible type is a twin building which satisfies the Moufang property. By Theorem (17.1) in [29], this implies that the order of the product of any two elements of S is $2, 3, 4, 6, 8$ or infinite.

3.4 Rank-one subgroups

For each $\alpha \in \Phi$ we denote by L_α the group generated by U_α and $U_{-\alpha}$ and the group $H.L_\alpha$ is denoted by P_α. It is a fact that P_α is the full stabilizer in G of every pair of opposite panels which belong to the wall $P(s_\alpha)$. Assume that the given TRD is locally split over algebraically closed fields $(\mathbb{K}_\alpha)_{\alpha\in\Phi}$. Then every panel $\pi \in P(s_\alpha)$ is isomorphic to the projective line $P_1(\mathbb{K}_\alpha)$ and the induced action of P_α on π is equivalent to the action of $PSL_2(\mathbb{K}_\alpha)$ on $P_1(\mathbb{K}_\alpha)$. (See for example Corollary 49 in [2].) This fact has the following consequence which we shall often use in the sequel: If $h \in H$ fixes some chamber of π which is not contained in the fundamental twin apartment of $\Delta(G)$, then h fixes π chamberwise. In other words, if h fixes three chambers of π, then it fixes every chamber of π.

4 From group isomorphisms to twin root datum isomorphisms

The purpose of this section is to present the proof of Theorem 2.2. The first subsection records a result related to the rigidity of Coxeter groups which is the crucial ingredient of this proof; the second subsection consists of the proof itself.

4.1 Rigidity of Coxeter groups

Let Σ be a set and let W be a group acting on Σ from the left. A subset $D \neq \emptyset$ of Σ is called **fundamental** (or a **fundamental domain**) if, for $w \in W$, we have $w = 1$ whenever $wD \cap D \neq \emptyset$ and if moreover $\bigcup_{w \in W} wD = \Sigma$.

Let now (W, S) be a Coxeter system, T the set of its reflections, $\Phi = \Phi(W, S)$ the set of its roots and $\Sigma = \Sigma(W, S)$ the corresponding thin building. A set $R \subseteq T$ is called **universal** if $(\langle R \rangle, R)$ is a Coxeter system. Let $\Psi \subseteq \Phi$ be a set of roots. We put $R(\Psi) = \{r_\psi | \psi \in \Psi\}$ (where r_ψ is the reflection associated to ψ). The set Ψ is called **universal** if $R(\Psi)$ is universal. It is called **weakly 2-geometric** if for all $\psi, \psi' \in \Psi$, at least one of the sets $\psi \cap \psi'$, $(-\psi) \cap (-\psi')$ is a fundamental domain for the action of $\langle r_\psi, r_{\psi'} \rangle$ on $\Sigma(W, S)$.

Notice that if α, β are two roots such that the product $s_\alpha s_\beta$ has infinite order, then the pair $\{\alpha, \beta\}$ is prenilpotent if and only if it is not weakly geometric. If the product $s_\alpha s_\beta$ has finite order, then there exists a spherical rank 2 residue R which is stabilized by s_α and s_β and the action of $\langle s_\alpha, s_\beta \rangle$ on R is faithful.

The following result is the main ingredient for the proof of Theorem 2.2. It is Theorem (3.3) in [5], but an equivalent version of that result, is already proved in [12] in a completely different setting (see also [10]).

Proposition 4.1. *Let (W, S) be a Coxeter system and let $\Psi \subseteq \Phi(W, S)$ be a finite, universal and weakly 2-geometric set of roots. Let $\Psi = \Psi_1 \cup \cdots \cup \Psi_n$ be the finest partition of Ψ such that the sets $R(\Psi_i)$ and $R(\Psi_j)$ centralize each other whenever $i \neq j$. Then, for each $1 \leq i \leq n$ there is a sign $\epsilon_i \in \{+, -\}$ such that $\bigcup_{i=1}^n \epsilon_i \Psi_i$ is a geometric set of roots.*

4.2 Proof of Theorem 2.2

We use the notation introduced in Section 1 and we add a superscript dash to denote the objects related to the second TRD (e.g. H', N').

By the definition of the fundamental torus the hypotheses imply $\xi(H) = xH'x^{-1}$. We may assume without loss of generality that $x = 1$. If α and β are distinct roots in Φ then there exists $w \in W$ such that $w\alpha$ is positive and $w\beta$ is negative. Hence $U_\alpha \cap U_\beta = \{1\}$ by 3.3. Therefore, there is a well defined bijection $\bar{\eta} : \Phi \to \Phi'$ such that $\xi(U_\alpha) = U'_{\bar{\eta}(\alpha)}$.

We claim that if $\beta \neq \pm\alpha$ then U_α and U_β are not conjugate in the group they generate. If the pair $\{\alpha, \beta\}$ is not prenilpotent, then the group $\langle U_\alpha \cup U_\beta \rangle$ is just the free product of U_α and U_β by the theorem of Section 3.5.3 in [18]. The claim follows. Now assume that the pair $\{\alpha, \beta\}$ is prenilpotent. Let $\pi \in P(s_\alpha)$ be such that $\pi \cap \alpha \subseteq \beta$, where α and β are seen as twin roots of $\Delta(G)$. Then the group $\langle U_\alpha \cup U_\beta \rangle$ stabilizes π, because it fixes the chamber in $\pi \cap \alpha$. Therefore, no element of $\langle U_\alpha \cup U_\beta \rangle$ maps $P(s_\alpha)$ on $P(s_\beta)$, which implies the claim.

Using the fact that U_α and $U_{-\alpha}$ are conjugate in $\langle U_\alpha \cup U_{-\alpha}\rangle$ by (TRD2) it follows from the claim above that $\bar\eta$ maps pairs of opposite roots of Φ to pairs of opposite roots of Φ'.

It is a well known fact that for a nontrivial element $u \in U_\alpha$ the elements $u', u'' \in U_\alpha$ which are used in the definition of $\mu(u)$ are uniquely determined by the requirement that U_α and $U_{-\alpha}$ are switched under conjugation by $\mu(u)$ (see Section 3.3 in [28]). By the previous paragraph this implies that $\xi(\mu(u)) = \mu(\xi(u))$. Therefore, we have $\xi(N) = N'$ and so ξ induces an isomorphism $\eta : W \to W'$ which sends the reflections of (W, S) to the reflections of (W', S'). Moreover, $\eta(S)$ is a universal set of reflections in W' because S is universal in W.

For each $s \in S$ we set $\beta_s := \bar\eta(\alpha_s)$. We have just seen that $\{\beta_s | s \in S\}$ is universal; we now prove that it is weakly 2-geometric.

Let s, t be distinct elements of S. Set $W'_{st} := \langle \eta(s), \eta(t)\rangle$. There are three cases.

- *The order of the product st is 2.* Then it is straightforward to see that $\beta_s \cap \beta_t$ is a fundamental domain for the action of W'_{st} on $\Sigma(W', S')$.

- *The order n of the product st is finite and at least 3.* We know by the last paragraph of Section 3.3 that $n \in \{3, 4, 6, 8\}$. There exists a rank 2 residue of spherical type R in $\Sigma(W', S')$ which is stabilized by W'_{st}. Moreover, since there is no reflection $r \in W$ such that $t \in W_{sr} := \langle s, r\rangle$ and that $|W_{sr}| > 2n$, we deduce $|R| = |W'_{st}| = 2n$.

  Since the order of the product $\eta(s)\eta(t)$ is n, the cardinality $d := |R \cap \beta_s \cap \beta_t|$ is prime to n. Moreover, we have $d \le n - 1$ since $|R \cap \beta_s| = |R|/2 = n$. Assume for a while that $d = n - 1$. Then we have $[\beta_s, \beta_t] = \{\beta_s, \beta_t\}$ and so $[U'_{\beta_s}, U'_{\beta_t}] = 1$ by (TRD1). On the other hand, we know that $[U_{\alpha_s}, U_{\alpha_t}] \ne \{1\}$ thanks to (5.7) in [29]. This shows that $d < n - 1$. Hence $d = 1$ if $n \ne 8$ because d is prime to n. Let us now consider the case $n = 8$. By the explicit formulas (16.9) in [29] we see that $[U_{\alpha_s}, U_{\alpha_t}]$ is non-abelian. On the other hand, the same formulas show that if d equals 3 or 5 then $[U'_{\beta_s}, U'_{\beta_t}]$ is an abelian group. Therefore, we conclude again that $d = 1$. Hence, in all cases $R \cap \beta_s \cap \beta_t$ is a fundamental domain for the action of W'_{st} on R. Using projections we deduce that $\beta_s \cap \beta_t$ is a fundamental domain for the action of W'_{st} on Σ'_+ which implies the claim in this case.

- *The order of the product st is infinite.* Since the pair $\{\alpha_s, \alpha_t\}$ is geometric it is not prenilpotent. We have already seen that this implies that the group $\langle U_{\alpha_s} \cup U_{\alpha_t}\rangle$ is the free product $U_{\alpha_s} * U_{\alpha_t}$. Hence, $\langle U'_{\beta_s} \cup U'_{\beta_t}\rangle = U'_{\beta_s} * U'_{\beta_t}$, and it is straightforward to deduce from (TRD1) that $\{\beta_s, \beta_t\}$ cannot be a prenilpotent pair of roots.

Hence the set $\{\beta_s | s \in S\}$ is universal and weakly 2-geometric. Now the conclusion follows from Proposition 4.1. □

5 Finite abelian subgroups

In this section we give the proof of Proposition 2.5.

Proof of Proposition 2.5. We keep the notation of the statement of 2.5. Moreover, we denote by $\Delta(G) = (\Delta_+, \Delta_-, \delta^*)$ the twin building associated with the given twin root datum.

Let A be a finite abelian subgroup of G whose order is prime to char$(\mathbb{K}_\alpha)$ for each $\alpha \in \Phi$. Let R_ϵ be a spherical residue of the building Δ_ϵ which is stabilized by A for $\epsilon \in \{+,-\}$; such a residue exists by the fixed point theorem for finite groups acting on buildings (see Corollary 11.9 in [9]). By Section 3.4 we know that R_ϵ is a spherical building in which each panel is a projective line over $\mathbb{K}_\alpha$ for some $\alpha \in \Phi$. In the following we show that R_ϵ is isomorphic to the direct product of buildings of the form $\Delta(\overline{G})$ for a simple algebraic group $\overline{G}$ over an algebraically closed field $\mathbb{K}$ isomorphic to $\mathbb{K}_\alpha$ for some $\alpha \in \Phi$. We proceed as follows. Recall from the proof of Lemma 6.3 that the only possible irreducible rank 2 residues of Δ_ϵ (and hence of R_ϵ) are the Moufang polygons associated with $PGL_3(\mathbb{K})$, $PSp_4(\mathbb{K})$ or $G_2(\mathbb{K})$, where $\mathbb{K}$ is isomorphic to $\mathbb{K}_\alpha$ and α is a root such that the reflection s_α stabilizes the rank 2 residue in question. The required isomorphism between R_ϵ and the direct product of the $\Delta(\overline{G})$ mentioned above is a consequence of Theorem 4.1.2 of [23] and Proposition 2 of [17].

Now, since a finite abelian subgroup of order prime to char$(\mathbb{K})$ is diagonalizable in a split semi-simple algebraic group over $\mathbb{K}$, we conclude that A fixes an apartment in R_ϵ. Hence A fixes at least one chamber d_ϵ in Δ_ϵ. Using again Section 3.4 together with the hypothesis on the order of A, it is easy to show that A fixes at least two chambers in each panel it stabilizes. This fact implies that the chambers d_+ and d_- can be chosen to be opposite in $\Delta(G)$. Hence, A fixes each chamber in the unique twin apartment which contains d_+ and d_-. Using now 3.3 we see that the chamberwise stabilizer of this apartment is conjugate to H in G. This proves the first statement of the proposition.

Suppose now that G is center-free. Then G acts faithfully on $\Delta(G)$ (see Section 3.3). Let c_+ and c_- be the fundamental chambers of $\Delta(G)$ and let $\Sigma = (\Sigma_+, \Sigma_-)$ be the fundamental twin apartment. For each $s \in S$ let π_s be the s-panel containing c_+. Let H_s be the group of permutations of π_s induced by H. By Section 3.4, we know that H_s acts sharply transitively on $\pi_s \backslash \Sigma_+$, and is isomorphic to $\mathbb{K}_{\alpha_s}^\times / \{\pm 1\} \simeq \mathbb{K}_{\alpha_s}^\times$. Hence, H_s is abelian. For each $s \in S$ we have a canonical epimorphism $\rho_s : H \to H_s$ induced by the restriction to π_s whose kernel consists of the elements of H acting trivially on π_s. By Proposition 3.1 and the fact that G is faithful the canonical homomorphism

$$\rho : H \to \prod_{s \in S} H_s,$$

direct product of the ρ_s, is injective. As the p-rank of $\prod_{s \in S} H_s$ is equal to the cardinality of S, the p-rank of H is finite. It is equal to the p-rank of G in view of the first part of the proposition.

Let now $A \leq H$ be an elementary abelian p-subgroup of rank $m_p(G)$. Then A is a characteristic subgroup of H which implies $N_G(H) \leq N_G(A)$. On the other hand, Σ is the unique twin apartment which is chamberwise stabilized by A as a consequence of Lemma 5.1 below. Thus, an element of G which normalizes A must stabilize Σ and therefore $N_G(A) \leq N_G(H)$ because $N_G(H)$ is the stabilizer of Σ in G by 3.3. Thus, to finish the proof, it remains to prove Lemma 5.1 below. $\square$

Lemma 5.1. *Let $(G, (U_\alpha)_{\alpha \in \Phi})$ be as in the statement of the main theorem and assume $Z(G) = 1$. Let p be a prime which is distinct from the characteristic of $\mathbb{K}_\alpha$ for each $\alpha \in \Phi$. Then an elementary abelian p-subgroup $A \leq H$ of p-rank equal to $m_p(G)$ fixes no chamber outside the fundamental twin apartment Σ.*

Proof. We keep the notation of the previous proof. Notice that $m_p(G) = m_p(H)$ by the first part of Proposition 2.5 (which is already proved completely). Since the p-rank of H_s is 1, we see that A fixes no chamber in $\pi\backslash\Sigma_+$ for each panel π containing the fundamental chamber c_+. Since A is normal in $N_G(H)$, and since $N_G(H)$ acts transitively on Σ_+, we conclude that A fixes no chamber in $\pi\backslash\Sigma_+$ for each panel π which intersects Σ_+. Now, using projections as in Section 3.4 we see that A fixes no chamber outside Σ_+. By symmetry, the analogous statement holds for Σ_-. $\square$

6 Characterization of rank-one subgroups

The purpose of this section is to present the proof of Proposition 2.6. Each of the first four subsections records an auxiliary result which we shall need in this proof; the proof itself is given in Section 6.5.

6.1 A fixed point lemma

The next lemma describes the fixed point structure of a group acting on a twin building which fixes all chambers of a twin apartment. Its proof is straightforward and will be omitted here. A detailed proof can be found in [2], Lemma 83.

Lemma 6.1. *Let A^1 be a group of automorphisms of a twin building $\Delta = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ fixing two opposite chambers $z_+ \in \Delta_+$ and $z_- \in \Delta_-$. Let Δ'_+ be the set of fixed points of A^1 in Δ_+ and let δ'_+ be the restriction of δ_+ to Δ'_+. Define similarly (Δ'_-, δ'_-) and denote by $\delta^*_|$ the restriction of δ^* to $(\Delta'_+ \times \Delta'_-) \cup (\Delta'_- \times \Delta'_+)$. Then $((\Delta'_+, \delta'_+), (\Delta'_-, \delta'_-), \delta^*_|)$ is a twin building of the same type as Δ (which is possibly neither thin nor thick).*

6.2 The thick frame of a weak twin building

The next result is a structure theorem for twin buildings in general. It says that, given a given non-thin twin building, then there exists a canonical thick twin building associated with it. In order to give a precise statement of this result we need some more terminology.

Let (Δ, δ) be a not necessarily thick building of type (W, S). Two chambers of Δ are called **thick-adjacent** if they are contained in some thick panel. A gallery $\gamma = (x_0, x_1, \ldots, x_n)$ is called **thin** if $\{x_{i-1}, x_i\}$ is a thin panel of Δ for each $1 \le i \le n$. The set of all ordered pairs of chambers (x, y) such that x can be joined to y by a thin gallery is an equivalence relation called the **thin-equivalence**. The corresponding equivalence classes are called **thin-classes**. If $c \in \Delta$ is any chamber, we denote by $\bar{c}$ the thin-class of c. For a set of chambers $\Gamma \subset \Delta$ we denote the corresponding set of thin-classes by $\bar{\Gamma}$. It is readily seen from the axioms that any apartment containing c contains the whole thin-class $\bar{c}$.

Proposition 6.2. *Let $\Delta = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ be a twinned pair of non-thin buildings of type (W, S). Let (c_+, c_-) be a fixed pair of opposite chambers in Δ and let $\Sigma = (\Sigma_+, \Sigma_-)$ denote the corresponding twin apartment. Let $\bar{S}$ be the set of reflections of Σ_+ corresponding to the thick panels which intersect $\bar{c}_+$ non-trivially and let $\bar{W}$ denote the group generated by $\bar{S}$. Then $(\bar{W}, \bar{S})$ is a Coxeter system and $(\bar{\Delta}_+, \bar{\Delta}_-)$ is naturally endowed* *with a structure of a thick twin building of type* $(\bar{W}, \bar{S})$*. This twin building is called the* ***thick frame*** *of* Δ *and it is denoted by* $\bar{\Delta}$*. If* (Σ'_+, Σ'_-) *is a twin apartment of* Δ*, then* $(\bar{\Sigma}'_+, \bar{\Sigma}'_-)$ *is a twin apartment of* $\bar{\Delta}$*; moreover, two thin-classes are opposite as chambers of* $\bar{\Delta}$ *if and only if they contain opposite chambers of* Δ*.*

Proof. This follows from the main result of [3]. ☐

6.3 A technical lemma on Moufang polygons

The following lemma is an important tool for the proof of Proposition 2.6. It concerns locally split root data over algebraically closed fields whose type is an irreducible spherical Coxeter system of rank 2. The building associated to such a TRD is a Moufang polygon.

Lemma 6.3. *Let* $(G, (U_\alpha)_{\alpha\in\Phi})$ *be a twin root datum of type* (W, S)*. For* $\alpha \in \Phi$ *we set* $H_\alpha := N_{L_\alpha}(U_\alpha) \cap N_{L_\alpha}(U_{-\alpha})$*. For* $\alpha, \beta \in \Phi$ *we set* $C_{\alpha\beta} := C_{H_\alpha . H_\beta}(U_\alpha)$*.*

Now assume that the given TRD is locally split over algebraically closed fields $(\mathbb{K}_\alpha)_{\alpha\in\Phi}$*, and that* (W, S) *has rank 2 and spherical type. If* $\alpha \neq \pm\beta$ *belong to* Φ *then* $|H_\alpha \cap H_\beta| \leq 3$ *and* $|C_{\alpha\beta} \cap C_{\beta\alpha}| \leq 3$*.*

Proof. If the Coxeter system (W, S) is reducible, then $G/Z(G) \simeq PSL_2(\mathbb{K}_\alpha) \times PSL_2(\mathbb{K}_\beta)$. Therefore, the necessary computations can be made in the group $PSL_2(\mathbb{K}_\alpha) \times PSL_2(\mathbb{K}_\beta)$, which makes this case easy.

Assume now that the type is irreducible. We first want to determine the different possibilities for the Moufang polygon $\Delta(G)$. Using explicit formulas from Chapter 16 in [29] together with a theorem by Hua (Proposition 8.12.3 in [23]) it follows that there are only three possibilities, namely $\Delta(G) \simeq \Delta(SL_3(\mathbb{K}))$, $\Delta(G) \simeq \Delta(Sp_4(\mathbb{K}))$ or $\Delta(G) \simeq \Delta(G_2(\mathbb{K}))$ where $\mathbb{K} \simeq \mathbb{K}_\alpha \simeq \mathbb{K}_\beta$. Once this is known, the desired result follows from computations using formulas (33.10), (33.13) and (33.16) in [29]. ☐

6.4 Fixed points of $SL_2(k)$ acting on a tree

We now recall a result of Tits which follows from Proposition 4 in [24] (see also the statement in §5.5 of *loc. cit.*).

Proposition 6.4. *Let* k *be a field and assume that* $SL_2(k)$ *is acting on a tree* T *without fixed point and without fixed end. Then there exists a well-defined non-archimidean valuation* η *of* k *and an* $SL_2(k)$*-equivariant isometric embedding of the Bruhat-Tits tree* T_η *into* T*.*

6.5 Proof of Proposition 2.6

In order to make future applications of the proposition easier, we prove a more technical but slightly stronger version of it.

Proposition 6.5. *Let* $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ *be a twin root datum of type* (W, S) *with* S *finite and let* H *be the fundamental torus. Assume that* $\mathcal{Z}$ *is locally split over algebraically closed fields* $(\mathbb{K}_\alpha)_{\alpha\in\Phi}$ *and that* G *is center-free. Let* $\mathbb{K}$ *be an algebraically closed field and let* p *be a prime such that* $p > 3$*,* $p \neq \mathrm{char}(\mathbb{K})$ *and* $p \notin \{\mathrm{char}(\mathbb{K}_\alpha) | \alpha \in \Phi\}$*. Let* l *be the* p*-rank of* G*, let* $A \leq G$ *be an elementary abelian* p*-subgroup of* p*-rank* l *and let* $x \in G$ *be such* *that $xAx^{-1} \le H$. Let L be a subgroup of G which is isomorphic to $SL_2(\mathbb{K})$ or $PSL_2(\mathbb{K})$ and which intersects the group A non-trivially. Then the following two statements are equivalent:*

(i) there exists a root $\alpha \in \Phi$ such that $xLx^{-1} \le \langle U_\alpha \cup U_{-\alpha} \rangle$;

(ii) $C_A(L)$ has order p^{l-1}.

Moreover, if (i) holds and if $\beta \in \Phi$ is a root such that $xLx^{-1} \le \langle U_\beta \cup U_{-\beta} \rangle$, then $\beta \in \{\alpha, -\alpha\}$.

Proof. In the language of buildings Part (i) of the proposition essentially means that the group L stabilizes some pair of opposite panels in the twin building $\Delta(G)$ associated with the given TRD. Our strategy is to reduce the problem to the special case where the twin root datum has *universal type*. This means that the product of any two distinct reflections in the corresponding Coxeter group has infinite order. In this special case the desired conclusion is deduced from Proposition 6.4.

We denote by $\Delta(G) = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ the twin building associated with the given twin root datum. Let $\Sigma = (\Sigma_+, \Sigma_-)$ be the fundamental twin apartment. We also set $A^1 := C_A(L)$.

(i) $\Rightarrow$ (ii): We may – and shall – assume that $x = 1$. Hence, we have $A \le H$ and $L \le L_\alpha := \langle U_\alpha, U_{-\alpha} \rangle$. We must show that $m_p(A^1) = l - 1$. Let (π_+, π_-) be a pair of opposite panels which belong to the wall $P(s_\alpha)$ and is thus stabilized by L_α. We know by Section 3.4 that the stabilizer of a chamber of π_ϵ in L_α is solvable for $\epsilon \in \{+, -\}$. Hence L has no fixed chamber in π_ϵ. Moreover, since $\pi_\epsilon \cap \Sigma_\epsilon$ has cardinality 2 and since L has no subgroup of index 2, we conclude that L does not stabilize $\{x_\epsilon, y_\epsilon\} := \pi_\epsilon \cap \Sigma_\epsilon$. Therefore, there exists $g \in L$ such that $g(x_\epsilon) \in \pi_\epsilon \backslash \Sigma_\epsilon$. Let now h be an arbitrary element of A^1. Then $z := g(x_\epsilon) = hgh^{-1}(x_\epsilon) = hg(x_\epsilon) = h(z)$ and h fixes a chamber which does not belong to Σ. By Section 3.4, this implies that A^1 is properly contained in A, namely $m_p(A^1) < l$.

The previous argument shows that any element of A^1 fixes a chamber of π_ϵ which does not belong to Σ. By Section 3.4, this implies that A^1 acts trivially on π_ϵ. On the other hand, any element of H which centralizes π_ϵ also centralizes L_α. Since $L \le L_\alpha$, we conclude that A^1 is the kernel of the action of A on π_ϵ.

Now, suppose that $m_p(A^1) < l - 1$. Then there exist two distinct cyclic subgroups of A, say C_1 and C_2, such that $\langle C_1 \cup C_2 \rangle \cap A^1 = \{1\}$ (and so $\langle C_1 \cup C_2 \cup A^1 \rangle \le A$). Since A^1 is the kernel of the action of A on π_ϵ, it follows that C_1 and C_2 act both faithfully on π_ϵ. It follows from Section 3.4 that they induce the same action on π_ϵ. Therefore, there is an isomorphism $\phi : C_1 \to C_2$ such that for each $c \in C_1$, the element $c\phi(c)$ acts trivially on π_ϵ. This implies that for each $c \in C_1$ the element $c\phi(c)$ centralizes L_α. Thus, the group $\langle C_1 \cup C_2 \rangle$ has a subgroup of order p which is contained in A^1. This is a contradiction and (ii) follows.

(ii) $\Rightarrow$ (i): Without loss of generality, we may – and shall – assume that $x = 1$. We have to prove that L is contained in $\langle U_\alpha \cup U_{-\alpha} \rangle$ for some $\alpha \in \Phi$ and that this property determines the pair $\{\alpha, -\alpha\}$ uniquely.

We set $A_L := A \cap L$. The group A_L is nontrivial by assumption. Moreover, it is not central in L because $p > 3$. We also set $A^1 := C_L(A)$. Thus, A^1 has order p^{l-1} and we have $A = A^1.A_L$ and $A^1 \cap A_L = \{1\}$.

By Lemma 5.1 we know that the set of fixed chambers of A in $\Delta(G)$ is the fundamental twin apartment Σ. Up to conjugation by some element of G we may assume that this twin apartment is Σ and that $A \leq H$.

Let Φ^1 denote the set of roots $\alpha \in \Phi$ such that A^1 centralizes U_α. Set also $G^1 := \langle U_\alpha | \alpha \in \Phi^1 \rangle \leq C_G(A^1)$.

The rest of the proof is divided into several steps.

Step 1: Φ^1 *is nonempty.*

It suffices to show that there is a panel of Σ whose chambers are all fixed by A^1. Suppose there is no such panel. Then it follows from Section 3.4 and an argument used in the proof of 5.1 that Σ is the set of chambers fixed by A^1. Since L centralizes A^1, we deduce that L stabilizes Σ and hence normalizes A by 3.3 and Proposition 2.5. As $A_L = A \cap L$ is not normal in L, we obtain a contradiction and we conclude that Φ^1 is not empty.

Step 2: $(G^1, (U_\alpha)_{\alpha \in \Phi^1})$ *is a twin root datum of universal type.*

For $\epsilon \in \{+,-\}$ let $\Delta^1_\epsilon \subset \Delta_\epsilon$ be the set of chambers fixed by A^1. Then (Δ^1_+, Δ^1_-) is naturally endowed with a structure of twin building by Lemma 6.1. Moreover, the buildings Δ^1_+ and Δ^1_- are not thin by the previous paragraph. We denote by $\bar{\Delta}^1 = (\bar{\Delta}^1_+, \bar{\Delta}^1_-, \bar{\delta}^*)$ the corresponding thick frame (see Section 6.2) and by (W^1, S^1) the type of $\bar{\Delta}^1$.

We now prove that (W^1, S^1) has universal type. Let s, t be distinct reflections of S^1; we have to prove that the order of the product st is infinite. Assume the contrary. Let R be a spherical residue of rank 2 which is stabilized by s and t. Let $c \in R \cap \Sigma$ and set $\alpha := H(s, c)$ and $\beta := H(t, c)$. Define now H_α and H_β as in the statement of Lemma 6.3 and let A_α and A_β be the subgroups of p-torsion elements of H_α and H_β respectively. By Lemma 6.3, we have $A_\alpha \cap A_\beta = \{1\}$ and so $A_\alpha.A_\beta$ has order p^2. Moreover, the same lemma also implies that the kernel of the action of $H_\alpha.H_\beta$ on $C(s)$ (resp. $C(t)$) has order at most 3. Therefore, no non-trivial element of $A_\alpha.A_\beta$ acts trivially on $C(s)$ (resp. $C(t)$). On the other hand, we know by definition that A^1 acts trivially on both $C(s)$ and $C(t)$. Hence $A^1 \cap (A_\alpha.A_\beta) = \{1\}$ and so the group $A^1.A_\alpha.A_\beta$ is an elementary abelian group of order p^{l+1} which is impossible by the definition of l. Thus, we obtain a contradiction and we conclude that (W^1, S^1) is of universal type.

We identify the roots in Φ with the twin roots of Σ as described in Section 3.3. If $\alpha = (\alpha_+, \alpha_-) \in \Phi$ is such that U_α centralizes A^1, then $\bar{\alpha} := (\bar{\alpha}_+, \bar{\alpha}_-)$ is a twin root of $\bar{\Delta}^1$ (see 6.2 for the notation $\bar{\alpha}_+$ and $\bar{\alpha}_-$) and U_α fixes $\bar{\alpha}$ thickly. Moreover, the action of U_α on any panel of $\Delta(G)$ belonging to $P(s_\alpha)$ is equivalent to the action of U_α on a panel of $\bar{\Delta}^1$ belonging to $P(s_{\bar{\alpha}})$. This follows from the construction of $\bar{\Delta}^1$ (see 6.2). Hence, the action of U_α on $\bar{\Delta}^1$ is faithful and we conclude that U_α is a root group of $\bar{\Delta}^1$. By identifying the twin roots of $(\bar{\Sigma}_+, \bar{\Sigma}_-)$ with the elements of $\Phi(W^1, S^1)$ as in Section 3.3 the mapping $\alpha \mapsto \bar{\alpha}$ provides a canonical bijection of Φ^1 onto $\Phi(W^1, S^1)$. Now, Proposition 7 of [28] yields that $(G^1, (U_\alpha)_{\alpha \in \Phi^1})$ is a TRD of universal type (W^1, S^1).

Step 3: *The buildings* $\bar{\Delta}^1_+$ *and* $\bar{\Delta}^1_-$ *are "trees".*

Any subgroup of G which centralizes A^1 acts on Δ^1 and $\bar{\Delta}^1$, where Δ^1 and $\bar{\Delta}^1$ are as in the proof of the claim. In particular, this is the case for L. Since $\bar{\Delta}^1$ has universal type, it follows that for $\epsilon \in \{+,-\}$ the building $\bar{\Delta}^1_\epsilon$ can be viewed as a tree $T(\bar{\Delta}^1_\epsilon)$ in a canonical way. The vertex set of $T(\bar{\Delta}^1_\epsilon)$ is the union of the set of chambers and the set of panels of $\bar{\Delta}^1_\epsilon$. Given a chamber x and a panel π, then $\{x, \pi\}$ is an edge of $T(\bar{\Delta}^1_\epsilon)$ if and only if $x \in \pi$ and each edge is of this form. Since the type of $\bar{\Delta}^1$ is universal, the graph $T(\bar{\Delta}^1_\epsilon)$ is a tree.

Step 4: *L has no fixed end in the tree $T(\bar{\Delta}^1_\epsilon)$.*

We first record some observations concerning the twin building $\bar{\Delta}^1$. We have already noticed that the group $C_G(A^1)$ acts on $\bar{\Delta}^1$. It follows from the construction of $\bar{\Delta}^1$ that an element of $C_G(A^1)$ fixes the fundamental twin apartment $\bar{\Sigma}$ of $\bar{\Delta}^1$ chamberwise if and only if it fixes Σ chamberwise. In particular, this shows that the chamberwise stabilizer of $\bar{\Sigma}$ in $C_G(A^1)$ is the fundamental torus H, which is abelian. We also deduce from the previous observation and from Proposition 2.5 that we have $m_p(C_G(A^1)/K) = m_p(H) - m_p(A^1) = 1$, where K denotes the kernel of the action of $C_G(A^1)$ on $\bar{\Delta}^1$. Therefore, by Lemma 5.1 we observe that A_L fixes no chamber outside $\bar{\Sigma}$ in $\bar{\Delta}^1$.

Recall that L is isomorphic to $SL_2(\mathbb{K})$ or $PSL_2(\mathbb{K})$. Hence there is an epimorphism $\lambda : SL_2(\mathbb{K}) \to G$ with kernel of order at most 2. We may moreover assume that $\lambda^{-1}(A_L)$ is diagonal in $SL_2(\mathbb{K})$. Set $n_L := \lambda(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$. We have $n_L \in N\backslash H$ and $(n_L)^4 = 1$. Now, since n_L normalizes A_L, it stabilizes $\bar{\Sigma}$ in $\bar{\Delta}^1$. The element n_L has finite order and therefore the fixed point theorem for finite groups acting on buildings implies that it stabilizes a spherical residue in $\bar{\Sigma}$. Since $\bar{\Delta}^1$ has universal type, the only spherical residues are panels or chambers. Therefore, n_L acts on $\bar{\Sigma}$ as a reflection because $\langle \{n_L\} \cup A_L \rangle$ is not abelian. Therefore n_L cannot fix all chambers of $\bar{\Sigma}$.

Assume now that L fixes an end e of $T(\bar{\Delta}^1_\epsilon)$. Since A_L fixes no chamber outside $\bar{\Sigma}$ we deduce that e is an end of $T(\bar{\Sigma}_\epsilon)$. But as n_L acts on $\bar{\Sigma}_\epsilon$ as a reflection, it does not fix any end of $T(\bar{\Sigma}_\epsilon)$ and we have a contradiction. Hence, L fixes no end in $T(\bar{\Delta}^1_\epsilon)$.

Step 5: *End of the proof.*

As the tree $T(\bar{\Delta}^1_\epsilon)$ is simplicial and as algebraically closed fields have no non-trivial discrete valuation, we conclude from Proposition 6.4 and from Step 4 that L fixes a vertex in $T(\bar{\Delta}^1_\epsilon)$.

A solvability argument as in the proof of '(i) $\Rightarrow$ (ii)' above shows that L fixes no chamber of $\bar{\Delta}^1_\epsilon$. This yields that L stabilizes a unique panel $\bar{\pi}_\epsilon$ of $\bar{\Delta}^1_\epsilon$. The group $A_L = A \cap L$ fixes no panel of $\bar{\Delta}^1$ which is not contained in $\bar{\Sigma}$. This follows from an argument as in Step 4 above. Hence, $\bar{\pi}_\epsilon$ is contained in $\bar{\Sigma}_\epsilon$. The panels $\bar{\pi}_+$ and $\bar{\pi}_-$ are opposite. Indeed, if $\bar{\pi}_+$ and $\bar{\pi}_-$ were not opposite, then there would exist $c \in \pi_-$ such that $\ell(\delta^*(x,c)) > \ell(\delta^*(x,y))$ for each $x \in \pi_+$ and each $y \in \pi_-\backslash\{c\}$. Hence L would fix c and thus also $\bar{c}$. We have seen above that this is impossible and therefore $\bar{\pi}_+$ and $\bar{\pi}_-$ are opposite.

Hence, L stabilizes a unique pair of opposite panels of $\bar{\Delta}^1$ and these panels belong to $\bar{\Sigma}$. By the construction of $\bar{\Delta}^1_\epsilon$ this yields a unique pair of opposite roots $\{\alpha, -\alpha\}$ of the fundamental twin apartment Σ such that L stabilizes every panel of $\Delta(G)$ which belongs to the wall $P(s_\alpha)$. Therefore $L \leq P_\alpha$ for some $\alpha \in \Phi$ by Section 3.4. Passing to derived groups, we finally obtain $L \leq L_\alpha$ as expected. This completes the proof. □

7 Proof of Theorem 2.3

Let us write $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ and $\mathcal{Z}' = (G', (U'_\alpha)_{\alpha\in\Phi'})$ and let H and H' be the respective fundamental tori. Let $\xi : G \to G'$ be an isomorphism.

Let p be a prime which is strictly greater than 3 and distinct from the characteristic of $\mathbb{K}_\alpha$ (resp. $\mathbb{K}'_{\alpha'}$) for all $\alpha \in \Phi$ (resp. $\alpha' \in \Phi'$). In view of Section 3.3, we may assume that $\mathcal{Z}$ and $\mathcal{Z}'$ are reduced, or in other words that G and G' are center-free. By Proposition 2.5 the p-ranks of G and G' are both finite. We have $m_p(G) = m_p(G')$ because G and

G' are isomorphic. Let A be the subgroup of p-torsion elements of H. Proposition 2.5 provides an element x of G' such that $\xi(A) \leq xH'x^{-1}$.

Set $L_\alpha := \langle U_\alpha \cup U_{-\alpha} \rangle$ and $L'_{\alpha'} := \langle U'_{\alpha'} \cup U'_{-\alpha'} \rangle$ for each $\alpha \in \Phi$ and $\alpha' \in \Phi'$. Given $\alpha \in \Phi$, then group $L_\alpha \cap A$ has order p. Therefore, Proposition 6.5 implies that $\xi(L_\alpha) \leq xL'_{\alpha'}x^{-1}$ for a root $\alpha' \in \Phi'$ which is uniquely determined up to a sign. A second application of Proposition 6.5 yields $L_\alpha \leq \xi^{-1}(xL'_{\alpha'}x^{-1}) \leq L_\beta$ for a root $\beta \in \Phi$ which is uniquely determined up to a sign. By the last statement of Proposition 6.5 applied to the inclusion $L_\alpha \leq L_\beta$ we obtain $\alpha = \pm\beta$. This implies finally $\xi(L_\alpha) = xL'_{\alpha'}x^{-1}$.

Let $\mathbb{K}$ be an algebraically closed field. Then the twin root datum of Example 2.1 is up to isomorphism the only twin root datum of rank 1 in which $SL_2(\mathbb{K})$ (resp. $PSL_2(\mathbb{K})$) is involved (see for example Theorem 31 in [22]; see also Theorem 7 of [2] for a more elementary treatment which is independent of the solution of the isomorphism problem for Chevalley groups). It follows that $\mathbb{K}_\alpha \simeq \mathbb{K}'_{\alpha'}$ and that $\mathrm{Ad}(x^{-1}) \circ \xi$ sends the conjugacy class $\mathbf{U}_\alpha$ of U_α in L_α onto the conjugacy class $\mathbf{U}'_{\alpha'}$ of $U'_{\alpha'}$ in $L'_{\alpha'}$, where $\mathrm{Ad}(x^{-1})$ denotes the conjugation by x^{-1}. The group A normalizes L_α and acts on $\mathbf{U}_\alpha$. By Lemma 5.1 the groups U_α and $U_{-\alpha}$ are the only fixed points of A in $\mathbf{U}_\alpha$. Hence $\mathrm{Ad}(x^{-1}) \circ \xi$ maps U_α on a fixed point of A' in $\mathbf{U'}_{\alpha'}$, namely $U'_{\alpha'}$ or $U'_{-\alpha'}$. This finally shows that ξ sends $\{U_\alpha | \alpha \in \Phi\}$ to $\{xU'_{\alpha'}x^{-1} | \alpha' \in \Phi'\}$. Now, the conclusion follows from Theorem 2.2. □

8 Kac-Moody groups

In this section we are interested in Kac-Moody groups in the sense of [25]. Our aim is to prove that Theorem 2.3 implies a factorization result for automorphisms of Kac-Moody groups over algebraically closed fields (see Theorem 2.4). We also discuss the existence of graph automorphisms of Kac-Moody groups over algebraically closed fields (see Proposition 8.3).

8.1 Definition and uniqueness

Let I be a finite set. Recall that a **generalized Cartan matrix** is a matrix $A = (A_{ij})_{i,j \in I}$ with integral coefficients such that $A_{ii} = 2$, $A_{ij} \leq 0$ if $i \neq j$ and $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$. It is called **symmetrizable** if it is the product of an invertible diagonal matrix and a symmetric matrix. A **Kac-Moody root datum** is a system $\mathcal{D} = (I, A, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ where I is a finite set, A is a generalized Cartan matrix indexed by I, Λ is a free $\mathbb{Z}$-module whose $\mathbb{Z}$-dual is denoted by $\Lambda^\vee$ and where the elements c_i of Λ and h_i of $\Lambda^\vee$ satisfy the relation $\langle c_i | h_j \rangle = A_{ji}$ for all $i, j \in I$.

Let us fix a Kac-Moody root datum $\mathcal{D}$. Following [25], a Kac-Moody group is defined as the value on a field of a certain group functor $\mathcal{G}$ called *Tits functor*, which is described by a series of axioms that we recall below. One of these axioms relates $\mathcal{G}$ to the complex Kac-Moody algebra $\mathfrak{g}_A$. We recall that this is the Lie algebra generated by the elements e_i, f_i and $\bar{h}_i$ ($i \in I$) with the following presentation:

$$[\bar{h}_i, e_j] = A_{ij} e_j, \qquad [\bar{h}_i, f_j] = -A_{ij} f_j, \qquad [\bar{h}_i, \bar{h}_j] = 0, \qquad [e_i, f_j] = -\bar{h}_i,$$

$$\text{for } i \neq j, \qquad [e_i, f_j] = 0, \qquad (\mathrm{ad}\ e_i)^{-A_{ij}+1}(e_j) = \mathrm{ad}\ (f_i)^{-A_{ij}+1}(f_j) = 0.$$

Let now $\mathcal{F} = (\mathcal{G}, (\phi_i)_{i \in I}, \eta)$ be a system consisting of a group functor $\mathcal{G}$ on the category of all commutative unitary $\mathbb{Z}$-algebras, a collection $(\varphi_i)_{i \in I}$ of morphisms of functors φ_i :

$SL_2 \to \mathcal{G}$, and a morphism of functors $\eta : \mathcal{T}_\Lambda \to \mathcal{G}$, where $\mathcal{T}_\Lambda$ is the split torus scheme, namely $\mathcal{T}_\Lambda(R) = \mathrm{Hom}_{\mathrm{gr}}(\Lambda, R^\times)$. The group functor $\mathcal{G}$ involved in such a system is called a **Tits functor** if it satisfies the following conditions, where r^{h_i} denotes the element $\lambda \mapsto r^{\langle \lambda, h_i\rangle}$ of $\mathcal{T}_\Lambda$:

(KMG1) if $\mathbb{K}$ is a field, $\mathcal{G}(\mathbb{K})$ is generated by the images of $\varphi_i(\mathbb{K})$ and $\eta(\mathbb{K})$;

(KMG2) for every ring R, the homomorphism $\eta(R) : \mathcal{T}_\Lambda(R) \to \mathcal{G}(R)$ is injective;

(KMG3) for $i \in I$ and $r \in R^\times$, one has $\varphi_i \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} = \eta(r^{h_i})$;

(KMG4) if ι is an injection of a ring R in a field $\mathbb{K}$, then $\mathcal{G}(\iota) : \mathcal{G}(R) \to \mathcal{G}(\mathbb{K})$ is injective;

(KMG5) there is a homomorphism $\mathbf{Ad} : \mathcal{G}(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g}_A)$ whose kernel is contained in $\eta(\mathcal{T}_\Lambda(\mathbb{C}))$, such that, for $c \in \mathbb{C}$,

$$\mathbf{Ad}\left(\varphi_i \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\right) = \exp \mathrm{ad}\, ce_i, \qquad \mathbf{Ad}\left(\varphi_i \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right) = \exp \mathrm{ad}\,(-cf_i),$$

and, for $t \in \mathcal{T}_\Lambda(\mathbb{C})$,

$$\mathbf{Ad}(\eta(t))(e_i) = t(c_i) \cdot e_i, \qquad \mathbf{Ad}(\eta(t))(f_i) = t(-c_i) \cdot f_i.$$

As above, let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum, and let $M(A) = (m_{ij})_{i,j\in I}$ be the Coxeter matrix over I defined as follows: $m_{ii} = 1$ and for $i \neq j$, $m_{ij} = 2, 3, 4, 6$ or ∞ according as the product $A_{ij}A_{ji}$ is equal to $0, 1, 2, 3$ or ≥ 4. Let (W, S) be the corresponding Coxeter system and set $\Phi := \Phi(W, S)$. In Section 3.6 of [25] Tits constructs a group functor $\mathcal{G}_\mathcal{D}$ (still on the category of commutative unitary $\mathbb{Z}$-algebras) and a family of morphisms of functors $\mathfrak{u}_\alpha : \mathfrak{Add} \to \mathcal{G}_\mathcal{D}$ ($\alpha \in \Phi$). This functor $\mathcal{G}_\mathcal{D}$ has the property that for each field $\mathbb{K}$ the system $(\mathcal{G}_\mathcal{D}(\mathbb{K}), (\mathfrak{U}_\alpha(\mathbb{K}))_{\alpha\in\Phi})$ is a $\mathbb{K}$-locally split twin root datum, where $\mathfrak{U}_\alpha(\mathbb{K})$ denotes the image of $\mathfrak{u}_\alpha(\mathbb{K})$. Moreover, there is a canonical morphism of functors $\mathcal{T}_\Lambda \to \mathcal{G}_\mathcal{D}$ such that the fundamental torus of this TRD is the image of $\mathcal{T}_\Lambda(\mathbb{K})$ in $\mathcal{G}_\mathcal{D}(\mathbb{K})$. The functor $\mathcal{G}_\mathcal{D}$ is called the **constructive Tits functor**.

The main result of [25] is that for a fixed Kac-Moody root datum $\mathcal{D}$ the value of any Tits functor on a field $\mathbb{K}$ is isomorphic to the value of the constructive Tits functor on $\mathbb{K}$. Here is the precise statement.

Theorem 8.1. *Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum and let $\mathcal{G}_\mathcal{D}$ be the corresponding constructive Tits functor. Set $\mathfrak{U}_{\pm i}(R) := \mathfrak{U}_{\pm\alpha_i}(R)$ for each simple root α_i. Let $\mathfrak{U}_+(R)$ (resp. $\mathfrak{U}_-(R)$) denote the subgroup of $\mathcal{G}_\mathcal{D}(R)$ generated by all $\mathfrak{U}_\alpha(R)$ for α positive (resp. negative), and let $\mathfrak{u}_+$ (resp. $\mathfrak{u}_-$) be the homomorphism $r \mapsto \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ (resp. $r \mapsto \begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}$) of $\mathfrak{Add}$ in SL_2. If $\mathcal{F} = (\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$ is a system which satisfies the axioms (KMG1) to (KMG5) for the Kac-Moody root datum $\mathcal{D}$, then:*

(i) there exists a unique morphism of group functors $\pi : \mathcal{G}_\mathcal{D} \to \mathcal{G}$ such that the canonical map $\mathcal{T}_\Lambda \to \mathcal{G}_\mathcal{D}$ followed by π coincides with η and that $\pi \circ \mathfrak{u}_{\pm\alpha_i} = \varphi_i \circ \mathfrak{u}_\pm$;

(ii) if $\mathbb{K}$ is a field, then $\pi(\mathbb{K})$ is an isomorphism unless $\varphi_i(SL_2(\mathbb{K}))$ is contained in $\pi(\mathfrak{U}_+(R))$ or in $\pi(\mathfrak{U}_-(R))$ for some i.

Is is mentioned in Section 3.10(b) of [25] that Tits functors do exist, which implies by the previous theorem, that the constructive Tits functor is indeed a Tits functor.

8.2 Sign, diagram, diagonal and field automorphisms

In this section we define several specific types of automorphisms of Kac-Moody groups. The terminology we introduce for these different types of automorphisms is taken over from [22] and [7]. Our definitions are constructive, except for graph automorphisms, which are discussed in the next subsection.

We keep the notation of Section 8.1. Moreover, we define $\varphi_i : SL_2 \to \mathcal{G}_\mathcal{D}$ by

$$\varphi_i \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \mathfrak{u}_{\alpha_i}(k) \qquad \text{and} \qquad \varphi_i \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} = \mathfrak{u}_{-\alpha_i}(k)$$

where k is an element of a field $\mathbb{K}$ and $\mathfrak{u}_\alpha(k)$ abusively denotes $\mathfrak{u}_{\alpha_i}(\mathbb{K})(k)$ (we shall again make this little abuse of notation in the following). Let $\eta : \mathcal{T}_\Lambda \to \mathcal{G}_\mathcal{D}$ be the canonical map. By the remark following Theorem 8.1, we know that the system $(\mathcal{G}_\mathcal{D}, (\varphi_i)_{i\in I}, \eta)$ satisfies the axioms (KMG1) – (KMG5). In particular, there exists a homomorphism $\mathbf{Ad} : \mathcal{G}_\mathcal{D}(\mathbb{C}) \to \mathfrak{g}_A$, such that the relations of (KMG5) are satisfied.

Sign automorphisms are defined as follows. Let τ denote the automorphism of SL_2 which is the conjugation by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For each $i \in I$, let $\widetilde{\varphi}_i := \varphi_i \circ \tau$. Let ι be the automorphism of $\mathcal{T}_\Lambda$ defined by $\iota(t) : \Lambda \to R^\times : \lambda \mapsto t(-\lambda) = (t(\lambda))^{-1}$. Set $\widetilde{\eta} := \eta \circ \iota$. One obtains a system $(\mathcal{G}_\mathcal{D}, (\widetilde{\varphi}_i)_{i\in I}, \widetilde{\eta})$, and it can be checked that it satisfies the axioms (KMG1) – (KMG4) (for (KMG3), the verification requires the use of Relation 3.6(6) of [25]). In order to verify (KMG5) we define a homomorphism $\widetilde{\mathbf{Ad}} : \mathcal{G}_\mathcal{D}(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g}_A)$ by $\widetilde{\mathbf{Ad}} = \omega \circ \mathbf{Ad} \circ \omega^{-1}$, where ω is the Chevalley involution of $\mathfrak{g}_A$ (namely $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$ and $\omega(\bar{h}_i) = -\bar{h}_i$). An easy computation shows that (KMG5) is also satisfied with $\widetilde{\mathbf{Ad}}$ playing the role of $\mathbf{Ad}$. Therefore, Theorem 8.1 insures (notice that, by 3.3, the condition of (ii) in that theorem is satisfied) the existence of a unique morphism of functors $\mathcal{G}_\mathcal{D} \to \mathcal{G}_\mathcal{D}$ (also denoted by ω) such that

$$\omega(R)(\mathfrak{u}_{\alpha_i}(r)) = \mathfrak{u}_{-\alpha_i}(-r), \qquad \omega(R)(\mathfrak{u}_{-\alpha_i}(r)) = \mathfrak{u}_{\alpha_i}(-r) \qquad \text{and} \qquad \omega(R)(\eta(t)) = \eta(\iota(t))$$

for each $i \in I, r \in R, t \in \mathcal{T}_\Lambda(R)$ and each ring R. Moreover, for every field $\mathbb{K}$ the morphism $\omega(\mathbb{K})$ is an automorphism called a **sign automorphism** of $\mathcal{G}_\mathcal{D}(\mathbb{K})$. The identity is considered to be also a sign automorphism.

Diagram automorphisms are defined as follows. *We assume in this paragraph that the lattice $\Lambda^\vee$ is generated by all the h_i's.* Let σ be a permutation of the index set I such that

$$A_{ij} = A_{\sigma i \sigma j} \tag{1}$$

for all $i, j \in I$. For each $i \in I$, let $\widetilde{\varphi}_i := \varphi_{\sigma i}$. By the above assumption, for every ring R, the torus $\mathcal{T}_\Lambda(R)$ is generated by elements of the form r^{h_i} with $r \in R$ and $i \in I$. Therefore, the relations $\widetilde{\eta}(r^{h_i}) = \eta(r^{h_{\sigma i}})$ induce a well defined homomorphism of functors $\widetilde{\eta} : \mathcal{T}_\Lambda \to \mathcal{G}_\mathcal{D}$. Again, one obtains a system $(\mathcal{G}_\mathcal{D}, (\widetilde{\varphi}_i)_{i\in I}, \widetilde{\eta})$ and it is readily verified that it satisfies the axioms (KMG1) – (KMG4). In order to verify (KMG5) we define a homomorphism $\widetilde{\mathbf{Ad}} : \mathcal{G}_\mathcal{D}(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g}_A)$ by $\widetilde{\mathbf{Ad}} = \boldsymbol{\sigma}^{-1} \circ \mathbf{Ad} \circ \boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the automorphism of $\mathfrak{g}_A$ defined by $\boldsymbol{\sigma}(e_i) = e_{\sigma i}$ and $\boldsymbol{\sigma}(f_i) = f_{\sigma i}$ (the fact that $\boldsymbol{\sigma}$ is indeed an automorphism of $\mathfrak{g}_A$

is an easily computable consequence of the condition $A_{ij} = A_{\sigma i \sigma j}$). Again, Theorem 8.1 insures the existence of a unique morphism of functors $\mathcal{G}_{\mathcal{D}} \to \mathcal{G}_{\mathcal{D}}$ (also denoted by $\boldsymbol{\sigma}$) such that

$$\boldsymbol{\sigma}(R)(\mathfrak{u}_{\alpha_i}(r)) = \mathfrak{u}_{\alpha_{\sigma i}}(r), \qquad \boldsymbol{\sigma}(R)(\mathfrak{u}_{-\alpha_i}(r)) = \mathfrak{u}_{-\alpha_{\sigma i}}(r) \qquad \text{and} \qquad \boldsymbol{\sigma}(R)(\eta(r^{h_i})) = \eta(r^{h_{\sigma i}})$$

for each $i \in I$ and $r \in R$. Moreover, for every field $\mathbb{K}$, the morphism $\boldsymbol{\sigma}(\mathbb{K})$ is an automorphism called a **diagram automorphism** of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$.

Remark 8.2. In the next subsection, we introduce *graph automorphisms*. We shall see that any diagram automorphism is a graph automorphism. The converse is not true in general.

Diagonal automorphisms are defined as follows. Let $\widetilde{\Lambda}$ be the $\mathbb{Z}$-lattice $\widetilde{\Lambda} = \bigoplus_{i \in I} \mathbb{Z}\widetilde{c}_i$. There is a canonical homomorphism $\widetilde{\Lambda} \to \Lambda : \widetilde{c}_i \mapsto c_i$, which induces a morphism of split tori $\mathcal{T}_\Lambda \to \mathcal{T}_{\widetilde{\Lambda}} := \mathrm{Hom}_{\mathrm{gr}}(\widetilde{\Lambda}, R^\times)$. For each $\overline{h} \in \mathcal{T}_{\widetilde{\Lambda}}$, we define an automorphism of $\mathcal{G}_{\mathcal{D}}(R)$, again denoted by $\overline{h}$, called a **diagonal automorphism**, and induced by the relations

$$\overline{h}|_H = \mathrm{id}_H \qquad \text{and} \qquad \overline{h}(\mathfrak{u}_\alpha(r)) = \mathfrak{u}_\alpha(\overline{h}(a_\alpha) r)$$

where α belongs to $\Phi(W, S)$ (see Section 8.1 for the definition of (W, S)) and a_α is the corresponding element of $\widetilde{\Lambda}$. We recall the definition of a_α. It is well known that the group W defined as in Section 8.1 acts on $\widetilde{\Lambda}$. This action is determined by the equations $s_i(c_j) = c_j - A_{ij} \cdot c_i$ where $i \in I$. If α is a root of (W, S), then $\alpha = w(\alpha_{s_i})$ for some $w \in W$ and $i \in I$. Thus $a_\alpha := w(c_i)$ is an element of $\widetilde{\Lambda}$ which depends only on α. Now, it can be verified that relations above indeed induce a well defined automorphism of $\mathcal{G}_{\mathcal{D}}(R)$ (using for example the explicit definition of the constructive Tits functor).

Ring automorphisms are defined as follows. Since $\mathcal{G}_{\mathcal{D}}$ is a functor, it is clear that to any automorphism f of R corresponds a well defined automorphism $\mathcal{G}_{\mathcal{D}}(f)$ of $\mathcal{G}_{\mathcal{D}}(R)$, called a **ring automorphism** (or a **field automorphism** according as R is a field). More explicitly we have $\mathcal{G}_{\mathcal{D}}(f)(t) = f \circ t$ for each $t \in \mathcal{T}_\Lambda(R)$ (which is identified with its image in $\mathcal{G}_{\mathcal{D}}(R)$) and $\mathcal{G}_{\mathcal{D}}(f)(\mathfrak{u}_\alpha(r)) = \mathfrak{u}_\alpha(f(r))$ for $\alpha \in \Phi$ and $r \in R$.

8.3 Graph automorphisms

It is well known that Chevalley groups of type B_2 and F_4 over a field of characteristic 2 admit graph automorphisms (see Proposition 12.3.3 in [6]) and these are not diagram automorphisms in the above sense. These automorphisms are exceptional in the sense that they exist only over a field of characteristic 2. In this subsection, we define graph automorphisms for arbitrary Kac-Moody groups. The question of the existence of these graph automorphisms is still open. We give necessary conditions for their existence (see Proposition 8.3); we do not know whether or not these conditions are sufficient. For graph automorphisms satisfying a certain restriction (see the remark below), similar conditions have been proved to be necessary and sufficient by A. Chosson [8] (see Theorem 8.5 below); another existence result had been obtained previously by J.-Y. Hée [11].

We keep the notation of Section 8.1, and denote by Π the set of simple roots of Φ. An automorphism θ of $\mathcal{G}_{\mathcal{D}}(R)$ is called a **graph automorphism** if there exists a permutation σ of Π such that

$$\theta(\mathfrak{U}_\alpha(R)) = \mathfrak{U}_{\sigma\alpha}(R) \qquad \text{and} \qquad \theta(\mathfrak{u}_\alpha(1)) = \mathfrak{u}_{\sigma\alpha}(1)$$

for every $\alpha \in \Pi$. In that situation we say that the permutation σ of Π **can be lifted** to a graph automorphism of $\mathcal{G}_{\mathcal{D}}(R)$.

The terminology of 'diagram' and 'graph' automorphisms is inspired by the distinction which is sometimes made between a Dynkin diagram (with arrows on multiple bonds) and its underlying Coxeter graph.

Proposition 8.3. *Let Π be a set indexing a generalized Cartan matrix A and let σ be a permutation of Π. Let G be a Kac-Moody group over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq 0$ whose underlying generalized Cartan matrix is A. If σ can be lifted to a graph automorphism of G, then the following conditions are satisfied for all $\alpha, \beta \in \Pi$:*

(i) $A_{\alpha\beta}A_{\beta\alpha} = A_{\sigma\alpha,\sigma\beta}A_{\sigma\beta,\sigma\alpha}$;

(ii) *if $A_{\alpha\beta} \neq 0$, then the number*

$$\frac{A_{\sigma\alpha,\sigma\beta}}{A_{\alpha\beta}}$$

either equals 1 if $p = 0$, or is an integral (possibly negative) power of p if $p > 0$;

(iii) *if $(\alpha_0, \alpha_1, \dots, \alpha_n = \alpha_0)$ is a loop in the Coxeter diagram $M(A)$ (see Section 8.1) then*

$$\prod_{i=1}^{n} \frac{A_{\sigma\alpha_{i-1},\sigma\alpha_i}}{A_{\alpha_{i-1}\alpha_i}} = 1.$$

Assume now that A is indecomposable and that σ can indeed be lifted to an automorphism θ of G (hence Conditions (i), (ii) and (iii) are satisfied). Fix an element $\alpha_0 \in \Pi$, and for each $\beta \in \Pi$, choose a path $(\alpha_0, \alpha_1, \dots, \alpha_n = \beta)$ from α_0 to β in the Coxeter graph associated to A. Let us define the number

$$c_\beta := \prod_{i=1}^{n} \frac{A_{\sigma\alpha_{i-1},\sigma\alpha_i}}{A_{\alpha_{i-1}\alpha_i}}$$

(by (iii), the number c_β is independent of the chosen path); by convention, we also put $c_{\alpha_0} := 1$. Then there exists a field automorphism f of G such that for $\theta' := \theta \circ f$, we have

$$\theta'(\mathfrak{u}_\beta(k)) = \mathfrak{u}_{\sigma\beta}(k^{c_\beta})$$

for each $\beta \in \Pi$.

This proposition is a consequence of the arguments used in the proof of Theorem 2.4, given below.

Remark 8.4. If the restriction of σ to every loop of the Coxeter graph $M(A)$ is a reflection preserving that loop, then Condition (iii) in the previous proposition is equivalent to the requirement that the generalized Cartan matrix A is symmetrizable. This follows from Exercise 2.1 on page 27 in [13].

As said in the introduction, Theorem 2.4 can be used to determine the automorphism group of a Kac-Moody over an algebraically closed field of characteristic 0. We also mentioned that a result from A. Chosson's doctoral thesis allows to determine the automorphism group in further cases (including the affine). For the sake of convenience, we reproduce this result here.

Theorem 8.5. (A. Chosson) *Let $(A_{ij})_{i,j} \in I$ be a generalized Cartan matrix and let $\mathcal{D} = (I, A, \bigoplus_{i\in I} \mathbb{Z}e_i, (c_i)_{i\in I}, (h_i)_{i\in I})$ be the Kac-Moody root datum defined by $c_i := \sum_{j\in I} A_{ji}e_j$ and $\langle e_i|h_j\rangle) = \delta_{ij}$ for all $i, j \in I$ (this is called the **simply connected** root datum). Let (W, S) be the associated Coxeter system (see Section 8.1). Let $\mathbb{K}$ be a field of characteristic $p \geq 0$ and let $(f_i)_{i\in I}$ be a collection of automorphisms of $\mathbb{K}$. Let σ be a permutation of I.*

Assume that $A_{ij} = A_{\sigma j,\sigma i}$ for all $i, j \in I$. Then σ can be lifted to a graph automorphism θ of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ such that $\theta(\mathfrak{u}_i(k)) = \mathfrak{u}_{\sigma i}(f_i(k))$ for all $i \in I$ and $k \in \mathbb{K}$ if and only if the following conditions are satisfied for all $i, j \in I$:

(1) if $p \neq 2$, then A_{ij} and A_{ji} have the same parity;

(2) if $A_{ij} = -1$, then either $A_{ji} = -1$ or $A_{ji} \equiv 0 \mathrm{mod} p$;

(3) for all $k \in K$ we have $f_i(k^{A_{ji}}) = f_j(k^{A_{ij}})$;

(4) if e_i and e_j are W-conjugate, then $\sigma_i = \sigma_j$ (the W-action on $\bigoplus_{i\in I} \mathbb{Z}e_i$ is defined as in the paragraph on diagonal automorphisms in the previous section).

Proof. This is Theorem 1.1 of Chapter 7 in [8]. □

8.4 Proof of Theorem 2.4

The main ingredient is of course Theorem 2.3. Let $\mathcal{Z} := (G, (U_\alpha)_{\alpha\in\Phi})$ be the natural twin root datum involving G. Here, $\Phi = \Phi(W, S)$ where (W, S) is the Coxeter system of type $M(A)$, defined as in Section 8.1. The twin root datum $\mathcal{Z}$ is thus locally split over the algebraically closed field $\mathbb{K}$. Let H be its fundamental torus and let U_+ (resp. U_-) be the group generated by all U_α with α positive (resp. $-\alpha$ positive).

Let us fix an automorphism ξ of G. Then Theorem 2.3 insures the existence of an inner automorphism ι of G such that $\xi_1 := \iota^{-1} \circ \xi$ leaves H invariant and maps U_+ and U_- respectively onto either U_+ and U_- or U_- and U_+. In both cases, we may compose ξ_1 with an appropriate sign automorphism ω in such a way that $\xi_2 = \omega \circ \xi_1$ stabilizes U_+ and U_-. Now, the automorphism of the group W induced by ξ_2 must leave S invariant. In particular, it induces a permutation σ of S according to the rule

$$\xi_2(\alpha_s) = \alpha_{\sigma s}$$

(we shall also write σ for the corresponding permutation of the simple roots) and the corresponding root groups are permuted accordingly.

It is now clear that we may choose a diagonal automorphism d in such a way that $\xi_3(\mathfrak{u}_\alpha(1)) = \mathfrak{u}_{\sigma\alpha}(1)$ for each simple root α, where $\xi_3 = d^{-1} \circ \xi_2$. Hence ξ_3 is a graph automorphism of G, and $\xi = \iota\omega d\xi_3$ as expected.

Now, we assume that either $\mathrm{char}(\mathbb{K}) = 0$ or every off-diagonal entry of A is prime to $\mathrm{char}(\mathbb{K})$.

Let us fix two simple roots α and β. There exist automorphisms f_α and f_β of $\mathbb{K}$ such that $\xi_3(\mathfrak{u}_\gamma(k)) = \mathfrak{u}_{\sigma\gamma}(f_\gamma(k))$ for all $k \in \mathbb{K}$, where $\gamma \in \{\alpha, \beta\}$. This follows from computations as in the proof of the assertion (4) on page 161 in [22]. Let us now consider the following special case of the relation 3.6(4) in [25]:

$$k^{h_\alpha}\mathfrak{u}_\beta(1)(k^{h_\alpha})^{-1} = \mathfrak{u}_\beta(k^{A_{\alpha\beta}}).$$

Transforming by ξ_3 and using the fact that $\xi_3(k^{h_\alpha}) = (f_\alpha k)^{h_{\sigma\alpha}}$, we obtain the following equality:

$$\mathfrak{u}_{\sigma\beta}\left((f_\alpha k)^{A_{\sigma\alpha\sigma\beta}}\right) = \mathfrak{u}_{\sigma\beta}\left((f_\beta k)^{A_{\alpha\beta}}\right),$$

which implies, since $\mathfrak{u}_{\sigma\beta}$ is bijective, that

$$(f_\beta^{-1} f_\alpha) k^{A_{\sigma\alpha\sigma\beta}} = k^{A_{\alpha\beta}}.$$

Now, the number of solutions of the polynomial equation $X^{|A_{\sigma\alpha\sigma\beta}|} = 1$ must be equal to the number of solutions of the polynomial equation $X^{|A_{\alpha\beta}|} = 1$. Since the polynomial $X^n - 1$ has exactly n distinct roots in $\mathbb{K}$ whenever n is prime to char($\mathbb{K}$), we see that the hypothesis on the characteristic of $\mathbb{K}$ implies that

$$A_{\sigma\alpha\sigma\beta} = A_{\alpha\beta}.$$

Now, unless $A_{\alpha\beta} = 0$, for any $k' \in \mathbb{K}^\times$, we may choose a $k \in \mathbb{K}^\times$ such that $k^{A_{\alpha\beta}} = k'$, and this implies that $f_\beta^{-1} f_\alpha$ is the identity on $\mathbb{K}$. In other words, the automorphisms f_α and f_β coincide. Since this holds for any pair of distinct simple roots, and since the generalized Cartan matrix A is supposed to be indecomposable, we deduce that the automorphisms f_γ all coincide. Finally, it follows from the previous discussion that we can choose a diagram automorphism g and a field automorphism f in such a way that $f^{-1} \circ g^{-1} \circ \xi_3$ is the identity on G. In other words, we have $\xi = \iota\omega dgf$. This concludes the proof of Theorem 2.4. $\square$

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