Isomorphisms of Kac-Moody groups which preserve bounded subgroups

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Abstract. A subgroup of a Kac-Moody group is called **bounded** if it is contained in the intersection of two finite type parabolic subgroups of opposite signs. In this paper, we study the isomorphisms between Kac-Moody groups over arbitrary fields of cardinality at least 4, which preserve the set of bounded subgroups. We show that such an isomorphism between two such Kac-Moody groups induces an isomorphism between the respective twin root data of these groups. As a consequence, we obtain the solution of the isomorphism problem for Kac-Moody groups over finite fields of cardinality at least 4.

1 Introduction

Kac-Moody groups are infinite-dimensional generalizations of Chevalley groups. It is known that each automorphism of a Chevalley group (of irreducible type and over a perfect field) can be written as a product of an inner, a diagonal, a graph and a field automorphism (see Theorem 30 in [14]). In [6] it was conjectured that the same statement holds for Kac-Moody groups over algebraically closed fields of characteristic 0 up to the addition of a so called sign automorphism. In [4] this conjecture is shown to be true for Kac-Moody groups over algebraically closed fields of any characteristic. This is achieved in loc. cit. by solving the isomorphism problem for those groups. In this paper, we study the isomorphism problem for Kac-Moody groups over arbitrary fields of cardinality at

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least 4. We restrict our attention to isomorphisms which preserve the set of bounded subgroups. In this context, a subgroup of a Kac-Moody group is called **bounded** if it is contained in the intersection of two finite type parabolic subgroups of opposite signs.

Throughout the paper we use Tits' definition for Kac-Moody groups over fields [15]. This definition does not only provide the abstract Kac-Moody group G but also a canonical system $(U_\alpha)_{\alpha\in\Phi}$ of root subgroups. The pair $(G,(U_\alpha)_{\alpha\in\Phi})$ is an example of a so called twin root datum. Twin root data have been introduced by Tits in order to give suitable axioms for these pairs arising from his definition of Kac-Moody groups.

Our main result says that if two Kac-Moody groups are isomorphic via such an isomorphism, then the groups are of the same type and defined over the same ground field. Here is a precise statement.

Theorem. *Let $\mathcal{D} = (G,(U_\alpha)_{\alpha\in\Phi})$ and $\mathcal{D}' = (G',(U'_{\alpha'})_{\alpha'\in\Phi'})$ be two twin root data associated with two Kac-Moody groups of non-spherical type over fields of cardinality at least 4. Let $\xi : G \to G'$ be a group isomorphism which maps bounded subgroups of G to bounded subgroups of G'. Then ξ induces an isomorphism of $\mathcal{D}$ to $\mathcal{D}'$.*

We refer to Section 2.3.2 below for the definition of an isomorphism between twin root data. Roughly speaking, it means that $(\xi(U_\alpha))_{\alpha\in\Phi}$ is 'nearly G'-conjugate' to $(U'_{\alpha'})_{\alpha'\in\Phi'}$.

As it is the case in the paper [4], the present work makes crucial use of the theory of twin buildings. A group endowed with a twin root datum is indeed naturally endowed with a strongly transitive action on a twin building, and the combinatorial properties of this action turn out to be the most appropriate tool in studying Kac-Moody groups from our point of view. However, we have tried to explain each crucial building-theoretic statement in more classical terms, without making reference to the language of buildings. We hope this will help the reader who is not familiar to the theory of buildings to understand the main ideas of this paper.

As a consequence of the theorem above, we obtain the following result on automorphisms of Kac-Moody groups.

Corollary A. *Let G be a Kac-Moody group over a field of cardinality at least 4. Let φ be an automorphism of G which preserves the set of bounded subgroups. Then φ splits as a product of an inner, a diagonal, a graph, a field and a sign automorphism.*

There are mainly two motivations to consider isomorphisms which preserve bounded subgroups.

The first motivation comes from the earlier work [7] by Kac and Wang. In this paper, automorphisms of Kac-Moody groups over fields of characteristic 0 and associated with symmetrizable Cartan matrices have been studied. One of the main results of loc. cit. is that, given such a Kac-Moody group G and its Kac-Moody algebra $\mathfrak{g}$, then an automorphism of G which preserves the set of $\mathrm{Ad}_{\mathfrak{g}'}$-finite elements splits as a product as in Corollary A above, where $\mathfrak{g}' := [\mathfrak{g},\mathfrak{g}]$. We recall that an element $g \in G$ is $\mathrm{Ad}_{\mathfrak{g}'}$-finite if and only if the subgroup generated by g is bounded (see [7], Theorem 2.10). Thus, Corollary A can be seen as a weaker version of Kac-Wang's result, which remains valid for Kac-Moody groups of arbitrary type and over fields of arbitrary characteristic.

The second motivation is the fact that, in the case of a Kac-Moody group over a finite field, a subgroup is bounded if and only if it is finite (see Corollary 3.8 below). Therefore, all isomorphisms preserve bounded subgroups in this case. Consequently, we obtain the following result.

Corollary B. *Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ and $\mathcal{D}' = (G', (U'_{\alpha'})_{\alpha'\in\Phi'})$ be two twin root data associated with two Kac-Moody groups over finite fields of cardinality at least 4 and let $\xi : G \to G'$ be an isomorphism. Then ξ induces an isomorphism of $\mathcal{D}$ to $\mathcal{D}'$ unless G and G' are both finite. Moreover, any automorphism of G splits as a product of an inner, a diagonal, a graph, a field and a sign automorphism.*

Kac-Moody groups over finite fields are finitely generated and some subclasses of them are known to be finitely presented. In the recent years these groups became important in geometric group theory for several reasons (see [10]). In this context, B. Rémy proved a factorization theorem for the automorphisms of certain Kac-Moody groups (see [9], Theorem 3.1). Corollary 2 covers this result as a special case.

Let us also mention the existence of exotic constructions of groups of Kac-Moody type, which are not Kac-Moody groups in the strict sense but which are also endowed with a twin root datum. For example, Rémy and Ronan [11] constructed examples of groups of Kac-Moody type defined simultaneously over different ground fields. It turns out that, provided the maximal tori are locally large enough, our methods extend also to these exotic cases, and the interested reader will have no difficulty to extend our arguments to this slightly more general situation (see also the introduction of [4] for other remarks and results related to the isomorphism problem of exotic groups of Kac-Moody type).

The paper is organized as follows. After a preliminary section where definitions are recalled, notation is fixed and some auxiliary results are proven, we discuss in Section 3 the Levi decomposition of the intersection of two parabolic subgroups of opposite sign in a group endowed with a twin root datum (a similar but slightly less general discussion had been done in [8]). The key result of this paper is contained in Section 4, where we prove that the maximal bounded subgroups coincide almost always with the Levi factors of the maximal parabolic subgroups of finite type. In the next section, we use this key result to state and prove a technical version of our main result, which is valid for a larger class of groups endowed with a twin root datum. Finally, the last two sections are devoted to the proof of the main theorem above and its corollaries.

2 Preliminaries

The main references are [1], [8], [12] and [17].

We start by fixing a general convention :

The ordered pair (W, S) is a Coxeter system and ℓ denotes the corresponding length function. For $J \subseteq S$ we set $W_J := \langle J \rangle$ and we call J ***spherical*** *whenever W_J is finite.*

2.1 Buildings

2.1.1 Definition

A **building** of type (W, S) is a set Δ, whose elements are called **chambers**, endowed with a map $\delta\ : \Delta \times \Delta \to W$ called the **W-distance** satisfying the following axioms, where $x, y \in \Delta$ and $w = \delta(x, y)$:

(Bu1) $w = 1 \Leftrightarrow x = y$;
(Bu2) if $z \in \Delta$ is such that $\delta(y, z) = s \in S,$ then $\delta(x, z) \in \{w, ws\}$;
if, furthermore, $\ell(ws) = \ell(w) + 1,$ then $\delta(x, z) = ws$;
(Bu3) if $s \in S,$ there exists $z \in \Delta$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

For any two chambers $x, y \in \Delta$, the natural number $\ell(\delta(x, y))$ is called the **numerical distance** between x and y.

An **isometry** between subsets of buildings of type (W, S) is a bijection preserving the W-distance.

2.1.2 Apartments

Given a Coxeter system (W, S), let $\delta : W \times W \to W$ be defined by $\delta : (x, y) \mapsto x^{-1}y$. In this way, we endow W with a canonical structure of a building of type (W, S). This building is denoted by $\mathcal{A}(W, S)$.

Any subset of a building (Δ, δ) of type (W, S) which is isometric to the canonical building $\mathcal{A}(W, S)$ is called an **apartment**. A fundamental property of buildings is that any two chambers lie in a common apartment (see [12], Theorem 3.7).

2.1.3 Panels, residues and galleries

Given $c \in \Delta$ and $s \in S$, then the set $\{x \in \Delta | \delta(x, c) \in \{1, s\}\}$ is called an s**-panel** of Δ or a **panel of type** s. A **panel** is an s-panel for some $s \in S$. More generally, for $c \in \Delta$ and $J \subseteq S$ the set

$$\mathrm{Res}_J(c) := \{x \in \Delta | \delta(x, c) \in W_J\}$$

is called the J**-residue** of Δ which contains c. Its **rank** is the cardinality of the set J; hence, residues of rank 0 are just chambers and the residues of rank 1 are panels.

It is an important fact that a J-residue is itself a building of type (W_J, J) with the W_J-distance induced by δ (see [12], Theorem 3.5). Moreover, given a residue R and an apartment Σ in a building Δ, the intersection $R \cap \Sigma$ is either empty or an apartment of R, and all apartments of R arise in this way. It is common and handy to say that R *is contained in* Σ whenever $R \cap \Sigma$ is nonempty.

A sequence of chambers such that two consecutive chambers are **adjacent**, namely contained in a common panel, is called a **gallery**. The gallery $\gamma = (x_0, x_1, \ldots, x_n)$ is called **minimal** if $n = \ell(\delta(x_0, x_n))$.

A building is called **thin** (resp. **thick**) if all of its panels have cardinality 2 (resp. at least 3). Any thin building of type (W, S) is isomorphic to the canonical building (W, S).

2.1.4 Projections and convexity in arbitrary buildings

A fundamental property of buildings, besides the existence of apartments, is the exitence of projections onto residues. We review here the main properties of projections in arbitratry buildings. The notion of a projection can be slightly refined in the case of twin buildings; we will come back to this refinement in Section 2.2.3 below.

Let (W, S) be a Coxeter system. We recall from [3] that if $J, K \subseteq S$ and $w \in W$, then there is a unique element of minimal length in the double coset $W_J w W_K$.

Let (Δ, δ) be a building of type (W, S) and let R_J, R_K be residues of Δ of respective type J and K. Then the set of all $\delta(c,d)$ for $c \in R_J$ and $d \in R_K$ is a double coset $W_J w W_K$. Its minimal element is denoted by $\delta(R_J, R_K)$.

The set

$$\mathrm{proj}_{R_J}(R_K) := \{c \in R_J | \exists d \in R_K \text{ such that } \delta(c,d) = \delta(R_J, R_K)\}$$

is called the **projection of R_K on R_J**. It is a residue of type $J \cap wKw^{-1}$, where $w := \delta(R_J, R_K)$; in particular, it is a spherical residue whenever J or K is spherical. Moreover, we have

$$\mathrm{proj}_{R_J}(R_K) = \{\mathrm{proj}_{R_J}(c) | c \in R_K\},$$

where we have written $\mathrm{proj}_{R_J}(c)$ for $\mathrm{proj}_{R_J}(\{c\})$.

If c is a chamber and R a residue, then $\mathrm{proj}_R(c)$ is a gate of c to R. This means that for any $x \in R$ there exists a minimal gallery joining c to x via $\mathrm{proj}_R(c)$. The chamber $\mathrm{proj}_R(c)$ is the unique chamber of R at minimal numerical distance from c.

A set $\mathcal{X}$ of chambers in a building Δ is called **convex** if the following property holds: *given chambers $x, x' \in \mathcal{X}$ and a spherical residue R containing x, then* $\mathrm{proj}_R(x') \in \mathcal{X}$. For example, apartments and residues are convex sets of chambers.

2.1.5 Spherical residues and spherical buildings

A building (Δ, δ) of type (W, S) is called **spherical** if W is finite. In that case, there exists a unique element w_0 of maximal length in W. Two chambers $x, y \in \Delta$ are called **opposite** if $\delta(x,y) = w_0$. Two residues R_J and R_K of Δ of type J and K respectively are called **opposite** if they contain opposite chambers and if $J = w_0 K w_0^{-1}$.

A residue R of type J in a building of arbitrary type (W, S) is called **spherical** if J is a spherical subset of S. Thus R is a spherical building and it makes sense to talk about opposite chambers and opposite residues of R.

The following lemma is a useful criterion of sphericity in terms of projections.

Lemma 2.1. *Let (Δ, δ) be a building of type (W, S), let $J \subseteq S$ and let R be a J-residue. Then J is spherical if and only if there exist $x, y \in R$ such that for every $j \in J$, we have* $\mathrm{proj}_{\pi_j}(x) \neq y$ *where π_j denotes the j-panel containing y.*

Proof. This follows from [12], Theorem (2.16). □

We end this subsection by recalling a celebrated fixed point theorem for finite groups acting on buildings.

Proposition 2.2. *Any finite group acting on a building of type (W, S), where S is finite, stabilizes a residue of spherical type.*

Proof. See [5], Corollary 11.9. □

2.2 Twin buildings

2.2.1 Definition

A **twinned pair of buildings** or **twin building** of type (W, S) is a pair $((\Delta_+, \delta_+), (\Delta_-, \delta_-))$ of buildings of type (W, S), endowed with a **W-codistance**

$$\delta^* \ : (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \to W$$

satisfying the following axioms, where $\epsilon \in \{+,-\}$, $x \in \Delta_\epsilon$, $y \in \Delta_{-\epsilon}$ and $w = \delta^*(x,y)$:

(Tw1) $\delta^*(y,x) = w^{-1}$;
(Tw2) if $z \in \Delta_{-\epsilon}$ is such that $\delta_{-\epsilon}(y,z) = s \in S$ and $\ell(ws) < \ell(w)$, then $\delta^*(x,z) = ws$;
(Tw3) if $s \in S$, there exists $z \in \Delta_{-\epsilon}$ such that $\delta_{-\epsilon}(y,z) = s$ and $\delta^*(x,z) = ws$.

In the sequel, we will often use the symbol Δ_+ to denote the building (Δ_+, δ_+) as well as its set of chambers and similarly for Δ_-. The meaning will be clear from the context.

A residue R of the twin building $\Delta = (\Delta_+, \Delta_-, \delta^*)$ is a residue of Δ_ϵ, for $\epsilon = +$ or $-$ and ϵ is called the **sign** of R. Two chambers x and y of opposite signs are called **opposite** if $\delta^*(x,y) = 1$. Two residues are called **opposite** if they are of the same type and contain opposite chambers. Given $J \subseteq S$, then a pair of opposite residues of type J endowed with the W-codistance induced from δ^* is itself a twin building of type (W_J, J).

Notice that we have defined the term *opposite* at two different places, namely in Section 2.1.5 above and here in Section 2.2.1. However, this terminology is standard and coherent. Indeed, the former notion applies to chambers or residues of the same sign and lying in a common spherical residue, while the latter applies to chambers or residues of opposite signs.

An **automorphism** of $\Delta = (\Delta_+, \Delta_-, \delta^*)$ is by definition a pair $\varphi = (\varphi_+, \varphi_-)$ of permutations of Δ_+ and Δ_- respectively preserving the W-distances δ_+, δ_- as well as the W-codistance δ^*. **Isomorphisms** of twin buildings are defined similarly. We recall from [16], Theorem 1, that if Δ_+ and Δ_- are thick, then an automorphism of Δ which fixes a pair of opposite chambers c, c' and all chambers adjacent to c is the identity.

2.2.2 Reflections and twin apartments

Let (Σ_+, δ_+) and (Σ_-, δ_-) be two copies of the canonical building $\mathcal{A}(W,S)$ of type (W,S) (see Section 2.1.2). Let $\delta^* : \Sigma_+ \times \Sigma_- \to W : (x,y) \mapsto x^{-1}y$; this makes sense since $\Sigma_+ = \Sigma_- = W$. Then $\Sigma(W,S) := ((\Sigma_+, \delta), (\Sigma_-, \delta), \delta^*)$ is a thin twin building of type (W,S), namely a twinned pair of thin buildings. It is the unique thin twin building of that type up to isomorphism.

The group W has a faithful action on $\Sigma(W,S)$ by automorphisms which is given by left multiplication on Σ_+ and Σ_-. Every automorphism of Σ is of this form.

A **reflection** is a non-trivial element of W which stabilizes a panel of $\Sigma(W,S)$. Conversely, to any panel of $\Sigma(W,S)$ corresponds a unique reflection of W which stabilizes it. Moreover, an element of W is a reflection if and only if it is conjugate to an element of S.

Let $\Delta = (\Delta_+, \Delta_-)$ be a twin buildings of type (W,S). A pair $\Sigma = (\Sigma_+, \Sigma_-)$ of subsets of Δ is called a **twin apartment** if it is isomorphic to the canonical twin building $\Sigma(W,S)$. Given a twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$, the restriction of the opposition relation of Δ to Σ is a one-one correspondence $\Sigma_+ \leftrightarrow \Sigma_-$. (It corresponds to the identity $\mathrm{id} : W \to W$ in the canonical twin building $\Sigma(W,S)$.) We denote it by op_Σ.

It is a fundamental fact that, given any two chambers $x \in \Delta_\epsilon$ and $y \in \Delta_{\epsilon'}$ in a twin building $\Delta = (\Delta_+, \Delta_-)$ of type (W,S), where $\epsilon, \epsilon' \in \{+,-\}$, there exists a twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$ such that $x \in \Sigma_\epsilon$ and $y \in \Sigma_{\epsilon'}$ (see [1], Lemma 2). It is common and handy to say that x and y are contained in Σ, and to write $x, y \in \Sigma$.

2.2.3 Projections and convexity in twin buildings

Let (W, S) be a Coxeter system and let $J, J \subset S$ be spherical subsets. Then there is a unique element of maximal length in $W_J w W_K$ ([1], Lemma 9).

Let $\Delta = (\Delta_+, \Delta_-, \delta^*)$ be a twin building of type (W, S) and let R_J, R_K be residues of Δ of respective type J and K. Assume moreover that J and K are spherical and that R_J and R_K have opposite signs. Then the set of all $\delta^*(c, d)$ for $c \in R_J$ and $d \in R_K$ is a double coset $W_J w W_K$. Its maximal element is denoted by $\delta^*(R_J, R_K)$. The set

$$\mathrm{proj}_{R_J}(R_K) := \{c \in R_J | \exists d \in R_K \text{ such that } \delta^*(c, d) = \delta^*(R_J, R_K)\}$$

is called the **projection of R_K on R_J**. It is a residue of type $w_J^0(J \cap wKw^{-1})w_J^0$, where $w := \delta^*(R_J, R_K)$ and w_J^0 denotes the maximal element of W_K ([1], Lemma 10). Moreover, we have

$$\mathrm{proj}_{R_J}(R_K) = \{\mathrm{proj}_{R_J}(c) | c \in R_K\}.$$

A set $\mathcal{X}$ of chambers in a twin building is called **convex** if the following condition holds: *given $x, x' \in \mathcal{X}$ and a spherical residue R containing x', then* $\mathrm{proj}_R(x) \in \mathcal{X}$. For example, twin apartments are convex sets of chambers in twin buildings. Actually, the convex hull of any pair of opposite chambers is a twin apartment containing them. This implies that two opposite chambers lie in a unique common twin apartment.

Notice that we have defined the terms *projections* and *convexity* at two different places, namely in Section 2.1.4 above and here in Section 2.2.3. The point is that the notion of projections in twin buildings is a generalization of the standard notion of projections in arbitrary buildings. There will be no confusion between both. Indeed, the meaning of the symbol $\mathrm{proj}_R(x)$ in the context of twin buildings depends on the respective signs of the residue R and the chamber x.

We end this subsection with a result is often useful to compute projections between residues of opposite signs using twin apartments.

Lemma 2.3. *Let $\Delta = (\Delta_+, \Delta_-, \delta^*)$ be a twin building and for each sign ϵ let R_ϵ be a spherical residue of Δ_ϵ. Let Σ is a twin apartment containing R_+ and R_- and let $\epsilon \in \{+, -\}$. Let R'_ϵ be the residue of Σ opposite $R_{-\epsilon}$. Then the residues* $\mathrm{proj}_{R_\epsilon}(R_{-\epsilon})$ *and* $\mathrm{proj}_{R_\epsilon}(R'_\epsilon)$ *are contained in Σ and opposite in R_ϵ (see Section 2.1.5).*

Proof. This is Proposition 4 in [1]. □

In brief, the statement of this lemma may be written as

$$\mathrm{proj}_{R_\epsilon}(R_{-\epsilon}) = \mathrm{op}_{\Sigma \cap R_\epsilon}(\mathrm{proj}_{R_\epsilon}(\mathrm{op}_\Sigma(R_{-\epsilon}))).$$

2.2.4 Parallelism

Let $\Delta = (\Delta_+, \Delta_-, \delta^*)$ be a twin building of type (W, S). Two residues R_J, R_K of Δ (assumed to be spherical if they have opposite signs) are called **parallel** if $\mathrm{proj}_{R_J}(R_K) = R_J$ and $\mathrm{proj}_{R_K}(R_J) = R_K$.

It follows from the definitions that $\mathrm{proj}_{R_J}(R_K)$ and $\mathrm{proj}_{R_K}(R_J)$ are always parallel.

Although parallel residues need not have the same type, they are nevertheless always 'almost isometric' in the following sense.

Lemma 2.4. *Let R_J (resp. R_K) be a residue of spherical type J (resp. K) and sign ϵ_J (resp. ϵ_K). Assume that R_J and R_K are parallel. Then there exists an isomorphism $\eta : W_J \to W_K$ with $\eta(J) = K$ such that*

$$\delta_{\epsilon_J}(\mathrm{proj}_{R_J}(x), \mathrm{proj}_{R_J}(y)) = \eta(\delta_{\epsilon_K}(x, y))$$

for all $x, y \in R_K$. In particular, if x and y are opposite in R_K, then so are $\mathrm{proj}_{R_J}(x)$ and $\mathrm{proj}_{R_J}(y)$ in R_J.

The proof of Lemma 2.4 is in the same spirit as the proof of Proposition 5.15 in [18] and is omitted here. We only mention that the isomorphism η of the lemma is actually induced by the conjugation by $\delta_{\epsilon_J}(R_J, R_K)$ if $\epsilon_J = \epsilon_K$ and by $w_J\delta^*(R_J, R_K)$ if $\epsilon_J = -\epsilon_K$. However, we do not need this fact here.

Lemma 2.5. *The spherical residues R_J and R_K of Δ are opposite if and only if $\mathrm{proj}_{R_J}(R_K)$ and $\mathrm{proj}_{R_K}(R_J)$ are opposite.*

Proof. Since opposite spherical residues are parallel, the implication '$\Rightarrow$' is obvious. The other implication follows from an easy computation using Lemma 9 of [1]. $\square$

Next, we give a rule for the composition of projections.

Lemma 2.6. *As before, Δ is a (possibly twin) building of type (W, S). Let R_I, R_J, R_K be residues of type I, J, K respectively and assume that $R_I \subseteq R_J$. Moreover, if Δ is a twin building and if R_J and R_K have opposite signs, then we also assume that I, J, K are spherical. Then we have*

$$\mathrm{proj}_{R_I}(R_K) = \mathrm{proj}_{R_I}(\mathrm{proj}_{R_J}(R_K)).$$

Proof. It suffices to prove the statement when the residue R_K is reduced to a single chamber, say c (or, in other words, when $K = \emptyset$). If R_J and c have the same sign, the result follows from the fact that $\mathrm{proj}_{R_J}(c)$ is a gate of c to R_J. If they have opposite signs, we may reduce ourselves to the preceding case in view of Lemma 2.3. $\square$

The following lemma characterizes the parallelism of spherical residues in thin buildings.

Proposition 2.7. *Let (Σ, δ) be the thin building of type (W, S). Let J, K be spherical subsets of S and let R_J, R_K be residues of type J, K respectively. Then the following statements are equivalent :*

(i) *R_J and R_K are parallel;*

(ii) *a reflection of Σ stabilizes R_J if and only if it stabilizes R_K;*

(iii) *there exist two sequences $R_J = R_0, R_1, \dots, R_n = R_K$ and $T_1, \dots, T_n$ of residues of spherical type such that for each $1 \leq i \leq n$ the rank of T_i is equal to $1+\mathrm{rank}(R_J)$, the residues R_{i-1}, R_i are contained and opposite in T_i and moreover, we have $\mathrm{proj}_{T_i}(R_J) = R_{i-1}$ and $\mathrm{proj}_{T_i}(R_K) = R_i$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is easy. The implication (iii) $\Rightarrow$ (i) follows from an obvious induction on n using the fact that opposite spherical residues are parallel. It remains to prove (i) $\Rightarrow$ (iii). Let $s \in S$ such that $\ell(s\delta(R_J, R_K)) < \ell(\delta(R_J, R_K))$. Clearly $s \notin J$. Let $x \in R_J$ and set $T_1 := \mathrm{Res}_{J\cup\{s\}}(x)$ and $R_1 := \mathrm{proj}_{T_1}(R_K)$. By definition of

T_1 we have $R_J \cap R_1 = \emptyset$ and so R_1 is properly contained in T_1. By Lemma 2.6 we have $\text{proj}_{R_J}(R_1) = R_J$. Therefore R_1 and R_J have the same rank and so they are parallel.

Let $x' \in R_1$ such that $\text{proj}_{R_J}(x') = x$ and choose y opposite to x' in R_1 (this makes sense since R_1, being the image of R_K under a projection, is spherical). Let now π be a panel containing x and contained in T_1. If the type of π is an element of J then $\text{proj}_\pi(y) \neq x$ by Lemma 2.1 and Lemma 2.4. If the type of π is s then the same inequality is still true by the definition of s and using Lemma 2.6. Therefore, T_1 is spherical by Lemma 2.1. Since $\delta(R_1, R_K)$ is shorter than $\delta(R_J, R_K)$ by construction, the desired result follows from an immediate induction. □

Corollary 2.8. *Let Δ be a (possibly twin) building of type (W, S) and let R_K be a spherical residue which is maximal with respect to that property. Then, for any residue R_J (assumed to be spherical if Δ is a twin building and if R_J and R_K have opposite signs) the projection of R_J on R_K is properly contained in R_K unless R_J and R_K are equal or opposite.*

Proof. Since $\text{proj}_{R_K}(R_J)$ and $\text{proj}_{R_J}(R_K)$ are parallel, the result clearly follows from the previous proposition, using also Lemma 2.3 if R_J and R_K have opposite signs. □

Corollary 2.9. *Let Δ be a twin building and let Σ be a twin apartment of Δ. Then the parallelism is an equivalence relation on the set of spherical residues of Σ.*

Proof. This follows from Lemma 2.3 and Proposition 2.7. □

2.2.5 Twin roots

Let $\Delta = (\Delta_+, \Delta_-, \delta^*)$ be a twin building of type (W, S).

A **twin root** of Δ is the convex hull of a pair of chambers 'at codistance 1', namely a pair $\{x, y\}$ such that $s := \delta^*(x, y) \in S$. Let π be the s-panel containing x. Then any chamber $x' \in \pi \backslash \{x\}$ is opposite y and determines therefore a twin apartment Σ which contains ϕ and x' (see Section 2.2.3). We say that ϕ is a twin root of Σ. The complement of ϕ in Σ is also a twin root; it is actually the convex hull of x' and $\text{proj}_\pi(x')$. This twin root is said to be **opposite** to ϕ in Σ and is denoted by $-\phi$ although its definition depends on Σ. A residue R of Δ is said to be **in the interior** of ϕ if it is contained in Σ and if $R \cap \Sigma$ is contained in ϕ. If $R \cap \phi$ and $R \cap (-\phi)$ are both nonempty, then R is said to be **on the boundary** of ϕ.

2.3 From groups to buildings: twin root data

2.3.1 Definition

Let $\Sigma = \Sigma(W, S)$ be the canonical twin building of type (W, S) (see Section 2.2.2) and let $\Phi(W, S)$ be the set of all its twin roots. We have already mentioned the action of W on Σ (see Section 2.2.2). Given a twin root $\phi \in \Phi(W, S)$, then all panels on the boundary of ϕ correspond to the same reflection of W. This reflection is denoted by s_ϕ and it permutes ϕ and $-\phi$. A pair $\{\phi, \psi\}$ of twin roots of $\Sigma = (\Sigma_+, \Sigma_-)$ is said to be **prenilpotent** if $\phi \cap \psi \cap \Sigma_+$ and $(-\phi) \cap (-\psi) \cap \Sigma_+$ are both nonempty; in that case, we denote by $[\phi, \psi]$ the set of all twin roots α of Σ such that $\alpha \supseteq \phi \cap \psi$ and $-\alpha \supseteq (-\phi) \cap (-\psi)$.

A **twin root datum** of type (W, S) is a system $\mathcal{D} := (G, (U_\phi)_{\phi \in \Phi(W,S)})$ consisting of a group G and a family of subgroups U_ϕ which satisfies the following axioms, where H and

$U(c)$ denote respectively the intersection of the normalizers of all U_ϕ's and the subgroup of G generated by the U_ϕ's such that ϕ contains the chamber c of Σ :

(TRD0) $U_\phi \neq 1$ for all $\phi \in \Phi(W,S)$;
(TRD1) if $\{\phi, \psi\}$ is a prenilpotent pair of distinct twin roots, the commutator $[U_\phi, U_\psi]$ is contained in the group generated by all U_γ's with $\gamma \in [\phi,\psi]\backslash\{\phi,\psi\}$;
(TRD2) if $\phi \in \Phi(W,S)$ and $u \in U_\phi\backslash\{1\}$, there exists elements u', u'' of $U_{-\phi}$ such that the product $\mu(u) = u'uu''$ conjugates U_ψ onto $U_{s_\phi(\psi)}$ for each $\psi \in \Phi(W,S)$;
(TRD3) if $\phi \in \Phi(W,S)$ and c is a chamber of Σ which is not contained in ϕ, then U_ϕ is not contained in $U(c)$;
(TRD4) the group G is generated by H and the U_ϕ's.

The group G is sometimes denoted by $G^{\mathcal{D}}$.

2.3.2 Isomorphisms of twin root data

Let $\mathcal{D} := (G, (U_\phi)_{\phi\in\Phi(W,S)})$ and $\mathcal{D}' := (G', (U'_\phi)_{\phi\in\Phi(W',S')})$ be twin root data. Let $S = S_1 \cup \cdots \cup S_n$ be the finest partition of S such that $[S_i, S_j] = 1$ whenever $1 \leq i < j \leq n$. Then $\mathcal{D}$ and $\mathcal{D}'$ are called **isomorphic** if there exist an isomorphism $\varphi : G \to G'$, an isomorphism $\pi : W \to W'$ with $\pi(S) = S'$, an element $x \in G'$ and a sign ϵ_i for each $1 \leq i \leq n$ such that

$$x\varphi(U_\phi)x^{-1} = U'_{\epsilon_i\pi(\phi)} \qquad \text{for every twin root } \phi \text{ with } s_\phi \in W_{S_i}, \tag{1}$$

where we denote by π the obvious bijection $\Phi(W,S) \to \Phi(W',S)$ induced by $\pi : W \to W'$. Thus, if (W,S) is irreducible, then either $x\varphi(U_\phi)x^{-1} = U'_{\pi(\phi)}$ or $x\varphi(U_\phi)x^{-1} = U'_{-\pi(\phi)}$ for all $\phi \in \Phi(W,S)$.

When (1) holds, we say that the isomorphism φ **induces an isomorphism** of $\mathcal{D}$ to $\mathcal{D}'$. In particular, this means, the isomorphism φ maps the union of conjugacy classes

$$\{gU_+g^{-1}|g \in G\} \cup \{gU_-g^{-1}|g \in G\}$$

to

$$\{gU'_+g^{-1}|g \in G'\} \cup \{gU'_-g^{-1}|g \in G\},$$

where U_+ (resp. U_-, U'_+, U'_-) denotes $U(c)$ (resp. $U(\mathrm{op}_\Sigma(c))$, $U(c')$, $U(\mathrm{op}_{\Sigma'}(c'))$) for some $c \in \Sigma = \Sigma(W,S)$ and some $c' \in \Sigma' = \Sigma(W',S')$.

A crucial fact on isomorphisms between twin root data we will need later is the following.

Proposition 2.10. *Let $\mathcal{D} := (G, (U_\phi)_{\phi\in\Phi(W,S)})$ and $\mathcal{D}' := (G', (U'_\phi)_{\phi\in\Phi(W',S')})$ be twin root data with S and S' finite and let $\varphi : G \to G'$ be an isomorphism. Assume there exists $g \in G'$ such that*

$$\{\varphi(U_\phi)|\ \phi \in \Phi(W,S)\} = \{gU'_\phi g^{-1}|\ \phi \in \Phi(W',S')\}.$$

Then φ induces an isomorphism of $\mathcal{D}$ to $\mathcal{D}'$.

Proof. This is Theorem (2.5) in [4]. □

2.3.3 Twin buildings from twin root data

Let $\mathcal{D} := (G, (U_\phi)_{\phi\in\Phi(W,S)})$ be a twin root datum of type (W, S). Let H be the intersection of the normalizers of all U_ϕ's and let N be the subgroup of G generated by H together with all $\mu(u)$ such that $u \in U_\phi\backslash\{1\}$, where $\mu(u)$ is as in (TRD2). Let c be a chamber of $\Sigma = \Sigma(W, S)$ of positive sign, let $c' := \mathrm{op}_\Sigma(c)$ and let $B_+ := H.U(c)$ and $B_- := H.U(c')$.

We recall from [17], Proposition 4, that (G, B_+, N) and (G, B_-, N) are both BN-pairs of type (W, S). Thus, we have corresponding Bruhat decompositions of G:

$$G = \coprod_{w\in W} B_+wB_+ \qquad \text{and} \qquad G = \coprod_{w\in W} B_-wB_-.$$

For each $\epsilon \in \{+, -\}$, the set $\Delta_\epsilon := G/B_\epsilon$ endowed with the map $\delta_\epsilon : \Delta_\epsilon \times \Delta_\epsilon \to W$ by

$$\delta_\epsilon(gB_\epsilon, hB_\epsilon) = w \Leftrightarrow B_\epsilon g^{-1}hB_\epsilon = B_\epsilon wB_\epsilon,$$

has a canonical structure of a thick building of type (W, S).

The twin root datum axioms imply that G also admits Birkhoff decompositions (see Lemma 1 in [1]):

$$G = \coprod_{w\in W} B_\epsilon wB_{-\epsilon}$$

for each $\epsilon \in \{+, -\}$. The pair $((\Delta_+, \delta_+), (\Delta_-, \delta_-))$ of buildings admits a natural twinning by means of the W-codistance δ^* defined by

$$\delta^*(gB_\epsilon, hB_{-\epsilon}) = w \Leftrightarrow B_\epsilon g^{-1}hB_{-\epsilon} = B_\epsilon wB_{-\epsilon}$$

for each $\epsilon \in \{+, -\}$. The triple $\Delta := ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ is a twin building of type (W, S).

We may and shall identify the chamber c (resp. c') of $\Sigma(W, S)$ with the chamber B_+ of $\Delta_+ = G/B_+$ (resp. B_- of Δ_-). We also identify $\Sigma(W, S)$ with the unique twin apartment of Δ containing c and c'; this twin apartment is denoted by Σ and is called the **fundamental twin apartment** of Δ (with respect to the twin root datum $\mathcal{D}$).

The diagonal action of G on $\Delta_+ \times \Delta_-$ by left multiplication is transitive on pairs of opposite chambers and, hence, on twin apartments.

2.3.4 Parabolic subgroups and root subgroups

We keep the notation of the previous subsection. We recall from the theory of BN-pairs (see [3], Chapter IV) that a subgroup P of G containing B_ϵ or any of its conjugates is called a **parabolic subgroup** of sign ϵ, where $\epsilon \in \{+, -\}$. If P contains B_ϵ, then there exists $J \subseteq S$ such that P has a Bruhat decomposition

$$P = \coprod_{w\in W_J} B_\epsilon wB_\epsilon;$$

the set J is called the **type** of the parabolic subgroup P. If J is spherical, then P is said to be **of finite type** (or of spherical type). A minimal parabolic subgroup (i.e. a parabolic subgroup of type $\emptyset$) such as B_+ or B_- is called a **Borel subgroup**.

For $\epsilon \in \{+,-\}$, let P_ϵ^J be the parabolic subgroup of type J containing B_ϵ. The geometric meaning of the groups B_+, B_-, P_+^J, P_-^J, H and N is as follows:

$$B_+ = \mathrm{Stab}_G(c), \quad B_- = \mathrm{Stab}_G(c'), \quad P_+^J = \mathrm{Stab}_G(\mathrm{Res}_J(c)), \quad P_-^J = \mathrm{Stab}_G(\mathrm{Res}_J(c'))$$

and

$$N = \mathrm{Stab}_G(\Sigma), \qquad H = B_+ \cap N = B_- \cap N = \mathrm{Fix}_G(\Sigma).$$

Given a twin root ϕ of Σ, then U_ϕ fixes chamberwise any panel in the interior of ϕ and is sharply transitive on the set of twin apartments containing ϕ. Moreover, it follows then from the axioms that for each $g \in G$ and each twin root ϕ of Σ, the group $U_{g(\phi)} := gU_\phi g^{-1}$ depends only on the twin root $g(\phi)$ and not on the choice of g and ϕ. Hence, for every twin root ψ of Δ, there is a well defined group U_ψ which fixes chamberwise any panel in the interior of ψ. The group U_ψ is sharply transitive on the set of twin apartments containing ψ; it is called the **root subgroup** associated with the twin root ψ.

2.3.5 The Conditions (P1)–(P3) and a technical lemma

Let $\mathcal{D} := (G, (U_\phi)_{\phi\in\Phi(W,S)})$ be a twin root datum. For each $\phi \in \Phi(W, S)$, we set $L_\phi := \langle U_\phi \cup U_{-\phi}\rangle$ and $H_\phi := N_{L_\phi}(U_\phi) \cap N_{L_\phi}(U_{-\phi})$. The group H_ϕ acts on the conjugacy class $\mathcal{C}_\phi$ of U_ϕ in L_ϕ. We shall be interested in the following three conditions (see Theorem 5.1 below) :

- **(P1)** for every $\phi \in \Phi(W, S)$, the group U_ϕ is nilpoptent;
- **(P2)** for every $\phi \in \Phi(W, S)$, the group L_ϕ is perfect;
- **(P3)** for every $\phi \in \Phi(W, S)$, the groups U_ϕ and $U_{-\phi}$ are the only fixed points of H_ϕ in $\mathcal{C}_\phi$.

The following lemma gives the geometric interpretation of Condition (P3).

Lemma 2.11. *Let $\mathcal{D} = (G, (U_\phi)_{\phi\in\Phi(W,S)})$ be a twin root datum of type (W, S) which satisfies Condition (P3). Let Δ be the twin building associated with $\mathcal{D}$ and let H be as above (see Section 2.3.3). Then H fixes no chamber outside the fundamental twin apartment of Δ.*

Proof. Let Σ be the fundamental twin apartment of Δ. Let $\phi \in \Phi$ and let H_ϕ be the intersection of the normalizers of U_ϕ and $U_{-\phi}$ in $\langle U_\phi \cup U_{-\phi}\rangle$. The group H_ϕ fixes Σ chamberwise and is therefore contained in H. Let π be any panel on the boundary of ϕ. Then π is stabilized by U_ϕ and $U_{-\phi}$. The condition (P3) means precisely that the only fixed chambers of H_ϕ in π are the two elements of $\pi \cap \Sigma$. This implies that for any panel π of Σ, the group H fixes no chamber in $\pi\backslash\Sigma$. Now, an easy induction on the gallery distance from an arbitrary chamber of Δ to Σ finishes the proof. □

2.3.6 Twin root data over fields

Let $\mathcal{D} = (G, (U_\phi)_{\phi\in\Phi(W,S)})$ be a twin root datum of type (W, S), let $H \leq G$ be as in the previous subsection and for each $\phi \in \Phi(W, S)$, let $\mathbb{K}_\phi$ be a field. We recall from [4] that the twin root datum $\mathcal{D}$ is called **locally split over the fields** $(\mathbb{K}_\phi)_{\phi\in\Phi(W,S)}$ if H is abelian and if for each $\phi \in \Phi(W, S)$, the twin root datum $\mathcal{D}_\phi := (H\langle U_\phi \cup U_{-\phi}\rangle, \{U_\phi, U_{-\phi}\})$ is isomorphic to the natural twin root datum of $SL_2(\mathbb{K}_\phi)$ or $PSL_2(\mathbb{K}_\phi)$. Of course, the natural twin root datum associated to a (split) Kac-Moody group over a field $\mathbb{K}$ is locally split over $\mathbb{K}$.

Notice that if $\mathcal{D}$ is locally split over the fields $(\mathbb{K}_\phi)_{\phi\in\Phi(W,S)}$, then $\mathcal{D}$ satisfies Condition (P1). Moreover, if $\mathbb{K}_\phi$ has cardinality at least 4 for every $\phi \in \Phi(W,S)$, then Conditions (P2) and (P3) are also satisfied.

3 Levi decomposition in twin root data

The purpose of this section is to obtain a Levi decomposition for intersections of finite type parabolic subgroups of opposite signs in a group with twin root datum (see Proposition 3.6). In the language of buildings, this group is the stabilizer of a pair of spherical residues of opposite signs.

3.1 Levi decomposition of parabolic subgroups

3.1.1 The setting

Let $\mathcal{D} := (G, (U_\alpha)_{\alpha\in\Phi(W,S)})$ be a twin root datum, let $\Delta = ((\Delta_+,\delta_+),(\Delta_-,\delta_-),\delta^*)$ be the corresponding twin building and let Σ_0 be the fundamental twin apartment (see Section 2.3.3).

Let Σ be any twin apartment of Δ. Let $c \in \Sigma$ be a chamber and let R be a spherical residue of Σ (i.e. $R \cap \Sigma \neq \emptyset$). Let Φ^Σ the set of all twin roots of Σ and let $\Phi^\Sigma(R)$ be the set of twin roots β of Σ such that $R \cap \beta$ and $R \cap (-\beta)$ are both nonempty, which means precisely that the reflection s_β stabilizes R (see Sections 2.2.2 and 2.2.5). We set

$$U^\Sigma(c) := \langle U_\phi |\ \phi \in \Phi^\Sigma, c \in \phi\rangle, \qquad U^\Sigma(R) := \bigcap_{x\in\Sigma\cap R} U^\Sigma(x)$$

and

$$L^\Sigma(R) := \mathrm{Fix}_G(\Sigma).\langle U_\phi |\ \phi \in \Phi^\Sigma(R)\rangle.$$

(See Section 2.3.4 for the definition of U_ϕ.)

We will see in Proposition 3.1 that $U^\Sigma(c)$ and $U^\Sigma(R)$ are actually independent of Σ. They will be denoted by $U(c)$ and $U(R)$ respectively.

Notice that $L^\Sigma(R) = L^\Sigma(\mathrm{op}_\Sigma(R))$ since $\Phi^\Sigma(R) = \Phi^\Sigma(\mathrm{op}_\Sigma(R))$. Moreover, if the residue R is reduced to a chamber c (i.e. if R is of type $\emptyset$) then $L^\Sigma(c) = \mathrm{Fix}_G(\Sigma)$, which is G-conjugate to the subgroup H of Section 2.3.3 (see Section 2.3.4).

3.1.2 Standard Levi decomposition: Levi factor and unipotent radical

The following result is the standard Levi decomposition of finite type parabolic subgroups of a group with a twin root datum. We state it in the language of buildings.

Proposition 3.1. *We have* $\mathrm{Stab}_G(R) = L^\Sigma(R) \ltimes U^\Sigma(R)$. *Moreover,* $U^\Sigma(R)$ *is sharply transitive on the set of residues which are opposite* R *in* $\Delta(G)$ *and*

$$L^\Sigma(R) = \mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(\mathrm{op}_\Sigma(R)).$$

In particular, the subgroup $U^\Sigma(R)$ *is independent of* Σ *and will be denoted by* $U(R)$.

Proof. This follows from the theorem of Section 6.2.2 in [8]. ∎

The group $U(R)$ is called the **unipotent radical** of the parabolic subgroup $\mathrm{Stab}_G(R)$ with respect to the twin root datum $\mathcal{D}$ and the group $L^\Sigma(R)$ is called a **Levi factor**.

3.2 Levi decomposition of parabolic intersections

3.2.1 More definitions and notation

We keep the notation of Section 3.1.1.

Let Σ be a twin apartment of Δ and for each $\epsilon \in \{+,-\}$, let R_ϵ be a residue of Σ of sign ϵ. We set

$$U^\Sigma(R_+, R_-) := \langle U_\beta |\ \beta \in \Phi^\Sigma \text{ and } R_\epsilon \cap \Sigma \subset \beta \rangle,$$

and, for $\epsilon \in \{+,-\}$,

$$\widetilde{U}^\Sigma(R_\epsilon, R_{-\epsilon}) := \langle U_\beta |\ \beta \in \Phi^\Sigma(R_\epsilon) \text{ and } \operatorname{proj}_{R_\epsilon}(R_{-\epsilon}) \subset \beta \rangle$$

(see Sections 2.2.5 and 2.3.4). Notice that $U^\Sigma(R_+, R_-) = U^\Sigma(R_-, R_+)$ while $\widetilde{U}^\Sigma(R_+, R_-)$ need not equal $\widetilde{U}^\Sigma(R_-, R_+)$. If R_+ (resp. R_-) is reduced to a chamber x_+ (resp. x_-), we have $\widetilde{U}^\Sigma(R_+, R_-) = \widetilde{U}^\Sigma(R_-, R_+) = \{1\}$. This remains true in the case where R_+ and R_- are parallel (see Section 2.2.4).

See Remark 3.4 below for an interpretation of the group $\widetilde{U}^\Sigma(R_\epsilon, R_{-\epsilon})$.

3.2.2 The intersection of a parabolic subgroup and unipotent radical

In order to obtain the Levi decomposition of the group $\operatorname{Stab}_G(\{R_+, R_-\}) = \operatorname{Stab}_G(R_+) \cap \operatorname{Stab}_G(R_-)$, we need a decomposition result for $\operatorname{Stab}_G(R_+) \cap U(R_-)$. We start with the case where both residues are reduced to single chambers.

Lemma 3.2. *Let $x_+ \in \Delta_+$ and $x_- \in \Delta_-$ be chambers of Δ. Let Σ be a twin apartment containing them both. Let $\epsilon \in \{+,-\}$, let $y_\epsilon := \operatorname{op}_\Sigma(x_{-\epsilon})$ and let $x_\epsilon = x_0, x_1, \dots, x_n = y_\epsilon$ be a minimal gallery joining x_ϵ to y_ϵ. For each $1 \leq i \leq n$, let β_i be the twin root of Σ containing x_{i-1} but not x_i. Then we have*

$$U(x_\epsilon) \cap \operatorname{Stab}_G(x_{-\epsilon}) = U^\Sigma(x_+, x_-) = U_{\beta_1}.U_{\beta_2} \dots U_{\beta_n}.$$

In particular, the product $U_{\beta_1}.U_{\beta_2} \dots U_{\beta_n}$ is a group which coincides with $U^\Sigma(x_+, x_-)$, and the latter does not depend on the twin apartment Σ. We will denote it by $U(x_+, x_-)$.

Proof. This follows from Lemma 1.5.2(iii) and Theorem 3.5.4 in [8]. □

The following lemma generalizes Lemma 3.2 to the case of spherical residues of higher rank.

Lemma 3.3. *Let $R_+ \subseteq \Delta_+$ and $R_- \subseteq \Delta_-$ be spherical residues of Δ. Let Σ be a twin apartment intersecting them both. Then, for each $\epsilon \in \{+,-\}$, we have*

$$U(R_{-\epsilon}) \cap \operatorname{Stab}_G(R_\epsilon) = \widetilde{U}^\Sigma(R_\epsilon, R_{-\epsilon}).U^\Sigma(R_+, R_-).$$

In particular, if R_+ and R_- are parallel, then

$$U(R_+) \cap \operatorname{Stab}_G(R_-) = \operatorname{Stab}_G(R_+) \cap U(R_-) = U^\Sigma(R_+, R_-).$$

Proof. The inclusion '$\supseteq$' of the first part is clear.

Let $R'_\epsilon := \mathrm{op}_\Sigma(R_{-\epsilon})$. Let us choose $z, z' \in \mathrm{proj}_{R'_\epsilon}(R_\epsilon) \cap \Sigma$ such that z and z' are opposite in the spherical residue $\mathrm{proj}_{R'_\epsilon}(R_\epsilon)$ (see Section 2.1.5). Set $x := \mathrm{op}_\Sigma(z)$ and $x' := \mathrm{op}_\Sigma(z')$. Notice that x and x' belong to $R_{-\epsilon} \cap \Sigma$. We also define $y := \mathrm{proj}_{R_\epsilon}(x)$.

We have $U(R_{-\epsilon}) \leq U(x')$ since $x' \in R_{-\epsilon}$. Moreover, $U(R_{-\epsilon}) \cap \mathrm{Stab}_G(R_\epsilon)$ fixes $y = \mathrm{proj}_{R_\epsilon}(x)$ because this group fixes x and stabilizes R_ϵ; hence, $U(R_{-\epsilon}) \cap \mathrm{Stab}_G(R_\epsilon) \leq \mathrm{Stab}_G(y)$. Therefore, we have $U(R_{-\epsilon}) \cap \mathrm{Stab}_G(R_\epsilon) \leq U(x') \cap \mathrm{Stab}_G(y) = U(x', y)$, where the latter equality follows Lemma 3.2.

We now choose a minimal gallery $y = y_0, \dots, y_j = \mathrm{proj}_{R_\epsilon}(z'), \dots, y_n = z'$ joining y to z' (see Section 2.1.4). Hence, for all $0 \leq i \leq n$, we have $y_i \in R_\epsilon$ if and only if $i \leq j$. (Notice that $j = 0$ or $j = n$ is possible.)

For each $1 \leq i \leq n$, let β_i be the twin root of Σ which contains y_{i-1} but not y_i. Thus $\{\beta_1, \dots, \beta_n\}$ is the set of twin roots containing x' and y or equivalently, y but not z' since $z' = \mathrm{op}_\Sigma(x')$. By definition, we have $U(x', y) = \langle U_{\beta_i} | 1 \leq i \leq n \rangle$. By Lemma 3.2 this group coincides with the product $U_{\beta_1}.U_{\beta_2} \dots U_{\beta_n}$.

Now we observe that by the definition of y, y_j and z' and by Lemmas 2.3 and 2.4, we have

$$\mathrm{proj}_{R'_\epsilon}(y) = \mathrm{proj}_{R'_\epsilon}(y_j) = z' \qquad \text{and} \qquad \mathrm{proj}_{\mathrm{proj}_{R_\epsilon}(R_{-\epsilon})}(z') = \mathrm{proj}_{\mathrm{proj}_{R_\epsilon}(R_{-\epsilon})}(y_j) = y.$$

In view of the properties of projections (see Section 2.1.4), this implies

$$\begin{aligned} \{\beta \in \Phi^\Sigma(R_\epsilon) |\ \mathrm{proj}_{R_\epsilon}(R_{-\epsilon}) \subset \beta\} &= \{\beta \in \Phi^\Sigma |\ y \in \beta \text{ and } y_j \notin \beta\} \\ &= \{\beta_1, \dots, \beta_j\} \\ &= \{\beta \in \Phi^\Sigma |\ y \in \beta \text{ and } \mathrm{op}_\Sigma(y_j) \in \beta\} \end{aligned}$$

and

$$\begin{aligned} \{\beta \in \Phi^\Sigma |\ R_\epsilon \cap \Sigma \subset \beta \text{ and } R_{-\epsilon} \cap \Sigma \subset \beta\} &= \{\beta \in \Phi^\Sigma |\ R_\epsilon \cap \Sigma \subset \beta \text{ and } R'_\epsilon \cap \Sigma \subset \beta\} \\ &= \{\beta \in \Phi^\Sigma |\ y_j \in \beta \text{ and } z' \notin \beta\} \\ &= \{\beta_{j+1}, \dots, \beta_n\} \\ &= \{\beta \in \Phi^\Sigma |\ y_j \in \beta \text{ and } x' \in \beta\}. \end{aligned}$$

We deduce from this, together with Lemma 3.2, that

$$\widetilde{U}^\Sigma(R_\epsilon, R_{-\epsilon}) = \langle U_i |\ 1 \leq i \leq j \rangle = U(y, \mathrm{op}_\Sigma(y_j)) = U_1 \dots U_j$$

and

$$U^\Sigma(R_\epsilon, R_{-\epsilon}) = \langle U_i |\ j+1 \leq i \leq n \rangle = U(x', y_j) = U_{j+1} \dots U_n.$$

In summary, we have shown that

$$\begin{aligned} U(R_{-\epsilon}) \cap \mathrm{Stab}_G(R_\epsilon) &\leq U(x', y) \\ &= \big(U_{\beta_1} \dots U_{\beta_j}\big).\big(U_{\beta_{j+1}} \dots U_{\beta_n}\big) \\ &= \widetilde{U}^\Sigma(R_\epsilon, R_{-\epsilon}).U^\Sigma(R_+, R_-). \end{aligned}$$

□

The lemma implies that, if for $\epsilon = +$ or $-$ we have $\mathrm{proj}_{R_\epsilon}(R_{-\epsilon}) = R_\epsilon$ (in particular, if R_+ and R_- are parallel) and hence $\widetilde{U}^\Sigma(R_\epsilon, R_{-\epsilon}) = \{1\}$, then the group $U^\Sigma(R_+, R_-)$ is independent of the twin apartment Σ. In that case, we may omit the superscript Σ and we shall write $U(R_+, R_-)$ rather than $U^\Sigma(R_+, R_-)$.

Remark 3.4. The group $\widetilde{U}^\Sigma(R_\epsilon, R_{-\epsilon})$ defined in the previous lemma actually coincides with $U(\mathrm{proj}_{R_\epsilon}(R_{-\epsilon})) \cap L^\Sigma(R_\epsilon)$. This can be seen as follows. First notice that $(L^\Sigma, (U_\beta)_{\beta\in\Phi^\Sigma(R_\epsilon)})$ is a twin root datum. Hence the group $\mathrm{Stab}_{L^\Sigma(R_\epsilon)}(\mathrm{proj}_{R_\epsilon}(R_{-\epsilon}))$ has a Levi decomposition in $L^\Sigma(R_\epsilon)$ by Proposition 3.1. The above claim is easily deduced from that fact: actually, the group $\widetilde{U}^\Sigma(R_\epsilon, R_{-\epsilon}) = U(\mathrm{proj}_{R_\epsilon}(R_{-\epsilon})) \cap L^\Sigma(R_\epsilon)$ is nothing but the unipotent radical of $\mathrm{Stab}_{L^\Sigma(R_\epsilon)}(\mathrm{proj}_{R_\epsilon}(R_{-\epsilon}))$ with respect to the above-mentioned twin root datum. We will not need that fact here.

The following is a consequence of the proof of the previous lemma.

Corollary 3.5. *Let $R_+ \subseteq \Delta_+$ and $R_- \subseteq \Delta_-$ be spherical residues. Then, for each $\epsilon \in \{+,-\}$ there exist chambers $x_+ \in R_+$ and $x_- \in R_-$ such that $U(R_{-\epsilon}) \cap \mathrm{Stab}_G(R_\epsilon) = U(x_+, x_-)$. In particular, if all root subgroups are nilpotent (i.e. if Condition (P1) holds), then $U^\Sigma(R_+, R_-)$ is nilpotent, where Σ is any twin apartment intersecting R_+ and R_-.*

Proof. The first statement was proved along the way. The second statement is a consequence of the first, using also Axiom (TRD1) and Lemma 3.2. □

3.2.3 The intersection of finite type parabolics of opposite signs

We are now able to prove the Levi decomposition for the intersection of two finite type parabolic subgroups of opposite signs.

Proposition 3.6. *Let $R_+ \subseteq \Delta_+$ and $R_- \subseteq \Delta_-$ be spherical residues of $\Delta(G)$. Let Σ be a twin apartment containing R_+ and R_-. For each sign ϵ set $R^\circ_\epsilon := \mathrm{proj}_{R_\epsilon}(R_{-\epsilon})$. Then, for all $\epsilon, \epsilon' \in \{+,-\}$ we have*

$$\begin{aligned} \mathrm{Stab}_G(R_+) \cap \mathrm{Stab}_G(R_-) &= L^\Sigma(R^\circ_{\epsilon'}) \ltimes U(R^\circ_+, R^\circ_-) \\ &= L^\Sigma(R^\circ_{\epsilon'}) \ltimes \big(\widetilde{U}^\Sigma(R_{-\epsilon}, R_\epsilon).U^\Sigma(R_+, R_-).\widetilde{U}^\Sigma(R_\epsilon, R_{-\epsilon})\big). \end{aligned}$$

Proof. Notice first that $\mathrm{Stab}_G(R_+) \cap \mathrm{Stab}_G(R_-) = \mathrm{Stab}_G(R^\circ_+) \cap \mathrm{Stab}_G(R^\circ_-)$. Moreover, since R°_+ and R°_- are parallel, we deduce from Lemma 2.3 and Proposition 2.7(ii) that $L^\Sigma(R^\circ_+) = L^\Sigma(R^\circ_-)$.

Now the inclusion $L^\Sigma(R^\circ_{\epsilon'}).U(R^\circ_+, R^\circ_-) \leq \mathrm{Stab}_G(R_+) \cap \mathrm{Stab}_G(R_-)$ is clear. On the other hand, we have the following:

$$\begin{aligned} \mathrm{Stab}_G(R^\circ_+) \cap \mathrm{Stab}_G(R^\circ_-) &= \big(L^\Sigma(R^\circ_\epsilon).U(R^\circ_\epsilon)\big) \cap \mathrm{Stab}_G(R^\circ_{-\epsilon}) \\ &\leq L^\Sigma(R^\circ_\epsilon).\big(U(R^\circ_\epsilon) \cap \mathrm{Stab}_G(R^\circ_{-\epsilon})\big) \\ &= L^\Sigma(R^\circ_{\epsilon'}).U(R^\circ_\epsilon, R^\circ_{-\epsilon}), \end{aligned}$$

where the last equality follows from Lemma 3.3. This proves that $\mathrm{Stab}_G(R_+) \cap \mathrm{Stab}_G(R_-) = L^\Sigma(R^\circ_{\epsilon'}).U(R^\circ_\epsilon, R^\circ_{-\epsilon})$.

Now $L^\Sigma(R^\circ_{\epsilon'}) = L^\Sigma(R^\circ_\epsilon)$ intersects $U(R^\circ_\epsilon)$ trivially and normalizes that group by Proposition 3.1. Since $L^\Sigma(R^\circ_{\epsilon'}) \leq \mathrm{Stab}_G(R^\circ_{-\epsilon})$ we also see that $L^\Sigma(R^\circ_{\epsilon'})$ normalizes $\mathrm{Stab}_G(R^\circ_{-\epsilon})$. Therefore, $L^\Sigma(R^\circ_{\epsilon'})$ normalizes $U(R^\circ_\epsilon) \cap \mathrm{Stab}_G(R^\circ_{-\epsilon}) = U(R^\circ_\epsilon, R^\circ_{-\epsilon})$ and intersects the latter group trivially. This proves the first equality of the lemma.

In order to establish the second equality, we first notice that $\widetilde{U}^\Sigma(R_{-\epsilon}, R^\circ_\epsilon) = \widetilde{U}^\Sigma(R_{-\epsilon}, R_\epsilon)$ by the definition of these groups. Now, (an argument as in) the proof of the previous lemma shows that

$$\begin{aligned} U(R^\circ_+, R^\circ_-) &= \widetilde{U}^\Sigma(R_{-\epsilon}, R^\circ_\epsilon).U(R^\circ_\epsilon, R_{-\epsilon}) \\ &= \widetilde{U}^\Sigma(R_{-\epsilon}, R_\epsilon).U^\Sigma(R_\epsilon, R_{-\epsilon}).\widetilde{U}^\Sigma(R_\epsilon, R_{-\epsilon}), \end{aligned}$$

from which the conclusion follows. □

Remark 3.7. The preceding proposition is proved in [8], Section 6.3.4, under an additional assumption called (NILP), defined in *op. cit.*, Section 6.3. Our proof shows that this extra assumption is not necessary for the result to hold.

Corollary 3.8. *Let (W,S) be a Coxeter system such that S is finite. Let $\mathcal{D} = (G,(U_\phi)_{\phi\in\Phi(W,S)})$ be a twin root datum such that each U_ϕ is finite and such that $H := \bigcap_{\phi\in\Phi(W,S)} N_G(U_\phi)$ is finite. Then the set of all bounded subgroups coincides with the set of all finite subgroups of G.*

Proof. Let Δ be the twin building associated with $\mathcal{D}$. The fact that each finite subgroup of G is bounded is an immediate consequence of Proposition 2.2. In order to prove that a bounded subgroup is finite, it suffices to prove that given a pair R_+, R_- of spherical residues of opposite signs, then the group $\mathrm{Stab}_G(R_+)\cap\mathrm{Stab}_G(R_-)$ is finite. Our hypotheses imply that every Levi factor of spherical type is finite. Hence, by Proposition 3.6, it just remains to show that $U(\mathrm{proj}_{R_+}(R_-),\mathrm{proj}_{R_-}(R_+))$ is finite. But this follows again from our hypotheses in view of Lemma 3.2 and Corollary 3.5. □

4 Maximal bounded subgroups

4.1 The main characterization

Let $\mathcal{D} = (G,(U_\alpha)_{\alpha\in\Phi})$ be a twin root datum of type (W,S). By a maximal bounded subgroup of G, we mean a bounded subgroup which is not properly contained in any other bounded subgroup of G. Let M be such a bounded subgroup. By definition M is the intersection of two finite type parabolic subgroups of opposite signs. The following theorem shows that there exists two canonical finite type parabolic subgroups P_+^M and P_-^M such that $M = P_+^M \cap P_-^M$. Case (i) of the theorem corresponds to the case where P_+^M and P_-^M are opposite; the group M is then the common Levi factor of P_+^M and P_-^M. To some extent, this is the generic case (see Proposition 4.2 and Remark 4.3 below).

Before stating the theorem, we need one more notation. Let $((\Delta_+,\delta_+),(\Delta_-,\delta_-),\delta^*)$ be the twin building associated with $\mathcal{D}$. Given a subgroup $M\leq G$, we denote by $\mathcal{S}_\epsilon(M)$ the set of all spherical residues of Δ_ϵ stabilized by M, where $\epsilon\in\{+,-\}$.

Theorem 4.1. *Let $(G,(U_\alpha)_{\alpha\in\Phi})$ be a twin root datum of type (W,S) and let $M\leq G$ be a maximal bounded subgroup. Then one of the following holds :*

(i) for $\epsilon\in\{+,-\}$ the set $\mathcal{S}_\epsilon(M)$ consists of a unique element R_ϵ, which is a maximal spherical residue of Δ_ϵ; moreover R_+ and R_- are opposite in $\Delta(G)$;

(ii) for $\epsilon\in\{+,-\}$ the set $\mathcal{S}_\epsilon(M)$ possesses two distinguished elements R_ϵ and $\overline{R}_\epsilon$ such that for every $T_\epsilon\in\mathcal{S}_\epsilon(M)$ we have $R_\epsilon\subseteq T_\epsilon\subseteq\overline{R}_\epsilon$ and $\mathrm{proj}_{T_\epsilon}(T_{-\epsilon}) = R_\epsilon$ and T_+ and T_- are not opposite; moreover $\overline{R}_\epsilon$ is the only maximal spherical residue containing R_ϵ.

In both cases we have $M = \mathrm{Stab}_G(R_+)\cap\mathrm{Stab}_G(R_-)$.

Conversely, let $R_+\subseteq\Delta_+$ and $R_-\subseteq\Delta_-$ be spherical residues such that either of the following conditions holds :

(i′) *R_+ and R_- are maximal spherical and opposite;*

(ii′) *R_+ and R_- are parallel; moreover, for each $\epsilon \in \{+,-\}$ the residue R_ϵ is properly contained in a unique maximal spherical residue $\overline{R}_\epsilon$ and we have $proj_{\overline{R}_\epsilon}(\overline{R}_{-\epsilon}) = R_\epsilon$.*

Then $M := \mathrm{Stab}_G(R_+) \cap \mathrm{Stab}_G(R_-)$ is a maximal bounded subgroup.

Proof. Let $M \leq G$ be a maximal bounded subgroup. For $\epsilon \in \{+,-\}$ let $R_\epsilon \in \mathcal{S}_\epsilon(M)$.

Assume first that R_+ and R_- are opposite. Hence, $M = \mathrm{Stab}_G(R_+) \cap \mathrm{Stab}_G(R_-)$ which implies by Proposition 3.1 that M does not stabilize any residue properly contained in R_+ or R_-. Moreover, the maximality of M implies that R_+ and R_- are maximal spherical residues. Let now $T_\epsilon \in \mathcal{S}_\epsilon(M)$. Then M stabilizes $\mathrm{proj}_{R_+}(T_\epsilon)$ and $\mathrm{proj}_{R_-}(T_\epsilon)$. Since these cannot be properly contained in R_+ and R_- respectively, we conclude from Corollary 2.8 that $T_\epsilon = R_\epsilon$. Hence, we are in Case (i). Notice that our discussion also proves the converse statement in the case (i′).

We now assume that R_+ and R_- are not opposite. For $\epsilon \in \{+,-\}$ we have $\mathrm{proj}_{R_\epsilon}(R_{-\epsilon}) \in \mathcal{S}_\epsilon(M)$. Moreover, we know by Lemma 2.5 that $\mathrm{proj}_{R_+}(R_-)$ and $\mathrm{proj}_{R_-}(R_+)$ are not opposite (this follows also from the first part of the present proof). Therefore, up to replacing R_ϵ by $\mathrm{proj}_{R_\epsilon}(R_{-\epsilon})$ we may assume that R_+ and R_- are parallel. Let J_+ and J_- be their respective types. Since R_+ and R_- are not opposite, Lemma 2.3 and Proposition 2.7 show that J_ϵ is not a maximal spherical subset of S.

Let now $s \in S \backslash J_\epsilon$ such that $J_\epsilon \cup \{s\}$ is spherical and let R^s_ϵ be the residue of type $J_\epsilon \cup \{s\}$ containing R_ϵ. Set also $T := \mathrm{proj}_{R^s_\epsilon}(R_{-\epsilon})$. We now prove that $T = R_\epsilon$.

Assume on the contrary that $T \neq R_\epsilon$. Then Lemma 2.6 and Proposition 2.7 imply that R_ϵ and T are opposite in R^s_ϵ. Let Σ be a twin apartment containing R_+ and R_-. There exists a twin root $\alpha = (\alpha_+, \alpha_-)$ of Σ such that $T \subseteq \alpha_\epsilon$, $R_\epsilon \subseteq -\alpha_\epsilon$ (hence the reflection s_α of Σ stabilizes R^s_ϵ) and $R_{-\epsilon} \subseteq \alpha_{-\epsilon}$. Now Proposition 3.1 implies that U_α acts freely on the residues opposite T in R^s_ϵ. In particular, we have $U_\alpha \cap M = \{1\}$ since M stabilizes R_ϵ. Therefore, the group $\langle U_\alpha \cup M \rangle$ contains M properly, and it stabilizes R^s_ϵ and $R_{-\epsilon}$ by construction. In other words, we have $M \lneq \langle U_\alpha \cup M \rangle$ is a bounded subgroup, which contradicts the maximality of M. This proves that $T = R_\epsilon$ as claimed.

Let S_ϵ be the set of all $s \in S \backslash J_\epsilon$ such that $J_\epsilon \cup \{s\}$ is spherical. Let $\overline{R}_\epsilon$ be the residue of type $J_\epsilon \cup S_\epsilon$ containing R_ϵ. We now prove that $\overline{R}_\epsilon$ is spherical.

To this end we consider, as before, a twin apartment Σ containing R_+ and R_-. Let $R'_\epsilon := \mathrm{op}_\Sigma(R_\epsilon)$. Choose a chamber $x \in R_\epsilon$ and a chamber z which is opposite to $\mathrm{proj}_{R'_\epsilon}(x)$ in R'_ϵ. Set $y := \mathrm{proj}_{\overline{R}_\epsilon}(z)$. Our aim is to apply the criterion of sphericity of Lemma 2.1 to x and y. Hence, let $j \in J_\epsilon \cup S_\epsilon$ and denote by π_j the j-panel containing x.

We know from Lemma 2.3 that R_ϵ and R'_ϵ are parallel. By Lemma 2.6, this implies that R_ϵ and $\mathrm{proj}_{\overline{R}_\epsilon}(R'_\epsilon)$ are parallel. By Lemma 2.4, we see that x and $\mathrm{proj}_{R_\epsilon}(y)$ are opposite in R_ϵ and we conclude from Lemma 2.1 that if $j \in J_\epsilon$ then $x \neq \mathrm{proj}_{\pi_j}(y) = \mathrm{proj}_{\pi_j}(\mathrm{proj}_{R_\epsilon}(y))$.

Let us now assume that $j \in S_\epsilon$. We have already proved that $\mathrm{proj}_{R^j_\epsilon}(R_{-\epsilon}) = R_\epsilon$, which implies that $\mathrm{proj}_{R^j_\epsilon}(R'_\epsilon)$ is opposite R_ϵ in R^j_ϵ by Lemma 2.3. Therefore, x and $\mathrm{proj}_{R^j_\epsilon}(y)$ are opposite in R^j_ϵ which implies by Lemma 2.1 that $x \neq \mathrm{proj}_{\pi_j}(y) = \mathrm{proj}_{\pi_j}(\mathrm{proj}_{R^j_\epsilon}(y))$.

Finally, Lemma 2.1 applied to x and y insures that $\overline{R}_\epsilon$ is spherical, i.e. that $J_\epsilon \cup S_\epsilon$ is a spherical subset of S. Moreover, it is clear by the definition of S_ϵ that $J_\epsilon \cup S_\epsilon$ is actually a *maximal* spherical subset of S.

By maximality of M, we have $M = \mathrm{Stab}_G(R_+) \cap \mathrm{Stab}_G(R_-)$. Therefore, we see by Proposition 3.6 that M does not stabilize any proper residue of R_ϵ for $\epsilon \in \{+,-\}$.

Moreover, the same result implies that if R is a residue contained in $\overline{R}_\epsilon$, then R is stabilized by M if and only if R contains R_ϵ.

Let now $T_\epsilon \in \mathcal{S}_\epsilon(M)$. Then $\mathrm{proj}_{R_\epsilon}(T_\epsilon) \in \mathcal{S}_\epsilon(M)$ and so $\mathrm{proj}_{R_\epsilon}(T_\epsilon) = R_\epsilon$. Therefore, R_ϵ and $\mathrm{proj}_{T_\epsilon}(R_\epsilon)$ are parallel. By the previous paragraph together with Proposition 2.7, this implies that $R_\epsilon = \mathrm{proj}_{T_\epsilon}(R_\epsilon)$, or in other words that $R_\epsilon \subseteq T \subseteq \overline{R}_\epsilon$.

Going now back to the first argument in the proof of (ii) above, we conclude moreover that $\mathrm{proj}_{T_\epsilon}(T_{-\epsilon}) = R_\epsilon$. The fact that T_+ and T_- are not opposite is now obvious.

It remains to prove the converse statement. For (ii′), let R_+, R_-, $\overline{R}_+$ and $\overline{R}_-$ be as in (ii′) and define $M := \mathrm{Stab}_G(R_+) \cap \mathrm{Stab}_G(R_-)$. Mimicking some of the arguments above we can prove again that if R is a residue contained in $\overline{R}_\epsilon$, then R is stabilized by M if and only if R contains R_ϵ. From this, we deduce as above that $R_\epsilon \subseteq T \subseteq \overline{R}_\epsilon$ whenever $T \in \mathcal{S}_\epsilon(M)$. Therefore, if $M \leq M_1$ and M_1 is bounded, then there exists $T_\epsilon \in \mathcal{S}_\epsilon(M_1)$ such that $R_\epsilon \subseteq T_\epsilon \subseteq \overline{R}_\epsilon$. But our hypotheses then imply that $\mathrm{proj}_{T_\epsilon}(T_{-\epsilon}) = R_\epsilon$. Therefore, we have $R_\epsilon \in \mathcal{S}_\epsilon(M_1)$ namely $M_1 \leq \mathrm{Stab}_G(R_\epsilon)$. Hence $M = M_1$. The proof is complete. □

4.2 Two specializations

4.2.1 Obstructions for Case (ii) of Theorem 4.1

In many interesting situations, only Case (i) of Theorem 4.1 occurs. The next result gives sufficient conditions on the Coxeter system (W, S) which imply that Case (ii) never happens.

Proposition 4.2. *Assume that the Coxeter system (W, S) satisfies one of the following conditions :*

(R1) for all $s, t \in S$, the order of st is not equal to 3;

(R2) for any pair J, K of spherical subsets of S such that J is properly contained in K and K is maximal spherical, there exists an $s \in S\backslash K$ such that $J \cup \{s\}$ is spherical but $K \cup \{s\}$ is not;

(R3) for every $j \in S$, there exists a unique maximal spherical subset J of S such that $j \in J$.

Then the case (ii) does not occur in the previous theorem.

Proof. We keep the notation of the proof of Theorem 4.1. It is clear that if (ii) holds in that theorem, then (R2) fails by choosing $J = J_\epsilon$ and $K = J_\epsilon \cup S_\epsilon$ where $\epsilon = +$ or $-$.

Now let Σ be a twin apartment containing R_+ and R_- as in the proof above, let $R'_\epsilon := \mathrm{op}_\Sigma(R_{-\epsilon})$ and let J''_ϵ be the type of $R''_\epsilon := proj_{\overline{R}_\epsilon}(R'_\epsilon)$. We know by Lemma 2.3 that R''_ϵ is opposite R_ϵ in $\overline{R}_\epsilon$ and, moreover, that $\overline{R}_\epsilon$ is the *unique* maximal spherical residue containing R_ϵ. On the other hand, since $\overline{R}_+$ and $\overline{R}_-$ are not opposite, we have $R'_\epsilon \neq R''_\epsilon$, and these two distinct residues are parallel. Therefore, it follows from Proposition 2.7 that there exists $s \in S\backslash(J_\epsilon \cup S_\epsilon)$ such that $J''_\epsilon \cup \{s\}$ is spherical. In particular, $J_\epsilon \cup S_\epsilon$ is *not* the unique maximal spherical subset of S containing J''_ϵ. Hence, (R3) fails and furthermore, we have $J_\epsilon \neq J''_\epsilon$. Since J_ϵ and J''_ϵ are the respective types of two opposite residues of $\overline{R}_\epsilon$, the latter inequality also implies that (R1) fails, using [13], Proposition 5.2.3 and the fact that there are no Moufang n-gons for odd n greater than 3. □

Remark 4.3. 1. Condition (R2) in the previous corollary is also equivalent to the following :

(R2′) for every non-maximal spherical subset J of S there exists (at least) two distinct maximal spherical subsets K_1 and K_2 of S containing J.

2. All affine and compact hyperbolic Coxeter diagrams satisfy Condition (R2) (notice that (R2) is empty for $\tilde{A}_1$ and so obviously satisfied; actually $\tilde{A}_1$ also satisfies (R1) and (R3)).

3. Condition (R3) in the previous corollary is also equivalent to each of the following ones :
   (R3′) for every maximal spherical subset J of S and all pairs j, s with $j \in J$ and $s \in S\backslash J$, the order of sj is infinite;
   (R3″) there is a partition $S = S_1 \cup \cdots \cup S_n$ of S into spherical subsets such that the order of st is infinite whenever $s \in S_i$, $t \in S_j$ and $i \neq j$.

4.2.2 A group theoretic description of Case (ii)

We end this section with a lemma which is will be used in the proof of Theorem 5.1.

Lemma 4.4. *Let $(G, (U_\alpha)_{\alpha \in \Phi})$ be a twin root datum of type (W,S) which satisfies Conditions (P1) and (P2) of Section 2.3.5. Let $M \leq G$ be a maximal bounded subgroup. A necessary and sufficient condition for M to have type (ii) in Theorem 4.1, is that*

$$M = U \rtimes (M \cap M'),$$

where U is a nontrivial nilpotent group and M' is a maximal bounded subgroup different from M.

Proof. We first prove that the condition is necessary. We keep the notation of Theorem 4.1 and assume that M satisfies Condition (ii). Let Σ denote a twin apartment containing R_+ and R_-. By Proposition 3.6, we have

$$\begin{aligned} M &= \mathrm{Stab}_G(R_+) \cap \mathrm{Stab}_G(R_-) \\ &= \mathrm{Stab}_G(\overline{R}_+) \cap \mathrm{Stab}_G(R_-) \\ &= L^\Sigma(R_+).\widetilde{U}^\Sigma(\overline{R}_+, R_-).U(\overline{R}_+, R_-).\widetilde{U}^\Sigma(R_-, \overline{R}_+) \\ &= L^\Sigma(R_+).\widetilde{U}^\Sigma(\overline{R}_+, R_-).U(\overline{R}_+, R_-), \end{aligned}$$

where the last equality follows from $\mathrm{proj}_{R_-}(\overline{R}_+) = R_-$ which implies $\widetilde{U}^\Sigma(R_-, \overline{R}_+) = \{1\}$.

Now, set $U := U(\overline{R}_+, R_-)$ and $M' := L^\Sigma(\overline{R}_+)$. By Theorem 4.1, the group M' is a maximal bounded subgroup. Clearly, the group U is nilpotent by 3.5 and nontrivial since $\mathrm{op}_\Sigma(R_-) \cap \overline{R}_+ = \emptyset$ (see the proof of Proposition 4.2). Moreover, we have $M' \cap U = \{1\}$ since $U \leq U(\overline{R}_+)$. On the other hand, we also have $L^\Sigma(R_+).\widetilde{U}^\Sigma(\overline{R}_+, R_-) \leq M'$ by definition. Therefore, we have $L^\Sigma(R_+).\widetilde{U}^\Sigma(\overline{R}_+, R_-) = M \cap M'$ and hence, $M = (M \cap M').U$.

It remains to prove that $M \cap M'$ normalizes U. Using again the fact that $\widetilde{U}^\Sigma(R_-, \overline{R}_+)$ is trivial, we deduce from Lemma 3.3 that $U = U(\overline{R}_+) \cap \mathrm{Stab}_G(R_-)$. Now the desired conclusion follows since M normalizes $\mathrm{Stab}_G(R_-)$ (because $M \leq \mathrm{Stab}_G(R_-)$) and M' normalizes $U(\overline{R}_+)$ (by Proposition 3.1).

We now show that the condition is sufficient. Assume that M has type (i) in Theorem 4.1 and that the condition of the lemma is satisfied. Let R_+ (resp. R_-) be the unique element of $\mathcal{S}_+(M)$ (resp. $\mathcal{S}_-(M)$) and let $T_+ \in \mathcal{S}_+(M')$. Since M and M' are distinct (or since U is nontrivial), the residues R_+ and T_+ are distinct. Now, $M \cap M'$ stabilizes $R := \mathrm{proj}_{R_+}(T_+)$, which is properly contained in R_+ by Corollary 2.8. In particular, the group U does not act trivially on R_+ since $M = (M \cap M').U$ does not stabilize R.

As before, let Σ be a twin apartment intersecting R_+ and R_-. Then $M = L^\Sigma(R_+)$ and we know that $(M, (U_\alpha)_{\alpha\in\Phi^\Sigma(R_+)})$ is a twin root datum of spherical type (W_J, J) for some $J \subseteq S$. Since U is normal in M and since it does not act trivially on R_+, we deduce from [3], Theorem 5 of Chapter IV that there exists a residue $R' \subseteq R_+$ such that U contains the group M_1 generated by all U_β with $\beta \in \Phi^\Sigma(R')$. But $\beta \in \Phi^\Sigma(R')$ implies $-\beta \in \Phi^\Sigma(R')$ and therefore M_1 is generated by subgroups of the form $\langle U_\beta \cup U_{-\beta}\rangle$. Since each group of the latter form is perfect by hypothesis, the group M_1 itself is perfect which contradicts the fact that it is contained in the nilpotent group U. This concludes the proof of the lemma. $\square$

5 The reduction theorem for isomorphisms which preserve bounded subgroups

In this section we state and prove a general theorem concerning isomorphisms between groups endowed with twin root data, which preserve bounded subgroups. Roughly speaking, it says that, under certain conditions, the isomorphism problem for groups with twin root data reduces to the isomorphism problem for groups of finite type. The main results of this paper will be deduced from it in the following two sections.

5.1 $\mathfrak{C}$-rigidity of twin root data

In order to make the statement of this theorem as precise and concise as possible, we introduce some additional terminology.

Let $\mathfrak{C}$ be a collection of twin root data. A twin root datum $\mathcal{D}$ is called **$\mathfrak{C}$-rigid** if the following holds (see Section 2.3.1 for the definition of $G^{\mathcal{D}}$):

If $\mathcal{D}' \in \mathfrak{C}$, then any isomorphism of $G^{\mathcal{D}}$ to $G^{\mathcal{D}'}$ induces an isomorphism of $\mathcal{D}$ to $\mathcal{D}'$.

Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi(W,S)})$ be a twin root datum, let Δ be the associated twin building, let Σ be the fundamental twin apartment and let c be a chamber of Σ. For each subset J of S, we set $L_J := L^\Sigma(\mathrm{Res}_J(c))$ and $\mathcal{D}_J := (L_J, (U_\alpha)_{\alpha\in\Phi^\Sigma(\mathrm{Res}_J(c))})$.

Given a collection $\mathfrak{C}$ as above, then we denote by $\mathfrak{C}_{\mathrm{sph}}$ the collection of all twin root data of the form $\mathcal{D}_J$ such that $\mathcal{D} \in \mathfrak{C}$ is of type $(W^{\mathcal{D}}, S^{\mathcal{D}})$ and J is a maximal spherical subset of $S^{\mathcal{D}}$. A twin root datum $\mathcal{D} \in \mathfrak{C}$ of type $(W^{\mathcal{D}}, S^{\mathcal{D}})$ is called **$\mathfrak{C}$-locally rigid** if for every every maximal spherical subset J of $S^{\mathcal{D}}$, the twin root datum $\mathcal{D}_J$ is $\mathfrak{C}_{\mathrm{sph}}$-rigid.

5.2 The result

Theorem 5.1. *Let $\mathcal{D}$ and $\mathcal{D}'$ be two twin root data which satisfy Conditions (P1), (P2) and (P3), and whose types are Coxeter systems of finite rank. Assume that $\mathcal{D}$ is $\{\mathcal{D}'\}$-locally rigid. If $\xi : G^{\mathcal{D}} \to G^{\mathcal{D}'}$ is an isomorphism which maps bounded subgroups of $G^{\mathcal{D}}$ to bounded subgroups of $G^{\mathcal{D}'}$, then ξ induces an isomorphism of $\mathcal{D}$ to $\mathcal{D}'$.*

Proof. Let M be a maximal bounded subgroup of $G^{\mathcal{D}}$. Then $\xi(M)$ is a maximal bounded subgroup of $G^{\mathcal{D}'}$ by the hypothesis on ξ. Moreover, it follows from Lemma 4.4 that M and $\xi(M)$ have the same 'type', where the 'type' of M, resp. $\xi(M)$ is given by either Case (i) or (ii) in Theorem 4.1.

Let now Σ be a twin apartment of the twin building Δ associated with $\mathcal{D}$ and let α be a twin root of Σ. Let also Δ' be the twin building associated with $\mathcal{D}'$. Choose a maximal residue of spherical type R intersecting Σ and such that $\alpha \in \Phi^{\Sigma}(R)$. By what we have just seen, we know that $\xi(L^{\Sigma}(R))$ has the form $L^{\Sigma'}(R')$ for some apartment Σ' of Δ' and some maximal residue of spherical type R' which intersects Σ'.

Since $\mathcal{D}$ is locally rigid, the restriction of ξ to $M = L^{\Sigma}(R)$ induces an isomorphism from the twin root datum $(L^{\Sigma}(R), (U_\beta)_{\beta\in\Phi^{\Sigma}(R)})$ to the twin root datum $(L^{\Sigma'}(R'), (U'_\beta)_{\beta\in\Phi^{\Sigma'}(R')})$, where we have used superscript $'$ to denote root groups of $\mathcal{D}'$. We may assume without loss of generality that

$$\{\xi(U_\beta) | \beta \in \Phi^{\Sigma}(R)\} = \{U'_\beta | \beta \in \Phi^{\Sigma'}(R')\}.$$

Let $H = \mathrm{Fix}_{G^{\mathcal{D}}}(\Sigma)$ and let $H' := \xi(H)$. Since $H = \bigcap_{\beta\in\Phi^{\Sigma}(R)} N_{L^{\Sigma}(R)}(U_\beta)$ (see Section 2.3.4) we deduce $H' = \bigcap_{\beta\in\Phi^{\Sigma'}(R')} N_{L^{\Sigma'}(R')}(U'_\beta)$. This implies that H' is the chamberwise stabilizer of Σ' in $G(\mathcal{D}')$.

Let now γ be a twin root of Σ which does not belong to $\Phi^{\Sigma}(R)$. Arguing as for α, we obtain that $\xi(U_\gamma) = U'_{\gamma'}$ where γ' is a twin root of Δ' which is contained in a twin apartment Σ'' whose chamberwise stabilizer is H'. On the other hand, our hypotheses imply that H' fixes a unique twin apartment chamberwise (see Lemma 2.11). In summary, we have shown that

$$\{\xi(U_\beta) | \beta \text{ is a twin root of } \Sigma\} = \{U'_\beta | \beta \text{ is a twin root of } \Sigma'\}.$$

Now the conclusion follows from Proposition 2.10. □

6 Kac-Moody groups over arbitrary fields

In this section and in the following one, we apply Theorem 5.1 to the case of Kac-Moody groups over fields in the strict sense.

It is known that a Kac-Moody group G over a field $\mathbb{K}$ naturally yields a twin root datum $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ which is locally split over $\mathbb{K}$ (namely which is locally split over $(\mathbb{K}_\alpha)_{\alpha\in\Phi}$, where $\mathbb{K}_\alpha = \mathbb{K}$ for each $\alpha \in \Phi$). We have also mentioned that if $\mathbb{K}$ has cardinality at least 4, then the conditions (P1), (P2) and (P3) of Section 2.3.5 are satisfied. In order to apply Theorem 5.1 to G, it remains to discuss the local rigidity of the twin root datum $\mathcal{D}$. This is done by using the classical theorems on isomorphisms between Chevalley groups, but the arguments are slightly different according as the ground field is finite or infinite.

6.1 Finite fields vs. infinite fields

The following result gives a handy criterion which distinguishes between these two cases.

Proposition 6.1. *Let G be a Kac-Moody group over a field $\mathbb{K}$. Then G is finitely generated if and only if $\mathbb{K}$ is finite.*

Proof. Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be the twin root datum which is naturally associated with G. Let $H := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$ and let $\Pi \subset \Phi$ be the (finite) set consisting of all roots $\phi \in \Phi = \Phi(W, S)$ such that the reflection s_ϕ belongs to S. It is known (and easy to see) that G is generated by the set $S_{\mathcal{D}} := H \cup \bigcup_{\alpha\in\Pi} U_\alpha$. Moreover, each U_α is isomorphic to the additive group $\mathbb{K}$ and the group H is a 'split $\mathbb{K}$-torus', namely it is isomorphic to a direct product of finitely many copies of the multiplicative group $\mathbb{K}^\times$.

If $\mathbb{K}$ is finite, then $S_{\mathcal{D}}$ is finite, whence G is finitely generated.

If G is finitely generated, then G is generated by a subset of $S_{\mathcal{D}}$. By [15], the defining relations satisfied by the elements of $S_{\mathcal{D}}$ in the group G involve only the ring structure of $\mathbb{K}$. Since no infinite field is a finitely generated ring, we deduce that $\mathbb{K}$ has to be finite. $\square$

6.2 The characteristic in the case of a finite ground field

The following result will spare us to worry about the exceptional isomorphisms between finite Chevalley groups.

Proposition 6.2. *Let G be an infinite Kac-Moody group over a finite field $\mathbb{K}$ of characteristic p. Let q be a prime. Then $p = q$ if and only if the set of orders of finite q-subgroups of G has no finite upper bound.*

Proof. Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be the twin root datum which is naturally associated with G.

Assume first that $p = q$. We must show that G possesses finite p-subgroups of arbitrary large orders. Since $\mathbb{K}$ is finite, the group U_α, which is isomorphic to the additive group $\mathbb{K}$, is finite for every $\alpha \in \Phi$. Since G is infinite by assumption, we deduce that the Coxeter group W is infinite. Let $\Delta = (\Delta_+, \Delta_-, \delta^*)$ be the twin building associated with $\mathcal{D}$ and let Σ be the fundamental twin apartment. Since Δ is of non-spherical type, we can find chambers x_+ and x_- of Σ such that $n := \ell(\delta^*(x_+, x_-))$ is arbitrarily large. On the other hand, we know by Lemma 3.2 that the group $U(x_+, x_-)$ may be written as a product of the form $U_{\beta_1}.U_{\beta_2}\dots U_{\beta_n}$ for certain twin roots $\beta_1, \dots, \beta_n$ of Σ. Since U_{β_i} is a finite p-group for each $1 \le i \le n$, it follows that $U(x_+, x_-)$ is a finite p-group of order at least p^n which yields the desired result.

We now assume that $p \neq q$. We must show that there is an upper-bound on the possible orders of finite q-subgroups of G. Let $Q \le G$ be such a finite q-group. By Proposition 2.2, the group Q is a bounded subgroup. Let $R_+ \subset \Delta_+$ and $R_- \subset \Delta_-$ be spherical residues which are stabilized by Q. Up to replacing R_+ and R_- by $\mathrm{proj}_{R_+}(R_-)$ and $\mathrm{proj}_{R_-}(R_+)$ respectively, we may assume that R_+ and R_- are parallel. Let Σ_Q be a twin apartment containing R_+ and R_-. By Proposition 3.6, we have $\mathrm{Stab}_G(R_+) \cap \mathrm{Stab}_G(R_-) = L^{\Sigma_Q}(R_+) \ltimes U(R_+, R_-)$. Hence there is a homomorphism $f : \mathrm{Stab}_G(R_+) \cap \mathrm{Stab}_G(R_-) \to L^{\Sigma_Q}(R_+)$. On the other hand, the order of every element of $U(R_+, R_-)$ is a power of p by Lemma 3.2 and Corollary 3.5. Since $p \neq q$ we deduce that Q and $f(Q)$ are isomorphic and, hence, we may assume that Q is contained in $L^{\Sigma_Q}(R_+)$. Now, up to conjugation by an element of G, we may assume that $L^{\Sigma_Q}(R_+) = L_J$ for some spherical subset J of S. Thus $|Q| \le \max\{|L_J| \mid J \subseteq S \text{ spherical}\}$ (note that each L_J is a finite Chevalley group over $\mathbb{K}$). The desired conclusion follows from the finiteness of S. $\square$

6.3 Isomorphisms of Kac-Moody groups

We are now able to apply Theorem 5.1 to Kac-Moody groups over fields. The following theorem is the main result of the introduction.

Theorem 6.3. *Let G and G' be infinite Kac-Moody groups over fields $\mathbb{K}$ and $\mathbb{K}'$ respectively, both of cardinality at least 4, and let $\mathcal{D}$ and $\mathcal{D}'$ be the corresponding twin root data. Let $\xi : G \to G'$ be a group isomorphism which maps bounded subgroups of G to bounded subgroups of G'. Then ξ induces an isomorphism of $\mathcal{D}$ to $\mathcal{D}'$.*

Proof. We have to show that ξ induces an isomorphism of $\mathcal{D}$ to $\mathcal{D}'$. By Theorem 5.1, it suffices to show that $\mathcal{D}$ is $\{\mathcal{D}'\}$-locally rigid. Let (W, S) and (W', S') be the respective types of $\mathcal{D}$ and $\mathcal{D}'$, let J and J' be maximal spherical subsets of S and S' and let $\varphi : L_J \to L_{J'}$ be a group isomorphism. We have to show that φ induces an isomorphism of $\mathcal{D}_J$ to $\mathcal{D}'_{J'}$.

By the definition of a Kac-Moody group, we know that L_J and $L_{J'}$ are Chevalley groups over $\mathbb{K}$ and $\mathbb{K}'$ respectively. Up to replacing L_J (resp. $L_{J'}$) by its derived subgroup modulo its center and $\mathcal{D}_J$ (resp. $\mathcal{D}'_{J'}$) by its *reduction* (see [17], Section 3.3 and [4], Section 3.13), we may assume that L_J (resp. $L_{J'}$) is an adjoint Chevalley group.

Since G and G' are isomorphic, it follows from Proposition 6.1 that $\mathbb{K}$ and $\mathbb{K}'$ are either both finite or both infinite. Moreover, if $\mathbb{K}$ and $\mathbb{K}'$ are finite, then they have the same characteristic in view of Proposition 6.2. Now, the desired result is a consequence of Theorem 31 in [14] if $\mathbb{K}$ and $\mathbb{K}'$ are finite and from Theorem 8.16 of [2] otherwise. □

The preceding theorem can be used to decompose automorphisms of a given Kac-Moody group in a product of automorphisms of five specific kinds, as mentioned in the introduction. For the precise definitions of these specific automorphisms and further comments on them, we refer the reader to Section 9 of [4].

Corollary 6.4. *Let G be a Kac-Moody group over a field $\mathbb{K}$ of at least 4 elements and associated with a generalized Cartan matrix A of indecomposable type. Then, any automorphism of G which preserve bounded subgroups can be written as a product of an inner, a sign, a diagonal, a graph and a field automorphism. Furthermore, if G is "simply connected in the weak sense" (see [15], Remark 3.7(c) p. 550) and if moreover, either* $\operatorname{char}(\mathbb{K}) = 0$ *or every off-diagonal entry of the generalized Cartan matrix A is prime to* $\operatorname{char}(\mathbb{K})$*, then the term 'graph automorphism' may be replaced by 'diagram automorphism' in the preceding statement.*

The proof goes along the lines of the proof of Theorem 2.7 of [4] and is omitted here.

7 Kac-Moody groups over finite fields

Corollary B of the introduction is a consequence of the following two results.

Theorem 7.1. *Let $\mathfrak{C}$ be the collection of all twin root data arising from Kac-Moody groups over finite fields of cardinality at least 4. Then any element of $\mathfrak{C}$ is $\mathfrak{C}$-rigid.*

Proof. Given any two elements $\mathcal{D}$ and $\mathcal{D}'$ of $\mathfrak{C}$, it is clear from Corollary 3.8 that every isomorphism of $G^{\mathcal{D}}$ to $G^{\mathcal{D}'}$ preserves bounded subgroups. The result is thus a consequence of Theorem 6.3. □

Corollary 7.2. *Let G be a Kac-Moody group over a finite field $\mathbb{K}$ of at least 4 elements and associated with a generalized Cartan matrix A of indecomposable type. Then, any automorphism of G can be written as a product of an inner, a sign, a diagonal, a graph* *and a field automorphism. Furthermore, if G is “simply connected in the weak sense” (see [15], Remark 3.7(c) p. 550) and if moreover, every off-diagonal entry of the generalized Cartan matrix A is prime to* $\mathrm{char}(\mathbb{K})$*, then the term ‘graph automorphism’ may be replaced by ‘diagram automorphism’ in the preceding statement.*

As for Corollary 6.4 above, the proof is omitted.

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