RANK ONE ISOMETRIES OF BUILDINGS AND QUASI-MORPHISMS OF KAC–MOODY GROUPS

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Abstract. Given an irreducible non-spherical non-affine (possibly non-proper) building X, we give sufficient conditions for a group $G < \mathrm{Aut}(X)$ to admit an infinite-dimensional space of non-trivial quasi-morphisms. The result applies in particular to all irreducible (non-spherical and non-affine) Kac–Moody groups over integral domains. In particular, we obtain finitely presented simple groups of infinite commutator width, thereby answering a question of Valerii G. Bardakov [MK99, Problem 14.13]. Independently of these considerations, we also include a discussion of rank one isometries of *proper* CAT(0) spaces from a rigidity viewpoint. In an appendix, we show that any homogeneous quasi-morphism of a locally compact group with integer values is continuous.

1. Introduction

Let G be a group. Recall that a **quasi-morphism** is a map $f : G \to \mathbf{R}$ such that

$$\sup_{g,h \in G} |f(gh) - f(g) - f(h)| < \infty.$$

A quasi-morphism is called **homogeneous** if its restriction to every cyclic subgroup is a homomorphism. The set $\mathrm{QH}(G)$ of all quasi-morphisms is naturally endowed with the structure of a real vector space. We denote by $\widetilde{\mathrm{QH}}(G)$ the vector space of **non-trivial quasi-morphisms**, namely

$$\widetilde{\mathrm{QH}}(G) = \mathrm{QH}(G) / \big(\ell^\infty(G) \oplus \mathrm{Hom}(G, \mathbf{R})\big).$$

The space $\widetilde{\mathrm{QH}}(G)$ naturally identifies to the kernel of the canonical map

$$H^2_{\mathrm{b}}(G, \mathbf{R}) \to H^2(G, \mathbf{R})$$

of the second bounded cohomology space with trivial coefficients to ordinary second cohomology. Groups G with vanishing $\widetilde{\mathrm{QH}}(G)$ include all amenable groups and all irreducible lattices in higher rank semisimple algebraic groups over local fields [BM02]. Opposite to these are groups with infinite-dimensional space of quasi-morphisms; they include non-elementary hyperbolic groups [EF97], mapping class groups of surfaces of higher genus [BF02] and outer automorphism groups of free groups [Ham08c], [BFe]. There exist groups G which have Kazhdan's property (T) such that $\widetilde{\mathrm{QH}}(G)$ is finite-dimensional but non-zero [MR06].

Before stating the main result of this paper, let us recall that a building of type (W, S) is a set X endowed with a map $\delta : X \times X \to W$ satisfying three simple axioms which are recalled in Sect. 5 below. The map δ is called the **Weyl distance**. A group Γ acting on X by automorphisms is said to be **Weyl-transitive** if for all $x, y, x', y' \in X$ with $\delta(x, y) = \delta(x', y')$ there exists $\gamma \in \Gamma$ such that $\gamma.x = x'$ and $\gamma.y = y'$.

Theorem 1.1. *Let (W, S) be an irreducible non-spherical and non-affine Coxeter system with S finite, X be a building of type (W, S) and G be a group acting on X by automorphisms. Assume that at least one of the following conditions is satisfied:*

(1) *The G-action on X is Weyl-transitive.*

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(2) *For some apartment* $A \subset X$, *the stabilizer* $\mathrm{Stab}_G(A)$ *acts cocompactly on* A.

Then $\widetilde{\mathrm{QH}}(G)$ *is infinite-dimensional.*

Remark 1.2.

(a) Notice that the building X is not assumed to be locally compact. Moreover X may contain flats of arbitrarily large dimension; in particular it need not be Gromov hyperbolic.

(b) The quasi-morphisms appearing in Theorem 1.1 take values in $\mathbf{Z}$ and extend to quasi-morphisms $\mathrm{Aut}(X) \to \mathbf{Z}$ defined over the full automorphism group of X. It is an amusing fact that *any homogeneous quasi-morphisms of a locally compact group with values in* $\mathbf{Z}$ *is continuous*; this will be shown in the appendix below. In the special case when X is locally compact, the bounded-open topology gives $\mathrm{Aut}(X)$ the structure of a locally compact (second countable) group. In particular, under the assumptions of Theorem 1.1, we deduce that the space of *continuous* non-trivial quasi-morphisms on $\mathrm{Aut}(X)$ is infinite-dimensional.

(c) Condition (1) in Theorem 1.1 implies in particular that G is far from discrete. We point out that, in the special case when X is Gromov hyperbolic, a related transitivity assumption (which is not logically correlated to Weyl transitivity, but which might look qualitatively stronger at a first sight) would automatically imply vanishing of $\widetilde{\mathrm{QH}}(G)$. Indeed, according to an unpublished result by N. Monod, independently established by Ursula Hamenstädt [Ham08a, Th. 4.1], if a group Γ admits a quasi-distance-transitive action on a Gromov-hyperbolic geodesic metric space, then $\widetilde{\mathrm{QH}}(\Gamma) = 0$ (for a related result see Corollary 3.2 below). By a **quasi-distance-transitive** action, we mean that there exists $C > 0$ such that for all $x, y, x', y' \in X$ with $d(x, y) = d(x', y')$, there exists $\gamma \in \Gamma$ such that $d(\gamma.x, x') \leq C$ and $d(\gamma.y, y') \leq C$.

The most important class of groups admitting Weyl-transitive actions on buildings of arbitrary type is provided by Kac–Moody groups. These groups are obtained by a functorial construction which associates a group functor on the category of commutative rings to any generalized Cartan matrix (or more generally to any Kac–Moody root datum), see [Tit87] and [Tit92] for the split case and [Rém02] for the almost split case.

Corollary 1.3. *Let* $\mathcal{G}$ *be a Kac–Moody-Tits functor whose Weyl group is irreducible non-spherical and non-affine and let* R *be an integral domain. Then* $\widetilde{\mathrm{QH}}(\mathcal{G}(R))$ *is infinite-dimensional. In particular* $\mathcal{G}(R)$ *possesses elements of strictly positive stable commutator length and is therefore of infinite commutator width.*

Recall that the **stable commutator length** of an element g, denoted by $\mathrm{scl}(g)$, of the commutator subgroup $[G, G]$ of a group G is defined as $\lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n}$, where $\mathrm{cl}(h)$ denotes the minimal number k such that $h \in [G, G]$ may be written as a product of k commutators. The **commutator width** of G is the supremum of the function cl on $[G, G]$. The connection between the second bounded cohomology of G, quasi-morphisms on G and the stable commutator length of elements in $[G, G]$ was discovered by Ch. Bavard [Bav91]. In particular, if $f(g) > 0$ for $f \in \widetilde{\mathrm{QH}}(G)$ and $g \in [G, G]$, then $\mathrm{scl}(g) > 0$. We refer to a recent monograph by D. Calegari [Cal] for more information on the stable commutator length.

A Kac–Moody group over a field of cardinality ≥ 4 is perfect. However even over the smallest fields, the abelianization is always finite (this follows from the last two paragraphs of the proof of Theorem 15 in [CR09]). The arguments of *loc. cit.* show that the group $\mathcal{G}(R)$ as in Corollary 1.3 generally admits no nontrivial finite quotient. In fact, when R is a finite field of order larger than the rank of the Weyl group, the group $\mathcal{G}(R)$ happens to be simple, and even finitely presented when the Weyl group is 2-spherical [CR09]. In particular, we obtain the following, which answers positively a question asked by Valerii G. Bardakov [MK99, Problem 14.13]:

Corollary 1.4. *There exists an infinite family of pairwise non-isomorphic finitely presented simple groups possessing elements of strictly positive stable commutator length; these groups have therefore infinite commutator width.*

Formerly finitely generated simple groups of infinite commutator width had been constructed by Alexey Muranov [Mur07].

As discussed in [CR09], properties of Kac–Moody groups over finite fields may be fruitfully compared to properties of higher rank arithmetic groups; this comparison highlights strong analogies between both families. Corollary 1.3 testifies for the fact that Kac–Moody groups also enjoy some form of hyperbolicity property, as opposed to the higher rank lattices. In order to stress this in a slightly different way, we include the following corollary, which follows immediately from Corollary 1.3:

Corollary 1.5. *Kac–Moody groups as in Corollary 1.3 do not have bounded generation. Moreover they are not boundedly generated by any family of torsion amenable subgroups.* □

The proof of Theorem 1.1 relies on a construction of quasi-morphisms for groups acting on CAT(0) spaces, elaborated by M. Bestvina and K. Fujiwara [BF07]. The conditions ensuring an infinite-dimensional space of quasi-morphisms are recalled in Sect. 2; the main one is the existence of *contracting isometries*, which by definition are isometries inducing a North-South dynamics on the boundary and generalize the *rank one isometries* as defined by W. Ballmann. The key geometric ingredients for the proof of Theorem 1.1 are a charaterization of contracting isometries (Theorem 5.1) and a criterion ensuring that two given contracting isometries are *independent* and *non-equivalent*. In fact, we believe that the hypotheses of Theorem 1.1 are unnecessarily strong for the existence of contracting isometries of buildings. To be more precise we propose the following:

Conjecture 1.6. *Let (W, S) be an irreducible non-spherical and non-affine Coxeter system with S finite, X be a building of type (W, S) and G be a group acting on X by automorphisms without fixing any point at infinity (in the* CAT(0) *realization of X). Then G either stabilizes a proper residue or contains a contracting isometry.*

This conjecture holds in the special case where W is Gromov hyperbolic, or more generally when W is relatively hyperbolic with respect to its maximal virtually Abelian subgroups. The latter condition is completely characterized in [Cap07] and therefore yields the following:

Proposition 1.7. *Assume that for any two infinite special subgroups $W_{J_1}, W_{J_2} < W$ such that $[W_{J_1}, W_{J_2}] = 1$, the group $\langle W_{J_1} \cup W_{J_2}\rangle$ is virtually abelian. Then any locally finite building of type (W, S) satisfies Conjecture 1.6.*

A major interest of a solution to Conjecture 1.6 is that, when combined with the Burger–Monod vanishing theorem [BM02, Theorems 20 and 21] and the Bestvina–Fujiwara construction, it would yield an interesting rigidity statement for higher rank lattices, in the same vein as those established in [BF02] and [Ham08c]. In order to illustrate this, we mention the following result, which should be compared to [Ham08b, Theorem 2].

Theorem 1.8. *Let X be a proper* CAT(0) *space and $G < \mathrm{Is}(X)$ be any group of isometries. Assume that G contains a rank one element. Let $\overline{G}$ be the closure of G in $\mathrm{Is}(X)$ with the compact-open topology. Then one of the following assertions holds, where Λ denotes the limit set of G:*

(1) *G either fixes a point in ∂X or stabilizes a geodesic line; in both cases, it possesses a subgroup of index at most 2 with infinite Abelianization.*

(2) *$\overline{G}$ acts transitively on $\Lambda \times \Lambda - \Delta$, where Δ denotes the diagonal.*
(3) *$\overline{G}$ does not act transitively on $\Lambda \times \Lambda - \Delta$, and the spaces $\widetilde{\mathrm{QH}}(G)$ and $\widetilde{\mathrm{QH}}_c(\overline{G})$ are both infinite-dimensional.*

Furthermore, if X has cocompact isometry group, then (1) *implies that $\overline{G}$ is amenable and* (2) *implies that the space of continuous nontrivial quasi-morphisms $\widetilde{\mathrm{QH}}_c(\overline{G})$ vanishes.*

Theorem 1.8 has the following consequence, which directly relates to Conjecture 1.6:

Corollary 1.9. *Let $\Gamma < G = \prod_{\alpha \in A} \mathbf{G}_\alpha(k_\alpha)$ be an irreducible lattice, where $|A| > 1$, $(k_\alpha)_{\alpha \in A}$ is a finite family of local fields and the $\mathbf{G}_\alpha$ are connected simply connected k_α-almost simple groups of k_α-rank > 1. Let X be a proper* CAT(0) *space and $\varphi : \Gamma \to \mathrm{Is}(X)$ be any homomorphism. Then $\varphi(\Gamma)$ does not contain any rank one element.*

In particular, combining Proposition 1.7 with Corollary 1.9, one obtains:

Corollary 1.10. *Let Γ be as in Corollary 1.9 and X be a locally finite building whose type satisfies the condition of Proposition 1.7. Then any Γ-action on X by automorphisms stabilises a residue of spherical or Euclidean type.* □

Let us finally mention that, independently of M. Bestvina and K. Fujiwara, Ursula Hamenstädt developed a slightly different approach providing a general axiomatic setting for groups acting on topological spaces by homeomorphisms to admit an infinite-dimensional space of non-trivial quasi-morphisms [Ham08c]. It turns out that her approach, when applied to the present context, would also provide information on the second bounded cohomology with nontrivial coefficients.

The paper is organized as follows. Section 2 is preliminary. The aim of Section 3 is the proof of Theorem 1.8 and its corollaries. It contains several geometrical results on rank one isometries of proper CAT(0) spaces which might be of some independent interest. Sect. 4 is devoted to Coxeter groups; its main purpose is to show that irreducible Coxeter groups which are not virtually abelian contain many contracting isometries in their natural action on the Davis complex. Finally, hyperbolic isometries of buildings are studied in Sect. 5.

Convention. In order to avoid any confusion, we remark that quasi-morphisms are sometimes called "quasi-homomorphisms" in the literature (for example in [BF02],[BF07]). This explains the notation QH for the space of quasi-morphisms; and $\widetilde{\mathrm{QH}}$ for the non-trivial ones in this paper, following [BF02],[BF07]. We note however that in the monograph [Cal], the space of quasi-morphisms is denoted by $\hat{Q}$ and the space of homogeneous quasi-morphisms is denoted by Q, while the latter is denoted by HQH (for '*homogeneous quasi-homomorphisms*') in [BF02],[BF07]. In the present paper, no special notation is used for the space of homogeneous quasi-morphisms.

Stable commutator length of g is denoted by $\mathrm{scl}(g)$ in this paper, but it is sometimes called '*stable length*', and also denoted by $\|g\|$ (as for example in [Bav91]).

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2. Rank one elements, contracting isometries and quasi-morphisms

Let X be a CAT(0) space. A geodesic line L in X is said to have **rank one** if it does not bound a flat half-plane. The line L is said to be B**-contracting** for some $B \geq 0$ if for every metric ball C disjoint from L the projection $\pi_L(C)$ has diameter at most B.

An isometry $\gamma \in \mathrm{Is}(X)$ is said to have **rank one** if it is hyperbolic and if some (and hence any) of its axes has rank one. Similarly γ is called B**-contracting** if it is hyperbolic and if some of its axes is B-contracting. It is called **contracting** if it is B-contracting for some $B \geq 0$.

Recall from [BF07, Thm. 5.4] that if X is proper, then an isometry has rank one if and only if it is B-contracting for some $B \geq 0$. We will see later (see Theorem 5.1) that for some class of finite-dimensional CAT(0) spaces, this assertion holds even without the properness assumption.

Following [BF07], we will use:

Definition 2.1. Let $\gamma_1, \gamma_2 \in \Gamma$ be hyperbolic elements and fix a base point $x_0 \in X$.

The elements γ_1 and γ_2 are called **independent** if the map

$$\mathbf{Z} \times \mathbf{Z} \to [0, \infty) : (m, n) \mapsto d(\gamma_1^m.x_0, \gamma_2^n.x_0)$$

is proper.

The elements γ_1 and γ_2 are called Γ**-equivalent** (notation: $\gamma_1 \sim_\Gamma \gamma_2$) if the following condition holds: there exist $\delta > 0$, a sequence (g_n) in Γ and two sequences of positive integers $m_1(n)$ and $m_2(n)$ tending to infinity with n, such that the geodesic segments $[x_0, \gamma_1^{m_1(n)}.x_0]$ and $g_n.[x_0, \gamma_2^{m_2(n)}.x_0]$ are δ-Hausdorff equivalent.

Notice that both properties are independent of the choice of the base point. Furthermore two elements γ_1 and γ_2 are independent if and only if γ_1 and γ_2^{-1} are independent. When the Γ-action is proper, both notions can be made more precise:

Lemma 2.2. *Let Γ act properly discontinuously on a complete* CAT(0) *space X. Then:*

(i) *Two hyperbolic elements γ_1, γ_2 are independent if and only if the canonical attracting fixed point γ_1^+ of γ_1 at infinity is distinct from both the attracting and the repulsive fixed point of γ_2 at infinity.*
(ii) *Two contracting elements γ_1, γ_2 satisfy $\gamma_1 \sim_\Gamma \gamma_2$ if and only if some positive powers of γ_1 and γ_2 are conjugate.*
(iii) *If two contracting elements γ_1 and γ_2 are not independent, then $\gamma_1 \sim_\Gamma \gamma_2$ or $\gamma_1 \sim_\Gamma \gamma_2^{-1}$.*
(iv) *If Γ is non-elementary and contains a contracting element, then it contains two contracting elements γ_1 and γ_2 such that $\gamma_1 \not\sim_\Gamma \gamma_2$ and $\gamma_1 \not\sim_\Gamma \gamma_2^{-1}$.*

Assertion (i) means in other words that given two rank one elements, either their axes are parallel (and the element are dependent) or the respective attracting and repulsive fixed points of the two elements at infinity form four distinct points of the visual boundary ∂X.

Proof. (i). The 'only if' part is clear. Let γ_1 and γ_2 be hyperbolic elements with non-parallel axes, say ℓ_1 and ℓ_2 respectively, and assume for a contradiction that γ_1 and γ_2 are not independent. Then ℓ_1 and ℓ_2 contain rays $\rho_1 \subset \ell_1$ and $\rho_2 \subset \ell_2$ which are at finite Hausdorff distance from one another. By properness of the Γ-action, it follows that there exist integers $m \neq m'$ and $n \neq n'$ such that $g_1^m g_2^n = g_1^{m'} g_2^{n'}$. Thus $g_1^{m''} = g_2^{n''}$ for some nonzero m'' and n''. This implies that ℓ_1 and ℓ_2 are at finite Hausdorff distance from one another, hence parallel. This is absurd.

(ii). See [BF07, Prop. 6.5(3)].

(iii). Follows from (i), (ii) and the fact that the stabilizer the parallel set of any axis is virtually cyclic by properness.

(iv). Follows from [BF07, Prop. 6.2 and 6.5(4)]. □

Remark 2.3. Important to observe is that, when the action is discrete as above, *the property of being* Γ*-inequivalent is conjugacy–invariant.* More precisely, given $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_1 \not\sim_\Gamma \gamma_2$ and $\gamma_1 \not\sim_\Gamma \gamma_2^{-1}$ as in (iv), then, for any $g \in \Gamma$, we have $\gamma_1 \not\sim_\Gamma g\gamma_2 g^{-1}$ and furthermore γ_1 and $g\gamma_2 g^{-1}$ are independent. This follows from (ii) and (iii) in Lemma 2.2.

The construction of quasi-morphisms that we will use was performed by M. Bestvina and K. Fujiwara [BF07, Th. 6.3]; notice that there is no discreteness assumption whatsoever on the action:

Proposition 2.4. *Let* $\Gamma < \mathrm{Is}(X)$ *be any group of isometries of a complete* CAT(0) *space* X*. Assume that* Γ *contains two independent rank-one elements which are not* Γ*-equivalent. Then* $\widetilde{\mathrm{QH}}(\Gamma)$ *is infinite-dimensional.* □

3. Proper CAT(0) spaces with rank one isometries and rigidity of higher rank lattices

The present section is aimed at proving Theorem 1.8; there is no logical dependence between this and the subsequent sections.

Given a proper CAT(0) space X, the compact-open topology gives $\mathrm{Is}(X)$ the structure of a locally compact second countable topological group.

3.A. The stabilizer of a point at infinity fixed by a rank one element.

Proposition 3.1. *Let* X *be a proper* CAT(0) *space with cocompact isometry group and* $G < \mathrm{Is}(X)$ *be any group of isometries. Let* $\xi \in \partial X$ *be a point fixed by some rank one element of* $\mathrm{Is}(X)$*. Then the stabilizer* $\overline{G}_\xi$ *is amenable.*

Proof. By the very nature of the statement to be established, there is no loss of generality in assuming that G is closed. If G stabilizes a geodesic line, then the desired conclusion clearly holds. We assume henceforth that G does not stabilize any line. In particular, the existence of a rank one element implies that G does have a global fixed point at infinity. Thus X possesses a nonempty minimal G-invariant closed convex subset $Y \subseteq X$ (see [CM08, Prop 4.1]). The point-wise stabilizer of Y being compact, hence amenable, there is no loss of generality in assuming that $Y = X$.

By [CM08, Th. 5.7] the group G is either an almost connected simple Lie group or totally disconnected. In the former case, the desired result follows from [CM08, Th. 7.4]. In the latter case, the result follows from [Cap09, Th. 1.5] since for any fixed point ξ of a rank one isometry, the transversal space X_ξ as defined in *loc. cit.* is bounded. □

The following statement parallels Proposition 6.4 in [Ham08b]:

Corollary 3.2. *Let* X *be a proper* CAT(0) *space with cocompact isometry group and* $G < \mathrm{Is}(X)$ *be a closed group of isometries with limit set* Λ*. Assume that* G *contains a rank one element. If* G *acts transitively on* $\Lambda \times \Lambda - \Delta$*, then the space of continuous nontrivial quasi-morphisms* $\widetilde{\mathrm{QH}}_c(G)$ *vanishes.*

Proof. Let $g \in G$ be the given rank one element and $a, b \in \partial X$ denote its fixed points. Let also $G_{\{a,b\}}$ denote the stabilizer of the pair $\{a, b\}$ in G. By assumption, for each $h \in G$ there exists $g' \in G$ such that $g'.h \in G_b$, and we may choose $g' \in G_a$ or $g' \in G_{\{a,b\}}$ according as $h(b) \neq a$ or $h(b) \neq a$. This shows that the group G is a product

$$G = G_a \cdot G_{\{a,b\}} \cdot G_b.$$

By Proposition 3.1 all subgroups G_a, G_b and $G_{\{a,b\}}$ are amenable. In particular the spaces $H^2_{\mathrm{cb}}(G_a, \mathbf{R})$, $H^2_{\mathrm{cb}}(G_b, \mathbf{R})$ and $H^2_{\mathrm{cb}}(G_{\{a,b\}}, \mathbf{R})$ vanish and any continuous nontrivial quasi-morphism of G_a, G_b or $G_{\{a,b\}}$ is bounded. Thus the same holds for G as desired. □

Remark 3.3. If X is a CAT(0) cell complex with finitely many types of cells and $G < \mathrm{Is}(X)$ acts by cellular transformations, then Proposition 3.1 and Corollary 3.2 remain true without the hypothesis that X has cocompact isometry group: this follows from the same reasoning as above, using a straightforward adaptation of the arguments in [Cap09].

3.B. **On the existence of independent rank one elements.**

Proposition 3.4. *Let X be a proper* CAT(0) *space and $G < \mathrm{Is}(X)$ be any subgroup. Assume that G contains a rank one element. Then one of the following assertions holds:*

(1) *G either fixes a point in ∂X or stabilize a geodesic line; in both cases, it possesses a subgroup of index at most 2 with infinite Abelianization. Furthermore, if X has cocompact isometry group, then $\overline{G} < \mathrm{Is}(X)$ is amenable.*
(2) *G contains two independent rank one elements; in particular $\overline{G}$ contains a discrete non-Abelian free subgroup.*

Proof. Any rank one element acts on the boundary at infinity with a North-South dynamics, see [Bal95, Lem. 3.3.3]. Therefore, if $\overline{G}$ contains two independent rank one elements then the existence of a discrete non-Abelian free subgroup follows from a standard ping-pong argument. We assume henceforth that G does not contain any pair of independent rank one elements. In particular any two rank one elements of G have a common fixed point in ∂X.

We claim that every triple of rank one elements of G have a common fixed point at infinity. Otherwise there would exist three rank one elements g_1, g_2, g_3 such that g_1 and g_2 have a common attracting fixed point, say a, and there respective repelling fixed points b_1 and b_2 are precisely the fixed points of g_3. Conjugating g_3 by a large positive power of g_1^{-1}, we then obtain a rank one element which is independent of g_2. This is a contradiction.

>From this claim, it follows that all rank one elements of G have a common fixed point at infinity, say ξ. In particular, the normal subgroup $N \lhd G$ generated by all rank one elements of G fixes ξ. Since any rank one element of G has exactly two fixed points at infinity, it follows that N has at most two fixed points as well. In particular G has a subgroup of index at most 2 which fixes ξ, and the Busemann character centered at ξ yields a homomorphism of this subgroup taking values in $\mathbf{R}$ (see *e.g.* [Cap09, §4.3]).

Passing to the closure, we deduce that $\overline{G}$ has a closed subgroup of index at most 2 which fixes ξ. Thus $\overline{G}$ is amenable as soon as $\mathrm{Is}(X)$ is cocompact by Proposition 3.1 and Assertion (1) holds. □

Proposition 3.5. *Let X be a proper* CAT(0) *space and $G < \mathrm{Is}(X)$. Assume that G contains two independent rank one elements. Then the set of pairs of fixed points of rank one elements of G is dense in $\Lambda \times \Lambda - \Delta$, where Λ denotes the limit set of G and $\Delta \subset \Lambda \times \Lambda$ the diagonal.*

Proof. The proof goes along the same lines as that of [BB95, Th. 4.1]. For the reader's convenience, we include the details of the argument.

Following *loc. cit.*, we shall say that a pair of points $\xi, \eta \in \partial X$ is **dual** (relative to G) if for all neighbourhoods U and V of ξ and η in the visual compactification $\overline{X}$, there exists $g \in G$ such that

$$g(\overline{X} - U) \subset V \quad \text{and} \quad g^{-1}(\overline{X} - V) \subset U.$$

Notice that the set of points which are dual to some fixed $\xi \in \partial X$ is closed (with respect to the cone topology).

The relevance of this notion comes from the following. The two fixed points of any given rank one element are dual to each other: this follows from [Bal95, Lem. 3.3.3]. Conversely, the results of [Bal95, Sect. III.3] imply that if $\{\xi, \eta\} \subset \partial X$ is a dual pair, then there exists a sequence of rank one elements $g_n \in G$ such that the attracting and repelling fixed points of g_n tend to ξ and η as n tends to infinity. All we need to show is thus that any point of Λ is dual to any other.

Let $\gamma_1, \gamma_2 \in G$ be two independent rank one elements, and a_1, a_2 (resp. b_1, b_2) denote their respective attracting (resp. repelling) fixed points. By considering products of the form $\gamma_1^m.\gamma_2^n$ for appropriately chosen integers m, n, one shows that any two distinct points in $\{a_1, a_2, b_1, b_2\}$ are dual to one another.

Let now $\xi \in \Lambda - \{a_1, a_2, b_1, b_2\}$. Pick a base point $x_0 \in X$ and choose a sequence $(g_n)_{n \geq 0}$ of elements of G such that $\lim_n g_n.x_0 \to \xi$. By [BB95, Lem. 4.4] we have $\lim_n g_n.a_1 = \xi$ or $\lim_n g_n.b_1 = \xi$ (or both). Thus $g'_n = g_n \gamma_1 g_n^{-1}$ is a sequence of rank one elements such that the corresponding sequence of attractive or repelling fixed points converges to ξ. Upon multiplying each g'_n by an appropriate power of γ_2 if necessary, we may extract a subsequence (g'_{n_k}) such that g'_{n_k} is independent from γ_1 for each $k \geq 0$ and which still enjoys the same convergence property of its attracting or repelling fixed points. The preceding paragraph then shows that ξ is dual to both a_1 and b_1. From a symmetric argument we deduce that ξ is also dual to both a_2 and b_2.

What we have done so far shows that for any four-tuple of points of Λ which are pairwise dual, any other point of Λ is dual to each of them. In view of the hypothesis, this implies that any two points of Λ are dual. □

3.C. **Quasi-morphisms and rigidity.**

Lemma 3.6. *Let X be a proper* CAT(0) *space and $G < \mathrm{Is}(X)$ be a closed subgroup with limit set Λ. For every hyperbolic isometry $g \in G$ with attracting and repelling fixed points $a \neq b \in \Lambda$, the G-orbit of (a, b) is a closed subset of $\Lambda \times \Lambda$.*

Proof. Identical to the proof of [Ham08b, Lem. 6.1]. □

Proof of Theorem 1.8. Assume that G is *non-elementary*, that is to say G does not fix a point at infinity and does not stabilize any geodesic line. By Proposition 3.4 it follows that G contains two independent rank one elements. Therefore the set of pairs of fixed points of rank one elements of G is dense in $\Lambda \times \Lambda - \Delta$.

Assume now that $\overline{G}$ does not act transitively on $\Lambda \times \Lambda - \Delta$. In view of Lemma 3.6, we deduce from the density assertion of the preceding paragraph that G contains two rank one elements g_1, g_2 with respective attracting and repelling fixed points (a_1, b_1) and (a_2, b_2), such that the $\overline{G}$-orbit of (a_1, b_1) is distinct from the $\overline{G}$-orbit of (a_2, b_2). It follows that g_1 and g_2 are $\overline{G}$-inequivalent.

Notice moreover that $b_1 \neq b_2$. Indeed, since $g_1^n.a_2$ tends to a_1 as n tends to infinity, the equality $b_1 = b_2$ would imply that (a_2, b_2) is in the same $\overline{G}$-orbit as (a_1, b_1), which is absurd. Similarly one shows that the four points a_1, a_2, b_1, b_2 are pairwise distinct. In particular g_1 and g_2 are independent.

Therefore, we may apply Proposition 2.4, which shows that $\widetilde{\mathrm{QH}}(G)$ and $\widetilde{\mathrm{QH}}(\overline{G})$ are infinite-dimensional. Since the Bestvina–Fujiwara construction yields quasi-morphisms with integer values, it follows from Theorem A.1 below that $\widetilde{\mathrm{QH}}_c(\overline{G})$ is infinite-dimensional as well.

The last two assertions of Theorem 1.8 follow from Proposition 3.1 and Corollary 3.2 since the limit set of G coincides with the limit set of its closure $\overline{G}$. □

Proof of Corollary 1.9. Suppose that $\varphi(\Gamma)$ contains a rank-one isometry. Since Γ has Kazhdan's property (T), every finite index subgroup of Γ has finite Abelianization. On the other hand, we know by [BM02, Theorems 20 and 21] that $\widetilde{\mathrm{QH}}(\Gamma) = 0$. Theorem 1.8 thus implies that $H = \overline{\varphi(\Gamma)}$ is transitive on the pair of distinct points of its limit set Λ.

Let $Y \subseteq X$ be a nonempty closed convex H-invariant subset; such a subspace exists since H has no fixed point at infinity in view of Proposition 3.4. The version of Monod's superrigidity theorem given in [CM08, Theorem 8.4] (which may be applied since Γ has property (T) and is square-integrable [Sha00]) provides a continuous homomorphism $\varphi : G \to \mathrm{Is}(Y)$ extending the given $\varphi : \Gamma \to \mathrm{Is}(Y)$.

Notice that since Y admits a rank one isometry by assumption, it is irreducible. Since Γ acts minimally, it follows from [CM08, Th. 1.6] that the continuous map $\varphi : G \to \mathrm{Is}(Y)$ factors through some simple factor of G, say G_α.

Given a semisimple element $h \in G_\alpha$ which is not *periodic* (*i.e.* the cyclic subgroup $\langle h \rangle$ is not relatively compact), the image $\varphi(h)$ is not an elliptic isometry, since any continuous nontrivial homomorphism of a simple algebraic group to a locally compact second countable group is proper [BM96, Lemma 5.3]. In particular the limit set of $\langle \varphi(h) \rangle$ in Y is a nonempty subset of Λ, thus consisting of 1 or 2 points.

Let now $P < G_\alpha$ be any proper parabolic subgroup. By [Pra77, Lemma 2.4] there exists a non-periodic semisimple element h such that

$$P = \mathscr{Z}_{G_\alpha}(h) \cdot U(h),$$

where $\mathscr{Z}_{G_\alpha}(h)$ is the centraliser of h in G_α and $U(h) \lhd P$ is the *contraction group* defined by

$$U(h) = \{g \in G_\alpha \mid \lim_{n\to\infty} h^n g h^{-n} = 1\}.$$

It follows that the limit point of the sequence $(h^{-n}.y_0)_{n\geq 0}$, where $y_0 \in Y$ is any base point, is P-invariant. We have thus established that any proper parabolic subgroup of G_α fixes some point of Λ. In particular the stabilizer of any $\xi \in \Lambda$ in G_α is a parabolic subgroup, since any subgroup containing a parabolic is itself parabolic. In view of [BB95, Lemma 4.4] the pointwise stabilizer of a triple of points of Λ in $\mathrm{Is}(Y)$ has a fixed point in Y and is thus compact. Since $\varphi : G_\alpha \to \mathrm{Is}(Y)$ is proper, it follows that any parabolic subgroup of G_α has at most 2 fixed points in Λ. Thus any minimal parabolic subgroup of G_α is contained in at most two maximal ones. This is absurd since G_α is simple and has k_α-rank ≥ 2. $\square$

Remark 3.7. It should be noted that the assumption that the group G has at least two simple factors accounts for the corresponding assumption in the version of the superrigidity theorem for higher rank lattices that we appeal to. It is reasonable to conjecture the conclusion of Corollary 1.9 still holds for lattices in higher rank *simple* groups; in fact, apart from the aforementioned superrigidity, all arguments of the proof remain valid in that more general context.

4. Rank one elements in Coxeter groups

Let (W, S) be a Coxeter system such that the Coxeter group W is finitely generated; equivalently S is finite. We denote by Σ the associated Davis complex; it is endowed with proper CAT(0) metric and a natural properly discontinuous cocompact W-action by isometries (See [Dav98]. This fact is proved by Moussong [Mou88]). We view the elements of W as isometries of Σ; thus by a *hyperbolic element* of W (resp. rank one, B-contracting, etc.) we wean an element which acts on Σ as a hyperbolic (resp. rank one, B-contracting, etc.) isometry.

Lemma 4.1. *Let $\gamma \in W$ and $x, y \in \Sigma$ such that $\gamma.x = y$. Then γ belongs to the group $W(x, y)$ generated by all those reflections which fix some point of the geodesic segment $[x, y]$.*

Proof. Let C_x be a chamber containing x and define $C_y = \gamma.C_x$. It is well known (and easy to see) that γ belongs to the group generated by all those reflections which fix a wall separating C_x from C_y. Clearly every wall separating C_x from C_y meets $[x, y]$; the result follows. $\square$

4.A. **Parabolic closures and essential elements.** Recall that a subgroup of W of the form W_J for some $J \subset S$ is called a **standard parabolic subgroup**. Any of its conjugates is called a **parabolic subgroup** of W. A basic fact on Coxeter groups is that any intersection of parabolic subgroups is itself a parabolic subgroup. This allows one to define the **parabolic closure** $\mathrm{Pc}(R)$ of a subset $R \subset W$: it is the smallest parabolic subgroup of W containing R (see [Kra09]).

An element $w \in W$ is called **standard** if its parabolic closure is a standard parabolic subgroup. It is called **cyclically reduced** if $\ell(w) = \min\{\ell(\gamma w \gamma^{-1}) \mid \gamma \in W\}$. The following elementary result shows in particular that any cyclically reduced element is standard:

Proposition 4.2. *Let $w \in W$. We have the following:*

(i) *If w is standard, then for any writing of w as a reduced word $w = s_1 s_2 \cdots s_{\ell(w)}$ with letters in S, we have $\mathrm{Pc}(w) = \langle s_1, s_2, \dots, s_{\ell(w)} \rangle$.*

(ii) *Let $x \in W$ be such that $x\,\mathrm{Pc}(w)x^{-1}$ is standard. Assume moreover that*
$$\ell(x) = \min\{\ell(\gamma) \mid \gamma \in W,\ \gamma\,\mathrm{Pc}(w)\gamma^{-1} \text{ is standard}\}.$$
If $x \neq 1$, then $\ell(xwx^{-1}) < \ell(w)$.

(iii) *Let $s \in S$. If w is standard, then either $\mathrm{Pc}(ws) \subset \mathrm{Pc}(w)$ or $\mathrm{Pc}(ws) = \langle \mathrm{Pc}(w) \cup \{s\} \rangle$ and ws is standard as well.*

Proof. (i). This is well known; it is an immediate consequence of the solution to the word problem [Tit69].

(ii). Set $P = \mathrm{Pc}(w)$. Let $J \subset S$ be such that $xPx^{-1} = W_J$ and set $R = x^{-1}W_J$. We view (the 1-skeleton of) Σ as a chamber system (see [Wei03]); the chambers are the elements of W. The coset R is the J-residue of Σ containing x^{-1}. Note that $\mathrm{Stab}_W(R) = P$. The condition that x minimizes the length of all elements which conjugate P to a standard parabolic subgroup means precisely that x^{-1} is the combinatorial projection of 1 onto R (see [Wei03, Th. 3.22] for the notion of projections onto residues). Thus $x^{-1} = \mathrm{proj}_R(1)$. Let also $y = \mathrm{proj}_R(w)$. Since R is P-invariant we have $w.x^{-1} = y$. Thus $d(x^{-1}, y) = \ell(xwx^{-1})$. Furthermore, in view of basic properties of the combinatorial projection [Wei03, Th. 3.22], every wall which separates x from y also separates 1 from w. This implies that $d(x^{-1}, y) \leq d(1, w) = \ell(w)$. Therefore, if $\ell(xwx^{-1}) \geq \ell(w)$, then we deduce $\ell(xwx^{-1}) = \ell(w)$ and the set $\mathscr{M}(x, y)$ of walls separating x from y coincides with the set $\mathscr{M}(1, w)$. We have to show that $x = 1$.

Let $s_1 \cdots s_{\ell(w)}$ be a reduced word representing w. Notice that the reflections associated to walls in $\mathscr{M}(1, w)$ are precisely
$$s_1,\ s_1 s_2 s_1,\ \dots\ ,\ s_1 \cdots s_{\ell(w)} \cdots s_1$$
since $w = s_1 s_2 \dots s_{\ell(w)}$. By the above, each of these reflections stabilizes R (see [Wei03, Prop. 4.10]) and, hence, belongs to P. This shows that $P \supset \langle s_1, s_2, \dots, s_{\ell(w)} \rangle$. Since the group generated by $s_1, s_2, \dots, s_{\ell(w)}$ is clearly a parabolic subgroup which contains w, we deduce $P = \mathrm{Pc}(w) = \langle s_1, s_2, \dots, s_{\ell(w)} \rangle$. In particular P is standard and hence x must be trivial, as desired.

(iii). By assumption w is standard. This implies that $Q := \langle \mathrm{Pc}(w) \cup \{s\} \rangle$ is a standard parabolic subgroup. Since w and s both belong to Q, it follows that $\mathrm{Pc}(ws) \subset Q$. If $s \in \mathrm{Pc}(w)$, then $Q = \mathrm{Pc}(w)$ and, hence $\mathrm{Pc}(ws) \subset \mathrm{Pc}(w)$. We assume henceforth that $s \notin \mathrm{Pc}(w)$. In particular $\mathrm{Pc}(w)$ is properly contained in Q; more precisely the ranks of $\mathrm{Pc}(w)$ and Q differ by 1.

If $wsw^{-1} \in \mathrm{Pc}(ws)$, then $w = wsw^{-1}.ws$ belongs to $\mathrm{Pc}(ws)$ and hence $\mathrm{Pc}(w) \subset \mathrm{Pc}(ws)$. Since s does not belong to $\mathrm{Pc}(w)$, we obtain
$$\mathrm{Pc}(w) \subsetneq \mathrm{Pc}(ws) \subset Q,$$
which implies that $\mathrm{Pc}(ws) = Q$ since these parabolic subgroups have the same rank. In particular ws is standard.

Assume now that $wsw^{-1} \notin \mathrm{Pc}(ws)$. Set $P' = \mathrm{Pc}(ws)$ and choose a residue R' whose stabilizer in W is P', in a similar way as in the proof of (ii). The condition that wsw^{-1} does not stabilize R' means that the projections $\mathrm{proj}_{R'}(w)$ and $\mathrm{proj}_{R'}(ws)$ must coincide. Arguing as in the proof of (ii), we deduce that every walls separating $\mathrm{proj}_{R'}(1)$ from $\mathrm{proj}_{R'}(ws)$ also separates 1 from w. Since w is standard, this implies in view of (i) that $\mathrm{Pc}(ws)$ is contained in $\mathrm{Pc}(w)$, as desired. $\square$

An element $\gamma \in W$ is called **essential** if $\mathrm{Pc}(\gamma) = W$. The following result appears in [Par07, Thm. 3.4]; we give an alternative argument:

Corollary 4.3. *Let $s_1, \dots, s_n$ be all the elements of S (in any order). Then $w = s_1 \cdots s_n$ is essential.*

Proof. For each $k = 1, \dots, n$, let $w_k = s_1 \cdots s_k$. An immediate induction using Proposition 4.2 shows that $\mathrm{Pc}(w_k) = \langle s_1, \dots, s_k \rangle$. □

4.B. **Walls separating a flat half-space.** Euclidean flats in Σ have been studied in [CH09]. The following parallels some results from *loc. cit.* in the case of flat half-planes:

Proposition 4.4. *Let H be a flat half-plane in Σ bounded by a periodic line L and denote by P the parabolic closure of the set of reflections fixing some point of L. Then:*

(i) *We have*
$$P \cong K \times P_1 \times \cdots \times P_k,$$
where each P_i is an infinite parabolic subgroup and K is a finite parabolic subgroup fixing L pointwise. Moreover, if none of the P_i's is of affine type and rank ≥ 3, then $k \geq 2$.

(ii) *The parabolic subgroup P contains every element $\gamma \in W$ which maps some point of L into L.*

Proof. Part (i) follows by adapting the arguments from [CH09] (see also [Cap07, Prop. 3.1]).

Part (ii) follows from (i) and Lemma 4.1. □

4.C. **Irreversible rank one elements of Coxeter groups.** There are obvious obstructions for a given element $\gamma \in W$ to have rank one: namely, if γ is contained in a parabolic subgroup $P < W$ which is of spherical or affine type, or which splits as a direct product of two infinite subgroups, then clearly γ cannot have rank one. The following shows that these are in fact the only obstructions:

Proposition 4.5. *An element $\gamma \in W$ does not have rank one if and only if γ is contained in a parabolic subgroup $P < W$ such that either P is finite, or P splits as $P = P_1 \times P_2$ where P_1 and P_2 are both infinite parabolic subgroups, or P splits as $P = K \times P_{\mathrm{aff}}$ where K is a finite parabolic subgroup and P_{aff} is an affine parabolic of rank ≥ 3.*

Proof. The 'if' part is clear; we focus on the 'only if' part. Let thus $\gamma \in W$ be an element which does not have rank one. If γ is not hyperbolic, then γ is of finite order and hence contained in a finite parabolic subgroup as is well known. We may therefore assume that γ is hyperbolic and the desired assertion is provided by Proposition 4.4. □

Corollary 4.6. *Let $\gamma \in W$ be an element of infinite order and $L \subset \Sigma$ be an axis of γ. Then:*

(i) *Either γ is rank-one or there exists $\gamma' \in W$ such that $\langle \gamma, \gamma' \rangle \cong \mathbf{Z} \times \mathbf{Z}$.*
(ii) *γ is rank-one if and only if its centralizer is virtually cyclic.*
(iii) *L is rank-one if and only if it is not contained in a periodic 2-flat.*

Proof. The first assertion follows from Proposition 4.5. The second assertion follows from the first and the easy fact that rank one elements have virtually cyclic centralizer. Assertion (iii) follows from (i) and the flat torus theorem [BH99, Thm. II.7.1]. □

Corollary 4.7. *The group W contains two rank-one elements γ_1, γ_2 such that $\gamma_1 \not\sim_W \gamma_2$ and $\gamma_1 \not\sim_W \gamma_2^{-1}$ if and only if W has an irreducible non-spherical non-affine parabolic subgroup of finite index.*

Notice that a parabolic subgroup of finite index in W is necessarily a direct factor.

Proof. Follows from Lemma 2.2, Corollary 4.3 and Proposition 4.5, and the fact that for proper CAT(0) spaces, a hyperbolic element is contracting if and only if it is rank one, see [BF07, Th. 5.4]. □

Given a hyperbolic element $\gamma \in W$, we say that γ is **irreversible** if $\gamma \not\sim_W \gamma^{-1}$. In the case of Coxeter groups, there is a simple algebraic criterion which may be used to detect irreversibility:

Lemma 4.8. *A rank one element $\gamma \in W$ is irreversible if and only if no positive power of γ can be written as a product $\gamma^k = a.b$ where $a, b \in W$ have order 2.*

Remark 4.9. Lemma 4.8 can be used to obtain the following refinement of Corollary 4.3: if W is infinite, irreducible and non-affine, then the Coxeter element is irreversible as soon as the Coxeter diagram of (W, S) is not a star, i.e. there is no element $s \in S$ such that the parabolic subgroup $W_{S\setminus\{s\}}$ is a finite elementary abelian 2-group.

Proof of Lemma 4.8. If $\gamma^k = a.b$ with a, b involutions for some $k > 0$, then γ^k is conjugate to γ^{-k} and, hence, $\gamma \sim \gamma^{-1}$ by [BF07, Prop. 6.5(3)]. Thus γ is not irreversible.

Suppose now that γ is not irreversible. Then by properness W possesses an element g which stabilises some γ-axis L and satisfies $g\gamma g^{-1}|_L = \gamma_L^{-1}$. By Selberg's lemma W possesses a torsion free normal subgroup of finite index, which acts thus freely on Σ. Let $k > 0$ be such that γ^k belongs to this finite index subgroup. Then $g\gamma^k g^{-1} = \gamma^{-k}$. Since W acts properly on Σ, the subgroup of W which stabilises L is virtually cyclic; thus we may and shall assume that g^2 acts trivially on L. In particular g is a torsion element of W of even order, say $2m$. Notice that $g^m\gamma g^{-m}|_L = \gamma_L^{-1}$, whence $g^m\gamma^k g^{-m} = \gamma^{-k}$. We set $a = g^m = g^{-m}$ and $b = g^m\gamma^k$. Then clearly $\gamma^k = a.b$ and $a^2 = 1 = b^2$, as desired. □

5. Rank one isometries of buildings

5.A. **Definitions and basic facts.** Let (W, S) be a Coxeter system. A **building** of type (W, S) is a set $\mathcal{C}$ endowed with a map $\delta : \mathcal{C} \times \mathcal{C} \to W$ submitted to the following conditions, where $x, y \in \mathcal{C}$ and $w = \delta(x, y)$:

- **(Bu1):** $w = 1$ if and only if $x = y$;
- **(Bu2):** if $z \in \mathcal{C}$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) = w$ or ws, and if, furthermore, $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$;
- **(Bu3):** if $s \in S$, there exists $z \in \mathcal{C}$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

The map δ is called the **Weyl distance**. An **automorphism** of $(\mathcal{C}, \delta)$ is a permutation of $\mathcal{C}$ which preserves the Weyl distance. The map $\delta_W : W \times W \to W$ defined by $\delta_W(x, y) = x^{-1}y$ turns canonically W into a building of type (W, S). Any subset of a building $(\mathcal{C}, \delta)$ of type (W, S) which is Weyl-isometric to (W, δ_W) is called an **apartment**. Given a subset $J \subseteq S$, we denote $W_J = \langle J \rangle$. Given any chamber $c \in \mathcal{C}$, the set

$$\mathrm{Res}_J(c) = \{x \in \mathcal{C} \mid \delta(c, x) \in W_J\}$$

is called the **residue** of type J containing c. An important fact is that a residue of type J, endowed with the appropriate restriction of the Weyl distance, is a building of type (W_J, J). We refer to [Wei03] for the general theory.

Another relevant fact is that any building of type (W, S) possesses a geometric realization as a CAT(0) metric space [Dav98]. In other words, given a building $\mathscr{B} = (\mathcal{C}, \delta)$ of type (W, S), there exists a CAT(0) space $X_{\mathscr{B}}$ and a canonical injection $\mathrm{Aut}(\mathscr{B}) \to \mathrm{Is}(X_{\mathscr{B}})$. We will identify all elements of $\mathrm{Aut}(\mathscr{B})$ to their image in $\mathrm{Is}(X_{\mathscr{B}})$.

5.B. **A characterization of rank one elements.**

Theorem 5.1. *Let $\mathscr{B} = (\mathcal{C}, \delta)$ be a building of type (W, S) and let $\gamma \in \mathrm{Aut}(\mathscr{B})$ be a hyperbolic element. Then the following assertions are equivalent:*

(i) *γ is a rank one isometry of $X_{\mathscr{B}}$.*
(ii) *γ is contracting.*
(iii) *γ does not stabilise any residue whose Weyl group is of the form $W_I \times W_J$, where either W_I and W_J are both infinite, or W_I is affine and W_J is finite.*

Proof. Let L be an axis of γ and A be an apartment containing L; such an apartment exists by [CH09, Thm. E]. Let π denote the nearest point projection to L and $C \subset L$ be a compact segment which is a fundamental domain for the $\langle\gamma\rangle$–action on L.

(i) $\Rightarrow$ (ii) Suppose for a contradiction that γ is a rank one isometry but that L is not B-contracting for any B. Then there exist sequences (x_n) and (y_n) in $X_{\mathscr{B}}$ such that $d(x_n, y_n) < d(x_n, L)$ and $d(\pi(x_n), \pi(y_n))$ tends to infinity with n. Since $L = \langle\gamma\rangle.C$, we may and shall assume that $\pi(x_n)$ is contained in C for all n. Upon extracting a subsequence, we may further assume that there exists a chamber c containing a point $p_0 \in L$ which does not meet C and which separates C from $\pi(y_n)$ for all n. We denote by

$$\rho = \rho_{A,c}$$

the retraction onto A centered at c. Recall that this map is the identity on A, it does not increase distances and its restriction to any geodesic segment emanating from a point of c (and more generally to any apartment containing c) is an isometry onto its image (see [AB08, §4.4]). A further extraction allows one to assume that the sequences $(\rho(x_n))$ and $(\rho(y_n))$ both converge to boundary points of A, say ξ and η respectively.

The choice of the base point p_0 and the contracting property of the retraction ρ guarantee that p_0 separates $\pi(\rho(x_n))$ from $\pi(\rho(y_n))$ for all n. Since moreover we have $d(\rho(x_n), \rho(y_n)) \leq d(x_n, y_n) < d(x_n, L) < d(x_n, p_0) = d(\rho(x_n), p_0)$, we deduce that the limit point $\xi = \lim \rho(x_n)$ does not belong to ∂L, since otherwise $\pi(\rho(x_n))$ and $\pi(\rho(y_n))$ would lie on the same side of p_0 for all large n.

We now claim that $\eta \in \partial L$. Indeed, consider $\eta' = \lim \pi(y_n) \in \partial L$. If $\eta \neq \eta'$, then there is a rank one geodesic line joining η to η' in A (see [Bal95, Lemma III.3.3]). In view of [Bal95, Lemma III.3.1], this implies the existence of some constant $K > 0$ such that $d(p_0, \pi(y_n)) + d(p_0, \rho(y_n)) - K \leq d(\pi(y_n), \rho(y_n))$ for all n. We infer that

$$d(p_0, \pi(y_n)) + d(p_0, y_n) - K \leq d(\pi(y_n), \rho(y_n)) \leq d(\pi(y_n), y_n) \leq d(p_0, y_n)$$

for all n, which contradicts the fact that $d(p_0, \pi(y_n))$ is unbounded. The claim stands proven.

The claim implies that the sequence of geodesic segments $[\rho(x_n), \rho(y_n)]$ converges to a rank one geodesic line joining ξ to η. Therefore there exists a constant $K' > 0$ such that $d(p_0, \rho(x_n)) + d(p_0, \rho(y_n)) - K' \leq d(\rho(x_n), \rho(y_n)) \leq d(x_n, y_n) < d(x_n, L) < d(x_n, p_0) = d(\rho(x_n), p_0)$ for all n. This is absurd since $d(p_0, \rho(y_n)) = d(p_0, y_n)$ tends to infinity with n.

(ii) $\Rightarrow$ (iii) Assume that γ stabilises a residue R whose Weyl group is of the form $W_I \times W_J$ as in (iii). Since R is a building of type $(W_I \times W_J, I \cup J)$, it follows that the Tits boundary of its CAT(0) realisation X_R has diameter π and, hence, does not contain any rank one isometry. In particular X_R has no B-contracting isometry. The result follows, since X_R is isometrically embedded in $X_{\mathscr{B}}$.

(iii) $\Rightarrow$ (i) Suppose that γ is not a rank one isometry; in other words some γ-axis L is contained in a flat half-plane, say H. The arguments of [CH09, Thm. 6.3] show that H is contained in an apartment A. Let $\rho = \rho_{A,c}$ be the retraction onto A centered at some chamber c intersecting L. Proposition 4.4 implies that $\rho \circ \gamma|_A$ is an isometry of A contained in a parabolic subgroup of the form $W_I \times W_J$ as in the statement of (iii). This implies that γ stabilises the residue of type $I \cup J$ containing c, thereby contradicting (iii). $\square$

5.C. **Existence of rank one elements in Weyl–transitive groups.** In order to deal with the question of existence, we shall transfer to the whole building the constructions performed so far at the level of apartments. An essential tool in doing this is the retraction that we have just considered.

As before, let $\mathscr{B} = (\mathcal{C}, \delta)$ be a building of type (W, S) and $g_1, g_2 \in \mathrm{Aut}(\mathscr{B})$ be rank one elements. For $i \in \{1, 2\}$ let also L_i be an axis of g_i, A_i be an apartment containing L_i, $c_i \in L_i$ be any point and $\rho_i = \rho_{A_i,c_i}$ be the retraction onto A_i centred at c_i. Then $\gamma_i := \rho_i \circ g_i|_{A_i} : A_i \to A_i$ is an automorphism of the apartment A_i.

Recall that any apartment is isomorphic to the Davis complex Σ, i.e. the standard CAT(0) realization of the thin building (W, δ_W). We now would like to compare γ_1 and γ_2 as elements of $W = \mathrm{Aut}(W, \delta_W) < \mathrm{Is}(\Sigma)$. In order to do this properly, we need to choose identifications $A_i \cong \Sigma$ and make sure that our considerations are independent of this choice.

Crucial to us is the following:

Lemma 5.2. *For $i \in \{1, 2\}$, let $f_i : A_i \cong \Sigma$ be any isomorphism (of thin buildings).*

If the elements g_1 and g_2 are $\mathrm{Aut}(\mathscr{B})$-equivalent, then $f_1\gamma_1 f_1^{-1}$ and $f_2\gamma_2 f_2^{-1}$, viewed as elements of W, are W-equivalent.

If the elements g_1 and g_2 are not independent, then $f_1\gamma_1 f_1^{-1} \sim_W f_2\gamma_2 f_2^{-1}$ or $f_1\gamma_1 f_1^{-1} \sim_W f_2\gamma_2^{-1} f_2^{-1}$.

Proof. The first thing to observe is that any modification of the isomorphism $f_1 : A_1 \cong \Sigma$ amounts to replacing the element $f_1\gamma_1 f_1^{-1} \in W$ by a W-conjugate. In view of Remark 2.3, the assertion of Lemma 5.2 is thus clearly independent of the choices of the f_i's. In order to avoid unnecessarily heavy notation, we shall henceforth identify both A_1 and A_2 to Σ by means of f_1 and f_2 respectively and, hence, omit to write the maps f_1 and f_2. In other words, the elements γ_1 and γ_2 will be viewed as elements of W acting on Σ.

Fix a chamber $c_i \subset A_i$ such that c_i meets L_i. Upon replacing respectively g_1 and g_2 by some positive powers, we may and shall assume further that

$$\delta_W(c_i, \gamma_i^n.c_i) = \delta_W(c_i, \gamma_i.c_i)^n \tag{5.i}$$

for all $n > 0$ and $i = 1, 2$. Since furthermore the chambers $g_i^n.c_i$ and $\gamma_i^n.c_i$ intersect in a point of L_i, the Weyl group element $\delta(g_i^n.c_i, \gamma_i^n.c_i)$ is contained in some standard finite parabolic subgroup of W for all $n > 0$ and $i = 1, 2$; in particular it is of uniformly bounded length. We deduce that there exist an element $\varepsilon_{i,n} \in W$ of uniformly bounded length such that

$$\delta(c_i, g_i^n.c_i) = \delta_W(c_i, \gamma_i^n.c_i)\varepsilon_{i,n} \tag{5.ii}$$

for all $n > 0$ and $i = 1, 2$.

Suppose now that g_1 and g_2 are $\mathrm{Aut}(\mathscr{B})$-equivalent. Then there exist a constant $D > 0$, a sequence (g_n) in $\mathrm{Aut}(\mathscr{B})$ and two sequences of positive integers $m_1(n)$ and $m_2(n)$ tending to infinity with n such that

$$\ell \circ \delta(g_n.c_1, c_2) < D \quad \text{and} \quad \ell \circ \delta\big(g_n g_1^{m_1(n)}.c_1,\ g_2^{m_2(n)}.c_2\big) < D \tag{5.iii}$$

for all $n > 0$, where $\ell : W \to \mathbf{N}$ denotes the word length with respect to the Coxeter generating set S. From (5.i), (5.ii) and (5.iii), we deduce that there exist two sequences (a_n), (b_n) of elements of W, of uniformly bounded length, such that

$$w_2^{m_2(n)} = a_n.w_1^{m_1(n)}.b_n$$

for all $n > 0$, where $w_i = \delta_W(c_i, \gamma_i.c_i) \in W$. Therefore, upon extracting a subsequence, we obtain two elements $a, b \in W$ such that $w_2^{m_2(n)} = a.w_1^{m_1(n)}.b$ for all $n > 0$.

Upon replacing γ_1 and γ_2 by a W-conjugate, we may assume that $c_1 = c_2$ and that this chamber corresponds to the identity element of W (recall that the 1-skeleton of Σ is nothing but the Cayley graph of (W, S)). This choice of parametrization yields $\gamma_1 = w_1$ and $\gamma_2 = w_2$.

Let now $L_1^+, L_2^+ \in \partial\Sigma$ be the respective attracting fixed points of w_1, w_2 at infinity. Set $c_0 := c_1 = c_2$. Since $w_2^n.c_0 \to L_2^+$ while $w_1^n b.c_0 \to L_1^+$ at the limit when n tends to infinity, if follows from the equality $w_2^{m_2(n)} = a w_1^{m_1(n)} b$ that $a.L_1^+ = L_2^+$. Thus $a w_1 a^{-1}$ and w_2 have the same attracting fixed point at infinity, namely L_2^+. By Lemma 2.2, this implies that $w_1 \sim_W w_2$.

Assume that that g_1 and g_2 are not independent. In other words the axes L_1 and L_2 contain respectively rays which are asymptotic to each other. It follows that upon replacing

g_1 and g_2 by appropriate nonzero powers (5.iii) holds with $g_n \equiv 1$ for some $D > 0$ and all $n \geq 0$. The same argument as above can be repeated and now yields either $w_1 \sim_W w_2$ or $w_1 \sim_W w_2^{-1}$. □

Proposition 5.3. *Let $\mathscr{B} = (\mathcal{C}, \delta)$ be a building of irreducible type (W, S) and $G < \mathrm{Aut}(\mathscr{B})$ be a group of automorphisms acting Weyl–transitively on the chambers. Then G contains two independent rank-one elements g_1, g_2 such that $g_1 \not\sim_{\mathrm{Aut}(\mathscr{B})} g_2$ if and only if W has an irreducible non-spherical non-affine parabolic subgroup of finite index.*

Proof. The 'only if' part is clear since, the Tits boundary of $X_{\mathscr{B}}$ is then either empty or of Tits diameter π. Suppose now that W has an irreducible non-spherical non-affine parabolic subgroup of finite index. Then, by Corollary 4.7, the group W contains two rank one elements γ_1, γ_2 such that $\gamma_1 \not\sim_W \gamma_2$ and $\gamma_1 \not\sim_W \gamma_2^{-1}$. Furthermore, the latter property remains valid if we replace γ_1 and γ_2 by any nonzero power or any W-conjugate, see Lemma 2.2 Remark 2.3. Therefore, we may and shall assume that some axis of γ_i $(i = 1, 2)$ contains a point in the relative interior of a fixed base chamber c of the CAT(0) realisation of (W, δ_W). Let $w_i = \delta_W(c, \gamma_i.c)$.

Fix now an apartment A of $\mathscr{B}$, which identify it with (W, δ_W). In this way we view $c_0, \gamma_i.c_0$ and $\gamma_i^2.c$ as chambers of $\mathscr{B}$, for $i = 1, 2$. By hypothesis G contains an element g_i such that $g_i.c = \gamma_i.c$ and $g_i^2.c = \gamma_i^2.c$. Since some γ_i-axis contains a point in the relative interior of c, there exists a point x_i in the relative interior of $\gamma_i.c$ such that $\angle_{x_i}(\gamma_i^{-1}.x_i, \gamma_i.x_i) = \pi$. This implies that $\angle_{x_i}(g_i^{-1}.x_i, g_i.x_i) = \pi$; in other words the points $\{g_i^n.x_i\}_{n\in\mathbf{Z}}$ are collinear, and hence g_i is a hyperbolic isometry, an axis of which contains x_i.

Let A_i be an apartment containing c and some axis L_i of γ_i; such an apartment exists by [CH09, Th. E]. Let $\rho_i = \rho_{A_i,c}$ be the retraction onto A centred at c. Then $\rho_i \circ g_i$ is an automorphism of A_i which maps c to $g_i.c = \gamma_i.c$ and, hence, coincides with γ_i if we identify A to A_i by an means of isomorphism which fixes c. In particular, it follows from Proposition 4.5 that the g_i-axis L_i is not contained in any residue whose Weyl group has the form $W_I \times W_J$ with W_I and W_J either both infinite or both virtually abelian. By Theorem 5.1, this implies that $g_i \in G$ is a rank one isometry of $X_{\mathscr{B}}$. Now the fact that g_1 and g_2 are independent and $\mathrm{Aut}(\mathscr{B})$-inequivalent follows from Lemma 5.2 in view of the definition of γ_1 and γ_2. □

We are now ready for the:

Proof of Theorem 1.1. If G is Weyl-transitive, then by Proposition 5.3, the group G contains two independent rank one elements which are not $\mathrm{Aut}(\mathscr{B})$-equivalent.

If $\mathrm{Stab}_G(A)$ acts cocompactly on some apartment A, we may choose two elements of $\mathrm{Stab}_G(A)$ whose action on A coincides with some powers of the elements provided by Corollary 4.7. These two elements of $\mathrm{Stab}_G(A)$ are rank one for the same reason as in the proof of Proposition 5.3 above; they are independent and $\mathrm{Aut}(\mathscr{B})$-inequivalent by Lemma 5.2.

In view of Theorem 5.1, we may apply Proposition 2.4, which yields the desired conclusion. □

Proof of Corollary 1.3. When R is a field, the Kac–Moody group $\mathcal{G}(R)$ acts Weyl-transitively on each of its two buildings. When R is a domain, we consider the action of $\mathcal{G}(R)$ on either of the two buildings $\mathscr{B}_+$ and $\mathscr{B}_-$ associated with $\mathcal{G}(k)$, where k is a field in which R embeds. Since $\mathcal{G}(R)$ already contains the Weyl group of $\mathcal{G}(k)$, it follows that $\mathcal{G}(R)$ acts transitively on the chambers of the standard apartment of both $\mathscr{B}_+$ and $\mathscr{B}_-$. In all cases, the fact that $\widetilde{\mathrm{QH}}(\mathcal{G}(R))$ is infinite-dimensional follows from Theorem 1.1.

The assertion on the stable commutator length now follows from [Bav91], while the assertion on the commutator width follows from a straightforward verification. □

Remark 5.4. It follows in particular that a rank one element of the Weyl group of $\mathcal{G}(R)$ acts as a contracting isometry on both $\mathscr{B}_+$ and $\mathscr{B}_-$.

Proof of Corollary 1.4. Immediate from Corollary 1.3, the simplicity result in [CR09] and the fact that Kac–Moody groups of different types over non-isomorphic finite fields are non-isomorphic [CM06, Cor. B]. □

5.D. **A special case: buildings with isolated residues.** The aim of this section is to prove Proposition 1.7. We first need an existence result for hyperbolic isometries of proper Gromov hyperbolic metric spaces. It is certainly well known to the experts; however we could not find a reference where it is explicitly stated in the literature. We therefore include a detailed proof.

Proposition 5.5. *Let X be a proper Gromov hyperbolic geodesic metric space and $G <$ $\mathrm{Is}(X)$ be any group of isometries. Then one of the following assertions holds:*

(1) *G contains a hyperbolic isometry.*
(2) *G has a bounded orbit.*
(3) *G has a unique fixed point at infinity.*

Proof. Let δ be a constant of hyperbolicity for the space X. We assume that G does not contain any hyperbolic isometry.

We start with the special case when G is countable. We may then write G as the union of an increasing chain of finite subsets $S_1 \subset S_2 \subset \dots$. By [Kou98, Proposition 3.2], for each n the set P_n consisting of those points $x \in X$ such that $d(g.x, x) \leq 100\delta$ for all $g \in S_n$, is nonempty. If each P_n meets some fixed bounded subset of X, then $\bigcap_n P_n$ is nonempty since X is proper and, hence, G has a bounded orbit. Otherwise, denoting by $\overline{X}$ the visual compactification $X \cup \partial X$, the intersection $\bigcap_n \overline{P_n}$ is a subset of ∂X which is pointwise fixed by G. If this subset contains more than 2 points then G has a bounded orbit; if it contains exactly two points then G acts by translation along the geodesic lines joining them and, since G has no hyperbolic element, we conclude again that G has a bounded orbit. Thus we are done in this special case.

We now turn to the general case and assume moreover that G has no bounded orbit. In view of the above we may assume that for every countable subgroup H of G the set P_H of those points $x \in X$ such that $d(g.x, x) \leq 100\delta$ for all $g \in H$, is nonempty. The same arguments as before then yield the desired conclusion. □

Proof of Proposition 1.7. By [Cap07, Corollary E] a building X of type (W, S) as in the statement possesses isolated Euclidean residues. Thus it admits a realization as a proper Gromov hyperbolic geodesic metric space $|X|$ on which $\mathrm{Aut}(X)$ acts by isometries, and such that the Euclidean residues correspond in a canonical way to the parabolic points at infinity of $|X|$, see [Bow99]. The desired result now follows from Proposition 5.5. □

Remark 5.6. The results of [Cap07] provide in fact a complete characterization of those buildings which are relatively hyperbolic with respect to some family of (non-necessarily Euclidean) residues. The arguments above show that Conjecture 1.6 holds in that more general context. The remaining open case of buildings whose Weyl group is *not* relatively hyperbolic with respect to any family of finitely generated subgroups is especially intriguing.

Appendix A. On homogeneous quasi-morphisms of locally compact groups with integer values

The purpose of this appendix is to prove the following.

Theorem A.1. *Let G be a locally compact group. Then any homogeneous quasi-morphism $\varphi : G \to \mathbf{Z}$ is continuous.*

It was observed by Roger Alperin that the solution to Hilbert fifth problem implies that any homomorphism of a locally compact group to $\mathbf{Z}$ is continuous (this follows from [Alp82, Corollary 3]). The above statement shows that this holds more generally for homogeneous quasi-morphisms. A remarkable result of a similar nature has been established in [BIW08, Lemma 7.4], asserting that for any locally compact group G, a homogeneous Borel quasi-morphism $G \to \mathbf{R}$ is continuous. Notice that non-homogeneous quasi-morphisms are generally discontinuous.

We start with a basic consequence of homogeneity.

Lemma A.2. *Let $\varphi : G \to \mathbf{R}$ be a homogeneous quasi-morphism of a group G, which vanishes on a normal subgroup N. Then φ descends to a homogeneous quasi-morphism of the quotient G/N.*

Proof. Given $g \in G$ and $n \in N$, we claim that $\varphi(g) = \varphi(g \cdot n)$. Indeed, for each integer $k > 0$, there exists $n_k \in N$ such that $(g \cdot n)^k = g^k \cdot n_k$. Therefore, we have

$$\begin{aligned} \varphi(g \cdot n) &= \lim_{k\to\infty} \frac{\varphi\big((gn)^k\big)}{k} \\ &= \lim_{k\to\infty} \frac{\varphi\big(g^k \cdot n_k\big)}{k} \\ &\leq \lim_{k\to\infty} \Big(\frac{\varphi(g^k)+D}{k}\Big) \\ &= \varphi(g), \end{aligned}$$

where D is a constant depending only on φ. In particular, we have also $\varphi(g \cdot n^{-1}) \leq \varphi(g)$, and hence $\varphi(gn) = \varphi(g)$. The desired conclusion follows. □

The next step is to consider totally disconnected groups, the key point being the compact case.

Lemma A.3. *Let G be a profinite group. Then any homogeneous quasi-morphism $\varphi : G \to \mathbf{Z}$ is constant.*

Proof. Assume for a contradiction that φ is not constant and let $g \in G$ be such that $\varphi(g) \neq 0$. Let H be the closure of $\langle g \rangle$ in G. Thus H is a pro-cyclic group. In particular it is Abelian and, hence, amenable as an abstract group. It follows that the restriction of φ to H is a homomorphism. Since $\mathbf{Z}$ is residually finite, the kernel of the restriction of φ to H is an intersection of finite index subgroups of H. By [Ser94, §4.2], any finite index subgroup of H is closed. This shows that the restriction of φ to H is continuous. Since H is compact, we deduce $\varphi(H) = 0$, a contradiction. □

Recall from [Kap54, p. 55] (see also [HRN56, Satz 4]) that a compact Abelian group is connected if and only if it is divisible (in fact, the latter holds for non-Abelian groups as well, see [Myc57, Corollary 2]). From this and the preceding two lemmas, we deduce the following.

Lemma A.4. *Let G be a compact group. Then any homogeneous quasi-morphism $\varphi : G \to \mathbf{Z}$ is constant.*

Proof. Let G° denote the neutral component of G. We claim that the restriction of φ to G° vanishes. As in the proof of Lemma A.3, it is enough to prove this fact in the case G° is Abelian. As recalled above, a compact connected Abelian group is divisible. Thus $\varphi(G^\circ) = 0$ since a divisible group admits no nonzero homogeneous quasi-morphism. It follows from Lemma A.2 that φ descends to a quasi-morphism of the group of components G/G°, and the desired conclusion now follows from Lemma A.3. □

The last and most important step is the following. It relies on the structure theory of locally compact connected groups.

Lemma A.5. *Let G be a connected locally compact group. Then any homogeneous quasi-morphism $\varphi : G \to \mathbf{Z}$ is constant.*

Proof. By [MZ55, Theorem 4.3], the group G possesses a compact normal subgroup K such that G/K is a Lie group. In view of Lemmas A.2 and A.4, there is no loss of generality in assuming $K = 1$. We suppose henceforth that G is a Lie group. Let R denote its soluble radical. Thus R is a connected Lie group which is amenable as an abstract group. In particular the restriction of φ to R is a homomorphism, and we deduce $\varphi(R) = 0$ since R is generated by one-parameter subgroups. By Lemma A.2, we may thus further assume that G is semi-simple. Appealing again to Lemma A.4, it is enough to deal with the case when G is simple and non-compact. Let P be a minimal parabolic subgroup of G. Then P is soluble-by-compact and, hence, as before the restriction of φ to P vanishes. The Bruhat decomposition now implies that every element of G is a bounded product of elements of a finite number of conjugates of P. Thus $\varphi(G)$ is bounded, whence constant by homogeneity. □

Proof of Theorem A.1. Let G° denote the neutral component of G. By Lemma A.5, the restriction of φ to G° vanishes. Thus Lemma A.2 ensures that φ descends to a homogeneous quasi-morphism of the group of components to $\mathbf{Z}$. In other words, it suffices to prove the theorem in the case when G is totally disconnected. It is enough to show that $\varphi^{-1}(0)$ is open. By [Bou71, III §4 No 6], the group G possesses some compact open subgroup Q. Thus Q is a profinite group and Lemma A.3 shows that $\varphi(Q) = 0$. Thus $\varphi^{-1}(0)$ is indeed open and we are done. □

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