Non-discrete simple locally compact groups

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Abstract. Simple Lie groups and simple algebraic groups over local fields are the most prominent members of the class $\mathscr{S}$ of compactly generated non-discrete simple locally compact groups. We outline a new trend, which emerged in the past decade, whose purpose is the study of $\mathscr{S}$ as a whole.

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C'est en quelque sorte une loi historique que les propriétés générales des groupes simples ont presque toutes été vérifiées d'abord sur les différents groupes et qu'on a ensuite cherché et trouvé une raison générale dispensant de l'examen des cas particuliers.[1]

Élie Cartan, 1936 ([29, p. 199])

1. Introduction

Through the history of their developments, locally compact groups provide a beautiful illustration of the unity of mathematics. Initiated at the turn of the 20th century under the impetus of Hilbert's fifth problem, their investigation led to the creation of topological algebra, laid the foundations of abstract harmonic analysis, and revealed the relevance of measure and integration, as well as ergodic theory, to classical number theory. In his 1946 article on the Future of Mathematics [73], André Weil underlines how the work of Siegel, continuing the great tradition of Dirichlet, Hermite and Minkowski, opened the way to a systematic study of discrete groups of arithmetic nature by means of the continuous groups in which they naturally embed. The tremendous developments in the study of lattices in semi-simple Lie and algebraic groups that occurred in the following 70 years show the extent to which Weil's statement was accurate (see [51] and references therein).

The fascinating properties of discrete subgroups of semi-simple Lie and algebraic groups have provided landmarks shaping the development of geometric group theory. Although the universe embraced by geometric group theory is endless and full of dark zones (see [9]), its investigation often refers to arithmetic lattices as a

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[1] *It is a kind of historical law that the general properties of the simple groups have been verified first in the various groups, and afterward one has sought and found general explanations that do not require the examination of special cases.*

reference frame guiding the intuition, to which the novelty of the phenomena observed can be confronted (see [36]). One may wish to formalize the special heuristic role played by higher rank arithmetic groups by finding purely algebraic properties characterizing them among all discrete groups. While this problem has been addressed and solved in characteristic 0 (see [44]), its resolution in characteristic $p > 0$ requires understanding which properties isolate, among non-discrete locally compact groups, the Lie and algebraic groups in which arithmetic groups embed as lattices. This leads us to the point of view adopted in this paper, which consists in considering the simple Lie groups and the simple algebraic groups over local fields as members of a much broader class, denoted by $\mathscr{S}$, comprising all non-discrete, compactly generated, locally compact groups that are **topologically simple**, i.e. whose only closed normal subgroups are the trivial subgroup and the whole group. The goal of this paper is to present an overview of results and problems pertaining to a new trend, which emerged in the past decade, whose purpose is the study of $\mathscr{S}$ as a whole.

1.1. Normal subgroups of lattices. The first compelling stride in the study of $\mathscr{S}$ is the following iconic theorem obtained by U. Bader and Y. Shalom in 2006.

Theorem 1.1 (Bader–Shalom [3]). *Let $G = G_1 \times \cdots \times G_n$ be the direct product of $n \geq 2$ compactly generated, non-discrete, locally compact groups without non-trivial closed normal subgroup isomorphic to $\mathbf{R}^d$. Let $\Gamma \leq G$ be a uniform lattice whose image under the natural projection $G \to G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n$ is dense for every i.*

If G_i is (hereditarily) just-non-compact for all i, then Γ is (hereditarily) just-infinite.

An abstract group is called **just-infinite** if it is infinite and all its proper quotients are finite. It is **hereditarily just-infinite** if every finite index subgroup is just-infinite. Similarly, a locally compact group is **just-non-compact** if it is non-compact and its proper Hausdorff quotients are compact, and **hereditarily just-non-compact** if that property is inherited by all open subgroups of finite index. Compactly generated just-non-compact locally compact groups are indeed intimately related to the class $\mathscr{S}$, as shown by the following.

Theorem 1.2 ([19, Th. E]). *A compactly generated just-non-compact locally compact group G satisfies exactly one of the following conditions:*

(i) G is discrete (hence just-infinite).

(ii) G has closed normal subgroup isomorphic to $\mathbf{R}^d$, and the quotient is a closed subgroup of $O(d)$ whose action on $\mathbf{R}^d$ is irreducible.

(iii) G has a cocompact closed normal subgroup which is a quasi-product[2] of

[2] A centerless locally compact group G is called the **quasi-product** of the locally compact groups $H_1, \dots, H_n$ if there is a continuous injective homomorphism of the direct product $H_1 \times \cdots \times H_n$ in G, whose image is dense. The respective images of the H_i in G are then called **quasi-factors**. The proof of [19, Th. E] ensures that the quasi-factors arising in Theorem 1.2(iii) are indeed topologically simple and non-discrete; the fact that they are compactly generated requires an extra argument similar to that in [25, Lem. 4.2].

finitely many pairwise isomorphic non-compact groups in $\mathscr{S}$.

In particular, a non-discrete hereditarily just-non-compact group without non-trivial closed normal subgroup isomorphic to $\mathbf{R}^d$ is an extension[3] *of a compact group by a group in $\mathscr{S}$.*

The structure of just-infinite discrete groups has been described by R. Grigorchuk and J. Wilson [33, Th. 3]: Every just-infinite group is either a branch group, or virtually a finite direct power of a simple group, or residually finite and virtually a finite direct power of a hereditarily just-infinite group.

Theorem 1.1 is a far-reaching extension of the Margulis Normal Subgroup theorem [45, Ch. VIII], whose scope is restricted to lattices in semi-simple Lie and algebraic groups. An amazing feature of Theorem 1.1 is that its proof does not require any algebraic information on the structure of G beyond the hypotheses made on the normal subgroups of the factors G_i. Understanding the mechanisms responsible for the existence of (irreducible) lattices in (products of) groups in $\mathscr{S}$ is however an extremely challenging problem that requires a much deeper understanding of groups in $\mathscr{S}$ and that goes far beyond the current state of knowledge. To illustrate this issue, we mention the following remarks.

- A **single group** in $\mathscr{S}$ may fail to contain any lattice whatsoever; see [1] and [41, Th. 1.4]. Others, like the non-linear Kac–Moody groups in $\mathscr{S}$, are subjected to a 'rank one behaviour' forcing all of their lattices to admit infinite proper quotients, and even to be SQ-universal; see [17, Cor. 3.6 and Rem. 3.7]. Yet other examples admit simple lattices [41, Th. 1.7]. One naturally asks whether the 'rank one versus higher rank' dichotomy governing the simple algebraic groups over local fields may be extended to a similarly meaningful partition of the whole class $\mathscr{S}$.

- A **product of two** non-linear groups in $\mathscr{S}$ can contain an irreducible lattice. This was first revealed by M. Burger and S. Mozes, who constructed groundbreaking examples of irreducible lattices in the product of two groups of $\mathscr{S}$ acting on trees, see [11]. More examples of a similar nature have been constructed by D. Rattaggi [58]. Theorem 1.1 also holds for non-uniform lattices under a technical condition called *integrability* (which requires in particular that the lattice be finitely generated, see [63, §2]). This was exploited to establish the simplicity of minimal Kac–Moody groups over finite fields, as well as more general twin building lattices, see [26] and [27]. No other irreducible lattice in a product of two non-linear groups in $\mathscr{S}$ is known as of today.

- The only known irreducible lattices in **products of more than two** groups in $\mathscr{S}$ are the S-arithmetic lattices in semi-simple algebraic groups. Finding non-arithmetic examples is a notorious open problem suggested by the seminal paper [11] of Burger–Mozes. Negative results in the context of Kac–Moody groups are established in [20]. Showing S-arithmetic lattices are the

[3] We say that a group G is an **extension** of a group Q by a group N if there is a short exact sequence $1 \to N \to G \to Q \to 1$.

only such examples requires discovering criteria that isolate the algebraic groups in $\mathscr{S}$.

1.2. Commensurated subgroups and dense embeddings. The Margulis Normal subgroup theorem and its extension by Bader–Shalom ensures that centerless irreducible lattices in semi-simple algebraic groups of higher rank or in non-trivial products of groups in $\mathscr{S}$ are hereditarily just-infinite. For an infinite residually finite group Γ, the hereditary just-infinite property can be interpreted as the property that Γ has 'as few normal subgroups as it can': the only non-trivial normal subgroups are those of finite index, which exist by residual finiteness.

A (hereditarily) just-infinite group may however contain infinite index subgroups that are close to normal, namely **commensurated subgroups**. Two subgroups L_1, L_2 of a group G are called **commensurate** if the indices $[L_1 : L_1 \cap L_2]$ and $[L_2 : L_1 \cap L_2]$ are both finite. The **commensurator** of a subgroup $L \leq G$ in G is defined by

$$\mathrm{Comm}_G(L) = \{g \in G \mid L \text{ and } gLg^{-1} \text{ are commensurate}\}.$$

We say that L is a **commensurated subgroup** of G if $\mathrm{Comm}_G(L) = G$. Clearly, the commensurator of L in G contains the normalizer $N_G(L)$. Obvious examples of commensurated subgroups are normal subgroups, finite subgroups and finite index subgroups. More generally, any subgroup of a group G that is commensurate to a normal subgroup of G is commensurated by G.

Arithmetic groups provide prototypical non-obvious examples: $\mathrm{GL}_d(\mathbf{Z})$ is a commensurated subgroup of $\mathrm{GL}_d(\mathbf{Q})$. It turns out that subgroup commensuration is indeed intimately related to arithmetic phenomena. One illustration of this is provided by Margulis' arithmeticity criterion, ensuring that a lattice Γ in a simple Lie group G is arithmetic if and only if its commensurator $\mathrm{Comm}_G(\Gamma)$ is dense in G, see [45, §IX(1.9)]. Moreover, depending on the nature of the ambient semi-simple group in which an arithmetic group embeds as a lattice, it may or may not admit infinite commensurated subgroups of infinite index. For example every commensurated subgroup of $\mathrm{PSL}_3(\mathbf{Z})$ is finite or of finite index (see [64]), while the group $\mathrm{PSL}_3(\mathbf{Z}[\frac{1}{2}])$ commensurates its infinite index subgroup $\mathrm{PSL}_3(\mathbf{Z})$. The Margulis–Zimmer conjecture predicts that the only source of commensurated subgroups in a higher rank arithmetic lattice are the totally disconnected factors of its ambient locally compact group. A discussion of that conjecture, which is open for uniform lattices, may be consulted in the paper [64] by Y. Shalom and G. Willis.

The approach to the Margulis–Zimmer conjecture adopted by Shalom–Willis consists in studying a commensurated subgroup L of an abstract group G via its **Schlichting completion**, i.e. the closure of the image of G in the full symmetric group on the coset space G/L with respect to the topology of pointwise convergence. This yields a totally disconnected locally compact (abbreviated by **t.d.l.c.**) group, denoted by $G/\!/L$, together with a homomorphism $G \to G/\!/L$ with dense image. It is shown in [64] that tools from the structure theory of t.d.l.c. groups may then be used to settle certain cases of the Margulis–Zimmer conjecture. The following result shows the relevance of the class $\mathscr{S}$ in this approach. A **dense embedding** of topological groups is defined as a continuous injective homomorphism with dense image.

Theorem 1.3 ([25, Prop. 3.6])**.** *For a finitely generated just-infinite group* Γ*, the following assertions are equivalent:*

(i) Γ *has a commensurated infinite subgroup of infinite index.*

(ii) Γ *has a dense embedding in a non-discrete compactly generated just-non-compact t.d.l.c. group* G*.*

If moreover Γ *is hereditarily just-infinite, then* Γ *has a virtually commensurated infinite subgroup of infinite index if and only if* Γ *virtually has a dense embedding in a t.d.l.c. group* G *which is an extension of a compact group by a totally disconnected group in* $\mathscr{S}$*.*

Some group properties, like amenability or Kazhdan's property (T), pass from dense subgroups to the ambient group and descend to cocompact closed normal subgroups. With Theorems 1.2 and 1.3 at hand, this means that general information on groups in the class $\mathscr{S}$ enjoying such a property will potentially yield information on commensurated subgroups of finitely generated just-infinite discrete groups satisfying that same property.

To conclude this discussion, let us finally mention that commensurated subgroups are also relevant to the study of **lattice embeddings** of discrete groups, a systematic treatment of which was recently initiated by Bader–Furman–Sauer [2]. While the Margulis–Zimmer conjecture predicts that a lattice in a higher rank connected simple Lie group does not have non-trivial commensurated subgroups, the absence of commensurated subgroups restricts drastically the structure of the possible lattice envelopes of a finitely generated group. This is illustrated by the following statement which was suggested to me by Ph. Wesolek. It can be obtained by combining [2, Prop. 1.1] with results from [19] (see [42, Proof of Th. 7.3]).

Theorem 1.4. *Let* Γ *be a finitely generated infinite group whose only commensurated subgroups are finite or of finite index.*

Any locally compact group H *in which* Γ *embeds as a lattice has a compact normal subgroup* Q *such that the quotient* $G = H/Q$ *satisfies one of the following assertions:*

(1) G *is discrete and just-infinite;*

(2) G *is virtually the direct product of connected non-compact centerless simple Lie groups;*

(3) G *is an extension of a compact group by a quasi-product of finitely many pairwise isomorphic totally disconnected groups in* $\mathscr{S}$*.*

The theorems discussed above highlight tight relations between the intrinsic structure of finitely generated just-infinite groups and the non-discrete simple locally compact groups. We view them as an invitation to study the class $\mathscr{S}$, on which we shall now focus.

2. Examples of simple groups

2.1. Semi-simple algebraic groups over local fields. The classification of the non-discrete locally compact fields, due independently to D. van Dantzig [72] and L. Pontryagin [55], is an early success of the development of the theory of locally compact groups. That classification ensures that a non-discrete locally compact field is isomorphic to $\mathbf{R}$, $\mathbf{C}$, or a finite extension of $\mathbf{Q}_p$ or of $\mathbf{F}_p((t))$. Among those, the disconnected ones (i.e. all of them except $\mathbf{R}$ and $\mathbf{C}$) are called **non-Archimedean local fields**. Given a locally compact field k, the group $\mathrm{GL}_d(k)$ inherits a locally compact topology from the product topology on $k^{d\times d}$. Any of its closed subgroups is thus locally compact. A linear algebraic group over a locally compact field is thus endowed with a locally compact group topology. The group is furthermore totally disconnected as soon as the field is.

The theory of semi-simple algebraic groups over local fields was initiated by N. Iwahori and H. Matsumoto, and gained full maturity in the monumental work of F. Bruhat and J. Tits. A complete classification of the simple algebraic groups over local fields was achieved by M. Kneser in characteristic 0 and Bruhat–Tits in general; see [70] for an extended overview.

Let k be non-Archimedean local field and $\mathbf{G}$ be a k-isotropic simply connected k-simple algebraic k-group. Then the quotient $G = \mathbf{G}(k)/Z$ of $\mathbf{G}(k)$ be its centre enjoys the following remarkable properties:

- G is a non-discrete compactly generated t.d.l.c. second countable group (see §2.3 in [45, Ch. I]).
- G acts continuously, properly, cocompactly on a locally finite Euclidean building whose dimension equals the k-rank of $\mathbf{G}$; see §2 in [70]. Moreover, the G-action is **strongly transitive**, i.e. transitive on pairs consisting of an apartment and a chamber in that apartment.
- G is **abstractly simple**, i.e. it is simple as an abstract group (see the main result of [67], combined with (2.3.1)(a) in [45, Ch. I]).
- Every proper open subgroup of G is compact (see [56, Th. (T)]).
- G has an open subgroup which is a hereditarily just-infinite pro-p group, where p is the residue characteristic of k (see [28, Th. 2.6]).
- G contains non-abelian discrete free subgroups (follows by a standard ping-pong argument in the rank-one case; the general case reduces to rank–one by [8, Th. 7.2]).

2.2. Kac–Moody groups over finite fields. Another important family of non-discrete simple locally compact groups of Lie theoretic origin is obtained by considering suitable completions of Kac–Moody groups over a finite ground field. This was observed for the first time by B. Rémy [60]. **Kac–Moody algebras** form a class of finitely generated complex Lie algebras whose main properties may be consulted in [38]. Their definition is constructive: it provides a list of Lie algebras $\mathfrak{g}_A$ attached to a parameter A via a presentation à la Chevalley–Serre. The parameter in question is a so-called **generalized Cartan matrix**, i.e. a square matrix $A = (A_{ij})_{1\leq i,j\leq d}$ of size d with integer coefficients, such that $A_{ii} = 2$, $A_{ij} \leq 0$ for $i \neq j$, and $A_{ij} = 0$ if and only if $A_{ji} = 0$. If A is a Cartan matrix in the usual sense, the Lie algebra $\mathfrak{g}_A$ is a finite-dimensional semi-simple complex Lie algebra; in any other case $\mathfrak{g}_A$ is infinite-dimensional. In [71], J. Tits defines a group functor $\mathcal{G}_A$ on the category of commutative rings, characterized by a small list of properties analogous to key features of Chevalley groups; one of them is that the group $\mathcal{G}_A(\mathbf{C})$ has a natural action by automorphisms on the Lie algebra $\mathfrak{g}_A$. The group obtained by evaluating the Tits functor $\mathcal{G}_A$ over a field k is called a **minimal Kac–Moody group over** k. Tits' construction equips the Kac–Moody group $\mathcal{G}_A(k)$ with a natural action by automorphisms on a product of two buildings $\mathcal{B}_+ \times \mathcal{B}_-$ whose type is determined by the Weyl group of the Kac–Moody algebra $\mathfrak{g}_A$. Those buildings are locally finite if and only if the field k is finite. In that case the group $\mathcal{G}_A(k)$ is finitely generated and its action on $\mathcal{B}_+ \times \mathcal{B}_-$ is proper. The full automorphism groups $\mathrm{Aut}(\mathcal{B}_+)$ and $\mathrm{Aut}(\mathcal{B}_-)$ are compactly generated t.d.l.c. groups, so the closure of the projection of $\mathcal{G}_A(k)$ onto a coordinate is also compactly generated and locally compact. The locally compact groups obtained in this way are respectively denoted by $\mathcal{G}_A^{rr+}(k)$ and $\mathcal{G}_A^{rr-}(k)$. They are called **maximal** or **complete Kac–Moody groups over** k. The letters rr stand for Rémy–Ronan, who introduced this completion of the minimal Kac–Moody group $\mathcal{G}_A(k)$ in [61]. Other completions of minimal Kac–Moody groups are described in the literature and yield potentially different groups; see [62, §6] and [46, Ch. 6] for more information and references. Let us merely mention the existence of one other completion, denoted by $\mathcal{G}_A^{ma+}(k)$ (or $\mathcal{G}_A^{ma-}(k)$), introduced by O. Mathieu [49] and developed by G. Rousseau [62], and also called **maximal Kac–Moody group**. Instead of relying on the action of $\mathcal{G}_A(k)$ on the building $\mathcal{B}_+$ (or $\mathcal{B}_-$), its definition rather uses the topology induced by the natural $\mathbf{Z}$-grading of the Lie algebra $\mathfrak{g}_A$. That completion is more naturally linked to the Kac–Moody algebra, and is thus more suited for algebraic investigations. The locally compact group $\mathcal{G}_A^{ma+}(k)$ also acts continuously, properly and cocompactly on the building $\mathcal{B}_+$.

The following statement summarizes properties of complete Kac–Moody groups established in [12], [18], [46], [47], [48] and [62].

Theorem 2.1. *Let $A = (A_{ij})_{1\leq i,j\leq d}$ be an indecomposable generalized Cartan matrix which is not of finite type (i.e. such that $\mathfrak{g}_A$ is infinite-dimensional). Set $G^{rr} = \mathcal{G}_A^{rr+}(\mathbf{F}_q)$ and $G^{ma} = \mathcal{G}_A^{ma+}(\mathbf{F}_q)$. Let Z' denote the kernel of the action of G^{ma} on the building $\mathcal{B}_+$.*

(i) G^{rr} and G^{ma} are non-discrete compactly generated t.d.l.c. second countable groups.

(ii) G^{rr} and G^{ma} act continuously, properly and strongly transitively (hence cocompactly) on the infinite locally finite building $\mathcal{B}_+$. Moreover G^{rr} is contained in G^{ma}/Z' (the latter being identified with its image in $\mathrm{Aut}(\mathcal{B}_+)$*).*

(iii) If the characteristic p of $\mathbf{F}_q$ is larger than the maximal off-diagonal entry of A in absolute value, then $G^{rr} = G^{ma}/Z'$. Otherwise the inclusion of G^{rr} in G^{ma}/Z' may be strict.

(iv) G^{rr} is abstractly simple, and so is G^{ma}/Z' provided the matrix A is non-affine.

(v) Every open subgroup of G^{rr} and G^{ma} is compactly generated (but not necessarily compact).

(vi) G^{rr} (resp. G^{ma}/Z') has an open pro-p subgroup U. If A has a proper submatrix which is not of finite type, then U has infinite closed normal subgroups of infinite index. If the characteristic p of $\mathbf{F}_q$ is larger than the maximal off-diagonal entry of A in absolute value, then U is topologically finitely generated.

(vii) G^{rr} and G^{ma} contain non-abelian discrete free subgroups.

Dwelling on the distinction between the completions G^{rr} and G^{ma} may seem artificially complicated and confusing, especially in view of the fact that the proper comparison between the groups $\mathcal{G}_A^{rr+}(\mathbf{F}_q)$ and $\mathcal{G}_A^{ma+}(\mathbf{F}_q)$ is a delicate and subtle question which is incompletely understood beyond the case covered by Theorem 2.1(iii). It should however be emphasized that, as stated in Theorem 2.1(iii), the simple groups G^{rr} and G^{ma}/Z' can be different. In fact, if one fixes the field $\mathbf{F}_q$ and let A run over all generalized Cartan matrices of size d, the completion $\mathcal{G}_A^{rr+}(\mathbf{F}_q)$ meets only finitely many isomorphism classes, while the number of isomorphism classes taken by $\mathcal{G}_A^{ma+}(\mathbf{F}_q)/Z'$ can be strictly larger (see [48, §6]). The Mathieu–Rousseau completion $\mathcal{G}_A^{ma+}(\mathbf{F}_q)/Z'$ thus affords more (and potentially infinitely many more) simple t.d.l.c. groups than the geometric completion $\mathcal{G}_A^{rr+}(\mathbf{F}_q)$.

2.3. Groups acting on trees. Historically, the first examples of non-discrete non-linear simple locally compact groups are due to J. Tits [68]. In that paper, J. Tits establishes a very flexible simplicity criterion for groups acting on trees (see Th. 4.5 in loc. cit.), which has been repeatedly exploited and generalized since then. The following result provides a first illustration of it.

Theorem 2.2. *Let T be an edge-transitive locally finite tree with vertex degrees ≥ 2 and at least one vertex of degree ≥ 3.*

Then the group $\mathrm{Aut}(T)^+$, generated by the pointwise edge-stabilizers in $\mathrm{Aut}(T)$, is a non-discrete compactly generated abstractly simple t.d.l.c. group. Moreover, for any n and any field F, the only homomorphism of $\mathrm{Aut}(T)^+$ to $\mathrm{GL}_n(F)$ is the trivial one.

Proof. The abstract simplicity is ensured by [68, Th. 4.5]. To prove the absence of finite-dimensional representations, it suffices to find a finitely generated subgroup of $\mathrm{Aut}(T)^+$ that is not residually finite. It is easy to see that the iterated wreath product $(C_2 \wr C_2) \wr \mathbf{Z}$ is isomorphic to a subgroup of the stabilizer of a geodesic line of T. That group is finitely generated, but not residually finite (see Prop. 5 in [27]). $\square$

In the introduction of [68], Tits writes that his simplicity criterion *provides a large variety of pairwise non-isomorphic simple locally compact groups* and illustrates that statement with explicit examples, see [68, Prop. 8.2 and Rem. 8.4]. Tits' remark on the variety of examples revealed to be utterly lucid: A remarkable list of examples of simple locally compact groups has been found since then with the help of Tits' criterion or natural generalizations thereof; see [4], [10, §3.2], [41], [50], [57], [65]. The scope of the criterion has also been extended to encompass groups acting on spaces more general than trees: see [35], [40, Appendix] for CAT(0) cube complexes and [13] for right-angled buildings.

The examples mentioned above are numerous and feature various properties. Giving an exhaustive account goes beyond the scope of this paper. We shall rather focus on one class of examples whose diversity is especially striking. This class is provided by Simon Smith's extension, described in [65], of a construction due to M. Burger and S. Mozes [10, §3].

Consider a non-empty (possibly infinite) discrete set Ω and a simplicial tree T all of whose vertices have a degree equal to the cardinality of Ω. Thus the set of edges $E(v)$ containing a vertex v is in one-to-one correspondence with the set Ω. A coherent choice of bijections of $E(v)$ to Ω for all v is afforded by a **legal coloring** of T, i.e. a map $i\colon E(T) \to \Omega$ whose restriction to each $E(v)$ is bijective. Clearly, legal colorings exist, and any two of them are transformed into one another by an automorphism of T. Given a legal coloring i on T, we associate to each automorphism $g \in \mathrm{Aut}(T)$ its **local action** at a vertex v

$$\sigma(g, v) = i|_{E(gv)} \circ g \circ \left(i|_{E(v)}\right)^{-1} \in \mathrm{Sym}(\Omega).$$

Fixing F any subgroup of $\mathrm{Sym}(\Omega)$, we define the group

$$\mathcal{U}(F) = \{g \in \mathrm{Aut}(T) \mid \forall v \in V(T),\ \sigma(g, v) \in F\}$$

and call it the **universal group** of automorphisms of T **with local action prescribed by** F. The conjugacy class of $\mathcal{U}(F)$ in $\mathrm{Aut}(T)$ is independent of the choice of the legal coloring i, which justifies the choice of a notation hiding the dependence on i. The definition of $\mathcal{U}(F)$ is due to Burger–Mozes [10, §3.2] in case Ω is finite, and to S. Smith [65] in general. Tits' simplicity criterion mentioned above ensures that the subgroup $\mathcal{U}(F)^+$ generated by the pointwise edge-stabilisers in $\mathcal{U}(F)$ is a simple group (unless trivial).

Let us now address the problem of endowing it with a compactly generated t.d.l.c. group topology. Given two permutation groups $F \leq F' \leq \mathrm{Sym}(\Omega)$, we have a natural inclusion $\mathcal{U}(F) \leq \mathcal{U}(F')$. Since the group $\mathcal{U}(\{1\})$ acts freely and transitively on $V(T)$, we see that $\mathcal{U}(F)$ is vertex-transitive for any F. When Ω is finite, the tree T is locally finite and the group $\mathrm{Aut}(T)$ is a compactly generated t.d.l.c. second countable group for the topology of pointwise convergence on the vertex-set. As a closed vertex-transitive subgroup of $\mathrm{Aut}(T)$, the universal group $\mathcal{U}(F)$ inherits the same topological properties. When Ω is infinite, the group

$\mathrm{Aut}(T)$ is no longer locally compact; it is however possible to construct a t.d.l.c. group topology on $\mathcal{U}(F)$ when F itself is a t.d.l.c. group acting continuously on the discrete set Ω. The key is to impose that the stabilisers in F of points in Ω are compact and open. Under that condition, every edge of T has a compact open stabiliser in $\mathcal{U}(F)$ (so that the latter is indeed a locally compact group), while the vertex-stabilisers are open but need not be compact. The properties of the group $\mathcal{U}(F)$ established in [10, §3.2] and [65] yield the following.

Theorem 2.3. *For any (possibly discrete) compactly generated t.d.l.c. group $F \neq \{1\}$ and any compact open subgroup $U \leq F$ such that $\bigcap_{g \in F} gUg^{-1} = \{1\}$ and $F = \langle gUg^{-1} \mid g \in F \rangle$, there is a non-discrete compactly generated abstractly simple t.d.l.c. group $\mathcal{U}(F)^+$ which has an open subgroup mapping continuously onto F with compact kernel.*

The class of groups F satisfying the condition of Theorem 2.3 is extremely broad, and the theorem suggests that the class of non-discrete compactly generated simple locally groups has a similar diversity (although it is not formally clear that two distinct groups F and F' yield two non-isomorphic simple groups $\mathcal{U}(F)^+$ and $\mathcal{U}(F')^+$). A precise illustration is provided in [65, §7]: an uncountable family of pairwise non-isomorphic non-discrete compactly generated simple locally compact groups is obtained by letting F run over the class of Olshanskii–Tarski monsters.

2.4. Groups almost acting on trees. Another fascinating class of simple groups closely related to groups acting on trees is provided by **groups of tree spheromorphisms**, also called **tree almost automorphisms**. The original definition goes back to the work of Y. Neretin [53]. The prototypical example of such groups is the following. Given a regular locally finite tree T of degree ≥ 3, we define the **Neretin group** $\mathrm{Ner}(T)$ by

$$\mathrm{Ner}(T) = \big\{ g \in \mathrm{Homeo}(\partial T) \mid \forall \xi \in \partial T,\ \exists h \in \mathrm{Aut}(T),\ \exists \alpha \subset \partial T \text{ clopen}, \\ \xi \in \alpha \text{ and } g|_\alpha = h|_\alpha \big\}.$$

In the language of topological dynamics, the group $\mathrm{Ner}(T)$ is the *topological full group* associated to the $\mathrm{Aut}(T)$-action on ∂T. The Neretin group $\mathrm{Ner}(T)$ is proved to be simple in [39]. It carries a unique t.d.l.c. group topology such that the natural inclusion $\mathrm{Aut}(T) \leq \mathrm{Ner}(T)$ is continuous and open; with respect to that topology it is a non-discrete compactly generated t.d.l.c. group. The group $\mathrm{Ner}(T)$ enjoys various striking properties, like the absence of any lattice subgroup established in [1]. Other non-discrete simple t.d.l.c. groups of a similar nature are discussed in [14, §6] and [5, Th. 4.16].

2.5. Beyond compact generation. We close this chapter by mentioning some of the pathologies that may occur as soon as one considers non-compactly generated groups, following [76, §3].

Let Ω be an infinite set. We endow $\mathrm{Sym}(\Omega)$ with the topology of pointwise convergence. The subgroup of finitely supported even permutations is denoted by $\mathrm{Alt}(\Omega)$.

Proposition 2.4. *Let $U \leq \mathrm{Sym}(\Omega)$ be an infinite closed subgroup acting with finite orbits. Then the group $G = \langle U \cup \mathrm{Alt}(\Omega)\rangle$ enjoys the following properties.*

(i) G carries a unique t.d.l.c. group topology such that the inclusion $U \to G$ is continuous and open. It is second countable if Ω is countable.

(ii) The compact open subgroups of G form a directed set whose union is G. In particular G is not compactly generated.

(iii) G is not abstractly simple.

(iv) If $U \cap \mathrm{Alt}(\Omega)$ is dense in U, then G is topologically simple.

Proof. (i) The hypotheses made on U imply that U is profinite, hence compact. The required property of G follows from the fact that U is a commensurated subgroup of G. Moreover, if Ω is countable, then U is metrizable and $\mathrm{Alt}(\Omega)$ is countable, so that G is indeed second countable.

(ii) Any compact subset of G preserves all but finitely many U-orbits, and is thus contained in a compact subgroup of G.

(iii) $\mathrm{Alt}(\Omega)$ is a non-trivial normal subgroup of $\mathrm{Sym}(\Omega)$, hence of G. We have $\mathrm{Alt}(\Omega) \neq G$ since U is infinite and closed, and thus contains permutations whose support is infinite. Thus G is not abstractly simple.

(iv) Assume now that $U \cap \mathrm{Alt}(\Omega)$ is dense in U. It follows that $\mathrm{Alt}(\Omega)$ is dense in G. Let now N be a non-trivial closed normal subgroup of G. Since the centralizer of $\mathrm{Alt}(\Omega)$ in $\mathrm{Sym}(\Omega)$ is trivial (because the $\mathrm{Alt}(\Omega)$-action is n-transitive for any n), we have $\{1\} \neq [N, \mathrm{Alt}(\Omega)] \leq N \cap \mathrm{Alt}(\Omega)$. Since $\mathrm{Alt}(\Omega)$ is simple, we infer that N contains $\mathrm{Alt}(\Omega)$. It follows that $N = G$ since $\mathrm{Alt}(\Omega)$ is dense. □

Any profinite group can be embedded as a closed subgroup in a product of finite groups, and any finite group can be embedded in a finite alternating group. Hence we obtain the following.

Corollary 2.5. *Any infinite direct product of finite groups can be continuously embedded as an open subgroup in a non-discrete topologically simple locally compact group.*

In particular any profinite group can be continuously embedded in a non-discrete topologically simple locally compact group.

The local structure of a non-discrete topologically simple t.d.l.c. group is thus quite flexible, although it is not completely arbitrary, since some profinite groups do not continuously embed as *open* subgroups of simple t.d.l.c. groups (e.g. a non-abelian p-adic analytic pro-p group whose $\mathbf{Q}_p$-algebra is not simple). That situation changes rather drastically under the assumption that the simple group is compactly generated, as we shall now see.

3. The class $\mathscr{S}$

Since the connected component of the identity is a closed normal subgroup of every locally compact group, it follows that a group in $\mathscr{S}$ is either connected or totally disconnected. The solution to Hilbert's fifth problem (see [66]) implies that the the connected members of $\mathscr{S}$ are all simple Lie groups. Thus $\mathscr{S}$ is naturally partitioned into two subclasses, denoted by $\mathscr{S}_{\mathrm{Lie}}$ (consisting of connected Lie groups) and $\mathscr{S}_{\mathrm{td}}$ (consisting of totally disconnected groups) respectively.

Among the examples of simple groups reviewed in the previous section, those derived from Tits' simplicity criterion or an extension thereof, i.e. those mentioned in Sections 2.3 and 2.4, seem intractable and certainly out of reach of any exhaustive understanding. It is however possible to identify formally common features that those groups share. One of these common features is that all those groups admit a non-trivial continuous action on a totally disconnected compact Hausdorff space with non-trivial elements of arbitrarily small support. This can be formalized with the concept of *micro-supported actions* introduced below.

3.1. Micro-supported actions. Consider a group G acting by homeomorphisms on a Hausdorff topological space X. The **rigid stabiliser** of a subset $U \subset X$, denoted by $\mathrm{Rist}_G(U)$, is defined as the pointwise stabiliser in G of the complement $X \setminus U$. The G-action on X is called **micro-supported** if for every non-empty open set $U \subset X$, the rigid stabiliser $\mathrm{Rist}_G(U)$ is non-trivial. Micro-supported actions occur naturally when considering *large* transformation groups, like the full homeomorphism group or diffeomorphism group of a manifold. Moreover, it has been observed long ago that a group with a micro-supported action which is 'sufficiently transitive' is often simple, or at least it has a simple derived subgroup; see [32] and references therein. The following result is an explicit illustration of that fact. A subset $\alpha \subset X$ is called **compressible** (under the G-action on X) if for every non-empty open set $\beta \subset \Omega$, there exists $g \in G$ such that $g\alpha \subset \beta$.

Proposition 3.1 ([25, Prop. H])**.** *Assume that the G-action on X is micro-supported.*

If there exists a non-empty compressible open set α in X, then the intersection S of all non-trivial normal subgroups of G is non-trivial.

If in addition there exits a non-empty open set α' in X which is compressible under the S-action on X, then S is simple.

It turns out that for groups in $\mathscr{S}_{\mathrm{td}}$, the existence of a non-trivial micro-supported action on a compact totally disconnected space is encoded in the *local* algebraic structure; in other words, it can be detected in arbitrarily small identity neighbourhoods. In order to explain this, we recall the construction of the **structure lattice**, a local invariant introduced in a joint work with Colin Reid and George Willis outlined in [22] and developed in [23, 25]. It is inspired by earlier work of John Wilson on the structure of just-infinite groups [77] and by Barnea–Ershov–Weigel on abstract commensurators of profinite groups [5].

Two subgroups L_1, L_2 of a t.d.l.c. group G are called **locally equivalent** if their intersection $L_1 \cap L_2$ is relatively open in both L_1 and L_2. The local class of L is denoted by $[L]$. When L_1 and L_2 are both compact, they are locally equivalent if and only if they are commensurate. A subgroup $L \leq G$ is called **locally normal** if its normalizer $N_G(L)$ is open. When G is a p-adic Lie group, two closed subgroups are locally equivalent if and only if their Lie algebras are the same subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$, and a closed subgroup is locally normal if and only if its Lie algebra is an ideal in $\mathfrak{g}$. The **structure lattice**[4] of a t.d.l.c. group G, denoted by $\mathcal{LN}(G)$, is the set of local equivalence classes of compact locally normal subgroups, equipped with the order relation induced by inclusion. It is a modular lattice on which G acts continuously by automorphisms. The structure lattice has a smallest element, denoted by 0 (namely the local class of the trivial subgroup) and a largest one, denoted by ∞ (namely the local class of compact open subgroups). Classical lattice theory associates geometric or topological structures to modular lattices that are **complemented**, i.e. for each α there exists β with $\alpha \wedge \beta = 0$ and $\alpha \vee \beta = \infty$ (see e.g. [7, Ch. IV]). Although the structure lattice is usually not a complemented lattice, there is a natural algebraic map whose properties suggest those of an orthocomplementation, namely the centralizer map. The centralizer map is order-reversing and maps a locally normal subgroup to a locally normal subgroup. However, there are two obstructions for the centralizer map to play the role of an orthocomplementation: the centralizers of two locally equivalent locally normal subgroups may be different, and a locally normal subgroup can have a non-trivial intersection with its centralizer. Concrete examples illustrating this are provided by the locally abelian topologically simple groups arising from Corollary 2.5. This issue is resolved in [23]: if a t.d.l.c. group G is **[A]-semi-simple**, which means that its only discrete normal subgroup and its only virtually solvable locally normal compact subgroups are the trivial subgroup, then for any locally normal compact subgroup $L \leq G$, we have $L \cap C_G(L) = 1$ and for any K locally equivalent to L, we have $C_G(L) = C_G(K)$. Therefore, the map

$$\mathcal{LN}(G) \to \mathcal{LN}(G) : [L] \mapsto [L]^\perp = [C_G(L)]$$

is well defined and satisfies $[L] \wedge [L]^\perp = 0$. The discrete G-set

$$\mathcal{LC}(G) = \{\alpha^\perp \mid \alpha \in \mathcal{LN}(G)\}$$

is then naturally endowed with the structure of a Boolean lattice. It is called the **centralizer lattice** of G.

Theorem 3.2 ([25, Theorem A])**.** *Every group $G \in \mathscr{S}_{\mathrm{td}}$ is [A]-semi-simple. In particular the centralizer lattice $\mathcal{LC}(G)$ is a well-defined Boolean algebra.*

The Stone duality theorem ensures that every Boolean lattice $\mathcal{A}$ is the lattice of clopen sets of its **Stone dual**, which is the compact totally disconnected space denoted by $\Omega_{\mathcal{A}}$ consisting of all lattice homomorphisms $\mathcal{A} \to \{0, 1\}$, endowed with

[4] We warn the reader that the word *lattice* has two different common acceptations in mathematics, both used in this paper: A lattice can mean a discrete subgroup of finite covolume (as in §1.1 above), or a poset in which any pair has a supremum and an infimum (as in the current section).

the topology of pointwise convergence. Thus Theorem 3.2 yields a canonical compact G-space. The following result shows that this space has remarkable dynamical properties and governs all micro-supported G-actions on compact totally disconnected spaces. A compact G-space Ω is called **minimal** if every G-orbit is dense, and **strongly proximal** if the closure of the G-orbit of every probability measure on Ω contains a Dirac mass.

Theorem 3.3 ([25, Th. K])**.** *Let $G \in \mathscr{S}_{\mathrm{td}}$ and Ω_G denote the Stone dual of $\mathcal{LC}(G)$.*

(i) The canonical G-action on Ω_G is continuous, minimal, strongly proximal, and micro-supported. Moreover there exists a non-empty compressible clopen set.

(ii) Every compact totally disconnected space with a continuous micro-supported G-action is a G-quotient of Ω_G. In particular, every such action is minimal, strongly proximal and has a non-empty compressible clopen set.

Corollary 3.4. *For a group $G \in \mathscr{S}_{\mathrm{td}}$, the following assertions are equivalent.*

(1) *Every continuous micro-supported G-action on a compact totally disconnected space is trivial.*

(2) $\mathcal{LC}(G) = \{0, \infty\}$.

(3) *For any pair of compact subgroups $L_1, L_2 \leq G$ with open normalizer, we have $[L_1, L_2] = 1$ if and only if $L_1 = 1$ or $L_2 = 1$.*

Let us close this section with what we view as a tentalizing analogy. The centralizer lattice of a group $G \in \mathscr{S}_{\mathrm{td}}$ is a local invariant that can be trivial or not, and thus provides an obvious partition of the class $\mathscr{S}_{\mathrm{td}}$ into two proper subclasses. Theorem 3.3 and Corollary 3.4 provide an interpretation of that subdivision in terms of a global property, namely the existence of a continuous micro-supported G-action on a compact totally disconnected space. There is an analogous subdivision of the class of finite simple groups: Indeed, the existence of a transitive action admitting elements with small support characterizes the alternating groups. This is expressed by the following beautiful result of Guralnick–Magaard, providing a sharp and surprising quantitative measure of what 'small' means in that context.

Theorem 3.5 (Guralnick–Magaard [34, Cor. 1])**.** *Let G be a finite simple group acting transitively*[5] *by permutations on a set X. If G contains a non-trivial element, the proportion of whose fixed points is greater than $4/7$, then G is an alternating group.*

According to the Classification of the non-abelian Finite Simple Groups, the complement of the subclass constituted by the alternating groups consists, up to twenty-six sporadic exceptions, of the groups of Lie type. At the time of this writing, the only known groups in $\mathscr{S}_{\mathrm{td}}$ with a trivial centralizer lattice are also

[5] The cited reference deals with primitive actions; the reduction from transitive to primitive actions is straightforward.

groups of Lie theoretic origin, see Section 2. It is a major challenge to elucidate the nature of the groups in $\mathscr{S}_{\mathrm{td}}$ whose centralizer lattice is trivial, and to determine their actual relation to Lie theory.

3.2. Characterizing algebraic groups. The simple algebraic groups over local fields are important members of the class $\mathscr{S}_{\mathrm{td}}$, due to their connections to other areas of mathematics. It is thus desirable to understand the specific properties that isolate algebraic groups within the class $\mathscr{S}_{\mathrm{td}}$.

3.2.1. Linearity. A first exceptional property of algebraic groups is that they are the only linear groups in $\mathscr{S}_{\mathrm{td}}$. A **linear group** is a locally compact group G admitting a continuous faithful representation $G \to \mathrm{GL}_d(k)$ over a locally compact field k. Linearity naturally unifies three important classes of simple locally compact groups:

Theorem 3.6 ([28, Cor. 1.6])**.** *A compactly generated topologically simple locally compact group is linear if and only if it belongs to one of the following families:*

- *Finite simple groups.*
- *Connected simple Lie groups.*
- *Simple algebraic groups over non-Archimedean local fields.*

Among non-discrete totally disconnected groups, the simple algebraic groups can be characterized locally:

Theorem 3.7 ([28, Cor. 1.4])**.** *A group $G \in \mathscr{S}_{\mathrm{td}}$ is a simple algebraic group over a local field if and only if G has a linear open subgroup.*

It is again important to note that such a result fails for non-compactly generated groups. By Corollary 2.5, a topologically simple locally compact group can be locally isomorphic to the additive group of the local field $\mathbf{F}_p(\!(t)\!)$. The proof of those results relies in an essential way on the results on the structure lattice from [23, 25], combined with R. Pink's advanced study of compact subgroups of linear groups over local fields [54].

3.2.2. Buildings and BN-pairs. Algebraic groups within $\mathscr{S}$ may also be characterized in geometric terms. Bruhat–Tits theory associates a locally finite Euclidean building to each simple algebraic group over a local field. Conversely, a classification theorem of Tits shows that all locally finite irreducible Euclidean buildings of dimension ≥ 3 arise from Bruhat–Tits theory. This suggests one may be able to characterize algebraic groups within $\mathscr{S}_{\mathrm{td}}$ in terms of their capability of acting sufficiently transitively on Euclidean buildings; this would yield a purely algebraic characterization via the concept of BN-pairs. However, dealing with the case of low dimensional buildings is especially challenging: for 2-dimensional buildings, the characterization was established only recently in [21, Cor. E], while for 1-dimensional buildings, namely trees, one has to cope with the numerous non-linear locally compact groups acting on trees. The following summarizes known results in this direction.

Theorem 3.8. *Let T be a locally finite tree with vertex degrees ≥ 2. Let $G \leq \mathrm{Aut}(T)$ be a closed subgroup belonging to $\mathscr{S}$, acting transitively on the set of ends ∂T.*

(i) ([15, Th. A]) G is isomorphic to $\mathrm{PSL}_2(k)$ with k a non-Archimedean local field if and only if the stabilizer G_ξ of an end $\xi \in \partial T$ is metabelian.

(ii) ([16, Cor. 1.2]) G is isomorphic to a rank one simple algebraic group over a non-Archimedean local field with abelian root groups if and only if the contraction group $\mathrm{con}(g) = \{x \in G \mid g^n x g^{-n} \to 1\}$ of an element $g \in G$ acting hyperbolically on T is abelian.

The interest of those statements is that the linearity of the group is deduced from an abstract/topological group property. An application of Theorem 3.8 to sharply-3-transitive locally compact groups is described in [16, §5.2].

The class of topologically simple groups acting properly on a tree T and transitively on ∂T may be viewed as a microcosmos reflecting some of the intriguing features of the class $\mathscr{S}_{\mathrm{td}}$. It contains rank one algebraic groups over local fields, rank two Kac–Moody groups over finite fields, as well as groups with a non-trivial centralizer lattice like (some of) those arising from Theorems 2.2 and 2.3. A breakthrough in the study of that microcosmos was accomplished by N. Radu [57], who obtained a remarkable classification theorem describing completely those groups under the extra hypothesis that the local action of a vertex stabiliser on its neighbours contains the full alternating group. The latter hypothesis happens to be redundant when the valency of the vertex in question avoids the sparse set of values constituted by the degrees of the finite 2-transitive groups different from the full symmetric or alternating groups (see [57, Th. B and Cor. D]).

3.3. Challenges.

> *Lorsque l'on veut parler de théorie des groupes, que ce soit au passé ou au présent, l'idée de* classification *se présente immanquablement à l'esprit, idée si obstinément attachée au sujet qui nous occupe qu'elle en a acquis mauvaise réputation auprès de bien des mathématiciens.*[6]
>
> Jacques Tits, 1975 [69]

The pantheon of classification theorems in mathematics includes some of the most salient results from group theory: the classification of the simple Lie groups by W. Killing and E. Cartan, the classification of the simple algebraic groups over algebraically closed fields by C. Chevalley (written up in famous seminar notes that have been nicknamed "The Bible" by the specialists — see [31] — and recently been reedited in [30]) and the Classification of the Finite Simple Groups.

[6] *When one wants to speak about group theory, whether past or present, the idea of a* classification *comes unavoidably to mind, an idea so intimately attached to the topic of our concern that it acquired a bad reputation among many mathematicians.*

Although still in its infancy, the study of the class $\mathscr{S}$ should not be aimed at a classification up to isomorphism: Theorem 2.3 indeed provides strong evidence that this classification problem is at least comparable in complexity to the classification of the finitely generated simple groups with torsion. This should not be viewed as an obstruction to the study of this class, but rather as a hint towards its proper calibration. We conclude this paper by suggesting directions that future study of this class could pursue.

Non-compact simple Lie groups are *non-positively curved*: they act properly on their associated symmetric spaces, which are simply connected Riemannian manifolds of non-positive sectional curvature. This fundamental feature, which is far from obvious from a contemplation of the axioms of Lie theory, influences deeply their global structure as well as the properties of their discrete subgroups. It has played a key role in the development of geometric group theory. Similarly, the non-positive curvature features of simple algebraic groups over local fields were unveiled by Bruhat–Tits' theory via Euclidean buildings. Understanding the extent to which those geometric features are shared by all non-compact groups in $\mathscr{S}$ is a general (and rather vague) problem, which however suggests specific questions based on the experience, acquired by geometric group theory, of the ways in which actions on geometric spaces of non-positive curvature influences the algebraic properties of a group. Non-elementary Gromov hyperbolic groups, as well as many other non-positively curved, have exponential growth. This suggests the following:

Question 3.9. *Can a group in $\mathscr{S}_{\mathrm{td}}$ be of subexponential growth?*

The **growth** means the word growth, i.e. the growth rate of the volume (in terms of Haar measure) of the ball of radius r around the identity in the word metric, with respect to a compact generating set, as a function of r. Losert's extension of Gromov's theorem (see [43]) implies that the growth of a group in $\mathscr{S}_{\mathrm{td}}$ is superpolynomial. All known examples in $\mathscr{S}_{\mathrm{td}}$ contain discrete free subgroups and thus have exponential growth. The following weakening of Question 3.9 is equally natural:

Question 3.10. *Can a group in $\mathscr{S}_{\mathrm{td}}$ be amenable?*

Additional motivation for this question is provided by the recent groundbreaking discovery of finitely generated infinite simple amenable groups by Juschenko–Monod [37]; V. Nekrashevych found examples that are moreover torsion groups of subexponential growth [52, Th. 1.2]. Theorem 3.3 implies a negative answer for all groups $G \in \mathscr{S}_{\mathrm{td}}$ with a non-trivial centralizer lattice. The negative answer to Question 3.10 has several striking implications: It implies that every finitely generated infinite simple amenable group has no non-trivial commensurated subgroups (by Theorem 1.3) and furthermore that the only lattice envelopes of such groups are compact-by-discrete (by Theorem 1.4).

Leaving the geometric aspects aside, we also mention another fundamental problem, that is very natural from a purely algebraic viewpoint, and is unavoidable from the point of view of the role of simple groups in the global structure of general t.d.l.c. groups (see [19] and [59]):

Question 3.11. *Can a group in $\mathscr{S}_{\mathrm{td}}$ have a proper dense normal subgroup?*

Again, when non-trivial, the centralizer lattice yields a partial negative answer (see [25, Th. Q]).

The theory of the scale function, initiated by G. Willis in [75], provides tools that are relevant to those a priori non-related problems. In order to illustrate this, let us first mention the following, which can be deduced from the results in [6]:

Theorem 3.12 (Baumgartner–Willis)**.** *Let G be a compactly generated t.d.l.c. group. If G contains a non-unimodular closed subgroup, then G has exponential growth.*

It is shown in [6] that the existence of a non-unimodular closed subgroup in an arbitrary t.d.l.c. group G implies the existence of an element $g \in G$ whose contraction group $\mathrm{con}(g) = \{x \in G \mid g^n x g^{-n} \to 1\}$ is non-trivial (indeed has non-compact closure). It turns out that contraction groups are directly relevant to Question 3.11:

Theorem 3.13 ([24])**.** *Let G be a t.d.l.c. group. Every dense subnormal subgroup of G contains the group $G^\dagger = \langle \overline{\mathrm{con}(g)} \mid g \in G \rangle$.*

In particular, if G is topologically simple and contains an element whose contraction group is non-trivial, then $G^\dagger$ is abstractly simple, and is the smallest dense normal subgroup of G.

Those considerations motivate the following.

Question 3.14. *Can a group in $\mathscr{S}_{\mathrm{td}}$ have all its closed subgroups unimodular? Can it exclusively consist of elements whose contraction group is trivial?*

The questions listed above should not be viewed as defining ultimate goals for the study of $\mathscr{S}$, but rather as illustrations of the limitations of the current state of knowledge. It is conceivable that a single new example of a group in $\mathscr{S}$ could provide a positive answer to all the above questions at once. A concrete strategy to find new examples is actually provided by Theorem 1.3: A positive answer to Question 3.10 (resp. Question 3.9) could be obtained by exhibiting a finitely generated just-infinite amenable group (resp. group of subexponential growth) with an infinite commensurated subgroup of infinite index. A recent result of Ph. Wesolek (elaborating on Theorem 1.3) implies that this won't work with the Grigorchuk group: Indeed a finitely generated just-infinite branch group does not have any infinite commensurated subgroup of infinite index (see [74]). This leads us naturally to a compelling open problem: What are the commensurated subgroups in the simple amenable groups constructed by Juschenko–Monod [37] and Nekrashevych [52]?

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References

[1] Uri Bader, Pierre-Emmanuel Caprace, Tsachik Gelander, and Shahar Mozes, *Simple groups without lattices*, Bull. Lond. Math. Soc. **44** (2012), no. 1, 55–67.

[2] Uri Bader, Alex Furman, and Roman Sauer, *On the structure and arithmeticity of lattice envelopes*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 5, 409–413.

[3] Uri Bader and Yehuda Shalom, *Factor and normal subgroup theorems for lattices in products of groups*, Invent. Math. **163** (2006), no. 2, 415–454.

[4] Christopher Banks, Murray Elder, and George A. Willis, *Simple groups of automorphisms of trees determined by their actions on finite subtrees*, J. Group Theory **18** (2015), no. 2, 235–261.

[5] Yiftach Barnea, Mikhail Ershov, and Thomas Weigel, *Abstract commensurators of profinite groups*, Trans. Amer. Math. Soc. **363** (2011), no. 10, 5381–5417.

[6] Udo Baumgartner and George A. Willis, *Contraction groups and scales of automorphisms of totally disconnected locally compact groups*, Israel J. Math. **142** (2004), 221–248.

[7] Garrett Birkhoff, *Lattice theory*, Third edition. American Mathematical Society Colloquium Publications, Vol. XXV, American Mathematical Society, Providence, R.I., 1967.

[8] Armand Borel and Jacques Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–150.

[9] Martin R. Bridson, *Non-positive curvature and complexity for finitely presented groups*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 961–987.

[10] Marc Burger and Shahar Mozes, *Groups acting on trees: from local to global structure*, Inst. Hautes Études Sci. Publ. Math. **92** (2000), 113–150 (2001).

[11] ———, *Lattices in product of trees*, Inst. Hautes Études Sci. Publ. Math. **92** (2000), 151–194 (2001).

[12] Inna Capdeboscq and Bertrand Rémy, *On some pro-p groups from infinite-dimensional Lie theory*, Math. Z. **278** (2014), no. 1-2, 39–54.

[13] Pierre-Emmanuel Caprace, *Automorphism groups of right-angled buildings: simplicity and local splittings*, Fund. Math. **224** (2014), no. 1, 17–51.

[14] Pierre-Emmanuel Caprace and Tom De Medts, *Simple locally compact groups acting on trees and their germs of automorphisms*, Transform. Groups **16** (2011), no. 2, 375–411.

[15] ———, *Trees, contraction groups, and Moufang sets*, Duke Math. J. **162** (2013), no. 13, 2413–2449.

[16] Pierre-Emmanuel Caprace and Matthias Grüninger, *Boundary moufang trees with abelian root groups of characteristic p*, Preprint `http://arxiv.org/abs/1406.5940`.

[17] Pierre-Emmanuel Caprace and David Hume, *Orthogonal forms of Kac-Moody groups are acylindrically hyperbolic*, Ann. Inst. Fourier (Grenoble) **65** (2015), no. 6, 2613–2640.

[18] Pierre-Emmanuel Caprace and Timothée Marquis, *Open subgroups of locally compact Kac-Moody groups*, Math. Z. **274** (2013), no. 1-2, 291–313.

[19] Pierre-Emmanuel Caprace and Nicolas Monod, *Decomposing locally compact groups into simple pieces*, Math. Proc. Cambridge Philos. Soc. **150** (2011), no. 1, 97–128.

[20] ______, *A lattice in more than two Kac-Moody groups is arithmetic*, Israel J. Math. **190** (2012), 413–444.

[21] ______, *An indiscrete Bieberbach theorem: from amenable* CAT(0) *groups to Tits buildings*, J. Éc. polytech. Math. **2** (2015), 333–383.

[22] Pierre-Emmanuel Caprace, Colin D. Reid, and George A. Willis, *Locally normal subgroups of simple locally compact groups*, C. R. Math. Acad. Sci. Paris **351** (2013), no. 17-18, 657–661.

[23] ______, *Locally normal subgroups of totally disconnected groups. Part I: General theory*, Forum Math. Sigma 5 (2017), e11, 76 pp.

[24] ______, *Limits of contraction groups and the Tits core*, J. Lie Theory **24** (2014), no. 4, 957–967.

[25] ______, *Locally normal subgroups of totally disconnected groups. Part II: Applications to simple groups*, Forum Math. Sigma 5 (2017), e12, 89 pp.

[26] Pierre-Emmanuel Caprace and Bertrand Rémy, *Simplicity and superrigidity of twin building lattices*, Invent. Math. **176** (2009), no. 1, 169–221.

[27] ______, *Simplicity of twin tree lattices with non-trivial commutation relations*, Topology and geometric group theory, 143–151, Springer Proc. Math. Stat., 184, Springer, 2016.

[28] Pierre-Emmanuel Caprace and Thierry Stulemeijer, *Totally Disconnected Locally Compact Groups with a Linear Open Subgroup*, Int. Math. Res. Not. IMRN **24** (2015), 13800–13829.

[29] Elie Cartan, *La topologie des espaces représentatifs des groupes de Lie.*, Enseign. Math. **35** (1936), 177–200.

[30] Claude Chevalley, *Classification des groupes algébriques semi-simples*, Springer-Verlag, Berlin, 2005, Collected works. Vol. 3, Edited and with a preface by P. Cartier, With the collaboration of Cartier, A. Grothendieck and M. Lazard.

[31] Jean Dieudonné and Jacques Tits, *Claude Chevalley (1909–1984)*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), no. 1, 1–7.

[32] David B. A. Epstein, *The simplicity of certain groups of homeomorphisms*, Compositio Math. **22** (1970), 165–173.

[33] Rostislav. I. Grigorchuk, *Just infinite branch groups*, New horizons in pro-p groups, Progr. Math., vol. 184, Birkhäuser Boston, Boston, MA, 2000, pp. 121–179.

[34] Robert Guralnick and Kay Magaard, *On the minimal degree of a primitive permutation group*, J. Algebra **207** (1998), no. 1, 127–145.

[35] Frédéric Haglund and Frédéric Paulin, *Simplicité de groupes d'automorphismes d'espaces à courbure négative*, The Epstein birthday schrift, Geom. Topol. Monogr., vol. 1, Geom. Topol. Publ., Coventry, 1998, pp. 181–248.

[36] Lizhen Ji, *Arithmetic groups, mapping class groups, related groups, and their associated spaces*, Fourth International Congress of Chinese Mathematicians, AMS/IP Stud. Adv. Math., vol. 48, Amer. Math. Soc., Providence, RI, 2010, pp. 127–175.

[37] Kate Juschenko and Nicolas Monod, *Cantor systems, piecewise translations and simple amenable groups*, Ann. of Math. (2) **178** (2013), no. 2, 775–787.

[38] Victor G. Kac, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990.

[39] Christophe Kapoudjian, *Simplicity of Neretin's group of spheromorphisms*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 4, 1225–1240.

[40] Nir Lazarovich, *On regular CAT(0) cube complexes*, Preprint `http://arxiv.org/abs/1411.0178`, 2014.

[41] Adrien Le Boudec, *Groups acting on trees with almost prescribed local action*, Comment. Math. Helv. **91** (2016), no. 2, 253–293.

[42] Adrien Le Boudec and Phillip Wesolek, *Commensurated subgroups in tree almost automorphism groups*, Preprint `http://arxiv.org/abs/1604.04162`, 2016.

[43] Viktor Losert, *On the structure of groups with polynomial growth*, Math. Z. **195** (1987), no. 1, 109–117.

[44] Alex Lubotzky and Tyakal Nanjundiah Venkataramana, *A group theoretical characterisation of S-arithmetic groups in higher rank semi-simple groups*, Geom. Dedicata **90** (2002), 1–28.

[45] Gregori A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 17, Springer-Verlag, Berlin, 1991.

[46] Timothée Marquis, *Topological Kac–Moody groups and their subgroups*, Ph.D. thesis, Université catholique de Louvain, 2013.

[47] ______, *Abstract simplicity of locally compact Kac-Moody groups*, Compos. Math. **150** (2014), no. 4, 713–728.

[48] ______, *Around the Lie correspondence for complete Kac–Moody groups and Gabber–Kac simplicity*, Preprint `http://arxiv.org/abs/1509.01976`, 2015.

[49] Olivier Mathieu, *Construction d'un groupe de Kac-Moody et applications*, Compositio Math. **69** (1989), no. 1, 37–60.

[50] Rögnvaldur G. Möller and Jan Vonk, *Normal subgroups of groups acting on trees and automorphism groups of graphs*, J. Group Theory **15** (2012), no. 6, 831–850.

[51] Dave Witte Morris, *Introduction to arithmetic groups*, Deductive Press, 2015.

[52] Volodymyr Nekrashevych, *Palindromic subshifts and simple periodic groups of intermediate growth*, Preprint `http://arxiv.org/abs/1601.01033`, 2016.

[53] Yu. A. Neretin, *Unitary representations of the groups of diffeomorphisms of the p-adic projective line*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 4, 92–93.

[54] Richard Pink, *Compact subgroups of linear algebraic groups*, J. Algebra **206** (1998), no. 2, 438–504.

[55] Lev Pontrjagin, *Über stetige algebraische Körper*, Ann. of Math. (2) **33** (1932), no. 1, 163–174.

[56] Gopal Prasad, *Elementary proof of a theorem of Bruhat-Tits-Rousseau and of a theorem of Tits*, Bull. Soc. Math. France **110** (1982), no. 2, 197–202.

[57] Nicolas Radu, *A classification theorem for boundary 2-transitive automorphism groups of trees*, Invent. Math. 209 (2017), no. 1, 1–60.

[58] Diego Attilio Rattaggi, *Computations in groups acting on a product of trees: Normal subgroup structures and quaternion lattices*, Ph.D. thesis, ETH Zürich, 2004.

[59] Colin D. Reid and Phillip R. Wesolek, *Dense normal subgroups and chief factors in locally compact groups*, Preprint `http://arxiv.org/abs/1601.07317`.

[60] Bertrand Rémy, *Topological simplicity, commensurator super-rigidity and non-linearities of Kac-Moody groups*, Geom. Funct. Anal. **14** (2004), no. 4, 810–852, With an appendix by P. Bonvin.

[61] Bertrand Rémy and Mark Ronan, *Topological groups of Kac-Moody type, right-angled twinnings and their lattices*, Comment. Math. Helv. **81** (2006), no. 1, 191–219.

[62] Guy Rousseau, *Groupes de Kac–Moody déployés sur un corps local, II. Masures ordonnées*, Bull. Soc. Math. France **144** (2016), no. 4, 613–692.

[63] Yehuda Shalom, *Rigidity of commensurators and irreducible lattices*, Invent. Math. **141** (2000), no. 1, 1–54.

[64] Yehuda Shalom and George A. Willis, *Commensurated subgroups of arithmetic groups, totally disconnected groups and adelic rigidity*, Geom. Funct. Anal. **23** (2013), no. 5, 1631–1683.

[65] Simon M. Smith, *A product for permutation groups and topological groups*, Duke Math. J. 166 (2017), no. 15, 2965–2999.

[66] Terence Tao, *Hilbert's fifth problem and related topics*, Graduate Studies in Mathematics, vol. 153, American Mathematical Society, Providence, RI, 2014.

[67] Jacques Tits, *Algebraic and abstract simple groups*, Ann. of Math. (2) **80** (1964), 313–329.

[68] ———, *Sur le groupe des automorphismes d'un arbre*, Essays on topology and related topics (Mémoires dédiés à Georges de Rham), Springer, New York, 1970, pp. 188–211.

[69] ———, *Leçon inaugurale de la Chaire de Théorie des groupes, prononcée au collège de france le 14 janvier 1975*, Coll. "Leçons inaugurales", Collège de France, 1976.

[70] ———, *Reductive groups over local fields*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69.

[71] ———, *Uniqueness and presentation of Kac–Moody groups over fields*, J. Algebra **105** (1987), no. 2, 542–573.

[72] David van Dantzig, *Studien over topologische algebra (proefschrift)*, Ph.D. thesis, Groningen, 1931.

[73] André Weil, *L'avenir des mathématiques*, Bol. Soc. Mat. São Paulo **1** (1946), 55–68.

[74] Phillip Wesolek, *Commensurated subgroups in finitely generated branch groups*, Preprint `http://arxiv.org/abs/1602.00213`, to appear in the J. Group Theory.

[75] George A. Willis, *The structure of totally disconnected, locally compact groups*, Math. Ann. **300** (1994), no. 2, 341–363.

[76] ———, *Compact open subgroups in simple totally disconnected groups*, J. Algebra **312** (2007), no. 1, 405–417.

[77] John S. Wilson, *Groups with every proper quotient finite*, Proc. Cambridge Philos. Soc. **69** (1971), 373–391.

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