Reflection triangles in Coxeter groups and biautomaticity

Pierre-Emmanuel Caprace* and Bernhard Mühlherr

18th August 2004

Département de Mathématiques,
Université libre de Bruxelles, CP216
Bd du Triomphe, B-1050 Bruxelles, Belgium
e-mail : *pcaprace@ulb.ac.be* and *bmuhlherr@ulb.ac.be*

Abstract. A Coxeter system (W, S) is called *affine-free* if its Coxeter diagram contains no affine subdiagram of rank ≥ 3. Let (W, S) be a Coxeter system of finite rank (i.e. $|S|$ is finite). The main result is that W is affine-free if and only if W has finitely many conjugacy classes of reflection triangles. This implies that the action of W on its Coxeter cubing (defined by Niblo-Reeves [11]) is cocompact if and only if (W, S) is affine-free. This result was conjectured in loc. cit. As a corollary, we obtain that affine-free Coxeter groups are biautomatic.

1 Introduction

1.1 The main result

Let (W, S) be a Coxeter system of finite rank (i.e. $|S|$ is finite) and let $t_1, t_2, t_3 \in S^W$ be 3 reflections. We say that $T := \{t_1, t_2, t_3\}$ is a **reflection triangle** if the order of $t_i t_j$ is finite for all $1 \leq i < j \leq 3$ and if T is not contained in a rank 2 parabolic subgroup of W. It is known that given a triangle T, there exists a triangle T' such that $\langle T \rangle = \langle T' \rangle$ and $(\langle T \rangle, T')$ is a Coxeter system. Moreover, the Coxeter diagram of $(\langle T \rangle, T')$ is uniquely determined by T and we denote it by $\mathcal{M}(T)$. We say that T is affine (resp. spherical, hyperbolic) if $\mathcal{M}(T)$ is affine (resp. spherical, hyperbolic).

Here is our main result.

*F.N.R.S. research fellow

AMS subject classification codes (2000) : 20F28, 20F55.
Keywords: Coxeter group, biautomatic.

Theorem 1.1. *Suppose* (W, S) *does not contain an affine reflection triangle. Then there are only finitely many conjugacy classes of reflection triangles.*

It is well-known that in an arbitrary Coxeter system of finite rank, there are finitely many conjugacy classes of spherical reflection triangles. On the other hand, it is obvious that Coxeter groups of type $\tilde{A}_2$, $\tilde{C}_2$ and $\tilde{G}_2$ possess infinitely many conjugacy classes of affine reflection triangles. Based on recent work of the first author, there is strong evidence that in an arbitrary Coxeter system of finite rank, there are only finitely many conjugacy classes of non-affine reflection triangles.

1.2 Affine reflection triangles

Using results from Daan Krammer's thesis [8] and some additional arguments (see Section 3 below), one obtains the following result.

Theorem 1.2. (D. Krammer) *Given an affine reflection triangle* T*, then there exists an irreducible affine parabolic subgroup* W_0 *of rank* ≥ 3 *such that* $\langle T \rangle$ *is conjugate to a subgroup of* W_0*.*

Using Doedhar [6], one can verify that a (standard) parabolic subgroup is not conjugate to a proper subgroup. Combining this observation with Theorem 1.2, one can improve our main result to the following.

Theorem 1.3. *Let* (W, S) *be a Coxeter system of finite rank. The following statements are equivalent:*
(i) there are only finitely many conjugacy classes of reflection triangles;
(ii) the Coxeter diagram of (W, S) *has no irreducible affine subdiagram of rank* ≥ 3*.*

1.3 Biautomaticity

It is proved in [3] that every Coxeter group of finite rank is automatic. The question of determining whether or not Coxeter groups satisfy the stronger condition of biautomaticity remains however open.

As before, let (W, S) be a Coxeter system of finite rank. In [11] it is proved that W acts properly discontinuously on a locally finite, finite-dimensional CAT(0) cube complex. This cube complex is called the **Coxeter cubing** associated with (W, S); we denote it by $\mathcal{X}(W, S)$. As noticed in loc. cit., it follows from a result of [10] that W is biautomatic whenever the action of W on $\mathcal{X}(W, S)$ is cocompact. Furthermore, the cocompactness of this action has the following characterization, due to B. Williams (see Theorem 6 in [11] and Theorem 5.16 in [14]).

Proposition 1.4. *The action of the Coxeter group* W *on the Coxeter cubing* $\mathcal{X}(W, S)$ *is cocompact if and only if* (W, S) *has only finitely many conjugacy classes of reflection triangles.*

Remark. The previous statement differs slightly from the original statement of that result (see Theorem 6 in [11] and Theorem 5.16 in [14]). Indeed, in the latter references, it is spoken about 'triangle sugroups' and the fact that these may be assumed to be generated by reflections is only implicit. Nevertheless, this is a consequence of the proof of that result, as it appears in [14].

Therefore, Theorem 1.3 has the following consequences.

Corollary 1.5. *The action of the Coxeter group W on the Coxeter cubing $\mathcal{X}(W,S)$ is cocompact if and only if the Coxeter diagram of (W,S) contains no subdiagram of affine type and rank ≥ 3.*

Corollary 1.6. *If the Coxeter diagram of (W,S) contains no subdiagram of affine type and rank ≥ 3, then the Coxeter group W is biautomatic.*

Remark. Corollary 1.5 and Corollary 1.6 had been proven independently by P. Bahls [1] for certain special classes of Coxeter groups which are all covered by our theorem.

ACKNOWLEDGEMENT. The authors are grateful to Daan Krammer who communicated the arguments of Section 3 to them.

2 Preliminaries

In this section, we recall the basic properties of Coxeter groups that are needed in the sequel. The main references are [2] and [12].

Let (W,S) be a Coxeter system. A **reflection** is an element of W that is conjugate to an element of S.

Let $\Sigma(W,S)$ be the **chamber system** associated with (W,S). We recall that $\Sigma(W,S)$ is a chamber system over S which is defined as follows : the **chambers** of $\Sigma(W,S)$ are the elements of W and for each $s \in S$ two chambers v,w are s**-adjacent** if and only if $v^{-1}w = s$. Two chambers are **adjacent** if they are s-adjacent for some $s \in S$.

The group W acts on $\Sigma(W,S)$ by left multiplication. This action is regular (i.e. sharply transitive) and preserves the s-adjacency for each $s \in S$.

For every subset $J \subseteq S$, we put $W_J := \langle J \rangle$. The subset $J \subseteq S$ is called **spherical** whenever W_J is finite. Let c be a chamber of $\Sigma(W,S)$. The J**-residue** (or the **residue of type** J) containing c is the left coset cW_J, viewed as a set of chambers. The cardinality of J is called the **rank** of that residue. Residues of rank 1 are called **panels**. An nontrivial element t of W stabilizes a panel if and only if t is a reflection. A **parabolic subgroup** of W is a subgroup of the form $\mathrm{Stab}_W(R)$ for some residue R. The **type** (resp. **rank**) of the parabolic subgroup $\mathrm{Stab}_W(R)$ is the type (resp. rank) of R.

Lemma 2.1. *Every finite subgroup of W is contained in a parabolic subgroup of spherical type.*

Proof. See [2]. ☐

A sequence of chambers $\Gamma = (c_0, c_1, \ldots, c_n)$ such that c_{i-1} is adjacent to c_i for $1 \leq i \leq n$ is called a **gallery of length** n joining c_0 to c_n. We say that Γ is **closed** if $x_0 = x_n$. Given $1 \leq i < j \leq n$, then the subsequence $(c_i, c_{i+1}, \ldots, c_j)$ of Γ is denoted by $\Gamma|_{c_i \to c_j}$. The gallery $(c_n, c_{n-1}, \ldots, c_1, c_0)$ is denoted by Γ^{-1}. The **distance** $d(x,y)$ between two chambers x and y is the minimal length of a gallery joining x to y. A gallery of length n joining x to y is called **minimal** whenever $n = d(x,y)$.

Given a chamber c and a residue R in $\Sigma(W,S)$, then there exists a unique chamber x of R such that $d(c,x) = d(c,R) = \min\{d(c,y) | y \in R\}$. This chamber is called the **projection** of c on R and is denoted by $\mathrm{proj}_R(c)$. Given any chamber y in R, then there exists a minimal gallery Γ joining c to y which goes through $\mathrm{proj}_R(c)$ and such that $\Gamma|_{\mathrm{proj}_R(c)\to y}$ is completely contained in R.

Given residues R_1, R_2 of $\Sigma(W,S)$, then the set $\mathrm{proj}_{R_1}(R_2) := \{\mathrm{proj}_{R_1}(c) | c \in R_2\}$ is itself a residue. We say that R_1 and R_2 are **parallel** if $\mathrm{proj}_{R_1}(R_2) = R_1$ and $\mathrm{proj}_{R_2}(R_1) = R_2$.

The following lemma gives two characterizations of the parallelism of residues of spherical type.

Lemma 2.2. *Let J, K be subsets of S and let R_J, R_K be residues of type J, K respectively. Then the following statements are equivalent:*
(i) *R_J and R_K are parallel;*
(ii) *a reflection stabilizes R_J if and only if it stabilizes R_K.*
Furthermore, if J or K is spherical, then (i) and (ii) above are also equivalent to the following:
(iii) *there exist two sequences $R_J = R_0, R_1, \dots, R_n = R_K$ and $T_1, \dots, T_n$ of residues of spherical type such that for each $1 \le i \le n$ the rank of T_i is equal to $1+\mathrm{rank}(R_J)$, the residues R_{i-1}, R_i are distinct, parallel and contained in T_i and moreover, we have $\mathrm{proj}_{T_i}(R_J) = R_{i-1}$ and $\mathrm{proj}_{T_i}(R_K) = R_i$.*

Proof. This follows from Proposition 2.7 in [4]. □

Let $s \in S$ and let $\pi = \{x, y\}$ be an s-panel of $\Sigma(W,S)$, namely a residue of type $\{s\}$. The set $\phi(x,y) = \{z | d(z,x) < d(z,y)\}$ is called a **root** of $\Sigma(W,S)$. The set $\phi(y,x)$ is also a root, complementary to $\phi(x,y)$, and the unique reflection that stabilizes the panel π interchanges $\phi(x,y)$ and $\phi(y,x)$. We denote this reflection by $r_{\phi(x,y)}$ or $r_{\phi(y,x)}$ and we write $\phi(y,x) = -\phi(x,y)$.

Let ψ be a root. We denote by $\partial\psi$ (resp. $\partial^2\psi$) the set of all panels (resp. spherical residues of rank 2) stabilized by r_ψ. We also set $\mathcal{C}(\partial\psi) := \bigcup_{\sigma\in\partial\psi} \psi$ and $\mathcal{C}(\partial^2\psi) := \bigcup_{\sigma\in\partial^2\psi} \psi$. The set $\partial\psi$ is called the **wall** associated to ψ. Let $\Gamma = (x_0, x_1, \dots, x_n)$ be a gallery. We say that Γ **crosses the wall** $\partial\psi$ if there exists $1 \le i \le n$ such that $\{x_{i-1}, x_i\}$ is a panel that belongs to $\partial\psi$. It is a fact that a gallery is minimal if and only if it crosses every wall at most once.

Lemma 2.3. *Let ψ be a root and let $x, y \in \psi \cap \mathcal{C}(\partial\psi)$. Then there exists a minimal gallery $\Gamma = (x = x_0, x_1, \dots, x_l = y)$ joining x to y such that $x_i \in \mathcal{C}(\partial^2\psi)$ for each $1 \le i \le l$.*

Proof. Let $\pi_x, \pi_y \in \partial\psi$ be panels such that $x \in \pi_x$ and $y \in \pi_y$. Then π_x and π_y satisfy Condition (ii) of Lemma 2.2. Therefore, there exist two sequences $\pi_x = \pi_0 \neq \pi_1 \neq \dots \neq \pi_n = \pi_y$ and $\sigma_1, \dots, \sigma_n$ such that for each $1 \le i \le n$ we have $\pi_i \in \partial\psi$, $\sigma_i \in \partial^2\psi$, $\mathrm{proj}_{\sigma_i}(\pi_x) = \pi_{i-1}$ and $\mathrm{proj}_{\sigma_i}(\pi_y) = \pi_i$.

For each $1 \le i \le n$ let Γ_i be the unique minimal gallery joining $\mathrm{proj}_{\sigma_i}(x)$ to $\mathrm{proj}_{\sigma_i}(y)$. Let Γ be the gallery obtained by concatenating the Γ_i's (namely $\Gamma = \Gamma_1 \sim \dots \sim \Gamma_n$). Then Γ is a gallery joining x to y and such that every chamber of Γ belongs to $\mathcal{C}(\partial^2\psi)$.

We claim that Γ is minimal. We prove this claim by induction on n; the result being obvious when $n = 1$.

Suppose that $n > 1$ and that Γ is not minimal. Then there exists a root ϕ containing π_x such that Γ crosses the wall $\partial\phi$ twice. By induction, we know that $\Gamma|_{x\to\mathrm{proj}_{\sigma_n}(x)}$ and $\Gamma|_{\mathrm{proj}_{\sigma_1}(y)\to y}$ are both minimal. By the construction of Γ, this implies that Γ crosses $\partial\phi$ exactly twice : once between x and $\mathrm{proj}_{\sigma_1}(y)$ and once between $\mathrm{proj}_{\sigma_n}(y)$ and y. We deduce that π_y is contained in the same side of $\partial\phi$ as π_x, namely $\pi_y \subseteq \phi$. This implies that $\text{proj}_{\sigma_1}(\pi_y) = \pi_x$. But we have seen above that $\text{proj}_{\sigma_1}(\pi_y) = \pi_1 \neq \pi_x$ and this is a contradiction. Therefore, Γ is minimal. □

Let Ψ be a set of roots. We set $R(\Psi) := \{r_\psi | \psi \in \Psi\}$ and $W(\Psi) := \langle R(\Psi)\rangle$. The set Ψ is called **geometric** if $\bigcap_{\psi\in\Psi} \psi$ is nonempty and if for all $\phi, \psi \in \Psi$, the set $\phi \cap \psi$ is a fundamental domain for the action of $W(\{\phi, \psi\})$ on $\Sigma(W, S)$. Here, a set D is called a **fundamental domain** for the action of a group G on a set E containing D if $\bigcup_{g\in G} gD = E$ and if $D \cap gD \neq \emptyset \Rightarrow g = 1$ for every $g \in G$.

The following result, due to Tits, is very useful.

Lemma 2.4. *Let Ψ be a geometric set of roots. Then $D := \bigcap \Psi$ is a fundamental domain for the action of $W(\Psi)$ on $\Sigma(W, S)$, and $(W(\Psi), R(\Psi))$ is a Coxeter system. The chambers of $\Sigma(W(\Psi), R(\Psi))$ may be identified with sets of chambers of $\Sigma(W, S)$, and more precisely with sets of the form wD with $w \in W(\Psi)$. Furthermore, two chambers C and C' of $\Sigma(W(\Psi), R(\Psi))$ are adjacent in $\Sigma(W(\Psi), R(\Psi))$ if and only if C and C', viewed as sets of chambers of $\Sigma(W, S)$, contain adjacent chambers of $\Sigma(W, S)$.*

Proof. This is essentially a consequence of Lemma 1 in [13]. See also Lemma 3.2 and Proposition 3.3 in [9]. □

Restated in other words, the last statement of Lemma 2.4 says that the Cayley graph of the Coxeter system $W((\Psi), R(\Psi))$ may be seen as a 'quotient' of the Cayley graph of (W, S).

3 Affine subgroups

In this section, we record an unpublished result due to Daan Krammer (see Theorem 3.3).

A Coxeter system (W, S) of finite rank is called **admissible** if each component of its Coxeter diagram is either spherical or affine of rank ≥ 3. A subset $I \subseteq S$ is called **admissible** if the Coxeter system (W_I, I) is admissible. If (W, S) is admissible but not of spherical type, then W is virtually a free abelian group of rank at least 2, i.e. W possesses a subgroup of finite index isomorphic to $\mathbb{Z}^n$, where $n \geq 2$.

Lemma 3.1. *Let (W, S) be an admissible Coxeter system. Then no quotient group of W is virtually infinite cyclic.*

Proof. If W is finite, the result is trivial. Otherwise, some component of S is affine of rank ≥ 3. Let $S_1, S_2, \ldots, S_k$ be the irreducible components of S. For each index i such that W_{S_i} is affine, let Z_i be the translation subgroup of W_{S_i}. Then W_{S_i} (and, hence, W) acts on Z_i by conjugation. Furthermore, the irreducibility of the geometric representation of finite irreducible Coxeter group (see [2]) implies that Z_i is an irreducible W_{S_i}-module. In particular, Z_i is an irreducible W-module. Therefore, given any normal subgroup N of W, we have either $N \cap Z_i = \{1\}$ or $N \cap Z_i$ is free abelian of rank n. The conclusion follows because each Z_i is a free abelian group of rank ≥ 2. □

Let (W, S) be a Coxeter group of finite rank and H be a subset of W. Let $\mathcal{S}(H)$ be the set of all residues stabilized by $\langle H\rangle$. Let $n(H)$ be the minimum of the set of ranks of elements of $\mathcal{S}(H)$. If R_1 and R_2 both belong to $\mathcal{S}(H)$, then so do $\text{proj}_{R_1}(R_2)$ and

$\operatorname{proj}_{R_2}(R_1) \in \mathcal{S}(H)$. We deduce that if $R_1, R_2 \in \mathcal{S}(H)$ are both of rank $n(H)$, then R_1 and R_2 are parallel. In particular, Lemma 2.2 gives $\operatorname{Stab}_W(R_1) = \operatorname{Stab}_W(R_2)$. We set $\operatorname{Pc}(H) := \operatorname{Stab}_W(R_1) = \operatorname{Stab}_W(R_2)$. The group $\operatorname{Pc}(H)$ is called the **parabolic closure** of H in W. It is the smallest parabolic subgroup of W containing H.

Proposition 3.2. *Let (W, S) be an irreducible, non-spherical and non-affine Coxeter system of finite rank. Let H be a subgroup of W with $\operatorname{Pc}(H) = W$ such that H is virtually $\mathbb{Z}^n$. Then $n = 1$.*

Proof. Let H_1 be a subgroup of finite index of H which is isomorphic to $\mathbb{Z}^n$. If $n = 0$ then H is finite and Lemma 2.1 implies that $\operatorname{Pc}(H)$ is spherical, in contradiction with our hypotheses. Thus H is infinite and $n > 0$.

The group $N_H(H_1)$ has finite index in H, which implies that H_1 has finitely many H-conjugates. Therefore, the group $H_0 := \bigcap_{h \in H} hH_1h^{-1}$ has finite index in H. Moreover, it is clear by definition that H_0 is normal in H and that H_0 is isomorphic to $\mathbb{Z}^n$. Since H normalizes H_0, it follows that H normalizes $\operatorname{Pc}(H_0)$. This gives

$$W = \operatorname{Pc}(H) \leq \operatorname{Pc}(N_W(\operatorname{Pc}(H_0)))$$

which implies that $\operatorname{Pc}(H_0) = W$ by Lemma 6.8.1 of [8] (notice that $\operatorname{Pc}(H_0)$ is not spherical since H_0 is infinite). Now Theorem 6.8.2 in [8] provides the desired conclusion. $\square$

Theorem 1.2 above is a consequence of the following result.

Theorem 3.3. (D. Krammer) *Let (W, S) and $(\tilde{W}, \tilde{S})$ be Coxeter systems of finite rank, such that $(\tilde{W}, \tilde{S})$ is admissible and $\tilde{W}$ is a subgroup of W. Then $\tilde{W}$ is contained in an admissible parabolic subgroup of W.*

Proof. Replacing $\tilde{W}$ be one of its conjugates if necessary, we may assume that $\operatorname{Pc}(\tilde{W}) = W_I$ for some $I \subseteq S$. We have to prove that I is admissible.

Let $I_1, I_2, \ldots, I_k$ be the irreducible components of I. Thus $\tilde{W} \leq W_I = W_{I_1} \times \cdots \times W_{I_k}$. For each $1 \leq i \leq k$, let $\tilde{W}_i$ be the projection of $\tilde{W}$ on W_{I_i}. Since $\operatorname{Pc}(\tilde{W}) = W_I$, we have $\operatorname{Pc}(\tilde{W}_i) = W_{I_i}$ for each $1 \leq i \leq k$.

We have to show that each I_i is either spherical or affine of cardinality at least 3.

Let i be such that $\tilde{W}_i$ is finite. Then $\operatorname{Pc}(\tilde{W}_i)$ is finite by Lemma 2.1, which implies that I_i is spherical in view of $\operatorname{Pc}(\tilde{W}_i) = W_{I_i}$.

Let i be such that $\tilde{W}_i$ is infinite. Then $\tilde{W}_i$, as a quotient of $\tilde{W}$, is virtually a free abelian group of rank m, and Lemma 3.1 gives $m \geq 2$. In particular, W_{I_i} is not infinite dihedral, which means that I_i is not of type $\tilde{A}_1$. Therefore, Proposition 3.2 implies that I_i is affine of cardinality at least 3. $\square$

4 Coxeter decompositions of hyperbolic triangles

This section is intended to recall the classification of the Coxeter decompositions of hyperbolic triangles. Our reference is [7].

Let P be a compact geodesic polygon of the hyperbolic plane $\mathbb{H}^2$. We recall that a Coxeter decomposition of P is a non-trivial decomposition of P into finitely many Coxeter polygons F_i, such that any two polygons F_1 and F_2 having a common side are symmetric

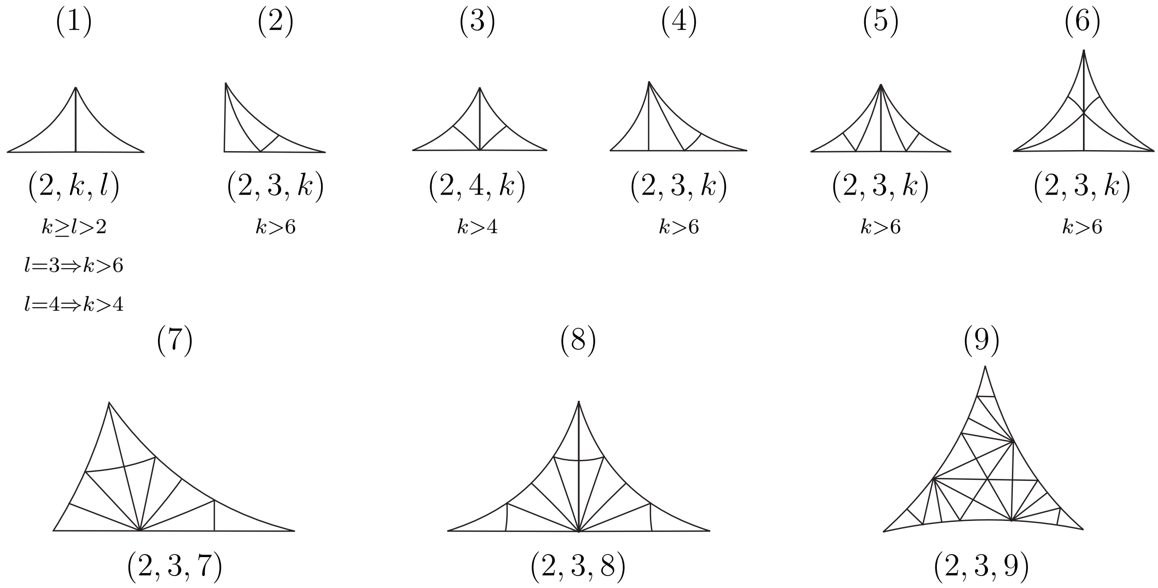

Figure 1: The Coxeter decompositions of hyperbolic triangles.

with respect to this common side. The polygons F_i are called the fundamental polygons of the Coxeter decomposition. It follows from the definition that they are all isometric.

If P is a triangle, then the fundamental polygons of any Coxeter decomposition of P are triangles (Lemma 1 in [7]). We say that a given Coxeter decomposition of P has **type** (k, l, m) if the angles of the fundamental triangles of that decomposition are $\frac{\pi}{k}$, $\frac{\pi}{l}$ and $\frac{\pi}{m}$. Let 0, 1 and 2 be the vertices of P. For each $0 \leq i \leq 2$ let μ_i be the number of fundamental triangles having a vertex that coincides with i. We say that (μ_0, μ_1, μ_2) are the **multiplicities** of the given Coxeter decomposition of P.

In Figure 1, some Coxeter decompositions of hyperbolic triangles are represented. The triple (k, l, m) under each picture gives the type of the fundamental triangle of the Coxeter decomposition that this picture represents.

Theorem 4.1. *Figure 1 exhausts the list of all possible Coxeter decompositions of hyperbolic triangles.*

Proof. See Sections 2 and 5 in [7]. □

5 Combinatorial triangles in $\Sigma(W, S)$

Definitions

Let (W, S) be a Coxeter system and let $\Sigma(W, S)$ be the associated chamber system.

A **combinatorial triangle** (or simply a **triangle**) of $\Sigma(W, S)$ is a set T of three roots which satisfy the following conditions:

(CT1) for all $\alpha, \alpha' \in T$, the order of $r_\alpha r_{\alpha'}$ is finite;
(CT2) the group $W(T)$ is not contained in any parabolic subgroup of rank 2;
(CT3) the set $\bigcap_{\alpha \in T} \alpha$ is nonempty.

Let T_1 and T_2 be combinatorial triangles. We say that T_1 is a **subtriangle** of T_2 if $\bigcap T_1 \subseteq \bigcap T_2$ and if there exists a triangle T_0 such that $W(T_1) \cup W(T_2) \subseteq W(T_0)$.

Let $T = \{\alpha_0, \alpha_1, \alpha_2\}$ be a combinatorial triangle and for each $0 \leq i \leq 2$, let λ_i be the order of $r_{\alpha_{i-1}} r_{\alpha_{i+1}}$, where the indices are taken modulo 3. We say that T is of **type** $(\lambda_0, \lambda_1, \lambda_2)$. In the case where T is geometric, we know from Lemma 2.4 that $(W(T), R(T))$ is a Coxeter system of rank 3, and the λ_i's are then the Coxeter numbers that appear on the Coxeter diagram of $(W(T), R(T))$.

A combinatorial triangle T is called **spherical** if $W(T)$ is finite. It is called **affine** if it is of type $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$, or if it is non-geometric and of type $(3, 6, 6)$. If T is geometric then T is affine if and only if the Coxeter diagram of $(W(T), T)$ is affine. A combinatorial triangle T is called **hyperbolic** if it is neither spherical nor affine.

Notice that that we have now two different kinds of hyperbolic triangles : the hyperbolic triangles as in Section 4 and the combinatorial hyperbolic triangles as in the preceding paragraph. In order to avoid any confusion between these two kinds, we sometimes refer to the former triangles as **genuine hyperbolic triangles**.

Let $T = \{\alpha_0, \alpha_1, \alpha_2\}$ be a combinatorial triangle. For each $0 \leq i \leq 2$, let σ_i be a spherical residue which belongs to $\partial^2\alpha_{i-1} \cap \partial^2\alpha_{i+1}$ (subscripted indices are taken modulo 3) and which is contained in α_i. The set $\{\sigma_0, \sigma_1, \sigma_2\}$ is called a **set of vertices** of T. Notice that the fact that $\sigma_i \in \partial^2\alpha_{i-1} \cap \partial^2\alpha_{i+1}$ automatically implies that $\sigma_i \subseteq \alpha_i$ or $\sigma_i \subseteq -\alpha_i$ because $\sigma_i \notin \partial^2\alpha_i$ in view of (CT2).

Let $T = \{\alpha_0, \alpha_1, \alpha_2\}$ be a combinatorial triangle and let $\{\sigma_0, \sigma_1, \sigma_2\}$ be a set of vertices of T. For each $0 \leq i \neq j \leq 2$ let $x_{i,j}$ be the unique chamber of $\mathrm{proj}_{\sigma_i}(\sigma_j)$ that belongs to α_k where $0 \leq k \leq 2$ and $i \neq k \neq j$. By Lemma 2.3, for each $0 \leq i \leq 2$ there exists a minimal gallery Γ_i joining $x_{i-1,i+1}$ to $x_{i+1,i-1}$ such that every chamber of Γ_i belongs to $\mathcal{C}(\partial^2\alpha_i)$ (indices are taken modulo 3). Let also $\tilde{\Gamma}_i$ be the unique minimal gallery joining $x_{i,i+1}$ to $x_{i,i-1}$. Finally, let Γ be the gallery obtained by concatenating the Γ_i's and the $\tilde{\Gamma}_i$'s. Hence

$$\Gamma = \Gamma_0 \sim \tilde{\Gamma}_1 \sim \Gamma_2 \sim \tilde{\Gamma}_0 \sim \Gamma_1 \sim \tilde{\Gamma}_2.$$

Notice that Γ is closed by construction. We say that the gallery Γ **skirts around** the triangle T and that the set of vertices $\{\sigma_0, \sigma_1, \sigma_2\}$ **supports** Γ. The **perimeter** of T is the minimum of the set of lengths of all galleries that skirt around T.

For $0 \leq i \leq 2$, we define $]\sigma_{i-1}, \sigma_{i+1}[_\Gamma$ to be the set of all $\sigma \in \partial^2\alpha_i$ that are crossed by Γ, i.e. such that Γ crosses a panel contained in σ. We also set $[\sigma_{i-1}, \sigma_{i+1}]_\Gamma :=]\sigma_{i-1}, \sigma_{i+1}[_\Gamma \cup \{\sigma_{i-1}, \sigma_{i+1}\}$, $[\sigma_{i-1}, \sigma_{i+1}[_\Gamma :=]\sigma_{i-1}, \sigma_{i+1}[_\Gamma \cup \{\sigma_{i-1}\}$ and $]\sigma_{i-1}, \sigma_{i+1}]_\Gamma :=]\sigma_{i-1}, \sigma_{i+1}[_\Gamma \cup \{\sigma_{i+1}\}$. We record that two distinct elements of $[\sigma_{i-1}, \sigma_{i+1}]_\Gamma$ are never parallel.

Lemma 5.1. *Let $T = \{\alpha_0, \alpha_1, \alpha_2\}$ be a combinatorial triangle, let $\{\sigma_0, \sigma_1, \sigma_2\}$ be a set of vertices and let Γ be a gallery supported by $\{\sigma_0, \sigma_1, \sigma_2\}$ and which skirts around T. Then for all $0 \leq i \neq j \leq 2$ we have the following.*

(1) For all $\sigma \in]\sigma_i, \sigma_j]_\Gamma$, r_{α_i} is the only reflection that stabilizes both σ_i and σ.

(2) If $\sigma \in]\sigma_i, \sigma_j[_\Gamma$ then $\sigma \subseteq \alpha_i \cap \alpha_j$;

(3) Let k be such that $\{0, 1, 2\} = \{i, j, k\}$, let $\sigma \in]\sigma_i, \sigma_j[_\Gamma$ and let $\tau \in]\sigma_i, \sigma_k[_\Gamma$. Assume that r is a reflection that stabilizes both σ and τ. Then σ_i is contained in one of the roots associated with r and every element of $[\sigma_j, \sigma_k]_\Gamma$ is contained in the other.

Proof. (1) If there existed a reflection $r \neq r_{\alpha_i}$ which stabilizes both σ_i and σ, then σ_i

and σ would be parallel. This contradicts the fact that $\sigma \in]\sigma_i, \sigma_j]_\Gamma$, which proves (1).

(2) By (1), we know that neither r_{α_i} nor r_{α_j} stabilizes σ.

Suppose that $\sigma \subseteq -\alpha_i$. Since $\sigma_i \subseteq \alpha_i$, it follows that Γ crosses the wall $\partial\alpha_i$. So, there exists $\sigma' \in]\sigma_i, \sigma[_\Gamma \cap \partial^2\alpha_i \subset]\sigma_i, \sigma_j[_\Gamma \cap \partial^2\alpha_i$. We have just seen that r_{α_i} does not stabilize any element of $]\sigma_i, \sigma_j[_\Gamma$. This contradiction shows that $\sigma \subseteq \alpha_i$ and by symmetry, it follows that $\sigma \subseteq \alpha_j$.

(3) Let ψ be the root containing $\mathrm{proj}_\sigma(\sigma_i)$ and such that $r_\psi = r$. Clearly, we have $\sigma_i \subseteq \psi$. Moreover, $\sigma_j \subseteq -\psi$ and $\sigma_k \subseteq -\psi$ because $\mathrm{proj}_\sigma(\sigma_j) \subseteq -\psi$ and $\mathrm{proj}_\sigma(\sigma_k) \subseteq -\psi$.

Let $\sigma' \in]\sigma_j, \sigma_k[_\Gamma$ and assume that $\sigma' \in \partial^2\psi$. Since σ_j and σ_k are both contained in $-\psi$, it follows that there exists a $\tau' \in]\sigma_j, \sigma_k[_\Gamma \cap \partial^2\psi$ with $\tau' \neq \sigma'$. Therefore, σ' and τ' are distinct and both are stabilized by r_ψ and r_{α_i}. Furthermore, by (2) we have $r_\psi \neq r_{\alpha_i}$ from which it follows that σ and τ' are parallel. This is impossible.

Thus r_ψ does not stabilize any element of $[\sigma_j, \sigma_k]_\Gamma$. We have seen above that σ_j and σ_k are both contained in $-\psi$. We deduce by an argument as in the proof of (2) above that every element of $[\sigma_j, \sigma_k]_\Gamma$ is contained in $-\psi$.

□

Coxeter decompositions of combinatorial triangles

Let $T = \{\alpha_0, \alpha_1, \alpha_2\}$ be a combinatorial triangle. For each $0 \leq i \leq 2$, let ϕ_i be a root such that $\{\alpha_i, \phi_i\}$ is a geometric pair and that $r_{\alpha_{i+1}} \in \langle r_{\alpha_i}, r_{\phi_i} \rangle$ (indices are taken modulo 3). Then we have $\alpha_i \cap \phi_i \subseteq \alpha_i \cap \alpha_{i+1}$ and we say that ϕ_i is a **decomposing root** of T. A decomposing root ϕ_i is called **standard** if $\langle r_{\alpha_i}, r_{\alpha_{i+1}} \rangle = \langle r_{\alpha_i}, r_{\phi_i} \rangle$. For each $0 \leq i \leq 2$, we put $\psi_i^0 := -\alpha_i$, $\psi_i^1 := \phi_i$ and $\psi_i^n := r_{\psi_i^{n-1}}(-\psi_i^{n-2})$ for $n \geq 2$. Let μ_i be the smallest integer n such that $\psi_i^n = \alpha_{i+1}$. The number μ_i is called the **multiplicity** of the decomposing root ϕ_i. In that situation, we say that the decomposing roots ϕ_0, ϕ_1 and ϕ_2 induce a **Coxeter decomposition** of T with multiplicities (μ_0, μ_1, μ_2). Notice that every triangle has standard decomposing roots. Notice that a triangle T is geometric if and only if it admits a Coxeter decomposition with multiplicities $(1, 1, 1)$.

The following lemma is a combinatorial analogue of Lemma 1 in [7].

Lemma 5.2. *Let $T = \{\alpha_0, \alpha_1, \alpha_2\}$ be a combinatorial triangle and let ϕ_0, ϕ_1, ϕ_2 be decomposing roots. Then there exists a geometric triangle $\tilde{T}$ such that $W(\tilde{T}) = \langle W(T) \cup \{r_{\phi_0}, r_{\phi_1}, r_{\phi_2}\} \rangle$.*

Proof. Let p be the perimeter of T. Let Γ be a closed gallery of length p that skirts around T and let $\{\sigma_0, \sigma_1, \sigma_2\}$ be a set of vertices that supports Γ.

The proof is by induction on p.

Assume that $p = 0$. Then σ_0, σ_1 and σ_2 have a chamber in common, which implies that T is geometric and that each decomposing root of T is standard. The result follows by choosing $\tilde{T} = T$.

Assume that $p > 0$. Then Γ is a nontrivial closed gallery and is *a fortiori* not minimal. If $\{\phi_0, \phi_1, \phi_2\}$ coincides with T, then the result follows again by choosing $\tilde{T} = T$. Otherwise, we may assume without loss of generality that the decomposing root ϕ_0 does not belong to T. Therefore, Γ crosses the wall $\partial\phi_0$ twice. By Lemma 5.1(1), it follows that there exists a residue $\sigma \in]\sigma_1, \sigma_2[_\Gamma$ which is stabilized by r_{ϕ_0}. Using Lemma 5.1(2) and (3), we deduce that $T_1 = \{\alpha_0, \phi_0, \alpha_2\}$ is a combinatorial triangle. By construction, we have $W(T_1) = \langle W(T) \cup \{r_{\phi_0}\}\rangle$. Moreover, it follows from the definition of Γ that the perimeter of T is strictly smaller than p. Now, we apply the induction hypothesis to T_1 endowed with a set of standard decomposing roots. This yields a geometric triangle $\tilde{T}_1$ such that $W(\tilde{T}_1) = W(T_1) = \langle W(T) \cup \{r_{\phi_0}\}\rangle$. If r_{ϕ_1} and r_{ϕ_2} belong to $W(\tilde{T}_1)$, then we define $\tilde{T} := \tilde{T}_1$ and we are done.

Suppose that $r_{\phi_1} \not\in W(T_1)$. Then $\tilde{T}_1$ possesses a $W(\tilde{T}_1)$-conjugate $T_2 = \{\beta_0, \beta_1, \beta_2\}$ such that $\langle r_{\beta_0}, r_{\beta_2}\rangle = \langle r_{\alpha_0}, r_{\alpha_2}\rangle$ and that ϕ_1 is a non-standard decomposing root of T_2. Since $\tilde{T}_1$ and T_2 have the same perimeter $< p$, the induction hypothesis yields a geometric triangle $\tilde{T}_2$ such that $W(\tilde{T}_2) = \langle W(T_2) \cup \{r_{\phi_1}\}\rangle = \langle W(T) \cup \{r_{\phi_0}, r_{\phi_1}\}\rangle$. If $r_{\phi_2} \in W(\tilde{T}_2)$, then we set $\tilde{T} := \tilde{T}_2$ and we are done. Otherwise, we apply to $\tilde{T}_2$ the argument we have just applied to $\tilde{T}_1$. This yields a geometric triangle $\tilde{T}$ such that $W(\tilde{T}) = \langle W(\tilde{T}_2) \cup \{r_{\phi_2}\}\rangle = \langle W(T) \cup \{r_{\phi_0}, r_{\phi_1}, r_{\phi_2}\}\rangle$. □

In the situation of Lemma 5.2, we say that $\tilde{T}$ is a **fundamental triangle** of the given decomposition of T. We define the **type** of that decomposition to be the type of $\tilde{T}$ (or the type of the Coxeter system $(W(\tilde{T}), R(\tilde{T}))$).

The following result justifies the similarities between the terminology introduced in the preceding section and in the current one.

Lemma 5.3. *Let T be a combinatorial triangle. Then every Coxeter decomposition of T of hyperbolic type corresponds canonically to a Coxeter decomposition of a genuine hyperbolic triangle (and thus to one of the decompositions represented in Figure 1). These two decompositions have the same type and the same multiplicities.*

Proof. Assume that T has a Coxeter decomposition of hyperbolic type with fundamental triangle $\tilde{T}$. Let (k, l, m) be the type of $\tilde{T}$. Since $\tilde{T}$ has hyperbolic type, the Coxeter group $W(\tilde{T})$ can be realized as the group generated by the reflection through the edge of a compact geodesic triangle of $\mathbb{H}^2$ whose angles are $\frac{\pi}{k}$, $\frac{\pi}{l}$ and $\frac{\pi}{m}$. Now, the result is a consequence of Lemma 2.4. □

Corollary 5.4. *Let T be a combinatorial triangle. Then the following statements are equivalent.*

(i) T is of spherical (resp. affine, hyperbolic) type.
(ii) every Coxeter decomposition of T is of spherical (resp. affine, hyperbolic) type.
(iii) T admits a Coxeter decomposition of spherical (resp. affine, hyperbolic) type.

Proof. In the spherical case, the equivalence between (i), (ii) and (iii) is an immediate consequence of the definitions and of Lemma 5.2. If T is affine, then T has no decomposition of hyperbolic type in view of Lemma 5.3. This proves that (i) implies (ii) in the affine case. The implication (ii) $\Rightarrow$ (iii) is immediate, because any triangle admits a Coxeter decomposition in view of from Lemma 5.2. A case by case consideration of the Coxeter complexes of affine type shows that if T admits a Coxeter decomposition of affine type, then T has to be affine. This proves that (iii) implies (i) in the affine case. Now, the desired equivalences in the hyperbolic case follow at once. □

A preparatory lemma

The following lemma collects a series of technicalities that are needed in the proof of Theorem 1.1.

Lemma 5.5. *Let $T = \{\alpha_0, \alpha_1, \alpha_2\}$ be a combinatorial triangle, $\{\sigma_0, \sigma_1, \sigma_2\}$ be a set of vertices of T and Γ be a gallery supported by $\{\sigma_0, \sigma_1, \sigma_2\}$ and which skirts around T. Let $\sigma \in]\sigma_1, \sigma_2[_\Gamma$ and let $n_\sigma := \frac{|\sigma|}{2}$. Then we have the following.*

(1) there exists an integer $1 \leq f_\sigma \leq n_\sigma - 1$ and a geometric pair of roots $\{\phi_\sigma, \psi_\sigma\}$ of σ such that $r_{\phi_\sigma} r_{\psi_\sigma}$ has order n_σ and that, up to interchanging α_1 and α_2 if necessary, one of the following situations occurs:

(i) $f_\sigma = n_\sigma - 1$, $T_\sigma := \{\alpha_0, \alpha_1, \phi_\sigma\}$ is a combinatorial triangle and $r_{\phi_\sigma}(-\psi_\sigma)$ is a decomposing root of T_σ of multiplicity f_σ; moreover, there exists a residue $\tau \in [\sigma_0, \sigma_2[_\Gamma$ such that $\{\sigma_2, \sigma, \tau\}$ is a set of vertices of T_σ;

(ii) $f_\sigma < n_\sigma - 1$, $T_\sigma := \{\alpha_0, \alpha_1, \phi_\sigma\}$ (resp. $T'_\sigma := \{\alpha_0, \alpha_2, \psi_\sigma\}$) is a combinatorial triangle and $r_{\phi_\sigma}(-\psi_\sigma)$ (resp. $r_{\psi_\sigma}(-\phi_\sigma)$) is a decomposing root of T_σ (resp. T'_σ) of multiplicity f_σ (resp. $n_\sigma - f_\sigma - 1$); moreover, there exists a residue $\tau \in [\sigma_0, \sigma_2[_\Gamma$ (resp. $\tau' \in]\sigma_0, \sigma_1[_\Gamma$) such that $\{\sigma_2, \sigma, \tau\}$ (resp. $\{\sigma_1, \sigma, \tau\}$) is a set of vertices of T_σ (resp. T'_σ).

(2) Assume that T is non-spherical and that T_σ and T'_σ (if it exists) are both hyperbolic. Assume also that no reflection of σ stabilizes σ_0. If $n_\sigma > 3$ then the product $r_{\alpha_0} r_{\alpha_2}$ is of order at least 3 except if $n_\sigma = 7$ and T has type $(2, 7, 7)$.

(3) Assume that T is non-spherical and geometric and that T_σ and T'_σ (if $f_\sigma < n_\sigma - 1$) are both hyperbolic and that $r_{\alpha_1} r_{\alpha_2}$ has order 2. Assume also that no reflection of σ stabilizes σ_0. Assume finally that there exists a root ϕ'_σ such that $r_{\phi'_\sigma}$ stabilizes σ, that $\alpha_0 \cap \phi'_\sigma \subseteq \alpha \cap \phi_\sigma$ and that $r_{\alpha_1} r_{\phi'_\sigma}$ has order 2. Then one of the following situations occur:

(a) $n_\sigma = 7$ and the pair $\{\alpha_0, \phi'_\sigma\}$ is not geometric;

(b) $n_\sigma = 4$, $f_\sigma = 1$ and $o(r_{\alpha_0} r_{\alpha_2}) \geq 5$;

(c) $n_\sigma = 3$, $f_\sigma = 2$ and $o(r_{\alpha_0} r_{\alpha_2}) \geq 4$ unless $\{\alpha_0, \alpha_2, -\phi_\sigma\}$ is an affine triangle;

(d) $n_\sigma = 3$ and $f_\sigma = 1$.

Proof. (1) Let $\phi_1, \phi_2 \dots \phi_{n_\sigma}$ be roots such that $\{r_{\phi_1}, \dots, r_{\phi_{n_\sigma}}\}$ is the set of all reflections that stabilize σ. We may choose the ϕ_i's in such a way that $\phi_{n_\sigma} = \alpha_0$ and that $\phi_1 \cap (-\phi_i) \subseteq \phi_1 \cap (-\phi_j)$ for all $1 \leq i \leq j \leq n_\sigma$ (see Figure 2). By the definition of Γ and by Lemma 5.1(3), it follows that Γ crosses every wall $\partial\phi_1, \dots, \partial\phi_{n_\sigma - 1}$ exactly twice. Up to interchanging α_1 and α_2 if necessary, we may assume that there exists a residue $\tau \in [\sigma_0, \sigma_2[_\Gamma \cap \partial^2\phi_1$. We define

$$f_\sigma := \max\{i | 1 \leq i \leq n_\sigma - 1, \text{ there exists } \tau \in [\sigma_0, \sigma_2[_\Gamma \cap \partial^2\phi_i\}.$$

We also set $\phi_\sigma := \phi_{f_\sigma}$ and $\psi_\sigma := -\phi_{f_\sigma + 1}$. By the definition of f_σ, we have $1 \leq f_\sigma \leq n - 1$ and if $f_\sigma < n - 1$ then there exists a residue $\tau' \in]\sigma_0, \sigma_1[_\Gamma \cap \partial^2\psi_\sigma$. Let $T_\sigma := \{\alpha_0, \alpha_1, \phi_\sigma\}$ and if $f_\sigma < n_\sigma - 1$ then let $T'_\sigma := \{\alpha_0, \alpha_2, \psi_\sigma\}$. Then T_σ and T'_σ satisfy (CT1). The fact that they also satisfy (CT2) and (CT3) is a consequence of Lemma 5.1. Now, all assertions of Part (1) follow at once.

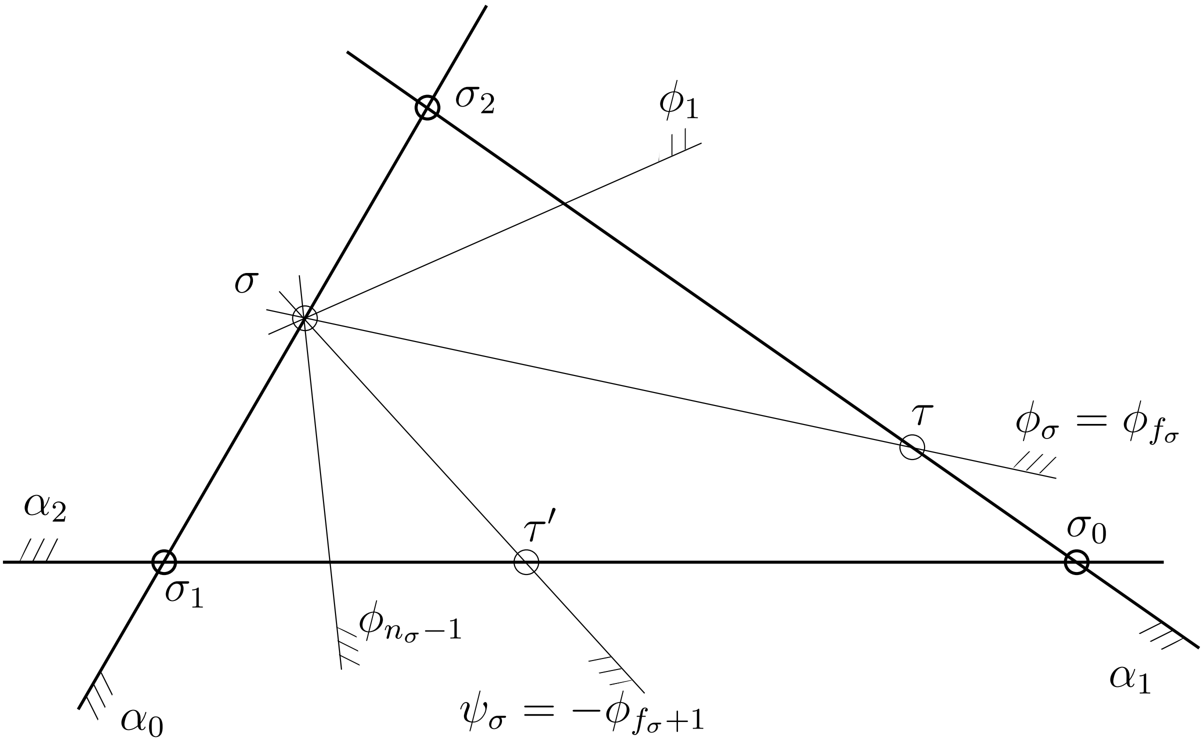

Figure 2: Proof of Lemma 5.5(1)

(2) Suppose by contradiction that $o(r_{\alpha_0} r_{\alpha_2}) = 2$. Since T_σ is hyperbolic and has a decomposing root of multiplicity f_σ, it follows from Lemma 5.3 that $f_\sigma \leq 4$ and that if $n_\sigma \leq 6$ then $f_\sigma \leq 2$. Moreover, if $f_\sigma = n_\sigma - 1$ (i.e. if we are in Situation (i) of Part (1)) then Lemma 5.3 implies that $n_\sigma = 3$, contradicting the hypothesis that $n_\sigma > 3$. Thus $f_\sigma < n_\sigma - 1$ and T'_σ is defined.

Hence T'_σ is hyperbolic (by hypothesis) and has a decomposing root of multiplicity $n_\sigma - f_\sigma - 1$ (by Part (1)). Since $r_{\alpha_0} r_{\alpha_2}$ has order 2, the combinatorial triangle T'_σ corresponds to a genuine hyperbolic triangle that possesses an angle $\frac{\pi}{2}$. By Lemma 5.3, this implies that $n_\sigma - f_\sigma - 1 \leq 2$ and that $n_\sigma - f_\sigma - 1 = 1$ if $n_\sigma \leq 6$.

Combining the conclusions of the preceding two paragraphs, we obtain $(n_\sigma, f_\sigma) \in \{(7,4), (4,2)\}$.

By Part (1) and the hypothesis that r_{ϕ_σ} does not stabilize σ_0, we know that there exists a residue $\tau \in]\sigma_0, \sigma_2[$ that is stabilized r_{ϕ_σ} and that the set $\{\sigma_2, \sigma, \tau\}$ is a set of vertices of T_σ.

Assume that $(n_\sigma, f_\sigma) = (7, 4)$. Then the Coxeter decomposition of T_σ induced by $r_{\phi_\sigma}(-\psi_\sigma)$ corresponds to the decomposition $\sharp 5$ in Figure 1 (see Lemma 5.3). This implies that $r_{\alpha_0} r_{\alpha_1}$ and $r_{\alpha_1} r_{\phi_\sigma}$ have both order 7 and that $\{\alpha_0, \alpha_1\}$ is a geometric pair. Now, we apply Part (1) to the triangle T and the residue $\tau \in]\sigma_0, \sigma_2[_\Gamma$. Since $o(r_{\alpha_1} r_{\phi_\sigma}) = 7$, we have $n_\tau \geq 7$. We may assume that $T_\tau = \{\alpha_0, \alpha_1, \phi_\tau\}$. Furthermore, T_σ is a subtriangle of T_τ. In view of Corollary 5.4 and the fact that T_σ is hyperbolic, it follows that T_τ is hyperbolic. Using again the fact T_σ is a subtriangle of T_τ combined with Lemma 5.3, we obtain $f_\tau \leq 2$. Hence, we have $f_\tau < n_\tau - 1$

and we are in Situation (ii) of (1) for τ. The triangle T'_τ has a decomposing root of multiplicity $n_\tau - f_\tau - 1$ and $n_\tau - f_\tau - 1 \geq 4$ because $n_\tau \geq 7$ and $f_\tau \leq 2$.

Assume that $n_\tau - f_\tau - 1 \geq 5$. All multiplicities of a Coxeter decomposition of a hyperbolic (resp. affine) triangle are lesser than or equal to 4. Thus T'_τ is spherical. The type of the Coxeter decomposition of T'_τ induced by $r_{\psi_\tau}(-\phi_\tau)$ is (n_τ, k, l) for some integers k, l. Since $n_\tau \geq 7$, Corollary 5.4 gives $k = l = 2$. We deduce that the order of $r_{\alpha_1} r_{\alpha_2} = 2$ which implies that T has type $(2, 2, 7)$. This implies that T has spherical type, in contradiction with one of the hypotheses.

Thus $n_\tau - f_\tau - 1 = 4$, $f_\tau = 2$ and $n_\tau = 7$. The Coxeter decomposition of T'_τ induced by the decomposing root $r_{\psi_\tau}(-\phi_\tau)$ corresponds to the decomposition $\sharp 5$ in Figure 1. This finally implies that $r_{\alpha_1} r_{\alpha_2}$ has order 7, and thus that T has type $(2, 7, 7)$. This proves the result in the case $n_\sigma = 7$.

It remains to treat the case $(n_\sigma, f_\sigma) = (4, 2)$. Let x be the order of $r_{\alpha_0} r_{\alpha_1}$ and y be the order of $r_{\alpha_2} r_{\psi_\sigma}$. The Coxeter decomposition of T_σ induced by the decomposing root $r_{\phi_\sigma}(-\psi_\sigma)$ corresponds to the decomposition $\sharp 1$ in Figure 1. Since T_σ is hyperbolic, we deduce that $x \geq 5$, that x also equals the order of $r_{\alpha_1} r_{\phi_\sigma}$ and that T_σ is geometric of type $(2, x, x)$. Now, using the solution of the word problem for Coxeter groups, an easy computation in the Coxeter system $(W(T_\sigma), R(T_\sigma))$, shows that the order of $r_{\psi_\sigma} r_{\alpha_1}$ is infinite.

On the other hand, the triangle T'_σ has type $(2, 4, y)$ and we have $y \geq 5$ because T'_σ is hyperbolic. Furthermore, we know by Part (1) that there exists a residue $\tau' \in]\sigma_0, \sigma_1[$ such that $\{\sigma_1, \sigma, \tau'\}$ is a set of vertices of T'_σ. Now, we apply Part (1) to the triangle T and the residue τ'. We may assume that $T_{\tau'} = \{\alpha_0, \alpha_1, \phi_{\tau'}\}$. Since $y \geq 5$ we have $n_{\tau'} \geq 5$. Moreover, since T'_σ is hyperbolic and is a subtriangle of $T_{\tau'}$, it follows from Corollary 5.4 that $T_{\tau'}$ is hyperbolic. Therefore, Lemma 5.3 implies that $f_{\tau'} \leq 2$ in view of the fact that $r_{\alpha_0} r_{\alpha_1}$ has order 2.

If $f_{\tau'} = 2$ then the Coxeter decomposition of $T_{\tau'}$ induced by $r_{\phi_{\tau'}}(-\psi_{\tau'})$ corresponds to decomposition $\sharp 2$ in Figure 1. Since T'_σ is a subtriangle of $T_{\tau'}$, this implies that T'_σ is of type $(2, 3, y)$ or $(2, y, 2y)$ where $y \geq 5$. But we have seen above that T'_σ has type $(2, 4, y)$. This is a contradiction. Hence, $f_{\tau'} = 1$.

Therefore, the triangle $T'_{\tau'}$ is defined and has a decomposing root of multiplicity $n_{\tau'} - f_{\tau'} - 1 = n_{\tau'} - 2$. As $n_{\tau'} \geq 5$, we deduce that $T'_{\tau'}$ is spherical. Since r_{α_1} and $r_{\phi_\sigma} = r_{\phi_{\tau'}}$ both belong to $W(T'_{\tau'})$, it follows that $r_{\alpha_1} r_{\psi_\sigma}$ has finite order. This is impossible because we have seen above that the order of $r_{\alpha_1} r_{\psi_\sigma}$ is infinite.

This finishes the proof of (2).

(3) Since the order of $r_{\alpha_1} r_{\alpha_2}$ equals 2, it follows that $\bar{T} = \{\alpha_0, \alpha_2, r_{\alpha_1}(\alpha_2)\}$ is a combinatorial triangle and that $\{\sigma_1, \sigma_2, r_{\alpha_1}(\sigma_1)\}$ is a set of vertices for $\bar{T}$. Let $x_{2,0}$ (resp. $x_{0,2}$) be the only chamber of $\mathrm{proj}_{\sigma_2}(\sigma_0)$ (resp. $\mathrm{proj}_{\sigma_0}(\sigma_2)$) that belongs to α_1. Let $\bar{\Gamma}$ be the gallery obtained by concatenating $\Gamma|_{x_{2,0} \to x_{0,2}}$ with $r_{\alpha_1}((\Gamma|_{x_{0,2} \to x_{2,0}})^{-1})$. Then $\bar{\Gamma}$ is a gallery supported by $\{\sigma_1, \sigma_2, r_{\alpha_1}(\sigma_1)$ which skirts around $\bar{T}$.

Since the order of $r_{\alpha_1} r_{\phi'_\sigma}$ equals 2, it follows that $r_{\alpha_1}(\sigma) \in]r_{\alpha_1}(\alpha_1), \sigma_2[_{\bar{\Gamma}}$ and that $\{\alpha_0, r_{\alpha_1}(\alpha_0), \phi'_\sigma\}$ is a triangle with decomposing root α_1 (see Figure 3).

Now, we apply Part (1) to the triangle $\bar{T}$ and the residue σ. This yields a number $1 \leq \bar{f}_\sigma \leq n_\sigma - 1$, a geometric pair of roots $\{\bar{\phi}_\sigma, \bar{\psi}_\sigma\}$ and a triangle $\bar{T}_\sigma = \{\alpha_0, r_{\alpha_1}(\alpha_0), \bar{\phi}_\sigma\}$

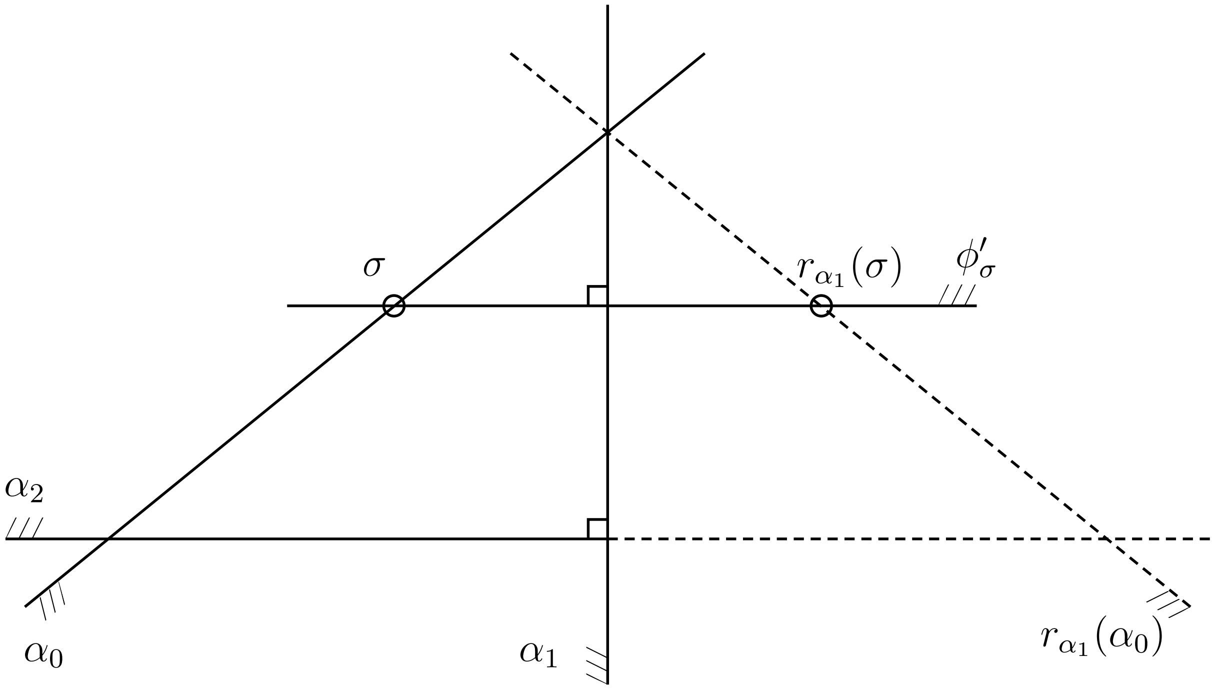

Figure 3: Proof of Lemma 5.5(3).

(and a triangle $\bar{T}'_\sigma = \{\alpha_0, \bar{\psi}_\sigma, \alpha_2\}$ if $\bar{f}_\sigma < n_\sigma - 1$). Since $\Gamma|_{x_{0,2}\to x_{2,0}}$ crosses $\partial\bar{\phi}_\sigma$, it follows that $\bar{f}_\sigma \leq f_\sigma$.

If T'_σ exists, namely if $f_\sigma < n_\sigma - 1$, then $\bar{f}_\sigma < n_\sigma - 1$ and $\bar{T}'_\sigma$ exists as well. In that case, T'_σ is a subtriangle of $\bar{T}'_\sigma$ and, hence, the fact that T'_σ is hyperbolic implies that $\bar{T}'_\sigma$ is hyperbolic as well in view of Corollary 5.4. Therefore, we obtain $n_\sigma - \bar{f}_\sigma - 1 \leq 4$.

On the other hand, since $\{\alpha_0, \alpha_1, \phi'_\sigma\}$ is a subtriangle of both T_σ and $\bar{T}_\sigma$, it follows from Corollary 5.4 and from the fact that T_σ is hyperbolic, that $\bar{T}_\sigma$ is hyperbolic. Therefore, by Lemma 5.3, either $\bar{f}_\sigma = 1$ and the Coxeter decomposition of $\bar{T}_\sigma$ induced by the decomposing roots α_1 and $r_{\bar{\phi}_\sigma}(-\bar{\psi}_\sigma)$ corresponds to decomposition $\sharp 1$ of Figure 1 or the Coxeter decomposition of $\bar{T}_\sigma$ induced by the decomposing roots α_1 and $r_{\bar{\phi}_\sigma}(-\bar{\psi}_\sigma)$ corresponds to decomposition $\sharp 6$ of Figure 1, $\bar{f}_\sigma = 2$ and $n_\sigma \geq 7$.

In the second case, we obtain $n_\sigma = 7$ because $n_\sigma - \bar{f}_\sigma - 1 \leq 4$. Moreover, $\bar{\phi}_\sigma = \phi'_\sigma$ and therefore $\{\alpha_0, \phi'_\sigma\}$ is not a geometric pair. Thus we are in Situation (a).

Now we may assume that $n_\sigma \leq 6$ and $\bar{f}_\sigma = 1$. We deduce that $\bar{\phi}_\sigma = \phi'_\sigma$.

If $n_\sigma = 5$ or $n_\sigma = 6$ then $\bar{T}'_\sigma$ exists and has a decomposing root of multiplicity 3 or 4. Since $\bar{T}'_\sigma$ is hyperbolic, Lemma 5.3 yields a contradiction. Therefore, we have $n_\sigma \leq 4$.

Assume that $n_\sigma = 4$. Then $f_\sigma = 1$ or $f_\sigma = 2$. If $f_\sigma = 1$ then the Coxeter decomposition of T'_σ induced by $r_{\psi_\sigma}(-\phi_\sigma)$ corresponds to decomposition $\sharp 1$ of Figure 1. Since T'_σ is hyperbolic, we deduce that the order of $r_{\alpha_0} r_{\alpha_2}$ is at least 5. Thus we are in Situation (b).

Now assume that $(n_\sigma, f_\sigma) = (4, 2)$ (see Figure 4). Then $\phi_\sigma = -\bar{\psi}_\sigma$ because $\bar{f}_\sigma = 1$. Let x (resp. y, y') be the order of $r_{\alpha_0} r_{\alpha_1}$ (resp. $r_{\alpha_0} r_{\alpha_2}$, $r_{\phi_\sigma} r_{\alpha_2}$). Since T_σ is hyper-

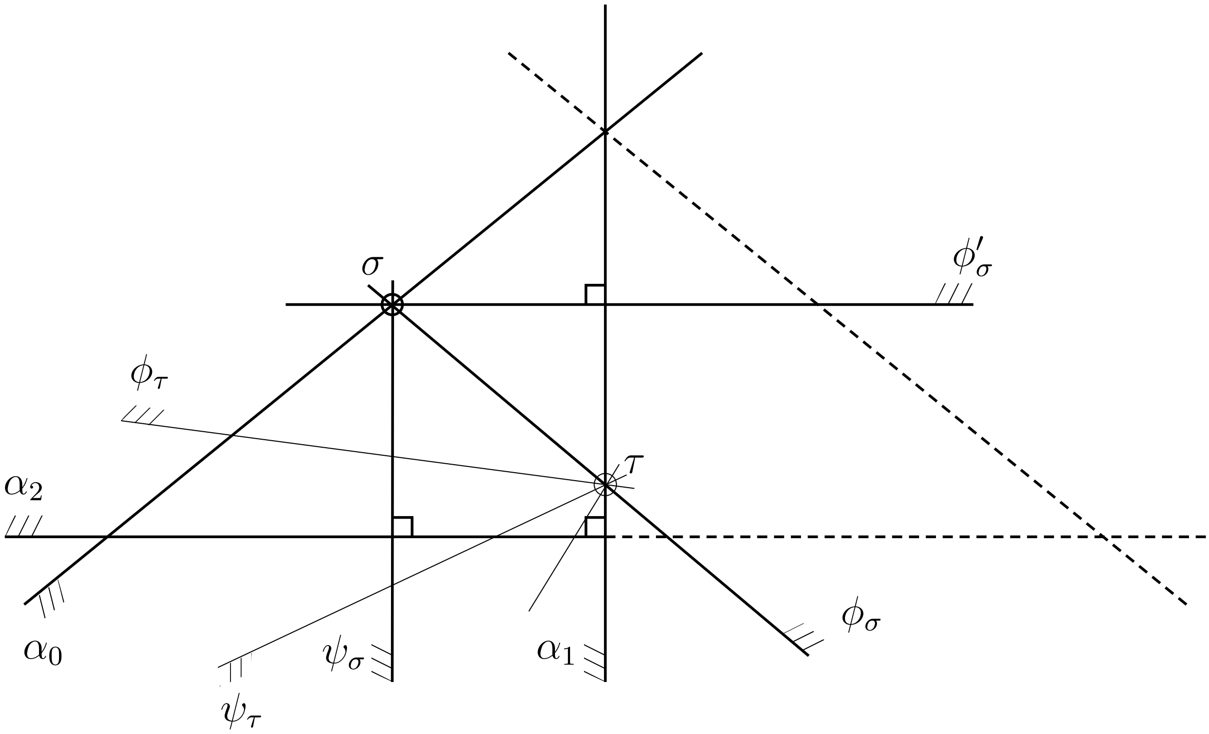

Figure 4: Proof of Lemma 5.5(3)– the case $(n_\sigma, f_\sigma) = (4, 2)$.

bolic, its Coxeter decomposition induced by the decomposing root ϕ'_σ corresponds to decomposition $\sharp 1$ if Figure 1. Hence T_σ has type $(2, x, x)$, we have $x \geq 5$ and x is also the order of $r_{\alpha_1} r_{\phi_\sigma}$. Since $\bar{T}'_\sigma$ is hyperbolic, its Coxeter decomposition induced by the decomposing root $r_{\bar{\psi}_\sigma}(-\bar{\phi}_\sigma)$ corresponds to decomposition $\sharp 1$ of Figure 1. It follows that $y = y' \geq 5$.

Let $\tau \in]\sigma_0, \sigma_2[_\Gamma$ be such that $\{\sigma_0, \sigma, \tau\}$ is a set of vertices of T_σ (see Part (1) and note that $\tau \neq \sigma_0$ in view of the hypothesis that r_{ϕ_σ} does not stabilize σ_0). Now we apply Part (1) to the triangle T and the residue τ. We may assume that $T_\tau = \{\alpha_0, \alpha_1, \phi_\tau\}$. Since T_σ is hyperbolic and is a subtriangle of T_τ, it follows that T_τ is hyperbolic by Corollary 5.4. Using Lemma 5.3, we obtain $f_\tau \leq 2$.

On the other hand, the set $\{-\phi_\sigma, -\alpha_1, \alpha_2\}$ is a geometric hyperbolic triangle of type $(2, x, y)$ and we have $x, y \geq 5$. Applying Lemma 5.3 to the triangle $\{-\phi_\sigma, \alpha_2, \psi_\tau\}$ (which contains T'_τ and $\{-\phi_\sigma, -\alpha_1, \alpha_2\}$ as subtriangles) and its decomposing root $-\alpha_1$, we deduce that $n_\tau - f_\tau - 1 = 1$.

Combining the conclusions of the preceding two paragraphs, we obtain $n_\tau \leq 4$ which contradicts the fact that the order of $r_{\alpha_1} r_{\phi_\sigma}$ equals $x \geq 5$. This proves that the case $n_\sigma = 4$ and $f_\sigma = 2$ is impossible.

It remains to consider the case $(n_\sigma, f_\sigma) = (3, 2)$. We have to prove that Situation (c) occurs. In other words, we have to prove that if $o(r_{\alpha_0} r_{\alpha_2}) \leq 3$ then $\{\alpha_0, \alpha_2, -\phi_\sigma\}$ is an affine triangle.

Notice that $\bar{\psi}_\sigma = -\phi_\sigma$, and thus $\{\alpha_0, \alpha_2, -\phi_\sigma\} = \bar{T}'_\sigma$. We define x, y, y' and τ as in the case $(n_\sigma, f_\sigma) = (4, 2)$ above. We assume that $y \leq 3$. Since T_σ is hyperbolic, its

Coxeter decomposition induced by the decomposing root ϕ'_σ corresponds to decomposition $\sharp 1$ if Figure 1. Hence the triangle $\{\alpha_0, \alpha_1, \phi'_\sigma\}$ has type $(2, 3, x)$, we have $x \geq 7$ and x is also the order of $r_{\alpha_1} r_{\phi_\sigma}$.

Notice that $y > 2$ because T is non-spherical and because the order of $r_{\alpha_0} r_{\alpha_1}$ equals 2. Thus we have $y = 3$.

Suppose by contradiction that $\bar{T}'_\sigma$ is not affine. Then $y' \geq 4$.

Now, as above, we apply Part (1) to the triangle T and the residue τ. We may assume that $T_\tau = \{\alpha_0, \alpha_1, \phi_\tau\}$. Since T_σ is hyperbolic and is a subtriangle of T_τ, it follows that T_τ is hyperbolic by Corollary 5.4. Using Lemma 5.3, we obtain $f_\tau \leq 4$.

On the other hand, the set $\{-\phi_\sigma, -\alpha_1, \alpha_2\}$ is a geometric hyperbolic triangle of type $(2, x, y')$ and we have $x \geq 7$ and $y' \geq 4$ (see Figure 4). Applying Lemma 5.3 to the triangle $\{-\phi_\sigma, \alpha_2, \psi_\tau\}$ (which contains T'_τ and $\{-\phi_\sigma, -\alpha_1, \alpha_2\}$ as subtriangles) and its decomposing root $-\alpha_1$, we deduce that $n_\tau - f_\tau - 1 = 1$.

Combining the conclusions of the preceding two paragraphs, we obtain $n_\tau \leq 6$ which contradicts the fact that the order of $r_{\alpha_1} r_{\phi_\sigma}$ equals $x \geq 7$. This shows that $\bar{T}'_\sigma$ is affine.

The proof is complete.

□

6 Proof of Theorem 1.1

Assume by contradiction that there are infinitely many conjugacy classes of reflection triangles.

We divide the proof into several steps.

Step 1: *there are infinitely many conjugacy classes of relfection triangles of type $\mathcal{M}$, where $\mathcal{M}$ is some fixed rank 3 Coxeter diagram of compact hyperbolic type.*
This follows directly from Lemma 2.1 and the pigeonhole principle.

Step 2: *there exist two geometric triangles T and U of type $\mathcal{M}$ such that $\bigcap U \subsetneq \bigcap T$.*
For each reflection triangle $\{a, b, c\}$ such that $(\langle a, b, c\rangle, \{a, b, c\})$ is a Coxeter system of type $\mathcal{M}$, there exists a geometric combinatorial triangle T such that $R(T) = \{a, b, c\}$. This follows from Step 1 and the main theorem of [5].

Hence, we know that $\Sigma(W, S)$ contains an infinite family $\{T_i\}_{i \in I}$ of geometric triangles of type $\mathcal{M}$ such that T_i is not W-conjugate to T_j whenever $i \neq j$.

For each $i \in I$ we write $T_i = \{\alpha_i, \beta_i, \gamma_i\}$ in such a way that the order of $r_{\alpha_i} r_{\beta_i}$ is constant when i varies in I. Using Lemma 2.1 and the finiteness of S again, we deduce from the pigeonhole principle that there exist a fixed geometric pair of roots $\{\alpha, \beta\}$ and an infinite subset $I' \subseteq I$ such that for every $i \in I'$, the roots α_i and β_i are simultaneously conjugate to α and β respectively. Replacing I by I' and some of the T_i's by one of their conjugate if necessary, we may thus assume that $T_i = \{\alpha, \beta, \gamma_i\}$ for each $i \in I$.

The fact that the T_i's are pairwise non-conjugate implies that the γ_i's are pairwise distinct. Thus, $\{r_{\gamma_i}\}_{i \in I}$ is an infinite family of reflections of W. By Lemma 3 in [11] (see also Proposition 1.4 in [3]) there exist indices $i, j \in I$ such that the order of $r_{\gamma_i} r_{\gamma_j}$ is infinite. Up to interchanging i and j, we may assume that the root γ_i is properly contained is the root γ_j.

We set $\gamma := \gamma_i$, $\gamma_T := \gamma_j$, $U := T_i = \{\alpha, \beta, \gamma\}$ and $T := T_j = \{\alpha, \beta, \gamma_T\}$. Since $\gamma \subsetneq \gamma_T$ we have $\bigcap_{\phi \in U} \phi \subsetneq \bigcap_{\phi \in T} \phi$.

Step 3: *there exists no geometric triangle $\tilde{T}$ such that $W(T) \cup W(U) \subseteq W(\tilde{T})$.*
Assume on the contrary that such a geometric triangle $\tilde{T}$ does exist. Since T and U are both hyperbolic, so is $\tilde{T}$ by Corollary 5.4.

Therefore, by Lemma 2.4, the combinatorial triangles T and U corresponds to genuine hyperbolic triangles $\mathcal{T}$ and $\mathcal{U}$ respectively.

Since the combinatorial triangles T and U are both geometric of type $\mathcal{M}$, the genuine hyperbolic triangles $\mathcal{T}$ and $\mathcal{U}$ have the same angles and, hence, they have the same area. On the other hand, Step 2 implies that $\mathcal{U} \subsetneq \mathcal{T}$, a contradiction.

Step 4.
Let $\{\pi, \sigma, \rho\}$ (resp. $\{\pi, \sigma_T, \rho_T\}$) be a set of vertices of U (resp. T), where π is stabilized by $\langle r_\alpha, r_\beta \rangle$, and σ (resp. σ_T) is stabilized by $\langle r_\alpha, r_\gamma \rangle$ (resp. $\langle r_\alpha, r_{\gamma_T} \rangle$). Let Γ be a gallery supported by $\{\pi, \sigma_T, \rho_T\}$ which skrts around T.

We have $\pi \subseteq \gamma$ and since the order of $r_\gamma r_{\gamma_T}$ is infinite (see Step 2), we deduce that σ_T and ρ_T are contained in $-\gamma$. It follows that $\sigma \in]\pi, \sigma_T[_\Gamma$ and $\rho \in]\pi, \rho_T[_\Gamma$.

Step 5: *there exists no reflection that stabilizes both σ and ρ_T (resp. ρ and σ_T).*
Assume there exists a root ϕ such that r_ϕ stabilizes both σ and ρ_T. By Lemma 5.1(2) we have $r_\phi \neq r_\beta$, which implies that $\pi \notin \partial^2 \phi$ by Lemma 5.1(1). Therefore, replacing ϕ by $-\phi$ if necessary, we may assume that $\pi \subseteq \phi$. Then $T_\phi := \{\alpha, \beta, \phi\}$ is a combinatorial triangle.

Let ψ_1 (resp. ψ_2) be a root such that $\{\phi, \psi_1\}$ (resp. $\{\phi, \psi_2\}$) is a geometric pair and that $\langle r_\phi r_{\psi_1} \rangle = \mathrm{Stab}_W(\sigma)$ (resp. $\langle r_\phi r_{\psi_2} \rangle = \mathrm{Stab}_W(\rho_T)$). Then $W(T) \cup W(U) \subseteq \langle W(T_\phi) \cup \{\psi_1, \psi_2\} \rangle$. But ψ_1 and ψ_2 are decomposing roots of T_ϕ, which implies by Lemma 5.2 that there exists a geometric triangle $\tilde{T}$ such that $W(T) \cup W(U) \subseteq W(\tilde{T})$. This contradicts Step 3.

The other assertion follows by symmetry.

Final step.
Now, we apply Lemma 5.5(1) to the triangle T and the residue σ. We may assume that $T_\sigma = \{\alpha, \beta, \phi_\sigma\}$. Since U is hyperbolic and is a subtriangle of T_σ, it follows from Corollary 5.4 that T_σ is hyperbolic.

If $f_\sigma = n_\sigma - 1$, then Lemma 5.3 applied to T_σ implies that $f_\sigma = 2$. Therefore, we obtain $n_\sigma = f_\sigma + 1 = 3$ in this case.

If $f_\sigma < n_\sigma - 1$, then T'_σ is hyperbolic as well. Indeed, if $\tilde{T}'$ is a fundamental triangle of the Coxeter decomposition of T'_σ induced by the decomposing root $r_{\psi_\sigma}(-\phi_\sigma)$, then r_{γ_T} and r_γ both belong to $W(\tilde{T}')$. Since $r_\gamma r_{\gamma_T}$ has infinite order (see Step 2) and since there is no affine triangle in $\Sigma(W, S)$, it follows that $\tilde{T}'$ is hyperbolic, and so is T'_σ in view of Corollary 5.4. Now, Lemma 5.3 applied to T_σ and to T'_σ implies that $f_\sigma \leq 4$ and that $n_\sigma - f_\sigma - 1 \leq 4$.

In all cases, we have $n_\sigma - 5 \leq f_\sigma \leq 4$ and, hence, $n_\sigma \leq 9$. The rest of the proof consists in a case by case discussion based on the value of n_σ which will eventually provide a contradiction.

Let x (resp. y, z, y_T, z_T) be the order of $r_\alpha r_\beta$ (resp. $r_\beta r_\gamma$, $r_\alpha r_\gamma$, $r_\beta r_{\gamma_T}$, $r_\alpha r_{\gamma_T}$). Thus (x, y, z) (resp. (x, y_T, z_T)) is the type of U (resp. T). Since T and U are of the same type, we deduce that

$$(y,z) = (y_T, z_T) \qquad \text{or} \qquad (y,z) = (z_T, y_T).$$

In the sequel, we will refer to the preceding fact as **Fact 1**.

Another relevant fact that will be useful in our discussion, is that

a combinatorial triangle of type (k,l,m) that is hyperbolic and geometric corresponds to a genuine hyperbolic triangle whose angles are $\frac{\pi}{k}$, $\frac{\pi}{l}$, $\frac{\pi}{m}$.

In the sequel, we will refer to the preceding fact as **Fact 2**.

We are now ready to start the discussion.

Case 1: $n_\sigma = 9$.

Then $f_\sigma = 4$. We apply Lemma 5.3 to T_σ. This implies that $x = 9$. Moreover, since U is geometric and is a subtriangle of T_σ, it follows from Fact 2 that $y = z = 3$. By Fact 1, we deduce that $y_T = z_T = 3$.

On the other hand, Lemma 5.3 applied to T'_σ implies that $z_T = 9$. This is a contradiction.

Case 2: $n_\sigma = 8$.

Then $f_\sigma = 3$ or $f_\sigma = 4$. We apply Lemma 5.3 to T_σ, which implies that $(x,y,z) \in \{(3,2,8),(3,3,4),(8,2,4),(8,8,2)\}$ and that there exists a root ϕ'_σ such that $r_{\phi'_\sigma}$ stabilizes σ, that $\alpha \cap \phi'_\sigma \subseteq \phi_\sigma$ and that the order of $r_{\phi'_\sigma} r_{\phi_\sigma}$ equals 2. If $(x,y,z) = (3,3,4)$, then $f_\sigma = 3$ and Lemma 5.3 applied to T'_σ shows that $z_T = 8$, contradicting Fact 1. In the other cases, the fact that U is geometric together with Fact 2 implies that $(y,z) \in \{(2,4),(8,2)\}$. By Fact 1 together with Lemma 5.5(2), it follows that $(y_T, z_T) \in \{(2,4),(2,8)\}$. Therefore, we may apply Lemma 5.5(3), which provides a contradiction.

Case 3: $n_\sigma = 7$.

Then $f_\sigma = 2, 3$ or 4.

By Fact 2 and the fact that U is geometric, it follows that $f_\sigma = 4$ is impossible.

If $f_\sigma = 3$, the same argument shows that $x = 3$, $y = 2$ and $z = 7$. On the other hand, Lemma 5.3 applied to T'_σ implies that $z_T = 3$ or $z_T = 7$. By Fact 1, we obtain $y_T = 2$ and $z_T = 7$. Thus T has type $(2,3,7)$. But we may now apply Lemma 5.5(3) by taking $\phi'_\sigma = \gamma$, which yields a contradiction.

If $f_\sigma = 2$ then by Fact 2, we obtain $(x,y,z) \in \{(x_0,2,7),(2,3,7)\}$ where $x_0 \geq 7$. Furthermore, Lemma 5.3 applied to T'_σ implies that $z_T = 7$ and $o(r_{\gamma_T} r_{\psi_\sigma}) = 7$.

If $y = 2$ then $y_T = 2$ by Fact 1. Thus we may apply Lemma 5.5(3) with $\phi'_\sigma = \gamma$. This provides a contradiction because U is geometric and, hence, so is $\{\alpha, \phi'_\sigma\}$.

Therefore, we have $y = y_T = 3$. By Lemma 5.5(1), there exists a residue $\tau' \in]\sigma_T, \rho_T[_\Gamma \cap \partial^2 \psi_\sigma$. Now, we apply Lemma 5.5(1) to T and the residue τ'. We have $n_{\tau'} \geq 7$ because $o(r_{\gamma_T} r_{\psi_\sigma}) = 7$. We may assume that $T_{\tau'} = \{\alpha, \gamma_T, \phi_{\tau'}\}$. Since T'_σ is a subtriangle of $T_{\tau'}$, it follows from Lemma 5.3 that $f_{\tau'} \leq 2$. Therefore, $n_\tau - f_\tau - 1 \geq 4$. Therefore, either the triangle T'_τ is spherical, in which case $y_T = 2$ because $n_{\tau'} \geq 7$ or T'_τ is hyperbolic, in which case $n_\tau - f_\tau - 1 = 4$ and $y_T = 7$ by Lemma 5.3. In both cases, we have a contradiction with $y_T = 3$.

Case 4: $n_\sigma = 6$.

Then $f_\sigma = 1, 2, 3$ or 4.

If $f_\sigma = 1$ or 2 (resp. $f_\sigma = 3$ or 4) then T'_σ (resp. T_σ) cannot be hyperbolic by Lemma 5.3. This is a contradiction.

Case 5: $n_\sigma = 5$.
Then $f_\sigma = 1, 2$ or 3.

If $f_\sigma = 1$ (resp. $f_\sigma = 3$) then T'_σ (resp. T_σ) cannot be hyperbolic by Lemma 5.3. Thus $f_\sigma = 2$.

By Fact 2 and the fact that U is geometric, it follows that $y = 2$ and $z = 5$. By Fact 1 together with Lemma 5.5(2), it follows that $y_T = 2$ and $z_T = 5$. But we may now apply Lemma 5.5(3) by taking $\phi'_\sigma = \gamma$, which yields a contradiction.

Case 6: $n_\sigma = 4$.
Then $f_\sigma = 1$ or 2. (Notice that the case $f_\sigma = 3$ is impossible by the second paragraph of the final step above.)

If $f_\sigma = 2$ then we have $x \geq 5$ since T_σ is hyperbolic. Moreover, since U is a subtriangle of T_σ, we obtain $(y, z) \in \{(2, 4), (x, 2)\}$. By Lemma 5.5(2) and Fact 1, we deduce that $(y_T, z_T) \in \{(2, 4), (2, x)\}$. In all cases, we may apply Lemma 5.5(3) by taking $\phi'_\sigma = r_{\phi_\sigma}(\psi_\sigma)$, and this yields a contradiction.

If $f_\sigma = 1$ then $z = 4$ and $\gamma = \phi_\sigma$. Lemma 5.3 applied to T'_σ implies that $z_T \geq 5$. It follows from Fact 1 that $y = z_T \geq 5$. This implies that $n_\rho \geq 5$. Therefore, if we interchange the roles of α and β and, hence, interchange σ and ρ, we are back in one of the preceding cases. This yields a contradiction.

Case 7: $n_\sigma = 3$.
Then $f_\sigma = 1$ or 2.

Assume that $f_\sigma = 2$. By Fact 2 and the fact that U is geometric and hyperbolic, it follows that $y = 2$, $z = 3$ and $x \geq 7$. Moreover, we have also $o(r_\beta r_{\phi_\sigma}) = x \geq 7$. By Fact 1, we deduce that $(y_T, z_T) \in \{(2, 3), (3, 2)\}$. Lemma 5.5(3) applied to T with $\phi'_\sigma = \gamma$ implies that the case $(y_T, z_T) = (2, 3)$ is impossible. Hence, we have $(y_T, z_T) = (3, 2)$.

By Lemma 5.5(1), there exists a residue $\tau \in]\pi, \rho_T[_\Gamma \cap \partial^2\phi_\sigma$ (the fact that $\tau \neq \rho_T$ follows from Step 5). Now, we apply Lemma 5.5(1) to the residue τ and the triangle $\bar{T} := \{\alpha, \beta, r_{\gamma_T}(\beta)\}$. Since $o(r_\beta r_{\phi_\sigma}) \geq 7$ we have $n_\tau \geq 7$. We may assume that $\bar{T}_\tau = \{\alpha, \beta, \phi_\tau\}$. Since T_σ is a subtriangle of $\bar{T}_\tau$, it follows from Corollary 5.4 that $\bar{T}_\tau$ is hyperbolic. Therefore, Lemma 5.3 implies that $\bar{f}_\tau \leq 4$.

We deduce that $\bar{f}_\tau < n_\tau - 1$ and, hence, the triangle $\bar{T}'_\tau$ is defined. Since $\bar{T}'_\tau$ is of type $(3, n_\tau, k)$ for some integer k and since $n_\tau \geq 7$, we deduce that $\bar{T}'_\tau$ is hyperbolic. Now, Lemma 5.3 applied to the Coxeter decomposition of $\bar{T}'_\tau$ induced by the decomposing roots $r_{\bar{\psi}_\tau}(-\bar{\phi}_\tau)$ and γ_T implies that $n_\tau - \bar{f}_\tau - 1 = 1$.

Combining the conclusions of the preceding two paragraphs, we obtain $n_\tau \leq 6$, a contradiction. This proves that the case $f_\sigma = 2$ does not occur.

Assume that $f_\sigma = 1$. Then $z = 3$. If $y \geq 4$ then $n_\rho \geq 4$, and by interchanging the roles of α and β we are back in one of the preceding cases. Thus we may assume that $y = 2$ or $y = 3$.

Let y' be the order of $r_{\gamma_T} r_{\psi_\sigma}$. If $z_T = 2$ then $r_\gamma r_{\gamma_T} = r_\alpha r_{\psi_\sigma} r_\alpha r_{\gamma_T} = r_\alpha r_{\psi_\sigma} r_{\gamma_T} r_\alpha$. It follows that $r_\gamma r_{\gamma_T}$ has order y', which contradicts that fact that $r_\gamma r_{\gamma_T}$ has infinite order (see Step 2). Thus $z_T \neq 2$.

By Fact 1, we deduce from the conclusions of the preceding two paragraphs that $z_T = 3$. Since T'_σ is hyperbolic, it follows that $y' \geq 4$.

By Lemma 5.5(1), there exists a residue $\tau' \in]\sigma_T, \rho_T[_\Gamma \cap \partial^2\psi_\sigma$. Now, we apply

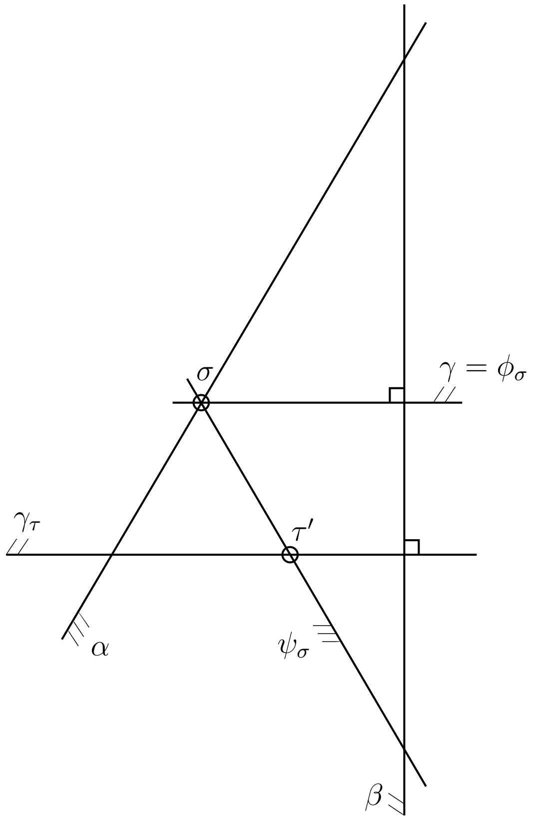

Figure 5: Proof of Theorem 1.1– the case $(n_\sigma, f_\sigma, y) = (3, 1, 2)$.

Lemma 5.5(1) to T and the residue τ'. We have $n_{\tau'} \geq 4$ because $y' \geq 4$. We may assume that $T_{\tau'} = \{\alpha, \gamma_T, \phi_{\tau'}\}$. Since T'_σ is a subtriangle of $T_{\tau'}$, it follows from Lemma 5.3 that $f_{\tau'} \leq 3$ and that $f_{\tau'} = 1$ if $n_{\tau'} \leq 6$. In particular $f_{\tau'} < n_{\tau'} - 1$ and the triangle $T'_{\tau'}$ is defined.

There are two cases: either $(y, y_T) = (2, 2)$ or $(y, y_T) = (3, 3)$.

Assume first that $(y, y_T) = (2, 2)$ (see Figure 5). Then $x \geq 7$ because U is hyperbolic. We remark that the order of $r_{\psi_\sigma} r_\beta$ equals $x \geq 7$ because $y = 2$ and $\psi_\sigma = r_\gamma(-\alpha)$. Thus $\hat{T} := \{\beta, -\gamma_T, -\psi_\sigma\}$ is a triangle of type $(2, x, y')$ where $x \geq 7$ and $y' \geq 4$. It follows that $\hat{T}$ is hyperbolic. Since there exists a geometric triangle that is a subtriangle of both $\hat{T}$ and $T'_{\tau'}$, we deduce from Corollary 5.4 that $T'_{\tau'}$ is hyperbolic.

Now, Lemma 5.3 together with the fact that $y_T = 2$ implies that $n_{\tau'} - f_{\tau'} - 1 \leq 2$ and that $n_{\tau'} - f_{\tau'} - 1 = 1$ if $n_{\tau'} \leq 6$. We deduce from $f_{\tau'} \leq 3$ that $n_{\tau'} \leq 6$. We have seen that this implies $f_{\tau'} = 1$ and $n_{\tau'} - f_{\tau'} - 1 = 1$. We deduce that $n_{\tau'} = 3$, which contradicts the fact that $n_{\tau'} \geq 4$. Thus the case $(y, y_T) = (2, 2)$ is impossible.

Now, assume that $(y, y_T) = (3, 3)$. An easy computation using the solution of the word problem for Coxeter groups in the geometric hyperbolic triangle U shows that the order of $r_\beta r_{\psi_\sigma}$ is infinite. This implies that $T'_{\tau'}$ is not spherical. Thus $T'_{\tau'}$ is hyperbolic since there are no affine triangles in $\Sigma(W, S)$. Therefore, Lemma 5.3 together with the fact that $y_T = 3$ implies that $n_{\tau'} - f_{\tau'} - 1 \leq 3$ and that $n_{\tau'} - f_{\tau'} - 1 = 1$ if $n_{\tau'} \leq 6$. We have seen above that $f_{\tau'} \leq 3$ and that $f_{\tau'} = 1$ if $n_{\tau'} \leq 6$. It follows that $(n_{\tau'}, f_{\tau'}) = (7, 3)$ and that $o(r_\alpha r_{\phi_{\tau'}}) = 7$.

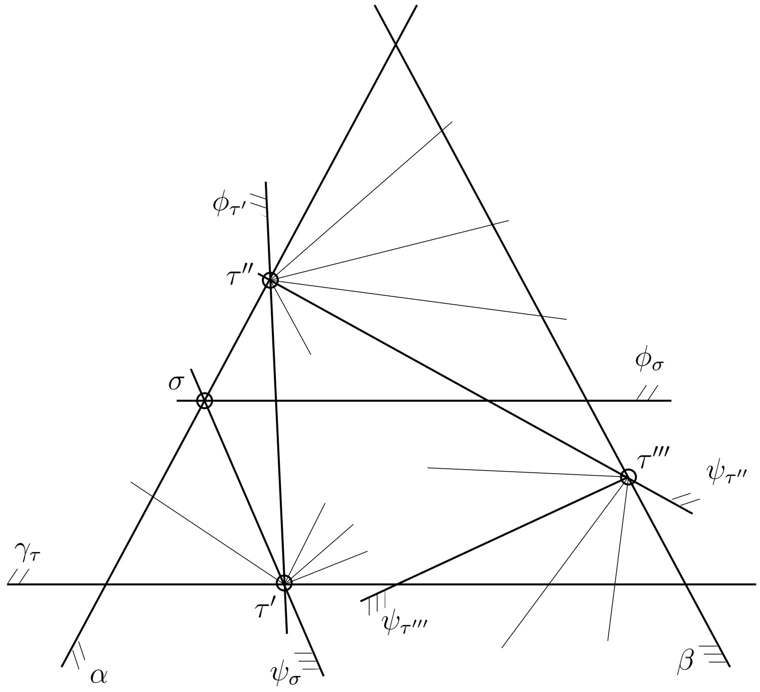

Figure 6: Proof of Theorem 1.1– the case $(n_\sigma, f_\sigma, y) = (3, 1, 3)$.

By Lemma 5.5(1) applied to T and τ', there exists a residue $\tau'' \in [\pi, \sigma_T[_\Gamma \cap \partial^2\phi_{\tau'}$. Moreover, if $\tau'' = \pi$ then $\beta = \phi_{\tau'}$ because $x \geq 7$ and $o(r_\alpha r_{\phi_{\tau'}}) = 7$. This contradicts Lemma 5.1(2). Thus $\tau'' \in]\pi, \sigma_T[_\Gamma$ (see Figure 6).

Now, we apply Lemma 5.5(1) to T and the residue τ''. Since $o(r_\alpha r_{\phi_{\tau'}}) = 7$, it follows that $n_{\tau''} \geq 7$. We may assume that $T_{\tau''} = \{\alpha, \gamma_T, \phi_{\tau''}\}$. Since $T_{\tau'}$ is a subtriangle of $T_{\tau''}$, we deduce from Corollary 5.4 that $T_{\tau''}$ is hyperbolic. Therefore, it follows from Lemma 5.3 that $f_{\tau''} \leq 2$. Thus the triangle $T'_{\tau''}$ is defined. It is of type $(n_{\tau''}, x, k)$ for some integer k, and we have $n_{\tau''}, x \geq 7$. Thus $T'_{\tau''}$ is hyperbolic. By Lemma 5.3, we have $n_{\tau''} - f_{\tau''} - 1 \leq 4$. We deduce that $(n_{\tau''}, f_{\tau''}) = (7, 2)$ and that $r_\beta r_{\psi_{\tau''}}$ has order 7. Furthermore, by Lemma 5.5(1), there exists a residue $\tau''' \in]\pi, \rho_T[_\Gamma \cap \partial^2\psi_{\tau''}$.

Now, we redo the discussion of the preceding paragraph with the residue τ'''. We apply Lemma 5.5(1) to T and the residue τ'''. Since $o(r_\beta r_{\psi_{\tau''}}) = 7$, it follows that $n_{\tau'''} \geq 7$. We may assume that $T_{\tau'''} = \{\alpha, \beta, \phi_{\tau'''}\}$. Since $T'_{\tau''}$ is a subtriangle of $T_{\tau'''}$, we deduce from Corollary 5.4 that $T_{\tau'''}$ is hyperbolic. Therefore, it follows from Lemma 5.3 that $f_{\tau'''} \leq 2$. Thus the triangle $T'_{\tau'''}$ is defined. It is of type $(n_{\tau'''}, y_T, k')$ for some integer k', and we have $n_{\tau'''} \geq 7$ and $y_T = 3$. Thus $T'_{\tau'''}$ is hyperbolic. By Lemma 5.3, we have $n_{\tau'''} - f_{\tau'''} - 1 \leq 3$ because $y_T = 3$. Combining the latter inequality with $f_{\tau'''} \leq 2$, we obtain $n_{\tau'''} \leq 6$, a contradiction.

Thus the case $(y, y_T) = (3, 3)$ is impossible as well.

Case 8: $n_\sigma = 2$.
Since U is not spherical, we deduce that $y \geq 3$. This implies that $n_\rho \geq 3$. By interchanging the roles of α and β, we are back to one of the preceding cases. $\square$

References

[1] P. Bahls. Some new biautomatic Coxeter groups. Preprint, 2004.

[2] N. Bourbaki. *Groupes et algèbres de Lie, Chapitres IV-VI.* Masson, 2nd edition, 1981.

[3] B. Brink and R. B. Howlett. A finiteness property and an automatic structure for coxeter groups. *Math. Ann.* **296**, pages 179–190, 1993.

[4] P.-E. Caprace and B. Mühlherr. On certain isomorphisms between Kac-Moody groups. Preprint, 2004.

[5] R. Charney and M. Davis. When is a Coxeter system determined by its Coxeter group? *J. of the London Math. Soc.* **61**, pages 441–461, 2000.

[6] V. V. Deodhar. On the root system of a Coxeter group. *Comm. Algebra* **10**, pages 611–630, 1982.

[7] A. Felikson. Coxeter decomposition of hyperbolic prolygons. *Europ. J. Combinatorics* **19**, pages 801–817, 1998.

[8] D. Krammer. *The conjugacy problem for Coxeter groups.* PhD thesis, Universiteit Utrecht, 1994.

[9] B. Mühlherr and R. Weidmann. Rigidity of skew-angled Coxeter groups. *Advances in Geometry* **2**, pages 391–415, 2002.

[10] G. Niblo and L. Reeves. The geometry of cube complexes and the complexity of their fundamental groups. *Topology* **37**, pages 621–633, 1998.

[11] G. Niblo and L. Reeves. Coxeter groups act on CAT(0) cube complexes. *J. Group Theory* **6**, pages 399–413, 2003.

[12] M. Ronan. *Lectures on buildings.* Academic Press, 1989.

[13] J. Tits. Structures et groupes de Weyl. *Séminaire Bourbaki, exposé no 288*, 1965.

[14] B. Williams. *Two topics in geometric group theory.* PhD thesis, University of Southampton, 1999. http://www.maths.soton.ac.uk/pure/preprints/thesis_btw.dvi.gz.