Reflection triangles and parallel walls in Coxeter complexes

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Abstract. Let (W, S) be a Coxeter system of finite rank (i.e. $|S|$ is finite). A **hyperbolic reflection triangle** is a set $T \subset S^W$ of 3 reflections such that the group $\langle T \rangle$ is isomorphic to a compact hyperbolic triangle group. Our main result is that W has finitely many conjugacy classes of hyperbolic reflection triangles. Using this result, we prove the strong parallel wall conjecture of Niblo and Reeves [10].

1 Introduction

Let (W, S) be a Coxeter system of finite rank (i.e. $|S|$ is finite). There are several ways to construct a geometric space equipped with a natural action of W. For example, one can consider the Cayley graph $\Sigma(W, S)$, the Coxeter complex $T(W, S)$ (see [12]) or the Davis complex $M(W, S)$ (see [5]). The Davis complex is a CAT(0) simplicial complex on which W acts properly discontinuously and cocompactly. It was used by Moussong [8] to give a characterization of word hyperbolic Coxeter groups. More recently, Niblo and Reeves constructed a new space on which W has a natural action: the Coxeter cubing $X(W, S)$. The latter is a CAT(0) cubical complex which is finite-dimensional, locally finite and properly discontinuously acted upon by W. Unfortunately, the action of W is not always cocompact; actually, one has the following characterization (see [4]): *the action of W upon $X(W, S)$ is cocompact if and only if the Coxeter diagram of (W, S) has no irreducible subdiagram of affine type and rank at least* 3.

In order to prove this result, the notion of a **reflection triangle** was used. We recall that a reflection triangle is a set $T := \{t_1, t_2, t_3\} \subset S^W$ of 3 reflections which is not contained in any parabolic subgroup of rank 2 and such that $o(t_i, t_j)$ is finite for

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$1 \le i < j \le 3$. It is known that for a given triangle T, there exists a triangle T' such that $\langle T \rangle = \langle T' \rangle$ and $(\langle T \rangle, T')$ is a Coxeter system. Moreover, the Coxeter diagram of $(\langle T \rangle, T')$ is uniquely determined by T; we call it the **type** of T and we denote it by $\mathcal{M}(T)$. We say that T is affine (resp. spherical, hyperbolic) if $\mathcal{M}(T)$ is affine (resp. spherical, hyperbolic).

Our main result is the following.

Theorem 1. *There are only finitely many conjugacy classes of non-affine reflection triangles.*

It is well known that W has finitely many conjugacy classes of finite subgroups, from which it follows that W has finitely many conjugacy classes of spherical triangles. Consequently, Theorem 1 amounts to the statement that W has finitely many conjugacy classes of hyperbolic reflection triangles. This improves Theorem 1.1 in [4].

The heuristic idea behind our strategy to prove Theorem 1 is the following. Let (W, S) be a Coxeter system and a, b, c, c' be four reflections of (W, S) such that $T = \{a, b, c\}$ and $T' = \{a, b, c'\}$ are both reflection triangles of the same type $\mathcal{M}$ and the product cc' is of infinite order. A set of four reflections which satisfy this condition, is called a *key set of type* $\mathcal{M}$. Clearly, affine Coxeter systems possess many key sets of reflections. On the other hand, if the Davis complex $M(W, S)$ can be realized as a Coxeter tiling of the hyperbolic plane $\mathbb{H}^2$, then (W, S) has no key set, because for such a key set, the triangles T and T' would correspond to geodesic triangles of $\mathbb{H}^2$ with the same angles, one properly contained in the other, which is impossible. Combining arguments from hyperbolic geometry with combinatorial considerations of the Davis complex, we prove that in any Coxeter system, every key set is of affine type (see Theorem 4.1). This is the main ingredient of the proof of Theorem 1.

Given a Coxeter system (W, S), there is a well known canonical way of constructing a *root system* $\Phi \subset$ contained in a real vector space of dimension $|S|$ and on which W acts linearly and faithfully (see [1], Chapitre V, §4). The notion of a spherical (resp. affine, hyperbolic) reflection triangle of (W, S) is essentially equivalent to the notion of a root subsystem of rank 3 of spherical (resp. affine, compact hyperbolic) type. In view of this, Theorem 1 can be reformulated as follows.

Theorem 1′. *Let (W, S) be a Coxeter system with S finite and Φ be the associated root system. There are only finitely many W-orbits of root subsystems of rank 3 and compact hyperbolic type in Φ.*

As a consequence of Theorem 1, we obtain the following.

Corollary 2. *There exists a constant $L = L(W, S)$ such that the following holds. Let $k_1, k_2, h_1, h_2, \ldots, h_n$ be hyperplanes in $M(W, S)$ (or in $T(W, S)$ or in $\Sigma(W, S)$) such that $k_1 \cap k_2 \cap h_1 \neq \emptyset$, the h_i's are pairwise non-intersecting and each h_i intersects both k_1 and k_2. If $n \ge L$ then $\{r_{k_1}, r_{k_2}, r_{h_i}\}$ is an affine reflection triangle for $i = 1, 2, \ldots, n$, where r_H denotes the reflection fixing the hyperplane H.*

A collection $\{k_1, k_2, h_1, h_2, \ldots, h_n\}$ of hyperplanes satisfying the conditions of the above corollary (plus some other minor conditions) is called a **ladder of hyperplanes** in [14]. Corollary 2 answers a question raised on p. 59 in loc. cit.

Using Theorem 1 and its corollary, we prove the following two results.

Theorem 3. ***(Parallel wall theorem)*** *For each positive integer n, there exists a constant $B(n) = B(n; W, S)$ such that the following holds. Given a hyperplane H and a point p in $M(W, S)$ (or in $T(W, S)$ or in $\Sigma(W, S)$) such that the distance from p to H is at least $B(n)$, then there exist n pairwise non-intersecting hyperplanes which separate p from H.*

This result was proved for $n = 1$ by Brink and Howlett and used to show that Coxeter groups are automatic (see [2]). The parallel wall theorem also implies the local finiteness of $X(W, S)$ (see §3.2 in [10]). We note that our proof of Theorem 3 is however independent of [2] and yields therefore a new approach to the parallel wall theorem. The main interest of the version of the parallel wall theorem stated above is that it allows us to prove Theorem 4, which was stated in [10] as the *strong parallel wall conjecture*.

Theorem 4. ***(Wall separating theorem)*** *There exists a constant $N = N(W, S)$ such that the following holds. Given two hyperplanes H_1 and H_2 in $M(W, S)$ (or in $T(W, S)$ or in $\Sigma(W, S)$) such that the distance from H_1 to H_2 is at least N, then there exists a hyperplane H which separates H_1 from H_2.*

Combined with Theorem 6.3 below, the separating wall theorem has the following consequence regarding the structure of the Coxeter cubing.

Corollary 5. *There exists a uniform bound on the size of a link of a vertex in $X(W, S)$.*

2 Preliminaries

We work in the Cayley graph $\Sigma = \Sigma(W, S)$ and we consider it as a chamber system over S. Our main reference for the language of chamber systems and for the standard properties of Σ is [13] (e.g. definition of a gallery, of a residue, existence of projections, ...).

Finite subgroups

Lemma 2.1. *A subgroup of W is finite if and only if it stabilizes a spherical residue of Σ.*

Proof. This is an exercise in [1]. It can be proven using the Tits cone (see Proposition 3.2.1 in [7]) or with the Davis complex (see Corollary 11.9 in [5]). □

Parallelism of residues

Given residues R_1, R_2 of $\Sigma(W, S)$, then the set $\mathrm{proj}_{R_1}(R_2) := \{\mathrm{proj}_{R_1}(c) | c \in R_2\}$ is itself a residue. We say that R_1 and R_2 are **parallel** if $\mathrm{proj}_{R_1}(R_2) = R_1$ and $\mathrm{proj}_{R_2}(R_1) = R_2$.

Lemma 2.2. *Let J, K be subsets of S and let R_J, R_K be residues of type J, K respectively. Then the following statements are equivalent:*
(i) R_J and R_K are parallel;
(ii) a reflection stabilizes R_J if and only if it stabilizes R_K.
Furthermore, if J or K is spherical, then (i) and (ii) above are also equivalent to the following:
(iii) there exist two sequences $R_J = R_0, R_1, \dots, R_n = R_K$ and $T_1, \dots, T_n$ of residues of spherical type such that for each $1 \leq i \leq n$ the rank of T_i is equal to $1+\mathrm{rank}(R_J)$, the residues R_{i-1}, R_i are distinct, parallel and contained in T_i and moreover, we have $\mathrm{proj}_{T_i}(R_J) = R_{i-1}$ and $\mathrm{proj}_{T_i}(R_K) = R_i$.

Proof. This follows from Proposition 2.7 in [3]. □

Roots and angles

Let ψ be a root. We denote by $\partial\psi$ or ∂r_ψ (resp. $\partial^2\psi$ or $\partial^2 r_\psi$) the set of all panels (resp. spherical residues of rank 2) stabilized by r_ψ. We also set $\mathcal{C}(\partial\psi) = \mathcal{C}(\partial r_\psi) := \bigcup_{\sigma\in\partial\psi}\psi$ and $\mathcal{C}(\partial^2\psi) = \mathcal{C}(\partial^2\psi) := \bigcup_{\sigma\in\partial^2\psi}\sigma$. The set $\partial\psi$ is called the **wall** or the **hyperplane** associated to ψ.

Lemma 2.3. *Let ψ be a root and let $x, y \in \mathcal{C}(\partial\psi)\cap\psi$. Then there exists a minimal gallery $\Gamma = (x = x_0, x_1, \dots, x_l = y)$ joining x to y such that $x_i \in \mathcal{C}(\partial^2\psi)$ for each $1 \leq i \leq l$.*

Proof. This is an easy consequence of Lemma 2.2. See Lemma 2.3 in [4]. □

Let ϕ and ψ be roots. We say that ϕ and ψ (or r_ϕ and r_ψ or $\partial\phi$ and $\partial\psi$) are **parallel** if $o(r_\phi r_\psi) = \infty$ and **incident** otherwise. Equivalently, ϕ and ψ are parallel if and only if $\partial\phi \subset \psi$ or $\partial\phi \subset -\psi$, while they are incident if and only if $\partial^2\phi \cap \partial^2\psi \neq \emptyset$.

Lemma 2.4. *There exists a constant $P = P(W, S)$ such that any collection of more than P walls contains a pair of parallel walls.*

Proof. See Lemma 3 in [10]. □

The following result, though elementary, is extremely useful.

Lemma 2.5. *Let ϕ, α, α' be roots, let $R \in \partial^2\alpha$ with $R \subset \phi$ and let $R' \in \partial^2\alpha'$ with $R' \subset -\phi$. Let $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \alpha'$ be a sequence of roots such that α_{i-1} is incident to α_i for $i = 1, \dots, n$. Then ϕ is incident to α_i for some $i \in \{0, 1, \dots, n\}$. Furthermore, if $n = 0$ then $\alpha = \alpha'$ and the result is true if $R \subset \phi$ and $R' \subset -\phi$ are panels stabilized by r_α.*

Proof. By induction on n.

Suppose $n = 0$. Then $\alpha = \pm\alpha'$ and by Lemma 2.3, there exists a gallery Γ joining a chamber of R to a chamber of R' and completely contained in $\mathcal{C}(\partial^2\alpha)$. Since $R \subset \phi$ and $R' \subset -\phi$, the gallery Γ must cross $\partial\phi$. Therefore, $\partial^2\phi \cap \partial^2\alpha \neq \emptyset$ and ϕ is incident to α, as expected.

Suppose the result is true for $n-1$. Since α is incident to α_1, there exists $R_1 \in \partial^2\alpha \cap \partial^2\alpha_1$. If $R_1 \in \partial^2\phi$ then ϕ is incident to α and we are done. If $R_1 \subset \phi$ then the induction hypothesis applies and yields the desired conclusion. Finally, if $R_1 \subset -\phi$ then an argument as in the case $n = 0$ shows that ϕ is incident to α. □

If ϕ and ψ are parallel, we define the **angle** $\angle(\phi, \psi)$ as follows:

$$\angle(\phi, \psi) := -\infty \text{ if } \phi \subset \psi \text{ or } \phi \supset \psi$$

and

$$\angle(\phi, \psi) := +\infty \text{ if } \phi \subset -\psi \text{ or } \phi \supset -\psi.$$

If ϕ and ψ are incident, let $R \in \partial^2\phi \cap \partial^2\psi$. We define the **angle** $\angle(\phi, \psi)$ as follows:

$$\angle(\phi, \psi) := 2\pi.\frac{|R \cap \phi \cap -\psi|}{|R|}.$$

Lemma 2.2 implies that $\angle(\phi, \psi)$ is independent of the choice of R.

Lemma 2.6. *Let $\alpha \neq \beta$ be roots and let $c \in \alpha \cap \beta$ be a chamber. Assume $c \in \mathcal{C}(\partial\beta)$ and let $n := d(c, \mathcal{C}(\partial\alpha))$. We have*

$$d(r_\beta(c), \mathcal{C}(\partial\alpha)) = n - 1 \ (\textit{resp. } n, n+1) \ \textit{if } \angle(\alpha, \beta) < \frac{\pi}{2} \ (\textit{resp. } = \frac{\pi}{2}, > \frac{\pi}{2}).$$

Proof. The statement is equivalent to Lemma 1.7 in [2], where an algebraic proof is given. Here is an alternative combinatorial argument.

Let $c_0 \in \mathcal{C}(\partial\alpha)$ be such that $d(c_0, c) = n$ and let $\Gamma = (c_0, c_1, \dots, c_n) = c$ be a minimal gallery.

If $\angle(\alpha, \beta) = -\infty$ then $\beta \subset \alpha$ and β is the unique root which contains c_n but not c_{n-1}. Hence $r_\beta(c) = c_{n-1}$ and $d(r_\beta(c), \mathcal{C}(\partial\alpha)) = n - 1$ as expected.

If $\angle(\alpha, \beta) = \frac{\pi}{2}$ then

$$n = d(c, (\alpha, \beta)) = d(r_\beta(c), r_\beta((\alpha, \beta))) = d(r_\beta(c), (\alpha, \beta))$$

as expected.

If $\angle(\alpha, \beta) \in]0, \frac{\pi}{2}[$, let $R \in \partial^2\alpha \cap \partial^2\beta$. Assume that Γ does not cross $\partial\beta$. Then Γ is completely contained in $\alpha \cap \beta$. In view of $\angle(\alpha, \beta) < \frac{\pi}{2}$, this implies that Γ crosses $\partial r_\beta(\alpha)$ because $\mathrm{proj}_R(c_0) \in \mathcal{C}(\partial\alpha) \cap \beta \cap R \subset -r_\beta(\alpha)$ while $\mathrm{proj}_R(c) \in \mathcal{C}(\partial\beta) \cap \alpha \cap R \subset r_\beta(\alpha)$ and hence $\partial r_\beta(\alpha)$ separates c_0 from c. Let $k = \max\{i | c_i \in \mathcal{C}(\partial r_\beta(\alpha))\}$. We have $k \geq 1$. Therefore, $\Gamma' := (r_\beta(c_k), r_\beta(c_{k+1}), \dots, r_\beta(c_n), c_n = c)$ is a gallery of length $n-k+1$ joining $r_\beta(c_k) \in r_\beta(\mathcal{C}(\partial r_\beta(\alpha))) = \mathcal{C}(\partial\alpha)$ to c. By the definition of n, we deduce $k = 1$. This shows that, up to replacing Γ by Γ', we may assume without loss of generality that Γ crosses $\partial\beta$. It follows that β is the unique root which contains c_n but not c_{n-1}. Hence $r_\beta(c) = c_{n-1}$ and $d(r_\beta(c), \mathcal{C}(\partial\alpha)) = n - 1$ as expected.

Finally, suppose $\angle(\alpha, \beta) > \frac{\pi}{2}$. Then $\angle(\alpha, -\beta) < \frac{\pi}{2}$ and $r_\beta(c) \in -\beta$. Thus, by what we have already proven, we have

$$n = d(c, \mathcal{C}(\partial\alpha)) = d(r_\beta(r_\beta(c)), \mathcal{C}(\partial\alpha)) = d(r_\beta(c), \mathcal{C}(\partial\alpha)) - 1$$

as expected. □

Fundamental domains and geometric sets

Let Ψ be a set of roots. We set $R(\Psi) := \{r_\psi | \psi \in \Psi\}$ and $W(\Psi) := \langle R(\Psi) \rangle$. The set Ψ is called **geometric** if $\bigcap_{\psi \in \Psi} \psi$ is nonempty and if for all $\phi, \psi \in \Psi$, the set $\phi \cap \psi$ is a fundamental domain for the action of $W(\{\phi, \psi\})$ on $\Sigma(W, S)$. Here, a set D is called a **fundamental domain** for the action of a group G on a set E containing D if $\bigcup_{g \in G} gD = E$ and if $D \cap gD \neq \emptyset \Rightarrow g = 1$ for every $g \in G$.

Lemma 2.7. *Let $\alpha \neq \beta$ be roots. The pair $\{\alpha, \beta\}$ is geometric if and only if either $\angle(\alpha, -\beta) = \frac{\pi}{n}$ for some integer $n \geq 2$ or $\angle(\alpha, \beta) = +\infty$ and $\alpha \cap \beta \neq \emptyset$.*

Proof. If α and β are parallel, the criterion is given by Lemma 4.5 in [9].

If α and β are incident, let $R \in \partial^2\alpha \cap \partial^2\beta$. The criterion follows from the faithfulness of the action of $\langle r_\alpha, r_\beta \rangle$ on R and from the following observation:

$$\alpha \cap \beta = \{c \in \Sigma | \mathrm{proj}_R(c) \in \alpha \cap \beta \cap R\}.$$

□

The following result, due to Tits, is very useful.

Lemma 2.8. *Let Ψ be a geometric set of roots. Then $D := \bigcap \Psi$ is a fundamental domain for the action of $W(\Psi)$ on $\Sigma(W, S)$, and $(W(\Psi), R(\Psi))$ is a Coxeter system. The chambers of $\Sigma(W(\Psi), R(\Psi))$ may be identified with sets of chambers of $\Sigma(W, S)$, and more precisely with sets of the form wD with $w \in W(\Psi)$. Furthermore, two chambers C and C' of $\Sigma(W(\Psi), R(\Psi))$ are adjacent in $\Sigma(W(\Psi), R(\Psi))$ if and only if C and C', viewed as sets of chambers of $\Sigma(W, S)$, contain adjacent chambers of $\Sigma(W, S)$.*

Proof. This is essentially a consequence of Lemma 1 in [11]. See also Lemma 3.2 and Proposition 3.3 in [9]. □

Restated in other words, the last statement of Lemma 2.8 says that the Cayley graph of the Coxeter system $W((\Psi), R(\Psi))$ may be seen as a 'quotient' of the Cayley graph of (W, S).

3 Triangles

Definition

In the introduction, we have defined the notion of a reflection triangle. In order to make our forthcoming developments easier, we need to consider a slightly different notion which we define now.

A **combinatorial triangle** (or simply a **triangle** is a set T of 3 roots which satisfy the following conditions:

- **(CT1)** the elements of T are pairwise incident;
- **(CT2)** the group $W(T)$ is not contained in any parabolic subgroup of rank 2;
- **(CT3)** for each $\alpha \in T$ there exists $\sigma \in \partial^2\beta \cap \partial^2\gamma$ such that $\sigma \subset \alpha$, where $\beta \neq \gamma \in T\backslash\{\alpha\}$.

Clearly, given a combinatorial triangle T, the set $R(T)$ is a reflection triangle. Conversely, let R be a reflection triangle. Then there exists a combinatorial triangle T such that $R(T) = R$. Moreover, this combinatorial triangle is unique if and only if R is of non-spherical type.

Let T_1 and T_2 be combinatorial triangles. We say that T_1 is a **subtriangle** of T_2 if $\bigcap T_1 \subseteq \bigcap T_2$ and if there exists a triangle T_0 such that $W(T_1) \cup W(T_2) \subseteq W(T_0)$.

The following lemma guarantees that every triangle admits a geometric subtriangle.

Lemma 3.1. *Let T be a combinatorial triangle. There exists a geometric triangle T' such that $W(T) = W(T')$.*

Proof. See Lemma 5.2 in [4]. □

Type of a triangle

The **type** of a combinatorial triangle T is the type of the reflection triangle $R(T)$ and we set $\mathcal{M}(\mathcal{T}) := \mathcal{M}(\mathcal{R}(\mathcal{T}))$. If T' is a geometric triangle such that $W(T) = W(T')$ (see Lemma 3.1) then the type of T is nothing but the type of the Coxeter system $(W(T), R(T'))$. We call T spherical, affine or hyperbolic if $\mathcal{M}(T)$ is spherical, affine or hyperbolic.

Let T be a combinatorial triangle. For each pair $\alpha \neq \beta \in T$, the angle $\angle(\alpha, -\beta)$ is called an **interior angle** (or simply an **angle**) of the triangle T.

Not surprisingly, we have the following characterization.

Lemma 3.2. *Let T be a combinatorial triangle, let*

$$A(T) := \sum_{\alpha\neq\beta\in T} \angle(\alpha, -\beta).$$

and let T' be a geometric triangle such that $W(T) \leq W(T')$.

The following assertions are equivalent:

(i) T *is spherical (resp. affine, hyperbolic);*

(ii) T' *is spherical (resp. affine, hyperbolic);*

(iii) *the Coxeter system* $(W(T'), R(T'))$ *is of spherical (resp. affine, compact hyperbolic) type;*

(iv) *the Coxeter complex* $\mathcal{T}(W(T'), R(T'))$ *is a tessellation of* $\mathbb{S}^2$ *(resp.* $\mathbb{E}^2$, $\mathbb{H}^2$*) by compact geodesic triangles;*

(v) $A(T) > \pi$ *(resp.* $A(T) = \pi$, $A(T) < \pi$*);*

(vi) *every subtriangle of* A *is spherical (resp. affine, hyperbolic).*

Proof. The equivalences $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ are clear. Since $W(T) \leq W(T')$, we may identify the combinatorial triangle T with a geodesic triangle of the Coxeter complex $\mathcal{T}(W(T'), R(T'))$, and this identification preserves the angles in view of Lemma 2.8. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (v)$ clearly follow. Finally, since every substriangle of T is (up to conjugation) a subtriangle of T', the equivalence $(i) \Leftrightarrow (vi)$ is a consequence of what we have already proven. $\square$

Vertices and perimeter

Let T be a combinatorial triangle.

A rank 2 spherical residue σ is called a **vertex** of T if there exist $\alpha \neq \beta \in T$ such that $\sigma \in \partial^2\alpha \cap \partial^2\beta$.

The following lemma gives a useful sufficient condition for a triangle to be non-spherical.

Lemma 3.3. *Let* $T = \{\alpha, \beta, \gamma\}$ *be a combinatorial triangle, let* ν *be a vertex of* T *such that* $\nu \subset \gamma$*. If there exists a root* ϕ *which is parallel to* γ *and such that* $\nu \in \partial^2\phi$*, then* T *is non-spherical.*

Proof. Suppose by contradiction that T is spherical. By Lemma 2.1, there exists a spherical residue R stabilized by $W(T)$. Let $\nu' \subset R$ be a vertex of T such that $\nu' \subset \gamma$. Thus ν and ν' are parallel, and by Lemma 2.2, we have $\nu' \in \partial^2\phi$. Thus r_ϕ stabilizes R. It follows that the product $r_\gamma r_\phi$ stabilizes the spherical residue R, which contradicts the hypothesis that γ and ϕ are parallel. $\square$

A set $\{\sigma_1, \sigma_2, \sigma_3\}$ is called a **set of vertices** of T if the following conditions are satisfied:

- for $i \in \{1, 2, 3\}$, σ_i is a vertex of T;
- for $i \in \{1, 2, 3\}$, there exists $\alpha_i \in T$ such that $\sigma_i \subset \alpha_i$;
- the α_i's are mutually distinct, i.e. $T = \{\alpha_1, \alpha_2, \alpha_3\}$.

The **perimeter** of T, denoted by $\mathrm{perim}(T)$, is defined by:

$$\mathrm{perim}(T) := \min\left\{ \sum_{\sigma \neq \tau \in V} d(\sigma, \tau) | V \text{ is a set of vertices of } T \right\} + \sum_{\alpha \neq \beta \in T} d(\mathcal{C}(\partial\alpha) \cap \alpha, \mathcal{C}(\partial\beta) \cap \beta).$$

Lemma 3.4. *Let* $T = \{\alpha_1, \alpha_2, \alpha_3\}$ *be a combinatorial triangle and let* $V = \{\sigma_1, \sigma_2, \sigma_3\}$ *be set of vertices such that* $\sigma_i \subset \alpha_i$ *for each* $i \in \{1, 2, 3\}$*. Assume* $\angle(\alpha_1, -\alpha_2) < \frac{\pi}{2}$ *and* $\angle(\alpha_2, \alpha_3) = \frac{\pi}{2}$*. Then* $\bar{T} := \{\alpha_1, r_{\alpha_2}(\alpha_1), \alpha_3\}$ *is a combinatorial triangle and* $\bar{V} := \{\sigma_2, r_{\alpha_2}(\sigma_2), \sigma_3\}$ *is a set of vertices of* $\bar{T}$*.*

Proof. Note that the elements of $\bar{T}$ are pairwise different because $\angle(\alpha_1, \alpha_2) \neq \frac{\pi}{2}$. We have $\angle(\alpha_3, r_{\alpha_2}(\alpha_1)) = \angle(r_{\alpha_2}(\alpha_3), \alpha_1) = \angle(\alpha_2, \alpha_1)$ because $\angle(\alpha_2, \alpha_3) = \frac{\pi}{2}$. It follows that $\bar{T}$ satisfies (CT1). For (CT2), it suffices to verify that σ_2 and σ_3 are not parallel (see Lemma 2.2), which follows from Lemma 3.5(1). Finally, we have $\emptyset \neq \alpha_1 \cap \alpha_2 \cap \sigma_3 \subset \alpha_1 \cap r_{\alpha_2}(\alpha_1) \cap \alpha_3$ which implies (CT3). Thus $\bar{T}$ is a combinatorial triangle.

Since $\angle(\alpha_1, -\alpha_2) < \frac{\pi}{2}$ and since σ_2 and σ_3 are not parallel, we have $\mathrm{proj}_{\sigma_3}(\sigma_2) \subset r_{\alpha_2}(\alpha_1)$ from which we deduce $\sigma_2 \subset r_{\alpha_2}(\alpha_1)$. Transforming by r_{α_2} we also obtain $r_{\alpha_2}(\sigma_2) \subset \alpha_1$. This shows that $\bar{V}$ is a set of vertices of $\bar{T}$. □

Circumscribing galleries

Let $T = \{\alpha_1, \alpha_2, \alpha_3\}$ be a combinatorial triangle and let $V = \{\sigma_1, \sigma_2, \sigma_3\}$ be a set of vertices of T such that $\sigma_i \subset \alpha_i$ for each $i \in \{1, 2, 3\}$. Let Γ be a gallery. We say that Γ is a (T, V)-**circumscribing** gallery if the following conditions are satisfied:

- Γ is closed;
- Γ is completely contained in $\bigcup_{i=1}^{3} \mathcal{C}(\partial^2 \alpha_i)$;
- the length of Γ equals $\sum_{1 \leq i < j \leq 3} d(\sigma_i, \sigma_j) + d(\mathcal{C}(\alpha_i) \cap \alpha_i, \mathcal{C}(\alpha_j) \cap \alpha_j)$.

The existence of a (T, V)-circumscribing gallery follows from Lemma 2.3.

As before, let $T = \{\alpha_1, \alpha_2, \alpha_3\}$ be a combinatorial triangle, let $V = \{\sigma_1, \sigma_2, \sigma_3\}$ be set of vertices such that $\sigma_i \subset \alpha_i$ for each $i \in \{1, 2, 3\}$ and let Γ be a (T, V)-circumscribing gallery. Let $i, j, k \in \{1, 2, 3\}$ be pairwise distinct. We denote by $]\sigma_i, \sigma_j[_\Gamma$ the set of all $\tau \in \partial^2 \alpha_k \backslash \{\sigma_i, \sigma_j\}$ that are crossed by Γ, i.e. that contain a panel crossed by Γ. We also set

$$[\sigma_i, \sigma_j]_\Gamma :=]\sigma_i, \sigma_j[_\Gamma \cup \{\sigma_i, \sigma_j\}, \quad [\sigma_i, \sigma_j[_\Gamma :=]\sigma_i, \sigma_j[_\Gamma \cup \{\sigma_i\} \quad \text{and} \quad]\sigma_i, \sigma_j]_\Gamma :=]\sigma_i, \sigma_j[_\Gamma \cup \{\sigma_j\}.$$

The basics

The following two lemmas collect several basic observations on combinatorial triangles which are all intuitively clear.

Lemma 3.5. *Let $T = \{\alpha_1, \alpha_2, \alpha_3\}$ be a combinatorial triangle, let $V = \{\sigma_1, \sigma_2, \sigma_3\}$ be set of vertices such that $\sigma_i \subset \alpha_i$ for each $i \in \{1, 2, 3\}$ and let Γ be a (T, V)-circumscribing gallery. Let $i, j, k \in \{1, 2, 3\}$ be pairwise distinct, let $\sigma \in]\sigma_i, \sigma_j[_\Gamma$ and let $r \neq r_{\alpha_k}$ be a reflection which stabilizes σ. We have the following:*

1. *Two distinct elements of $[\sigma_i, \sigma_j]_\Gamma$ cannot be parallel.*
2. *$\sigma \subseteq \alpha_i \cap \alpha_j$.*
3. *σ_i is contained in one of the roots associated with r, say ψ, and σ_j is contained in the other.*
4. *If r stabilizes some residue $\tau \in]\sigma_i, \sigma_k[_\Gamma$, then every element of $[\sigma_j, \sigma_k]_\Gamma$ is contained in $-\psi$.*
5. *There exists a unique residue $\tau \in]\sigma_i, \sigma_k]_\Gamma \cup]\sigma_j, \sigma_k]_\Gamma$ which is stabilized by r.*
6. *If $\tau \in]\sigma_i, \sigma_k]_\Gamma$ (resp. $\tau \in]\sigma_j, \sigma_k]_\Gamma$) then $\{\alpha_j, \alpha_k, \psi\}$ (resp. $\{\alpha_i, \alpha_j, -\psi\}$) is a combinatorial triangle.*
7. *If ϕ is a root such that $\sigma_i \in \partial^2 \phi$ and $\angle(\alpha_k, -\phi) < (\alpha_k, -\alpha_j)$ then there exists a unique $\rho \in]\sigma_j, \sigma_k[\cap \partial^2 \phi$. Moreover, $\{\alpha_i, \alpha_k, \phi\}$ and $\{\alpha_i, \alpha_j, -\phi\}$ are combinatorial triangles which are both subtriangles of T.*

Proof. (1) By (CT2), σ_i and σ_j are not parallel. By the definition of a (T, V)-circumscribing gallery, no element of $]\sigma_i, \sigma_j[_\Gamma$ is parallel to σ_i or σ_j and no two elements of $]\sigma_i, \sigma_j[_\Gamma$ are parallel.

(2) By (1), we know that neither r_{α_i} nor r_{α_j} stabilizes σ.

Suppose that $\sigma \subseteq -\alpha_i$. Since $\sigma_i \subseteq \alpha_i$, it follows that Γ crosses the wall $\partial\alpha_i$. Hence, there exists $\sigma' \in]\sigma_i, \sigma_j[_\Gamma \cap \partial^2\alpha_i$. However, r_{α_i} does not stabilize any element of $]\sigma_i, \sigma_j[_\Gamma$ by (1). This contradiction shows that $\sigma \subseteq \alpha_i$ and by symmetry, we obtain $\sigma \subseteq \alpha_j$.

(3) Let ψ be the root associated with r and containing $\mathrm{proj}_\sigma(\sigma_i)$. Then, in view of (1), we have $\mathrm{proj}_\tau(\sigma_i) \subset \psi$ and $\sigma_i \subseteq \psi$. Similarly, $\sigma_j \subseteq -\psi$ because $\mathrm{proj}_\sigma(\sigma_j) \subset -\psi$.

(4) By (3), we have $\sigma_k \subseteq -\psi$.

Let $\sigma' \in]\sigma_j, \sigma_k[_\Gamma$ and assume that $\sigma' \in \partial^2\psi$. Since σ_j and σ_k are both contained in $-\psi$, it follows that there exists a $\tau' \in]\sigma_j, \sigma_k[_\Gamma \cap \partial^2\psi$ with $\tau' \neq \sigma'$. Therefore, σ' and τ' are distinct and both are stabilized by r_ψ and r_{α_i}. Furthermore, we have $r_\psi \neq r_{\alpha_i}$ because r_ψ does not stabilize σ_j. It follows that σ and τ' are parallel, which contradicts (1).

Thus r_ψ does not stabilize any element of $[\sigma_j, \sigma_k]_\Gamma$. We have seen above that σ_j and σ_k are both contained in $-\psi$. We deduce by an argument as in the proof of (2) that every element of $[\sigma_j, \sigma_k]_\Gamma$ is contained in $-\psi$.

(5) (Compare Lemma 2.5). The existence of τ follows from the fact that Γ is closed and crosses thus $\partial\psi$ at least twice. The uniqueness of τ follows from (1) and (4).

(6) Assume $\tau \in]\sigma_i, \sigma_k]_\Gamma$. It is clear that $\{\alpha_j, \alpha_k, \psi\}$ satisfies (CT1). Moreover, σ_i and σ are not parallel by (1), whence (CT2). Finally, we have $\emptyset \neq \sigma_i \cap \alpha_j \cap \alpha_k \subseteq \psi \cap \alpha_j \cap \alpha_k$ by (3), whence (CT3). The case $\tau \in]\sigma_j, \sigma_k]_\Gamma$ follows by symmetry.

(7) The existence of ρ follows from (1) combined with an argument as in (6). Applying now (6) to ρ, we deduce that $\{\alpha_i, \alpha_k, \phi\}$ and $\{\alpha_i, \alpha_j, -\phi\}$ are combinatorial triangles. Let now ϕ' be the root such that $\sigma_i \in \partial^2\phi'$ and $\angle(\alpha_k, -\phi') = \frac{2\pi}{|\sigma_i|}$. By what we have just proven, $\{\alpha_i, \alpha_k, \phi'\}$ is a combinatorial triangle. Moreover, it is clear from the definition of ϕ' that $W(\{\alpha_i, \alpha_k, \phi'\})$ contains $W(T)$, $W(\{\alpha_i, \alpha_k, \phi\})$ and $W(\{\alpha_i, \alpha_j, -\phi\})$ as subgroups. Whence the conclusion.

□

The previous lemma allows us to introduce some notation which will be used intensively in Section 4.

Let $T = \{\alpha_1, \alpha_2, \alpha_3\}$, $V = \{\sigma_1, \sigma_2, \sigma_3\}$, Γ be as in the statement of Lemma 3.5. Let $\sigma \in]\sigma_1, \sigma_2[_\Gamma$. We set

$$\Phi_T(\sigma, \sigma_1) := \{\phi | \phi \text{ is a root}, \sigma \in \partial^2\phi, \sigma_1 \subset \phi \text{ and }]\sigma_1, \sigma_3]_\Gamma \cap \partial^2\phi \neq \emptyset\}.$$

Note that Lemma 3.5(5) implies that $\Phi_T(\sigma, \sigma_1) \cup \Phi_T(\sigma, \sigma_2)$ is nonempty. Actually, we have $\frac{|\sigma|}{2} - 1 \leq |\Phi_T(\sigma, \sigma_1) \cup \Phi_T(\sigma, \sigma_2)| \leq \frac{|\sigma|}{2}$.

If $\Phi_T(\sigma, \sigma_1)$ is nonempty, we denote by

$$\phi_T(\sigma, \sigma_1)$$

the root $\phi \in \Phi_T(\sigma, \sigma_1)$ such that $\angle(\alpha_3, -\phi)$ is maximal.

Decompositions of triangles

An essential tool in our study of triangles is the possibility of determining all subtriangles of a given non-spherical combinatorial triangle T. Since every triangle contains a geometric subtriangle (see Lemma 3.1), this determination is equivalent to the determination (up to conjugation), for a given geometric triangle F, of all combinatorial triangles T such that $T \subset W(F)$. This is the purpose of the following result.

Proposition 3.6. *Let F be a geometric combinatorial triangle of non-spherical type. Let T be a combinatorial triangle such that $W(T) \subset W(F)$. There are only finitely many possibilities for the angles of T. All possibilities are listed in Table 1.*

Proof. The first assertion is a direct consequence of Lemma 2.1. The second is clear if F is affine and follows from the results of [6] if F is hyperbolic. □

Affine triangles

Crucial to our arguments is the following result on affine triangles.

Proposition 3.7. (D. Krammer) Let T be an affine combinatorial triangle. Then there exists an irreducible residue of affine type and rank ≥ 3 which is stabilized by $W(T)$.

Proof. See Theorem 1.2 in [4]. □

4 The key configuration

Let α, β, γ and γ_T be roots of Σ such that the following conditions hold:
- γ is properly contained in γ_T;
- $T := \{\alpha, \beta, \gamma_T\}$ and $U := \{\alpha, \beta, \gamma\}$ are combinatorial triangles;
- T and U are geometric and of the same non-spherical type $\mathcal{M}$.

In this situation, we say that α, β, γ and γ_T are in the **key configuration**.

The aim is to prove the following.

Theorem 4.1. *$\mathcal{M}$ is affine.*

This is the key result on which our proof of Theorem 1 rests.

The proof of Theorem 4.1 works by contradiction, so we assume from now on that $\mathcal{M}$ is compact hyperbolic. We aim at obtaining a contradiction. There are several intermediate steps, which we present in the following three technical lemmas.

Throughout, we consider vertex sets $V_T := \{\nu, \sigma_T, \rho_T\}$ and $V_U := \{\nu, \sigma, \rho\}$ of T and U respectively, which are such that:
- ν is contained in γ;
- σ and σ_T are contained in β;
- ρ and ρ_T are contained in α.

Without loss of generality, we may and shall assume that $\text{perim}(T) = d(\nu, \sigma_T) + d(\nu, \rho_T) + d(\sigma_T, \rho_T) + d(\mathcal{C}(\partial\alpha) \cap \alpha, \mathcal{C}(\partial\beta) \cap \beta) + d(\mathcal{C}(\partial\alpha) \cap \alpha, \mathcal{C}(\partial\gamma_T) \cap \gamma_T) + d(\mathcal{C}(\partial\beta) \cap \beta, \mathcal{C}(\partial\gamma_T) \cap \gamma_T)$. We also consider a (T, V_T)-circumscribing gallery Γ.

Given a spherical residue R, we set

$$n_R := \frac{|R|}{2}.$$

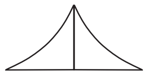
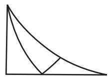
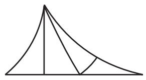
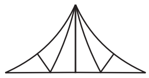
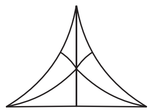
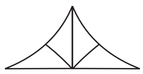
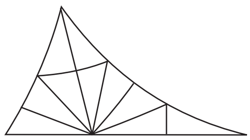
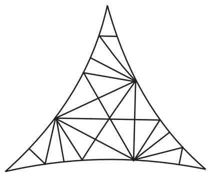
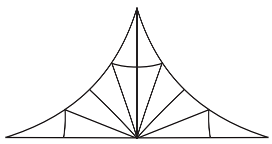
Type	Angles of F	Angles of T	Picture	Symbol
Affine	$(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$	$(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$		$(A1)$
	$(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$	$(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$		$(A2)$
	$(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$	$(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$		$(A3)$
		$(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$		$(A4)$
		$(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$		$(A5)$
Hyperbolic	$(\frac{\pi}{2}, \frac{\pi}{k}, \frac{\pi}{l})$	$(\frac{\pi}{k}, \frac{\pi}{k}, \frac{2\pi}{l})$		(H1)
	$(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{k})$	$(\frac{\pi}{2}, \frac{\pi}{k}, \frac{2\pi}{k})$		(H2)
		$(\frac{\pi}{3}, \frac{\pi}{k}, \frac{3\pi}{k})$		(H3)
		$(\frac{\pi}{k}, \frac{\pi}{k}, \frac{4\pi}{k})$		(H4)
		$(\frac{2\pi}{k}, \frac{2\pi}{k}, \frac{2\pi}{k})$		(H5)
	$(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{k})$	$(\frac{\pi}{k}, \frac{\pi}{k}, \frac{2\pi}{k})$		(H6)
	$(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7})$	$(\frac{\pi}{3}, \frac{\pi}{7}, \frac{2\pi}{7})$		(H7)
		$(\frac{\pi}{7}, \frac{\pi}{7}, \frac{\pi}{7})$		(H8)
	$(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{8})$	$(\frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{8})$		(H9)

Table 1: Decompositions of triangles

Lemma 4.2. *We have* $\operatorname{perim}(U) < \operatorname{perim}(T)$.

Proof. Let Γ' be a (U, V_U)-circumscribing gallery. Let $C_T := d(\mathcal{C}(\partial\alpha) \cap \alpha, \mathcal{C}(\partial\beta) \cap \beta) + d(\mathcal{C}(\partial\alpha) \cap \alpha, \mathcal{C}(\partial\gamma_T) \cap \gamma_T) + d(\mathcal{C}(\partial\beta) \cap \beta, \mathcal{C}(\partial\gamma_T) \cap \gamma_T)$ and $C_U := d(\mathcal{C}(\partial\alpha) \cap \alpha, \mathcal{C}(\partial\beta) \cap \beta) + d(\mathcal{C}(\partial\alpha) \cap \alpha, \mathcal{C}(\partial\gamma) \cap \gamma) + d(\mathcal{C}(\partial\beta) \cap \beta, \mathcal{C}(\partial\gamma) \cap \gamma)$. We have

$$\begin{aligned} \operatorname{perim}(T) &= \ell(\Gamma) \\ &= d(\nu, \sigma_T) + d(\nu, \rho_T) + d(\sigma_T, \rho_T) + C_T \\ &= d(\nu, \sigma) + d(\sigma, \sigma_T) + d(\nu, \rho) + d(\rho, \rho_T) + d(\sigma_T, \rho_T) + n_\sigma - 1 + n_\rho - 1 + C_T \end{aligned}$$

and

$$\ell(\Gamma') = d(\nu, \sigma) + d(\nu, \rho) + d(\sigma, \rho) + C_U.$$

Since the numerical distance d is a pseudo-metric on the residues, we have $d(\sigma, \rho) \leq d(\sigma, \sigma_T) + d(\sigma_T, \rho_T) + d(\rho_T, \rho)$. Furthermore, since $\sigma \in \partial^2\alpha \cap \partial^2\gamma$ and $\rho \in \partial^2\beta \cap \partial^2\gamma$, we have $d(\mathcal{C}(\partial\alpha) \cap \alpha, \mathcal{C}(\partial\gamma) \cap \gamma) < n_\sigma - 1$ and $d(\mathcal{C}(\partial\beta) \cap \beta, \mathcal{C}(\partial\gamma) \cap \gamma) < n_\rho - 1$. We deduce

$$\operatorname{perim}(T) - \ell(\Gamma') > d(\mathcal{C}(\partial\alpha) \cap \alpha, \mathcal{C}(\partial\gamma_T) \cap \gamma_T) + d(\mathcal{C}(\partial\beta) \cap \beta, \mathcal{C}(\partial\gamma_T) \cap \gamma_T) \geq 0,$$

whence the result, since $\operatorname{perim}(U) \leq \ell(\Gamma')$. □

Lemma 4.3. *The triangle U is not a subtriangle of T. In particular, there exists no reflection stabilizing both σ and ρ_T (resp. ρ and σ_T).*

Proof. Assume U is a subtriangle of T. Then there exists a geometric triangle $\widetilde{U}$ such that $W(T) \cup W(U) \subseteq W(\widetilde{U})$. By Lemma 3.2, the triangle $\widetilde{U}$ is hyperbolic since T and U are. Now, T and U may be identified with geodesic triangles of the Coxeter complex $\mathcal{T}(W(\widetilde{U}), R(\widetilde{U}))$ which is a tessellation of $\mathbb{H}^2$, and this identification preserves the angles. Since T and U are of the same type, they are identified with triangles of the same area, which contradicts the fact that $\bigcap_{\phi \in U} \phi$ is properly contained in $\bigcap_{\phi \in T} \phi$.

The last assertion is now a direct consequence of Lemma 3.5(7). □

The proofs of the following four lemmas are technical and require repeated application of Proposition 3.6 and, thereby, of Table 1.

Lemma 4.4. *We have*

$$\angle(\beta, \gamma) = \frac{\pi}{2} \Rightarrow \angle(\beta, \gamma_T) \neq \frac{\pi}{2}.$$

Proof. Suppose T is a triangle of minimal perimeter among all triangles which contradict the lemma.

Thus $\angle(\beta, \gamma_T) = \frac{\pi}{2}$ and it follows that $\angle(\alpha, -\beta) = \frac{\pi}{n}$ for some integer $n \geq 3$ because T is geometric and hyperbolic (see Lemma 2.7). Let $\bar{T} := \{\alpha, \gamma_T, r_\beta(\alpha)\}$ and $\bar{V} := \{\sigma_T, \nu, r_\beta(\sigma_T)\}$. By Lemma 3.4, $\bar{T}$ is a combinatorial triangle and $\bar{V}$ is a set of vertices of $\bar{T}$. Let $\bar{\Gamma}$ be a $(\bar{T}, \bar{V})$-circumscribing gallery such that $r_\beta(\sigma) \in]\nu, r_\beta(\sigma_T)[_{\bar{\Gamma}}$ (see the paragraph preceding Lemma 3.5 for the notation). Using the invariance of $\bar{T}$ under r_β, it is easily seen that such a circumscribing gallery exists.

Since $r_\beta(\sigma) \in]\nu, r_\beta(\sigma_T)[_{\bar{\Gamma}}$, the set $\Phi_{\bar{T}}(\sigma, \nu)$ is nonempty (see the paragraph immediately following Lemma 3.5 for the notation). Let $\bar{\phi}_\sigma := \phi_{\bar{T}}(\sigma, \nu)$ and $\bar{T}_\sigma := \{\alpha, \bar{\phi}_\sigma, r_\beta(\alpha)\}$.

Since $\bar{T}_\sigma$ and U have a common subtriangle and since U is hyperbolic, we deduce from Lemma 3.2 that $\bar{T}_\sigma$ is hyperbolic. Let us apply Proposition 3.6 to $\bar{T}_\sigma$. There are two cases: either $\bar{T}_\sigma$ corresponds to (H1) in Table 1 and then $\phi_\sigma = \gamma$ and $\angle(\alpha, -\bar{\phi}_\sigma) = \frac{\pi}{n_\sigma}$

or $\bar{T}_\sigma$ corresponds to (H5) in Table 1 and then $\angle(\alpha, -\bar{\phi}_\sigma) = 2.\angle(\alpha, -\bar{\phi}_\sigma) = \frac{2\pi}{n_\sigma}$. Thus we obtain

$$\text{either } \angle(\alpha, -\bar{\phi}_\sigma) = \frac{\pi}{n_\sigma} \text{ and } n_\sigma \geq 3 \quad \text{or} \quad \angle(\alpha, -\bar{\phi}_\sigma) = \frac{2\pi}{n_\sigma} \text{ and } n_\sigma \geq 7, \tag{1}$$

where the lower bounds on n_σ follow from the fact that $\bar{T}_\sigma$ is hyperbolic. In both cases, the set $\Phi_{\bar{T}}(\sigma, \sigma_T)$ is nonempty. Let $\bar{\phi}'_\sigma := \phi_{\bar{T}}(\sigma, \sigma_T)$ and $\bar{T}'_\sigma := \{\alpha, \bar{\phi}'_\sigma, \gamma_T\}$. The latter is a triangle by Lemma 3.5(6); it is non-spherical because γ and γ_T are parallel. Applying Proposition 3.6 to $\bar{T}'_\sigma$, we obtain

$$\angle(\alpha, -\bar{\phi}'_\sigma) \leq \frac{4\pi}{n_\sigma}. \tag{2}$$

From (2) and the inequality $\angle(\alpha, -\bar{\phi}_\sigma) + \angle(\alpha, -\bar{\phi}'_\sigma) \geq \nu - \frac{\pi}{n_\sigma}$, we deduce $n_\sigma = 7$ in the second case of (1). This implies $\bar{\phi}_\sigma = \gamma$ by Proposition 3.6. Since $\gamma \in \Phi_{\bar{T}}(\sigma, \nu)$ because γ and γ_T are parallel, U is a subtriangle of $\bar{T}_\sigma$ and, by Proposition 3.6, it has either an angle $\frac{2\pi}{3}$ or an angle $\frac{2\pi}{7}$. This contradicts the fact that U is geometric (see Lemma 2.7).

We deduce $n_\sigma \leq 6$, $\angle(\alpha, -\bar{\phi}_\sigma) = \frac{\pi}{n_\sigma}$ and $\bar{\phi}_\sigma = \gamma$. Since U and T have the same angles, we obtain

$$\angle(\alpha, \gamma) = \angle(\alpha, \gamma_T). \tag{3}$$

Since $\bar{T}_\sigma$ is non-spherical, we have $n_\sigma \neq 2$ and since $\bar{T}'_\sigma$ is non-spherical, Proposition 3.6 implies $n_\sigma \neq 5$. For U is hyperbolic, Proposition 3.6, Lemma 3.2 and (3) also yield

$$\begin{array}{lll} n_\sigma = 3 & \Rightarrow & \angle(\alpha, -\beta) \leq \frac{\pi}{7}, \angle(\alpha, -\gamma) = \frac{\pi}{3} \text{ and } \angle(\gamma_T, -\bar{\phi}'_\sigma) \leq \frac{\pi}{3} \\ n_\sigma = 4 & \Rightarrow & \angle(\alpha, -\beta) \leq \frac{\pi}{5}, \angle(\alpha, -\gamma) = \frac{\pi}{4} \text{ and } \angle(\gamma_T, -\bar{\phi}'_\sigma) = \frac{\pi}{4} \\ n_\sigma = 6 & \Rightarrow & \angle(\alpha, -\beta) \leq \frac{\pi}{4}, \angle(\alpha, -\gamma) = \frac{\pi}{6} \text{ and } \angle(\gamma_T, -\bar{\phi}'_\sigma) = \frac{\pi}{6}. \end{array} \tag{4}$$

Assume $]\nu, \rho_T]_\Gamma \cap \partial^2 \bar{\phi}'_\sigma \neq \emptyset$. Let $\tau \in]\nu, \rho_T]_\Gamma \cap \partial^2 \bar{\phi}'_\sigma \neq \emptyset$. By Lemma 4.3, we have $\tau \neq \rho_T$. We have $\bar{\phi}'_\sigma \in \Phi_T(\tau, \nu)$. Let $\phi_\tau := \phi_T(\tau, \nu)$. By Lemma 3.5(6), the set $T_\tau := \{\alpha, \beta, \phi_\tau\}$ is a combinatorial which is hyperbolic as it contains U as a subtriangle. By (4), Proposition 3.6 and the equality $\angle(\alpha, -\beta) = \angle(\beta, -\bar{\phi}'_\sigma)$ we obtain

$$\begin{array}{lll} n_\sigma = 3 & \Rightarrow & \angle(\beta, -\phi_\tau) \leq \frac{4\pi}{n_\tau} \text{ and } n_\tau \geq 7 \\ n_\sigma = 4 & \Rightarrow & \angle(\beta, -\phi_\tau) \leq \frac{2\pi}{n_\tau} \text{ and } n_\tau \geq 5 \\ n_\sigma = 6 & \Rightarrow & \angle(\beta, -\phi_\tau) = \frac{\pi}{n_\tau} \text{ and } n_\tau \geq 4, \end{array} \tag{5}$$

and in each case, the set $\Phi_T(\tau, \rho_T)$ is nonempty. Let $\phi'_\tau := \phi_T(\tau, \rho_T)$.

By Lemma 3.5(6), the sets $T'_\tau := \{\beta, \gamma_T, \phi'_\tau\}$ and $\bar{T}_\tau := \{-\beta, \gamma_T, \bar{\phi}'_\sigma\}$ are combinatorial triangles. Moreover, they are both contained as subtriangles in the combinatorial triangle $\{\gamma_T, \phi'_\tau, \bar{\phi}'_\sigma\}$. For $\angle(\beta, \bar{\phi}'_\sigma) = \angle(\alpha, -\beta)$, Lemma 3.2 implies that $\bar{T}_\tau$ is hyperbolic in view of (4). We deduce from Lemma 3.2 that T'_τ is hyperbolic. Therefore, since $\angle(\beta, \gamma_T) = \frac{\pi}{2}$, Proposition 3.6 implies

$$\text{either } \angle(\beta, -\phi'_\tau) = \frac{\pi}{n_\tau} \quad \text{or} \quad \angle(\beta, -\phi'_\tau) = \frac{2\pi}{n_\tau} \text{ and } n_\tau \geq 7. \tag{6}$$

Combining (5) and (6) with the inequality $\angle(\beta, -\phi_\tau) + \angle(\beta, -\phi'_\tau) \geq \nu - \frac{\pi}{n_\tau}$, we obtain

$$\angle(\alpha, -\beta) = \frac{\pi}{7}, n_\sigma = 3, n_\tau = 7, \angle(\beta, -\phi_\tau) = \frac{4\pi}{7} \text{ and } \angle(\gamma_T, -\phi'_\tau) = \frac{\pi}{7}. \tag{7}$$

Let now $\tau' \in [\sigma_T, \rho_T[_\Gamma \cap \partial^2 \phi'_\tau$. Since $\angle(\alpha, -\gamma_T) = \angle(\alpha, -\gamma) = \frac{\pi}{3}$ (see (3), (4) and (7)) while $\angle(\gamma_T, -\phi'_\tau) = \frac{\pi}{7}$, we have $\tau' \neq \sigma_T$. Proposition 3.6 implies that $\Phi_T(\tau', \rho_T) =$

$\{\phi'_\tau\}$. Therefore, the set $\Phi_T(\tau', \sigma_T)$ contains at least 5 roots because $n_{\tau'} \geq 7$ and thus $\angle(\gamma_T, -\phi_T(\tau', \sigma_T)) \geq \frac{5\pi}{n_{\tau'}}$. Proposition 3.6 now implies that triangle $\{\alpha, \gamma_T, \phi_T(\tau', \sigma_T)\}$ is spherical, and for $n_{\tau'} \geq 7$, we deduce $\angle(\alpha, -\gamma_T) = \frac{\pi}{2}$. This contradicts $\angle(\alpha, -\gamma_T) = \angle(\alpha, -\gamma) = \frac{\pi}{3}$ (see (3), (4) and (7)).

This shows $]\nu, \rho_T]_\Gamma \cap \partial^2 \bar{\phi}'_\sigma = \emptyset$.

We claim that the roots $-\bar{\phi}'_\sigma, \beta, -\gamma_T$ and $-\gamma$ are in the key configuration and that $\bar{T} := \{-\bar{\phi}'_\sigma, \beta, -\gamma\}$ and $\bar{U} := \{-\bar{\phi}'_\sigma, \beta, -\gamma_T\}$ are hyperbolic. Since $\mathrm{perim}(\bar{T}) = \mathrm{perim}(r_\gamma(U)) = \mathrm{perim}(U) < \mathrm{perim}(T)$ (see Lemma 4.2), this claim is in contradiction with the minimality of $\mathrm{perim}(T)$ and yields the desired conclusion.

It remains to prove the claim. We have $\bar{T} = r_\gamma(U)$ which implies that $\bar{T}$ is a combinatorial triangle of hyperbolic type. Thus the claim will be proven once we show that $\bar{U}$ is a combinatorial triangle of the same type as $\bar{T}$.

Let $\tau \in]\sigma_T, \rho_T]_\Gamma \cap \partial^2 \bar{\phi}'_\sigma$. By Lemma 4.3, we have $\tau \neq \rho_T$. Thus $\bar{U}$ satisfies (CT2). It is clear by the definition of $\bar{U}$ that (CT1) is satisfied. Moreover, we have $\emptyset \neq \tau \cap -\gamma_T \cap -\bar{\phi}'_\sigma \subset \beta$ (see Lemma 3.5(2)) from which (CT3) follows. Thus $\bar{U}$ is a combinatorial triangle.

It remains to show $\angle(-\gamma_T, \bar{\phi}'_\sigma) = \angle(-\gamma, \bar{\phi}'_\sigma)$ or equivalently $\angle(\gamma_T, -\bar{\phi}'_\sigma) = \angle(\alpha, -\gamma)$. By (4), this is true for $n_\sigma = 4$ or 6 and we may assume $n_\sigma = 3$, $\angle(\alpha, -\beta) \leq \frac{\pi}{7}$, $\angle(\alpha, -\gamma) = \frac{\pi}{3}$ and $\angle(\gamma_T, -\bar{\phi}'_\sigma) \leq \frac{\pi}{3}$.

Suppose by contradiction $\angle(\gamma_T, -\bar{\phi}'_\sigma) < \frac{\pi}{3}$, whence $n_\tau \geq 4$. By (3), we have $\angle(\alpha, -\gamma_T) = \frac{\pi}{3}$. Applying Proposition 3.6 to $\bar{T}'_\sigma$, we obtain

$$\text{either } |\Phi_T(\tau, \sigma_T)| = 1 \quad \text{or} \quad |\Phi_T(\tau, \sigma_T)| = 2 \text{ and } n_\tau \geq 7. \tag{8}$$

In all cases, the $\Phi_T(\tau, \rho_T)$ is nonempty. Let $\phi_\tau := \phi_T(\tau, \rho_T)$. By Lemma 3.5(6), the set $T_\tau := \{\beta, \phi_\tau, -\bar{\phi}'_\sigma\}$ is a combinatorial triangle. Moreover, T_τ is non-spherical because it contains $\bar{U}$ as a subtriangle, and $\bar{U}$ itself is non-spherical because $\angle(\beta, \bar{\phi}'_\sigma) = \angle(\alpha, -\beta) \leq \frac{\pi}{7}$ and $\angle(\gamma_T, \bar{\phi}'_\sigma) < \frac{\pi}{3}$. Applying Proposition 3.6 to T_τ yields now

$$\text{either } |\Phi_T(\tau, \rho_T)| = 1 \quad \text{or} \quad |\Phi_T(\tau, \rho_T)| = 2 \text{ and } n_\tau \geq 7. \tag{9}$$

Combining (8) and (9) with the equality $|\Phi_T(\tau, \sigma_T)| + |\Phi_T(\tau, \rho_T)| + 1 = n_\tau$, we finally obtain a contradiction, which finishes the proof. $\square$

Lemma 4.5. *We have*
$$\angle(\beta, \gamma_T) = \frac{\pi}{2} \Rightarrow \angle(\beta, \gamma) = \frac{\pi}{2}.$$

Proof. Suppose by contradiction that $\angle(\beta, \gamma_T) = \frac{\pi}{2}$ and $\angle(\beta, \gamma) \neq \frac{\pi}{2}$. Since T and U are geometric and of the same type, they have the same angles and we deduce

$$\angle(\alpha, \gamma) = \frac{\pi}{2} \quad \text{and} \quad \angle(\beta, -\gamma) = \frac{\pi}{n}, n \in \mathbb{N}, n \geq 3 \tag{10}$$

(see Lemma 2.7).

As γ and γ_T are parallel, we have $\partial^2\gamma \cap]\nu, \sigma_T[_\Gamma \neq \emptyset$. Moreover, as $\angle(\beta, \gamma_T) = \frac{\pi}{2}$, it follows that $\beta(\gamma) \subset \gamma_T$ and thus $\partial^2 r_\beta(\gamma) \cap]\nu, \sigma_T[_\Gamma \neq \emptyset$. This shows

$$\{\gamma, r_\beta(\gamma)\} \subseteq \Phi_T(\rho, \nu) \tag{11}$$

Let $\phi_\rho := \phi_T(\rho, \nu)$. By (11), we have $\angle(\beta, -\phi_\rho) \geq \angle(\beta, -r_\beta(\gamma))$. This yields

$$\angle(\beta, -\gamma) + \angle(\beta, -\phi_\rho) \geq \nu \tag{12}$$

because $\angle(\beta, -\gamma) + \angle(\beta, -r_\beta(\gamma)) = \nu$.

Let $T_\rho := \{\alpha, \beta, \phi_\rho\}$. This is a combinatorial triangle (see Lemma 3.5(6)) which is hyperbolic because it contains U as a subtriangle (see Lemma 3.2). Applying Proposition 3.6 to T_ρ yields

$$\begin{array}{l} \text{either } \angle(\beta, -\phi_\rho) = 2\angle(\beta, -\gamma) \\ \text{or } \angle(\beta, -\phi_\rho) = \frac{3\pi}{n_\rho}, \angle(\beta, -\gamma) \leq \frac{2\pi}{n_\rho} \text{ and } n_\rho \geq 7. \end{array} \tag{13}$$

We deduce from (10), (12) and (13) that $\angle(\beta, -\gamma) = \frac{\pi}{3}$ and $\phi_\rho = r_\beta(\gamma) = r_\gamma(\beta)$. Thus, we have

$$\angle(\alpha, -\gamma_T) = \frac{\pi}{3} \quad \text{and} \quad \angle(\alpha, -\beta) = (\alpha, -\phi_\rho) \leq \frac{\pi}{7} \tag{14}$$

because T and U are geometric hyperbolic and have the same angles.

Let $\tau \in]\nu, \sigma_T]_\Gamma \cap \partial^2\phi_\rho$. By Lemma 4.3, we have $\tau \neq \sigma_T$. Moreover (14) implies $n_\tau \geq 7$.

Let $\bar{T} := \{\alpha, r_\beta(\alpha), \gamma_T\}$ and $\bar{V} := \{\nu, r_\beta(\sigma_T), \sigma_T\}$. By Lemma 3.4, $\bar{T}$ is a combinatorial triangle and $\bar{V}$ is a set of vertices of $\bar{T}$.

Since $\phi_\rho = r_\beta(\gamma)$ and γ_T are parallel, we have $\phi_\rho \in \Phi_{\bar{T}}(\tau, \nu)$. Let $\phi_\tau := \phi_{\bar{T}}(\tau, \nu)$ and $T_\tau := \{\alpha, r_\beta(\alpha), \phi_\tau\}$. Applying Proposition 3.6 to the hyperbolic triangle T_τ, we obtain

$$\angle(\alpha, -\phi_\tau) \leq \frac{2\pi}{n_\tau}. \tag{15}$$

As $n_\tau \geq 7$ we deduce $\Phi_{\bar{T}}(\tau, \sigma_T) \neq \emptyset$. Let $\phi'_\tau := \phi_{\bar{T}}(\tau, \sigma_T)$ and $T'_\tau := \{\alpha, \gamma_T, \phi'_\tau\}$. Since $\phi_\rho = r_\beta(\gamma)$ and γ_T are parallel, the combinatorial triangle T'_τ is non-spherical (see Lemma 3.3) and we deduce from Proposition 3.6 that

$$\angle(\alpha, -\phi'_\tau) \leq \frac{3\pi}{n_\tau} \tag{16}$$

because $\angle(\alpha, -\gamma_T) = \frac{\pi}{3}$ (see (14)). Combining (15) and (16) with the inequality $\angle(\alpha, -\phi_\tau) + \angle(\alpha, -\phi'_\tau) \geq \nu - \frac{\pi}{n_\tau}$, we obtain a contradiction with $n_\tau \geq 7$. □

Lemma 4.6. *At least one of the angles $\angle(\alpha, \beta)$, $\angle(\alpha, \gamma)$, $\angle(\beta, \gamma)$ equals $\frac{\pi}{2}$.*

Proof. Suppose by contradiction that U is not a right triangle. We assume that the roots $\alpha, \beta, \gamma, \gamma_T$ are chosen among all roots in the key configuration which contradict the lemma in such a way that T is of minimal perimeter. Moreover, we may and shall assume without loss of generality that

$$\angle(\alpha, -\beta) = \frac{\pi}{n_\pi}, \quad \angle(\alpha, -\gamma) = \frac{\pi}{n_\sigma} \quad \text{and} \quad \angle(\beta, -\gamma) = \frac{\pi}{n_\rho}, \tag{17}$$

which implies

$$\text{if } \widetilde{U} \text{ is a combinatorial triangle which contains } U \text{ as a subtriangle, then } \widetilde{U} = U. \tag{18}$$

The latter is a consequence of Prop 3.6. It can also be obtained in a direct way by easy computations in the Coxeter system $(W(U), R(U))$.

¿From (18), we deduce $\phi_T(\sigma, \nu) = \gamma = \phi_T(\rho, \nu)$.

Since U is not right, the sets $\Phi_T(\sigma, \sigma_T)$ and $\Phi_T(\rho, \rho_T)$ are both nonempty. Let $\phi_\sigma := \phi_T(\sigma, \sigma_T)$, $T_\sigma := \{\alpha, \gamma_T, \phi_\sigma\}$,$\phi_\rho := \phi_T(\rho, \rho_T)$ and $T_\rho := \{\beta, \gamma_T, \phi_\rho\}$. By Lemma 3.3, T_σ and T_ρ are non-spherical. By Proposition 3.6, we obtain, in view of the preceding paragraph, that n_σ and n_ρ both belong to $\{3, 4, 6\}$ and moreover

$$\begin{array}{lll} n_\sigma = 6 & \Rightarrow & \angle(\alpha, -\gamma_T) = \angle(\phi_\sigma, -\gamma_T) = \frac{\pi}{6} \text{ and } \angle(\alpha, -\phi_\sigma) = \frac{2\pi}{3} \\ n_\sigma = 4 & \Rightarrow & \angle(\alpha, -\gamma_T) = \angle(\phi_\sigma, -\gamma_T) = \frac{\pi}{4} \text{ and } \angle(\alpha, -\phi_\sigma) = \frac{\pi}{2} \end{array} \tag{19}$$

and similarly for ρ. Since U and T have the same angles and since T_σ and T_ρ are non-spherical, we deduce, using Lemma 3.2(v),

$$n_\sigma = 3 \;\Rightarrow\; \angle(\alpha, -\gamma_T) = \angle(\alpha, -\phi_\sigma) = \tfrac{\pi}{3} \text{ and } \angle(\phi_\sigma, -\gamma_T) \leq \tfrac{\pi}{3} \tag{20}$$

and similarly for ρ.

There are two cases.

Case 1: $r_\gamma(\nu) \subset \gamma_T$.
Let $U'' := \{\gamma_T, -r_\gamma(\alpha), -r_\gamma(\beta)\}$. The set U'' is a combinatorial triangle: it clearly satisfies (CT1), while (CT2) and (CT3) are easy to deduce from $r_\gamma(\nu) \subset \gamma_T$. By (19) and (20), the angles $\angle(\gamma_T, -\phi_\sigma)$ and $\angle(\gamma_T, -\phi_\rho)$ are both $\leq \frac{\nu}{3}$. Moreover, we have $\angle(\phi_\sigma, -\phi_\rho) = \angle(r_\gamma(\alpha), -r_\gamma(\beta)) = \angle(\alpha, -\beta) \leq \frac{\pi}{3}$ because U is not right. In view of Lemma 3.2(v), it follows that U'' is non-spherical. Furthermore, the assumption (17) implies that each root $\psi \notin \{\pm\phi_\sigma, \pm\phi_\rho\}$, we have

$$r_\gamma(\nu) \in \partial^2\psi \Rightarrow \partial\psi \subset \gamma_T. \tag{21}$$

Notice also that, in view of (18),

$$\text{the roots } \alpha \text{ and } \phi_\rho \text{ are parallel.} \tag{22}$$

Assume $n_\sigma = 4$. By (19) this implies $\angle(\alpha, \phi_\sigma) = \frac{\pi}{2}$ and it follows from (22) that α and $r_{\phi_\sigma}(\phi_\rho)$ are parallel. By Lemma 3.5(5) applied to T_σ and $r_\gamma(\nu)$, this implies that γ_T and $r_{\phi_\sigma}(\phi_\rho)$ are incident, which contradicts (21). Thus $n_\sigma \neq 4$ and by symmetry $n_\rho \neq 4$.

Let $n := n_\nu = n_{r_\gamma}(\nu)$. Since $n \geq 3$, the set $\Phi_{T_\sigma}(r_\gamma(\nu))$ is nonempty. Let $\phi := \phi_{T_\sigma}(r_\gamma(\nu))$. It follows from (21) that $\angle(\phi, -\phi_\sigma) = \frac{n-2}{n}\nu$.

Assume $n_\sigma = 6$. By (19) this implies $\angle(\alpha, \phi_\sigma) = \frac{2\pi}{3}$. By Lemma 3.3 and (22), the combinatorial triangle $\{\alpha, \phi_\sigma, \phi\}$ is non-spherical and Lemma 3.2(5) yields $(\alpha, -\phi_\sigma) + (\phi, -\phi_\sigma) < \nu$. This contradicts $n \geq 3$. Thus $n_\sigma \neq 6$ and by symmetry $n_\rho \neq 6$.

Hence $n_\sigma = 3 = n_\rho$. Then $(\alpha, -\phi_\sigma) = \frac{\pi}{3}$. Since U is hyperbolic, we deduce $n \geq 4$. Moreover, a computation as in the case $n_\sigma = 6$ yields here $n < 6$. Thus $n = 4$ or $n = 5$. In both cases, an application of Proposition 3.6 to the non-spherical triangle $\{\alpha, \phi_\sigma, \phi\}$ yields a contradiction.

Case 2: $r_\gamma(\nu) \subset -\gamma_T$.
Let $\bar{T} := \{-\gamma, -\phi_\sigma, -\phi_\rho\}$ and $\bar{U} := \{-\gamma_T, -\phi_\sigma, -\phi_\rho\}$. Since $\bar{T} = r_\gamma(U)$, it follows that $\bar{T}$ is a combinatorial triangle and $\{\sigma, \rho, r_\gamma\}$ is a set of vertices of $\bar{T}$. Similarly, the set $\bar{U}$ is a combinatorial triangle: (CT1) is clearly satisfied, while (CT2) and (CT3) are easy to deduce from $r_\gamma(\nu) \subset -\gamma_T$. Moreover, we have $\angle(\phi_\sigma, -\phi_\rho) = \frac{\pi}{n}$ while $\angle(\gamma_T, -\phi_\sigma)$ and $\angle(\gamma_T, -\phi_\sigma)$ are both $\leq \frac{\pi}{3}$ by (19) and (20). It follows from Lemma 3.2(v) that U is non-spherical.

We claim that if $n_\sigma = 3$ then $\angle(\gamma_T, -\phi_\sigma) = \frac{\pi}{3}$.

Assume $n_\sigma = 3$. Let τ be the unique element of $\partial^2\phi_\sigma \cap]\sigma_T, \rho_T[_\Gamma$. Since $\angle(\gamma_T, -\phi_\sigma) \leq \frac{\pi}{3}$, we have $n_\tau \geq 3$.

Applying Proposition 3.6 the non-spherical triangle T_σ, we get

$$\text{either } \angle(\gamma_T, -\phi_\sigma) = \frac{\pi}{n_\tau} \quad \text{or} \quad \angle(\gamma_T, -\phi_\sigma) = \frac{2\pi}{n_\tau} \text{ and } n_\tau \geq 6. \tag{23}$$

The set $\Phi_{\bar{T}}(\tau, r_\gamma(\nu))$ is nonempty as is contains $-\gamma_T$. Let $\phi_\tau := \phi_{\bar{T}}(\tau, r_\gamma(\nu))$. Applying Proposition 3.6 to the triangle $\{-\phi_\sigma, -\phi_\rho, \phi_\tau\}$ which is non-spherical as it contains $\bar{U}$ as a subtriangle, we obtain (using also (23) and the inequality $\angle(\gamma_T, -\phi_\rho) \leq \frac{\pi}{3}$)

$$\begin{aligned}&\text{either } \angle(\phi_\tau, \phi_\sigma) = \tfrac{\pi}{n_\tau} \quad \text{or} \quad \angle(\phi_\tau, \phi_\sigma) \in \{\tfrac{2\pi}{n_\tau}, \tfrac{3\pi}{n_\tau}\}, \angle(\gamma_T, -\phi_\sigma) = \tfrac{2\pi}{n_\tau} \text{ and } n_\tau \geq 6;\\ &\text{moreover, if } \angle(\phi_\tau, \phi_\sigma) = \tfrac{3\pi}{n_\tau} \text{ then } \angle(\phi_\sigma, -\phi_\rho) = \angle(\gamma_T, -\phi_\rho) = \tfrac{\pi}{3}.\end{aligned} \tag{24}$$

In all cases, the set $\Phi_{\bar{T}}(\tau,\sigma)$ is nonempty because $n_\tau \geq 3$. Let $\phi'_\tau := \phi_{\bar{T}}(\tau,\sigma)$. Applying Proposition 3.6 to the triangle $\{-\gamma,-\phi_\sigma,\phi'_\tau\}$ which is non-spherical by Lemma 3.3, we obtain (using also the equality $\angle(\gamma,-\phi_\sigma)=\frac{\pi}{3}$)

$$\text{either } \angle(\phi'_\tau,\phi_\sigma)=\frac{\pi}{n_\tau} \quad \text{or} \quad \angle(\phi'_\tau,\phi_\sigma)\in\{\frac{2\pi}{n_\tau},\frac{3\pi}{n_\tau}\} \text{ and } n_\tau \geq 6. \tag{25}$$

Combining (24) and (25) with the inequality $\angle(\phi_\tau,\phi_\sigma)+\angle(\phi'_\tau,\phi_\sigma)\geq \nu-\frac{\pi}{n_\tau}$, we finally obtain

$$\text{either } \angle(\gamma_T,-\phi_\sigma)=\frac{\pi}{3} \quad \text{or} \quad n_\tau=7, \angle(\phi_\sigma,-\phi_\rho)=\angle(\gamma_T,-\phi_\rho)=\frac{\pi}{3}.$$

In the second case, we deduce $\angle(\alpha,-\beta)=\angle(\alpha,-\gamma)=\frac{\pi}{3}$ and (19) implies $\angle(\beta,-\gamma)=\frac{\pi}{3}$. This contradicts the fact that U is hyperbolic.

This proves the claim. By symmetry, if $n_\rho=3$ then $\angle(\gamma_T,-\phi_\rho)=\frac{\pi}{3}$.

These facts, together with (19), imply that the triangles U, $\bar{T}$ and $\bar{U}$ have the same angles. Thus the roots $-\phi_\sigma$, $-\phi_\rho$, $-\gamma_T$ and $-\gamma$ are in the key configuration. Moreover, we have $\mathrm{perim}(\bar{T})=\mathrm{perim}(r_\gamma(U))=\mathrm{perim}(U)<\mathrm{perim}(T)$ by Lemma 4.2, which contradicts the minimality of $\mathrm{perim}(T)$. □

Lemma 4.7. *We have*

$$\angle(\alpha,\beta)\neq\frac{\pi}{2}.$$

Proof. Suppose by contradiction that $\angle(\alpha,\beta)=\frac{\pi}{2}$.

Let $\phi_\sigma := \phi_T(\sigma,\nu)$ and $T_\sigma := \{\alpha,\beta,\phi_\sigma\}$. Since U is a subtriangle of T_σ, the latter is hyperbolic (Lemma 3.2) and Proposition 3.6 yields

$$\text{either } \angle(\alpha,-\phi_\sigma)=\frac{\pi}{n_\sigma} \quad \text{or} \quad \angle(\alpha,-\phi_\sigma)=\frac{2\pi}{n_\sigma}, \angle(\beta,-\phi_\sigma)=\frac{\pi}{n_\sigma} \text{ and } n_\sigma\geq 7. \tag{26}$$

In both cases, the $\Phi_T(\sigma,\sigma_T)$ is nonempty. Let $\phi'_\sigma := \phi_T(\sigma,\sigma_T)$. Thus $T'_\sigma := \{\alpha,\gamma_T,\phi'_\sigma\}$ is a combinatorial triangle which is non-spherical by Lemma 3.3. Applying Proposition 3.6 to T'_σ yields

$$\angle(\alpha,-\phi'_\sigma)\leq\frac{4\pi}{n_\sigma} \quad \text{and} \quad \angle(\alpha,-\phi'_\sigma)\geq\frac{3\pi}{n_\sigma}\Rightarrow n_\sigma\geq 6. \tag{27}$$

Combining (26) and (27) with the equality $\angle(\alpha,-\phi_\sigma)+\angle(\alpha,-\phi'_\sigma)=\nu-\frac{\pi}{n_\sigma}$, we obtain $n_\sigma\leq 7$ and $n_\sigma\neq 5$.

By symmetry between α and β, we deduce $n_\rho\neq 5$. Moreover, we may and shall assume without loss of generality that $\angle(\alpha,-\gamma)\leq\angle(\beta,-\gamma)$. Since U is hyperbolic this implies $\angle(\alpha,-\gamma)\leq\frac{\pi}{5}$.

The conclusions of the preceding two paragraphs imply

$$\text{either } \angle(\alpha,-\gamma)=\frac{\pi}{6} \quad \text{or} \quad \angle(\alpha,-\gamma)=\frac{\pi}{7}.$$

Suppose $\angle(\alpha,-\gamma)=\frac{\pi}{7}$. By (26) and (27), we deduce $\angle(\beta,-\phi_\sigma)=\frac{\pi}{7}$. Let $\tau\in\,]\nu,\rho_T]_\Gamma\cap\partial^2\phi_\sigma$. By Lemma 4.3, we have $\tau\neq\rho_T$. Moreover, $\phi_\sigma\in\Phi_T(\tau,\nu)$ and Proposition 3.6 applied to $\{\alpha,\beta,\phi_T(\tau,\nu)\}$ yields $\Phi_T(\tau,\nu)=\{\phi_\sigma\}$. Since $\angle(\beta,-\phi_\sigma)=\frac{\pi}{7}$, we have $n_\tau\geq 7$ and we deduce that $\angle(\beta,-\phi_T(\tau,\rho_T)=\frac{5\pi}{7}$. By Proposition 3.6, this implies that the combinatorial triangle $\{\beta,\gamma_T,\phi_T(\tau,\rho_T)\}$ is spherical. Therefore, its type is not irreducible and we obtain $\angle(\beta,\gamma_T)=\frac{\pi}{2}$. This contradicts the hypothesis $\angle(\alpha,\beta)=\frac{\pi}{2}$ because U is hyperbolic.

Suppose $\angle(\alpha, -\gamma) = \frac{\pi}{6}$. By (26) and (27), we deduce $\angle(\alpha, -\phi'_\sigma) = \frac{2\pi}{3}$ and Proposition 3.6 applied to T'_σ implies $\angle(\alpha, -\gamma_T) = \frac{\pi}{6}$. Since U is hyperbolic and $\angle(\alpha, -\gamma) \leq \angle(\beta, -\gamma)$, we have $\angle(\beta, -\gamma) \in \{\frac{\pi}{4}, \frac{\pi}{6}\}$ (note that the case $\angle(\beta, -\gamma) = \frac{\pi}{5}$ is impossible for we have seen above $n_\rho \neq 5$). In both cases, the set $\Phi_T(\rho, \rho_T)$ is nonempty and $T_\rho := \{\beta, \gamma_T, \phi_T(\rho, \rho_T)\}$ is a well defined combinatorial triangle which is non-spherical (see Lemma 3.3). Applying Proposition 3.6 to T_ρ, we obtain that the angles of T_ρ are either $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2})$ or $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$.

Let $\bar{T} := \{\alpha, \gamma_T, r_\beta(\gamma_T)\}$. By Lemma 3.4, $\bar{T}$ is a combinatorial triangle. Furthermore, since γ and γ_T are parallel, it follows from Lemma 3.5(6) that $\bar{U} := \{\alpha, \gamma, r_\beta(\gamma_T)\}$ is also a combinatorial triangle. An easy computation in the affine triangle group $W(T_\rho)$ shows moreover that $\angle(\gamma, r_\beta(\gamma_T)) = \angle(\gamma_T, r_\beta(\gamma_T))$. It follows that the roots α, $r_\beta(\gamma_T)$, γ and γ_T are in the key configuration, and that the triangle $\bar{T}$ and $\bar{U}$ are hyperbolic. For $\angle(\beta, -\gamma) = \frac{\pi}{4}$, this contradicts Lemma 4.4. For $\angle(\beta, -\gamma) = \frac{\pi}{6}$, this contradicts Lemma 4.6. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. We have assume $\mathcal{M}$ is compact hyperbolic and we want to obtain a contradiction. Since the triangles T and U are geometric and of the hyperbolic type, they have the same angles. By Lemma 4.6, one of these angles equals $\frac{\pi}{2}$. By symmetry between α and β, Lemma 4.4 and Lemma 4.5 imply that none of the angles $\angle(\alpha, -\gamma)$ and $\angle(\beta, -\gamma)$ equals $\frac{\pi}{2}$. Therefore $\angle(\alpha, \beta) = \frac{\pi}{2}$. This contradicts Lemma 4.7. □

5 Finitely many conjugacy classes of hyperbolic triangles

Theorem 5.1. *There are at most finitely many conjugacy classes of hyperbolic combinatorial triangles.*

Proof. By Lemma 3.1 and Proposition 3.6, it suffices to prove the statement for geometric triangles.

Suppose by contradiction that there are infinitely many conjugacy classes of geometric hyperbolic triangles. It follows from Lemma 2.1 that the angle between two incident roots can take only a finite number of distinct values. By the pigeonhole principle, this implies that there are infinitely many conjugacy classes of hyperbolic triangles which are all of the same type, say $\mathcal{M}$. By Lemma 2.1 and the pigeonhole principle again, we deduce there exists an infinite family $(T_i)_{i \in I}$ (where I is some infinite parameter set) of combinatorial triangle which are all of type $\mathcal{M}$ and which contain all a common geometric pair of roots $\{\alpha, \beta\}$. For each $i \in I$, let $\gamma_i \in T_i \backslash \{\alpha, \beta\}$. Thus $(\gamma_i)_{i \in I}$ is an infinite family of roots, and by Lemma 2.4, this family contains a pair of pairwise parallel roots, say γ and γ_T. We may assume without loss of generality $\gamma \subset \gamma_T$. We conclude that the roots α, β, γ and γ_T are in the key configuration, which contradicts Theorem 4.1. □

Theorem 1 of the introduction is a consequence of Theorem 5.1 because, by Lemma 2.1, there are at most finitely many conjugacy classes of spherical triangles. As for Corollary 2 of the introduction, it is a consequence of the following.

Corollary 5.2. *There exists a constant $L = L(W, S)$ such that the following holds. Let $\alpha, \beta, \gamma_0, \ldots, \gamma_n$ be roots, let $\nu \in \partial^2\alpha \cap \partial^2\beta$. Suppose*

- *$T_n := \{\alpha, \beta, \gamma_n\}$ is a combinatorial triangle,*
- *$\nu \subset \gamma_n$,*
- *$\nu \in \partial^2\gamma_0$,*
- *$\gamma_{i-1} \subset \gamma_i$ for all $i = 1, \ldots, n$.*

If $n \geq L$ then T_n is affine and for each $i = 1, \ldots, n$ we have $\angle(\alpha, \gamma_i) = (\alpha, \gamma_0)$ and $\angle(\beta, \gamma_i) = \angle(\beta, \gamma_0)$.

Proof. Let $L := \frac{3}{2} + \frac{1}{2}.\max\{\mathrm{perim}(T) | T \text{ is a non-affine combinatorial triangle}\}$. By Theorem 5.1, L is a well defined integer. The hypotheses imply that for each set of vertices V of T_n, we have $\sum_{\sigma \neq \rho \in V} d(\sigma, \rho) \geq 2.(n-1)$. It follows from the definition of L that T_n is an affine triangle. By Theorem 3.7, there exists an irreducible residue of affine type which is stabilized by $W(T_n)$. The other assertions follow. □

6 Parallel walls

Many pairwise parallel walls

Lemma 6.1. *For each integer $k \geq 2$, there exists a constant $C(k) = C(k; W, S)$ such that any collection of at least $C(k)$ walls contains a subcollection of k walls which are pairwise parallel.*

Proof. This is a straightforward consequence of Lemma 2.4 together with Ramsey's theorem. □

A technical lemma

The main tool for the proofs of Theorem 3 and Theorem 4 is provided by the following result.

Lemma 6.2. *Let α be a root, let $x \in \mathcal{C}(\partial\alpha)$ and $y \in \alpha$ be chambers such that $d(x, y) = d(\mathcal{C}(\partial\alpha), y)$. Let $\gamma_0, \ldots, \gamma_n$ be pairwise distinct roots such that $x \in \gamma_0 \subset \gamma_1 \subset \cdots \subset \gamma_n \not\ni y$. Assume each γ_i is incident to α. If $n \geq L$ (where L is as in Corollary 5.2), then we have the following:*

- *(i) there is an infinite dihedral group D_1 which contains r_{γ_i} for each $i = 0, \ldots, n$;*
- *(ii) if γ is a root such that $r_\gamma \in D_1$ and $\gamma \subset \gamma_1$ or $\gamma_1 \subset \gamma$, then $\angle(\alpha, \gamma) = \angle(\alpha, \gamma_1)$;*
- *(iii) there exist $m := \lfloor \frac{n}{2} \rfloor$ reflections $r_1, \ldots, r_m$ which are pairwise parallel and which separate α from y;*
- *(iv) the group $D_2 := \langle r_1, \ldots, r_m \rangle$ is infinite dihedral and contains r_α.*

Proof. Let $i \in \{1, \ldots, n\}$ and let ϕ be a root such that $\gamma_{i-1} \subset \phi \subset \gamma_i$. Lemma 2.5 implies that ϕ is incident to α. Thus we may assume without loss of generality that, for every $i \in \{1, \ldots, n\}$, the only such ϕ's are γ_{i-1} and γ_i.

Let Γ be a minimal gallery joining x to y. Thus Γ crosses $\partial\gamma_i$ for every $i \in \{0, \ldots, n\}$.

Let $\sigma_n \in \partial^2\alpha \cap \partial^2\gamma_n$, let $x' = \mathrm{proj}_{\sigma_n}(x)$ and let Γ' be a minimal gallery from x to x' which is contained in $\mathcal{C}(\partial^2\alpha)$ (see Lemma 2.3). For each $i \in \{0, 1, \ldots, n-1\}$, let $\sigma_i \in \partial^2\alpha \cap \partial^2\gamma_i$ be a residue which is crossed by Γ'.

In view of Lemma 2.6, the hypotheses imply $\angle(\alpha, \gamma_0) > \frac{\pi}{2}$. It follows that $n_{\sigma_0} \geq 3$. Let β be a root such that $\sigma_0 \in \partial^2\beta$ and $\angle(\alpha, -\beta) = \frac{\pi}{n_\sigma}$. Using Lemma 2.6 again, we see that β does not separate x from y. Thus Γ does not cross $\partial\beta$. On the other hand, the gallery Γ' does cross $\partial\beta$ as well as γ_i for every $i \in \{0, \ldots, n\}$. Therefore, for every

$i \in \{0, \ldots, n\}$, there exist two panels of $\partial\gamma_i$, one contained in β and the other in $-\beta$. We deduce from Lemma 2.5 that β is incident to γ_i for every $i \in \{0, \ldots, n\}$.

It follows that $T_i := \{\alpha, \beta, \gamma_i\}$ is a combinatorial triangle for every $i \in \{1, \ldots, n\}$. Since $n \geq L$, Corollary 5.2 implies that all T_i's are of the same affine type. Let R be an irreducible affine residue stabilized by $W(T_n)$. Then R is stabilized by each r_{γ_0} and r_{γ_n}. Thus there exists panels $P_1 \in \partial\gamma_0$ and $P_2 \in \partial\gamma_n$ which are completely contained in R. Since R is convex and since any gallery joining a chamber of $\mathcal{C}(\partial\gamma_1)$ to a chamber $\mathcal{C}(\partial\gamma_n)$ crosses each wall $\partial\gamma_i$ for $i = 1, \ldots, n-1$, we conclude that each r_{γ_i} stabilizes R. Moreover, since no root separates γ_{i-1} from γ_i, we deduce $W(T_1)$ contains r_{γ_i} for each $i \in \{0, \ldots, n\}$. Assertions (i) and (ii) follow.

Since T_n is affine, we have $n_{\sigma_0} \in \{3, 4, 6\}$ and $\angle(\alpha, -\gamma_0) \in \{\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}\}$.

Case 1: $\angle(\alpha, -\gamma_0) = \frac{\pi}{3}$.
Then $\angle(\alpha, -\beta) = \frac{\pi}{3}$ or $\frac{\pi}{6}$ and, up to replacing β by $r_\beta(\alpha)$, we may assume without loss of generality that $\angle(\alpha, -\beta) = \frac{\pi}{3}$. For every $i \in \{1, \ldots, n\}$, let $\beta_i := r_{\gamma_i}(\beta)$. We have $-\alpha \subset \beta_1 \subset \cdots \subset \beta_n$, and the reflections r_{β_i} generate an infinite dihedral group which contains r_α. Thus (iii) and (iv) will be proven if we show that each $\partial\beta_i$ separates x from y.

Let $z \in \mathcal{C}(\partial\gamma_i) \cap -\gamma_i$ be a chamber which is crossed by the gallery Γ. Let $z' = \mathrm{proj}_{\sigma_n}(z)$. We have $\{x, z'\} \subset \mathcal{C}(\partial^2\alpha) \subset \beta_i$. Moreover, we have already seen that $\partial\beta$ separates each chamber crossed by Γ from σ_i. In view of Lemma 2.3, this implies that $\partial\beta_i$ separates z from z', whence $z \in -\beta_i$. Therefore, the gallery Γ crosses $\partial\beta_i$, as was to be shown.

Case 2: $\angle(\alpha, -\gamma_0) = \frac{\pi}{4}$.
Then $\angle(\alpha, -\beta) = \frac{\pi}{4}$. By Lemma 2.6, the wall $\partial r_\beta(\alpha)$ does separate x from y. We set $\beta_i := r_{\gamma_i} r_\beta(\alpha)$ for every $i \in \{1, \ldots, n\}$. We have $-\alpha \subset \beta_1 \subset \cdots \subset \beta_n$, and the reflections r_{β_i} generate an infinite dihedral group which contains r_α. Assertions (iii) and (iv) follow by an argument as in Case 1.

Case 3: $\angle(\alpha, -\gamma_0) = \frac{\pi}{6}$.
Then $\angle(\alpha, -\beta) = \frac{\pi}{6}$ and $\angle(\beta, -\gamma_i) = \frac{2\pi}{3}$ for $i \in \{0, \ldots, n\}$. For every $i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, let $\beta_i := r_{\gamma_i} r_\alpha r \gamma_i r_\beta(\alpha) = -r_{\gamma_{2i}} r_{\gamma_i}(\alpha)$. Here again, we have $-\alpha \subset \beta_1 \subset \cdots \subset \beta_n$, and the reflections r_{β_i} generate an infinite dihedral group which contains r_α. Assertions (iii) and (iv) follow by an argument as in Case 1. $\square$

Proof of Theorem 3

We may assume without loos of generality $n \geq L$ where L is as in Corollary 5.2. We define $B(n) := C(3n-1)$ where C is as in Lemma 6.1.

Let α be a root and let $y \in \alpha$ be a chamber such that $d(y, \mathcal{C}(\partial\alpha)) \geq H(k)$. Let $x \in \mathcal{C}(\partial\alpha)$ be such that $d(x, y) = d(\mathcal{C}(\partial\alpha), y)$. By Lemma 6.1, there exist $3n-1$ pairwise parallel walls which separate x from y. If n of these walls are parallel to $\partial\alpha$, then we are done. Otherwise, there are at least $2n$ of these walls which are incident to $\partial\alpha$. In that case, Lemma 6.2(iii) yields the desired conclusion. $\square$

Proof of Theorem 4

We define $N := B(2L+1)$ where B is as in Theorem 3 and L as in Corollary 5.2.

Let α and α' be roots such that $-\alpha \subset \alpha'$, $-\alpha' \subset \alpha$ and $n := d(\mathcal{C}(\partial\alpha), \mathcal{C}(\partial\alpha')) \geq N$. Assume by contradiction that no wall separates $\partial\alpha$ from $\partial\alpha'$.

Let $x \in \mathcal{C}(\partial\alpha)$ and $x' \in \mathcal{C}(\partial\alpha')$ be such that $d(x, x') = n$. Since $d(x, \mathcal{C}(\partial\alpha')) = n$, Theorem 3 implies that there exist $2L+1$ pairwise parallel walls which separate x from

$\partial\alpha'$. Since no wall separates $\partial\alpha$ from $\partial\alpha'$, it follows that each of these $2L+1$ walls is incident to α. By Lemma 6.2(iii), this implies that there exist L roots $\beta_1 \subset \cdots \subset \beta_L$ which are pairwise parallel and separate x' from $\partial\alpha$. Since no wall separates $\partial\alpha$ from $\partial\alpha'$, it follows that each β_i is incident to α'. Now Lemma 6.2(i) yields an infinite dihedral group D which contains r_{β_i} for every $i \in \{1, \dots, L\}$. By Lemma 6.2(iv), we have $r_\alpha \in D$. By Lemma 6.2(ii), we obtain $\angle(\alpha', \beta_1) = \angle(\alpha', \alpha)$. This contradicts the fact that α and α' are parallel. □

Another finiteness property related to parallel walls

In order to apply Theorem 4 to obtain information on the Coxeter cubing of Niblo-Reeves, we will need the following result.

Theorem 6.3. *For each $k \in \mathbb{N}$ there exists a constant $U(k) = U(W, S; k)$ such that the following holds:*
Let $\mathcal{H}$ be a collection of half-spaces such that

(i) $\bigcap_{\phi\in\mathcal{H}} \phi \neq \emptyset$;

(ii) for all $\phi, \psi \in \mathcal{H}$, the hyperplanes $\partial\phi$ and $\partial\psi$ are parallel.

If $\mathcal{H}$ is of cardinality at least $U(n)$ then there exist $\phi, \psi \in \mathcal{H}$ such that $d(\partial\phi, \partial\psi) > k$.

Remark. Let W_0 be a universal Coxeter group of rank r which is contained as a reflection subgroup in W. It is well known that r can be arbitrarily large. Qualitatively, the preceding theorem says the following: the higher the rank r, the larger the index $[W : W_0]$.

The proof will use the following lemmas.

Lemma 6.4. *For each $k \in \mathbb{N}$, the group W has finitely many orbits on pairs of hyperplanes which are at distance at most k.*

Proof. Clear since S is finite. □

Lemma 6.5. *Let $\Phi = \Phi(W, S)$ be the standard root system associated with (W, S). For each $k \in \mathbb{N}$ there exists a constant $T(k)$ such that given $\phi, \psi \in \Phi$ with $|(\phi, \psi)| > T(k)$ (where $(\cdot, \cdot)$ denotes the standard inner product), we have $d(\partial\phi, \partial\psi) > k$.*

Proof. Immediate consequence of the previous lemma. □

Proof of Theorem 6.3. We view $\mathcal{H}$ as a subset of Φ. Let $\Pi \subset \mathcal{H}$ be a basis of the vector space V spanned by $\mathcal{H}$. Clearly the restriction of the inner product to V is non-degenerate. Therefore, the set B of all $v \in V$ such that $(v, \phi) \in [-1, -N]$ for all $\phi \in \Pi$ is compact (where $N > 1$ is an arbitrary real number). Since Φ is discrete, the set $B \cap \Phi$ is finite. We deduce that when $\mathcal{H}$ is sufficiently large, there exists $\phi \in \mathcal{H}$ and $\psi \in \Pi \subset \mathcal{H}$ such that $(\phi, \psi) < -N$. By the previous lemma, this implies the desired result when N is large enough. □

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