Some properties of non-positively curved lattices

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Abstract

We announce results on the structure of CAT(0) groups, CAT(0) lattices and of the underlying spaces. Our statements rely notably on a general study of the full isometry groups of proper CAT(0) spaces. Classical statements about Hadamard manifolds are established for singular spaces; new arithmeticity and rigidity statements are obtained.

Résumé

Quelques propriétés des groupes CAT(0). Nous présentons des résultats de structure sur les groupes CAT(0), les réseaux CAT(0) et sur les espaces sous-jacents. Nos énoncés reposent notamment sur une étude générale des groupes d'isométries pleins des espaces CAT(0) propres. Nous démontrons des résultats qui généralisent des énoncés classiques sur les variétés de Hadamard et proposons de nouveaux théorèmes d'arithméticité et rigidité.

Version française abrégée

Nous conviendrons qu'un **groupe CAT(0)** est un couple (Γ, X) où X est un espace CAT(0) propre et Γ un groupe d'isométries de X dont l'action est proprement discontinue et cocompacte. Il s'agit d'étudier les relations entre la géométrie de X et les propriétés algébriques de Γ. Ce cadre permet un traitement unifié de nombreuses situations classiques (groupes fondamentaux de variétés compactes à courbure négative, réseaux uniformes des groupes algébriques semisimples, en particulier groupes S-arithmétiques anisotropes) et moins classiques liées à la théorie géométrique des groupes (réseaux non linéaires associés aux arbres, immeubles exotiques et non euclidiens, nombreux groupes Gromov-hyperboliques). Cette note annonce quelques résultats généraux qui relèvent de ce contexte. Nous considérons parfois le cas plus général des **réseaux CAT(0)** qui, à défaut de meilleure définition, consistent des couples (Γ, X) formés d'un réseau Γ dans un groupe d'isométries cocompact de X.

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Voici un premier exemple des relations entre X et Γ : la dimension du facteur euclidien de X est égale au rang maximal des sous-groupes abéliens libres normaux dans Γ (**Cor. 1.2** ; ces numéros renvoient à la partie anglaise). Dans le cas des variétés riemanniennes, c'est là un résultat d'Eberlein [3] qui peut être vu comme une réciproque (partielle) au « théorème du tore plat » [5], [7], [2, §II.7].

Pour obtenir des énoncés de rigidité, il est utile (et bien souvent nécessaire) de supposer que X soit **géodésiquement complet**, ce qui n'exclut aucun des exemples classiques (immeubles, variétés, ...). Nous montrons alors que X possède une isométrie parabolique si et seulement si X admet une décomposition isométrique $X = M \times X'$ où X' est CAT(0) et M est un espace symétrique de type non compact. (**Cor. 2.3** ; cet énoncé est faux pour des espaces X sans réseau Γ). Plus précisément, soit (Γ, X) un groupe CAT(0) avec Γ irréductible et X géodésiquement complet. Supposons que X admette une isométrie parabolique. Si Γ est résiduellement fini, alors X est isométrique à un produit d'espaces symétriques et d'immeubles de Bruhat–Tits. Sinon, l'intersection Γ_D de tous les sous-groupes d'indice fini de Γ n'est pas de type fini et le quotient Γ/Γ_D est un groupe arithmétique (**Thm. 2.2**).

Notre travail s'appuie sur une analyse des groupes d'isométries des espaces CAT(0) propres indépendante de l'existence de réseaux. On se place dans le cadre où un groupe $G < \mathrm{Is}(X)$ agit **minimalement** et sans point fixe à l'infini (il convient de montrer qu'il est possible de se restreindre à ce cas). Lorsque le bord à l'infini de X, muni de la métrique de Tits, est de dimension finie, on montre que X possède une décomposition canonique en un produit d'un facteur euclidien et d'un nombre fini de facteurs irréductibles non euclidiens ; c'est là une variante de la décomposition de de Rham obtenue dans [4]. En particulier le groupe d'isométries complet de X se décompose virtuellement comme produit des groupes d'isométries de chaque facteur de X. En outre, le groupe d'isométries de tout facteur irréductible non euclidien est soit un groupe de Lie simple presque connexe de centre trivial, soit un groupe totalement discontinu dont le radical moyennable est trivial. Dans tous les cas, ce groupe est irréductible en ce sens qu'aucun sous-groupe fermé d'indice fini ne se scinde en un produit direct non trivial (**Thm. 4.2**). En fait, tout groupe d'isométries d'un espace CAT(0) irréductible non euclidien dont le bord est de dimension finie transmet à chacun de ses sous-groupes normaux non triviaux la propriété d'agir minimalement et sans point fixe au bord sur l'espace en question (**Thm. 4.3**). De cette propriété de « densité géométrique » des sous-groupes normaux, que l'on peut interpréter comme une forme faible de simplicité, découlent des énoncés purement algébriques : un sous-groupe normal, et plus généralement un sous-groupe sous-normal, ne se scinde pas en produit direct non trivial, son radical moyennable et son centralisateur sont tous deux triviaux. Lorsque non seulement le groupe d'isométries $\mathrm{Is}(X)$, mais aussi chacun de ses sous-groupes ouverts, agit sans fixer de point à l'infini, ces différentes conclusions peuvent être considérablement renforcées (**Thm. 4.6**).

Le phénomène de « densité géométrique » qu'on vient de décrire pour les sous-groupes normaux d'un groupe d'isométries d'un espace CAT(0) irrédutible se manifeste également pour les réseaux (**Thm. 3.1**). Le terme de « densité » est ici particulièrement approprié : de ces considérations, on déduit en outre un preuve du résultat classique, dû à Borel, de densité de Zariski dans le cas particulier des réseaux de groupes semisimples.

Mentionnons finalement que ces différents développements permettent d'appliquer le théorème de superrigidité de Margulis pour certains groupes arithmétiques tels que $\mathrm{SL}_n(\mathbf{Z})$ dans un contexte purement CAT(0) (**Thm. 4.5**). Par ailleurs, on montre qu'au sein des espaces CAT(0) propres géodésiquement complets, les espaces symétriques et immeubles euclidiens fortement homogènes sont caractérisés par le fait que le fixateur de tout point à l'infini est cocompact (**Thm. 4.7**).

1. CAT(0) groups and lattices

We define a **CAT(0) group** as a pair (Γ, X) where X is a proper CAT(0) space and Γ a properly discontinuous cocompact group of isometries of X. The study of these objects centers around the following general question:

($\star$) *What is the interplay between the geometry of X and the algebraic properties of Γ?*

The motivation for considering CAT(0) groups is that they provide a common framework for at least four classical topics: closed manifolds of non-positive curvature; uniform lattices in semi-simple groups over local fields, in particular anisotropic S-arithmetic groups; non-linear cousins such as tree lattices, exotic and non-Euclidean buildings; general geometric group theory. Actually, the classical situations alluded to above naturally suggest to relax the cocompactness condition in order to cover *all* lattices in semi-simple groups, as well as larger families of non-linear relatives, including non-uniform lattices arising from Kac–Moody theory. In the general CAT(0) setting, there is (as yet) no consensus on a good definition for discrete groups of finite covolume; we shall content ourselves with the following *ad hoc* definition:

A **CAT(0) lattice** is a pair (Γ, X) where X is a proper CAT(0) space whose isometry group $\mathrm{Is}(X)$ acts cocompactly and Γ is a lattice in $\mathrm{Is}(X)$. We emphasise that a CAT(0) group is in particular a finitely generated (uniform) CAT(0) lattice.

The goal of this note is to announce a few general results on CAT(0) lattices, of relevance to each of the above themes.

A first example towards question ($\star$) regards the maximal Euclidean factor. We recall that the *Flat Torus theorem*, originating in the work of Gromoll–Wolf [5] and Lawson–Yau [7], associates Euclidean subspaces $\mathbf{R}^n \subseteq X$ to subgroups $\mathbf{Z}^n \subseteq \Gamma$ (see [2, §II.7]). The converse is a well-known open problem (Gromov [6, §6.B_3]; for manifolds see Yau, problem 65 in [9]). We propose the following.

Theorem 1.1 *Let (Γ, X) be a finitely generated CAT(0) lattice and let $X \cong \mathbf{R}^n \times X'$ be the Euclidean decomposition. Then there is a finite index subgroup Γ_0 which splits as a direct product $\Gamma_0 \cong \mathbf{Z}^n \times \Gamma'$.*

In the special case of cocompact Riemannian manifolds, the above was the main result of Eberlein's article [3]. A CAT(0) space X is called **minimal** if $\mathrm{Is}(X)$ acts minimally, i.e. without stabilising any non-empty closed convex proper subset.

Corollary 1.2 *If X is minimal* (e.g. *geodesically complete*)*, then the dimension of the Euclidean factor of X equals the maximal $\mathbf{Q}$-rank of an Abelian normal subgroup of Γ.*

In the sequel, an abstract group Γ is called **irreducible** if no finite index subgroup splits as a non-trivial direct product. We generalise to all finitely generated CAT(0) lattices the Margulis irreducibility criterion for lattices in semi-simple groups.

2. Geometric arithmeticity

A **neutral** parabolic isometry is a fixed-point free isometry with zero translation length.

Theorem 2.1 *Let (Γ, X) be an irreducible CAT(0) group. If X admits any neutral parabolic isometry, then either:*

(i) $\mathrm{Is}(X)$ is a rank one simple Lie group with trivial centre; or:

(ii) Γ possesses a normal subgroup Γ_D such that Γ/Γ_D is an arithmetic group. Moreover, Γ_D is either finite or infinitely generated.

Assuming in addition that every geodesic segment of X can be extended to a bi-infinite geodesic line (which need not be unique), we obtain geometric information on X and at the same time drop the neutrality assumption.

Theorem 2.2 *Let (Γ, X) be an irreducible* $\mathrm{CAT}(0)$ *group with X geodesically complete. Assume that X possesses some parabolic isometry.*

If Γ is residually finite, then X is a product of symmetric spaces and Bruhat–Tits buildings. In particular, Γ is an arithmetic lattice unless X is a real or complex hyperbolic space.

If Γ is not residually finite, then X still splits off a symmetric space factor. Moreover, the finite residual Γ_D of Γ is infinitely generated and Γ/Γ_D is an arithmetic group.

Corollary 2.3 *Let (Γ, X) be a* $\mathrm{CAT}(0)$ *group with X geodesically complete. Then X possesses a parabolic isometry if and only if $X \cong M \times X'$, where M is a symmetric space of non-compact type.*

For lattices in products of groups that are *simple*, or have few factors, an arithmeticity/non-linearity alternative was proved in [8]. In our geometric setting, we can establish it without any assumption on the factors, and moreover establish geometric superrigidity.

Theorem 2.4 *Let (Γ, X) be an irreducible* $\mathrm{CAT}(0)$ *group with X geodesically complete. Assume that Γ possesses some faithful finite-dimensional linear representation (in characteristic $\neq 2, 3$).*

If X is reducible, then Γ is an arithmetic lattice and X is a product of symmetric spaces and Bruhat–Tits buildings.

3. A geometric Borel density theorem

A recurring theme of our work is that *minimality* — a much weaker assumption, upon adjustments, than the familiar notions of cocompactness or of full limit sets — is a valuable geometric notion, similar to Zariski density in algebraic groups. The following corresponds to Borel's classical result [1] (and contains it, indeed).

Theorem 3.1 *Let G be a locally compact group with a continuous isometric action on a proper* $\mathrm{CAT}(0)$ *space X without Euclidean factor.*

If G acts minimally and without global fixed point in ∂X, then any closed subgroup with finite invariant covolume in G still has these properties.

One deduces generalisations of some facts known in the case of lattices in semi-simple groups.

Corollary 3.2 *Let X be a proper* $\mathrm{CAT}(0)$ *space without Euclidean factor such that $G = \mathrm{Is}(X)$ acts minimally without fixed point at infinity, and let $\Gamma < G$ be a closed subgroup with finite invariant covolume. Then:*

(i) *Γ has trivial amenable radical.*
(ii) *The centraliser $\mathscr{Z}_G(\Gamma)$ is trivial.*
(iii) *If Γ is finitely generated, then is has finite index in its normaliser $\mathscr{N}_G(\Gamma)$ and the latter is a finitely generated lattice in G.*

As pointed out by P. de la Harpe, (ii) implies in particular that any lattice in G is ICC and hence its von Neumann algebra is a factor.

4. Isometry groups and their normal subgroups

Our results on CAT(0) groups and lattices require some groundwork on the full isometry group of the underlying CAT(0) spaces. A common first step for many of our result is the following fact.

Theorem 4.1 *Let X be a proper* CAT(0) *space with finite-dimensional boundary and no Euclidean factor. Let $G < \mathrm{Is}(X)$ be a closed subgroup acting minimally and without fixed point at infinity. Then the amenable radical of G is trivial.*

Next, we establish a group decomposition, supplemented by a de Rham decomposition of the space which is a variant of [4].

Theorem 4.2 *Let X be a proper minimal* CAT(0) *space with finite-dimensional boundary. If $G = \mathrm{Is}(X)$ has no global fixed point at infinity, then there is a canonical finite index open characteristic subgroup $G^* \lhd \mathrm{Is}(X)$ which admits a canonical decomposition*

$$G^* \cong\ S_1 \times \cdots \times S_p \times \big(\mathbf{R}^n \rtimes \mathbf{O}(n)\big) \times D_1 \times \cdots \times D_q \qquad (p, q, n \geq 0)$$

where S_i are almost connected simple Lie groups with trivial centre and D_j are totally disconnected irreducible groups. Furthermore, there is a canonical equivariant isometric splitting

$$X \cong\ X_1 \times \cdots \times X_p \times \mathbf{R}^n \times Y_1 \times \cdots \times Y_q$$

with componentwise minimal action; all X_i and Y_j are irreducible. Any other product decomposition of G^* or X is a regrouping of the above factors.

The conclusion of Theorem 4.2 can be considerably strengthened by a further description of the factors D_i. Indeed, they satisfy the following geometric form of simplicity.

Theorem 4.3 *Let $X \neq \mathbf{R}$ be an irreducible proper* CAT(0) *space with finite-dimensional Tits boundary and $G < \mathrm{Is}(X)$ any subgroup whose action is minimal and does not have a global fixed point in ∂X.*

Then every non-trivial normal subgroup $N \lhd G$ still acts minimally and without fixed point in ∂X. Moreover, the amenable radical of N and its centraliser $\mathscr{Z}_{\mathrm{Is}(G)}(N)$ are both trivial; N does not split as a product.

These conclusions hold more generally when $N < G$ is any non-trivial subnormal or even ascending subgroup.

Notice that Theorem 4.3 can be gainfully combined with Theorem 3.1 and Corollary 3.2 in order to obtain for instance information on normal subgroups of lattices, or lattices in normal subgroups.

For the study of the totally disconnected factors of isometry groups, the following *smoothness* result is a basic link between the topology and the algebra.

Theorem 4.4 *Let X be a geodesically complete proper* CAT(0) *space and $G < \mathrm{Is}(X)$ a totally disconnected closed subgroup acting minimally. Then the pointwise stabiliser in G of every bounded set is open.*
(The conclusion can indeed fail when X is not geodesically complete.)

At this point, we notice in passing that we have gathered enough information about general cocompact CAT(0) spaces to apply Margulis' superrigidity at least for some lattices.

Theorem 4.5 *Let X be a proper* CAT(0) *space whose isometry group acts cocompactly and without global fixed point at infinity. Let $\Gamma = \mathrm{SL}_n(\mathbf{Z})$ with $n \geq 3$ and $G = \mathrm{SL}_n(\mathbf{R})$.*

For any isometric Γ-action on X there is a non-empty Γ-invariant closed convex subset $Y \subseteq X$ on which the Γ-action extends uniquely to a continuous isometric action of G.

(*The corresponding statement applies to all those lattices in semi-simple Lie groups that have virtually bounded generation by unipotents. It also applies to S-arithmetic lattices such as $\mathrm{SL}_n(\mathbf{Z}[\frac{1}{m}])$, $n \geq 3$.*)

Observe that the above theorem has no assumptions whatsoever on the action; cocompactness is an assumption on the given CAT(0) space. It may happen that Γ fixes points in ∂X, but its action on Y is always minimal and without fixed points at infinity.

Recall that the **quasi-centre** $\mathscr{QZ}(G)$ of a group G is the union of the discrete conjugacy classes; it is always a (topologically) characteristic subgroup. The following result establishes in particular the existence of a minimal non-trivial normal subgroup; to state it, we define the **socle** $\mathrm{soc}(\cdot)$ as the subgroup generated by the (possibly empty) collection of all minimal non-trivial closed normal subgroups.

Theorem 4.6 *Let X be a proper geodesically complete* CAT(0) *space without Euclidean factor and $G < \mathrm{Is}(X)$ a closed subgroup acting cocompactly. Suppose that no open subgroup of G fixes a point at infinity.*

(i) X admits a canonical equivariant splitting

$$X \cong X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q$$

where each X_i is a symmetric space and each Y_j possesses a G-invariant locally finite decomposition into compact convex cells.

(ii) Every compact subgroup of G is contained in a maximal one; the maximal compact subgroups fall into finitely many conjugacy classes.

(iii) $\mathscr{QZ}(G) = \{1\}$; in particular G has no non-trivial discrete normal subgroup.

(iv) $\mathrm{soc}(G^*)$ *is a direct product of $p+q$ non-discrete characteristically simple groups.*

Finally, we present a result of a different vein. Symmetric spaces and Bruhat–Tits buildings have in common the property that the stabilisers of points at infinity are cocompact (being always parabolic; the case of Euclidean type is obvious). This property is further shared by *Bass–Serre* trees, *i.e.* edge-transitive trees (which are in particular regular or bi-regular).

Theorem 4.7 *Let X be a geodesically complete proper* CAT(0) *space. Suppose that the stabiliser of every point at infinity acts cocompactly on X.*

Then X is isometric to a product of symmetric spaces, Euclidean buildings and Bass–Serre trees.

The Euclidean buildings appearing in the preceding statement admit an automorphism group that is strongly transitive, *i.e.* acts transitively on pairs (c, A) where c is a chamber and A an apartement containing c. This property characterises the Bruhat–Tits buildings, except perhaps for some two-dimensional cases where this is a known open question.

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