

# Erratum to ‘Buildings with isolated subspaces and relatively hyperbolic Coxeter groups’

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The goal of this note is to correct two independent errors in [Cap09], respectively in Theorems A and B from loc. cit. I am indebted to Alessandro Sisto, who pointed them out to me. Those corrections affect neither the characterization of toral relatively hyperbolic Coxeter groups (Cor. D and E from [Cap09]), nor the other intermediate results from the original paper.

We keep the notation and terminology from loc. cit. Moreover all Coxeter groups under consideration are assumed to be finitely generated. The first correction concerns Theorem A: its assertions (ii), (iii), (iv) are indeed equivalent, but a third condition (RH3) has to be added to (RH1) and (RH2) in assertion (i), as in the following reformulation.

**Theorem A’.** *Let  $(W, S)$  be a Coxeter system and  $\mathcal{T}$  be a set of subsets of the Coxeter generating set  $S$ . Then  $W$  is hyperbolic relative to  $\mathcal{P} = \{W_J \mid J \in \mathcal{T}\}$  if and only if the following three conditions hold:*

**(RH1)** *For each irreducible affine subset  $J \subset S$  of cardinality  $\geq 3$ , there exists  $K \in \mathcal{T}$  such that  $J \subset K$ . Similarly, for each pair of irreducible non-spherical subsets  $J_1, J_2 \subset S$  with  $[J_1, J_2] = 1$ , there exists  $K \in \mathcal{T}$  such that  $J_1 \cup J_2 \subset K$ .*

**(RH2)** *For all  $K_1, K_2 \in \mathcal{T}$  with  $K_1 \neq K_2$ , the intersection  $K_1 \cap K_2$  is spherical.*

**(RH3)** *For each  $K \in \mathcal{T}$  and each irreducible non-spherical  $J \subset K$ , we have  $J^\perp \subset K$ .*

*Proof.* The necessity of (RH1) and (RH2) is established in [Cap09]. The condition (RH3) is also necessary, as pointed out by Alessandro Sisto: if there is a reflection  $s \in S$  and a set  $K \in \mathcal{T}$  such that  $s \notin K$  and  $s$  commutes with an irreducible non-spherical subset  $J \subset K$ , then the cosets  $W_K$  and  $sW_K$  of the parabolic subgroup  $W_K$  are distinct, but the intersection of their respective 1-neighbourhoods in the Cayley graph of  $(W, S)$  is unbounded, since it contains  $W_J$ . This contradicts that  $W$  is hyperbolic relative to  $\mathcal{P}$ .

Assume conversely that (RH1), (RH2) and (RH3) hold. As in [Cap09], we need to show that the set  $\mathcal{F}$ , consisting of all residues of the Davis complex of  $(W, S)$  whose type belongs to  $\mathcal{T}$ , satisfies the isolation conditions (A) and (B) from loc. cit. The arguments given there show that (RH1) is sufficient to ensure that (A) holds. Moreover it is shown that if  $\mathcal{F}$  does not satisfy (B), then there exists two distinct residues  $F, F' \in \mathcal{F}$  whose respective stabilisers  $P, P'$ , which are parabolic subgroups of  $W$ , share a common infinite dihedral reflection subgroup. The mistake in [Cap09] lies in the sentence: ‘By (RH2), this implies that  $P$  and  $P'$  coincide.’ The corrected argument, which requires also invoking (RH3), goes as follows. We may write  $P = gW_Kg^{-1}$  and  $P' = g'W_{K'}(g')^{-1}$  for some  $K, K' \in \mathcal{T}$  and  $g, g' \in W$ . Since  $P \cap P'$  contains an infinite dihedral reflection subgroup, it also contains the parabolic closure of that subgroup, say  $Q$ , which is of irreducible non-spherical type by [Cap09, Lem. 2.1]. Therefore there is an irreducible non-spherical subset  $J \subset K$  (resp.  $J' \subset K'$ ) such that  $Q$  is conjugate to  $gW_Jg^{-1}$  in  $P$  (resp. to  $g'W_{J'}(g')^{-1}$  in  $P'$ ). It follows that

$W_J$  is conjugate to  $W_{J'}$  and, hence, that  $J$  and  $J'$  are conjugate in  $W$ . By [Deo82, Prop. 5.5], it follows that  $J = J'$ , so that  $K = K'$  by (RH2). In particular  $P$  and  $P'$  are conjugate. Let  $p \in P$  be an element which conjugates  $gW_Jg^{-1}$  to  $Q$ . Upon replacing  $g$  by  $pg$ , we may assume that  $Q = gW_Jg^{-1}$ . Similarly we may assume that  $Q = g'W_J(g')^{-1}$ . It follows that  $g^{-1}g'$  normalises  $W_J$ . By [Deo82, Prop. 5.5], the normaliser of  $W_J$  coincides with  $W_{J \cup J^\perp}$ , and is thus contained in  $W_K$  by (RH3). Hence  $g^{-1}g'$  normalizes  $W_K$ , so that  $P = P'$ . Condition (RH3) together with [BH99, Prop. 2.1] and [Deo82, Prop. 5.5] also implies that  $P$  is self-normalising, which implies that there is a unique residue in the Davis complex, whose full stabiliser is  $P$ . We deduce that  $F = F'$ , a contradiction. This confirms that (B) holds.  $\square$

We next remark that Corollaries D and E from [Cap09] are not affected by the above correction: indeed, in the respective settings of those corollaries, the condition (RH3) holds automatically. In Corollary C, for all three conditions (RH1)–(RH3) to be satisfied, the definition of  $\mathcal{T}$  has to be adapted as follows:

$$\mathcal{T} = \{S \setminus \{s_0\}\} \cup \{J \cup J^\perp \mid J \text{ is irreducible affine of cardinality } \geq 3 \text{ and contains } s_0\}.$$

We now turn to the second error, which lies in Theorem B from [Cap09]. The purpose of that statement was to answer the following question: *assuming that  $W$  is hyperbolic with respect to some peripheral subgroups  $H_1, \dots, H_m$ , can one relate those peripheral subgroups to the parabolic subgroups of  $W$  (in the usual Coxeter group theoretic sense)?* Theorem B asserted that those peripheral subgroups are always parabolic in the Coxeter group theoretic sense. This is not true in general: indeed, any Gromov hyperbolic group is also relatively hyperbolic with respect to any malnormal collection of quasi-convex subgroups, see [Bow12, Th. 7.11]. Therefore, even if  $W$  is Gromov hyperbolic, one can always make it relatively hyperbolic by adding maximal self-normalising cyclic subgroups as peripheral subgroups, and those are not parabolic in the Coxeter sense. The correct statement can be phrased as follows: *if  $W$  is relatively hyperbolic with respect to some peripheral subgroups  $H_1, \dots, H_m$ , then it is also relatively hyperbolic with respect to a (possibly empty) collection of Coxeter-parabolic subgroups  $P_1, \dots, P_k$ , and moreover, each  $P_i$  is conjugate to a subgroup of some  $H_j$ .* In particular every Coxeter group admits a canonical, minimal, relatively hyperbolic structure, whose peripheral subgroups are indeed parabolic in the Coxeter group theoretic sense. The latter result has been obtained in a joint work with Jason Behrstock, Mark Hagen and Alessandro Sisto. In that work, we also provide various characterizations of the canonical parabolic subgroups  $P_1, \dots, P_k$ , and describe necessary and sufficient conditions on a Coxeter presentation of  $W$  ensuring that  $W$  is not relatively hyperbolic with respect to any collection of proper subgroups. Those results appear in the Appendix to [BHS13].

## References

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