REGULAR ELEMENTS IN CAT(0) GROUPS

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Abstract. Let X be a locally compact geodesically complete CAT(0) space and Γ be a discrete group acting properly and cocompactly on X. We show that Γ contains an element acting as a hyperbolic isometry on each indecomposable de Rham factor of X. It follows that if X is a product of d factors, then Γ contains $\mathbf{Z}^d$.

Let X be a proper CAT(0) space and Γ be a discrete group acting properly and cocompactly by isometries on X. The *flat closing conjecture* predicts that if X contains a d-dimensional flat, then Γ contains a copy of $\mathbf{Z}^d$ (see [Gro93, Section 6.B_3]). In the special case $d = 2$, this would imply that Γ is hyperbolic if and only if it does not contain a copy of $\mathbf{Z}^2$. This notorious conjecture remains however open as of today. It holds when X is a real analytic manifold of non-positive sectional curvature by the main result of [BS91]. In the classical case when X is a non-positively curved symmetric space, it can be established with the following simpler and well known argument: by [BL93, Appendix], the group Γ must contain a so called **R-regular** semisimple element, *i.e.* a hyperbolic isometry γ whose axes are contained in a unique maximal flat of X. By a lemma of Selberg [Sel60], the centraliser $\mathscr{Z}_\Gamma(\gamma)$ is a lattice in the centraliser $\mathscr{Z}_{\mathrm{Is}(X)}(\gamma)$. Since the latter centraliser is virtually $\mathbf{R}^d$ with $d = \mathrm{rank}(X)$, one concludes that Γ contains $\mathbf{Z}^d$, as desired.

It is tempting to try and mimick that strategy of proof in the case of a general CAT(0) space X: if one shows that Γ contains a hyperbolic isometry γ which is **maximally regular** in the sense that its axes are contained in a unique flat of maximal possible dimension among all flats of X, then the flat closing conjecture will follow as above. The main result of this note provides hyperbolic isometries satisfying a weaker notion of regularity.

Theorem. *Assume that X is geodesically complete.*

Then Γ contains a hyperbolic element which acts as a hyperbolic isometry on each indecomposable de Rham factor of X.

Every CAT(0) space X as in the theorem admits a canonical de Rham decomposition, see [CM09a, Corollary 5.3(ii)]. Notice that the number of indecomposable de Rham factors of X is a lower bound on the dimension of all maximal flats in X, although two such maximal flats need not have the same dimension in general. As expected, we deduce a corresponding lower bound on the maximal rank of free abelian subgroups of Γ.

Corollary 1. *If X is a product of d factors, then Γ contains a copy of $\mathbf{Z}^d$.*

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We believe that those results should hold without the assumption of geodesic completeness; in case X is a CAT(0) cube complex, this is indeed so, see [CS11, §1.3].

The proof of the theorem and its corollary relies in an essential way on results from [CM09a] and [CM09b]. The first step consists in applying [CM09a, Theorem 1.1], which ensures that X splits as

$$X \cong \mathbf{R}^d \times M \times Y_1 \times \cdots \times Y_q,$$

where M is a symmetric space of non-compact type and the factors Y_i are geodesically complete indecomposable CAT(0) spaces whose full isometry group is totally disconnected. Moreover this decomposition is canonical, hence preserved by a finite index subgroup of $\mathrm{Is}(X)$ (and thus of Γ). The next essential point is that, by [CM09b, Theorem 3.8], the group Γ virtually splits as $\mathbf{Z}^d \times \Gamma'$, and the factor Γ' (resp. $\mathbf{Z}^d$) acts properly and cocompactly on $M \times Y_1 \times \cdots \times Y_q$ (resp. $\mathbf{R}^d$). Therefore, our main theorem is a consequence of the following.

Proposition 2. *Let $X = M \times Y_1 \times \cdots \times Y_q$, where M is a symmetric space of non-compact type and Y_i is a geodesically complete locally compact* CAT(0) *space with totally disconnected isometry group.*

Any discrete cocompact group of isometries of X contains an element acting as an $\mathbf{R}$-regular hyperbolic element on M, and as a hyperbolic element on Y_i for all i.

As before, this yields a lower bound on the rank of maximal free abelian subgroups of Γ, from which Corollary 1 follows.

Corollary 3. *Let $X = M \times Y_1 \times \cdots \times Y_q$ be as in the proposition. Then any discrete cocompact group of isometries of X contains a copy of $\mathbf{Z}^{\mathrm{rank}(M)+q}$.*

Proof. Let $\Gamma < \mathrm{Is}(X)$ be a discrete subgroup acting cocompactly. Upon replacing Γ by a subgroup of finite index, we may assume that Γ preserves the given product decomposition of X (see [CM09a, Corollary 5.3(ii)]). Let $\gamma \in \Gamma$ be as in Proposition 2 and let γ_M (resp. γ_i) be its projection to $\mathrm{Is}(M)$ (resp. $\mathrm{Is}(Y_i)$). Then $\mathrm{Min}(\gamma_M) = \mathbf{R}^{\mathrm{rank}(M)}$ and for all i we have $\mathrm{Min}(\gamma_i) \cong \mathbf{R} \times C_i$ for some CAT(0) space C_i, by [BH99, Theorem II.6.8(5)]. Hence the desired conclusion follows from the following lemma. □

Lemma 4. *Let $X = X_1 \times \cdots \times X_p$ be a proper* CAT(0) *space and Γ a discrete group acting properly cocompactly on X. Let also $\gamma \in \Gamma$ be an element preserving some d_i-dimensional flat in X_i on which it acts by translation, for all i.*

Then Γ contains a free abelian group of rank $d_1 + \cdots + d_p$.

Proof. By assumption γ preserves the given product decomposition of X. We let γ_i denote the projection of γ on $\mathrm{Is}(X_i)$. Observe that

$$\mathrm{Min}(\gamma) = \mathrm{Min}(\gamma_1) \times \cdots \times \mathrm{Min}(\gamma_p).$$

By hypothesis, we have $\mathrm{Min}(\gamma_i) \cong \mathbf{R}^{d_i} \times C_i$ for some CAT(0) space C_i. Therefore $\mathrm{Min}(\gamma) \cong \mathbf{R}^{d_1+\cdots+d_p} \times C_1 \times \cdots \times C_p$. By [Rua01, Theorem 3.2] the centraliser $\mathscr{Z}_\Gamma(\gamma)$ acts cocompactly (and of course properly) on $\mathrm{Min}(\gamma)$. Therefore, invoking [CM09b, Theorem 3.8], we infer that $\mathbf{Z}^{d_1+\cdots+d_p}$ is a (virtual) direct factor of $\mathscr{Z}_\Gamma(\gamma)$. □

It remains to prove Proposition 2. We proceed in three steps. The first one provides an element $\gamma_Y \in \Gamma$ acting as a hyperbolic isometry on each Y_i. This combines an argument of E. Swenson [Swe99, Theorem 11] with the phenomenon of **Alexandrov angle rigidity**, described in [CM09a, Proposition 6.8] and recalled below. The latter requires the hypothesis of geodesic completeness. The second step uses that Γ has subgroups acting properly cocompactly on M, and thus contains an element γ_M acting as an $\mathbf{R}$-regular isometry of M by [BL93]. The last step uses a result from [PR72] ensuring that for all elements δ' in some Zariski open subset of $\mathrm{Is}(M)$ and all sufficiently large $n > 0$, the product $\gamma_M^n \delta'$ is $\mathbf{R}$-regular. Invoking the Borel density theorem, we finally find an appropriate element $\delta \in \Gamma$ such that the product $\gamma = \gamma_M^n \delta \gamma_Y$ has the requested properties. We now proceed to the details.

Proposition (Alexandrov angle rigidity). *Let Y be a locally compact geodesically complete* CAT(0) *space and G be a totally disconnected locally compact group acting continuously, properly and cocompactly on Y by isometries.*

Then there is $\varepsilon > 0$ such that for any elliptic isometry $g \in G$ and any $x \in X$ not fixed by g, we have $\angle_c(gx, x) \geq \varepsilon$, where c denotes the projection of x on the set of g-fixed points.

Proof. See [CM09a, Proposition 6.8]. □

Proposition 5. *Let $Y = Y_1 \times \cdots \times Y_q$, where Y_i is a geodesically complete locally compact* CAT(0) *space with totally disconnected isometry group, and G be a locally compact group acting continuously, properly and cocompactly by isometries on Y.*

Then G contains an element acting on Y_i as a hyperbolic isometry for all i.

Proof. Upon replacing G by a finite index subgroup, we may assume that G preserves the given product decomposition of Y, see [CM09a, Corollary 5.3(ii)]. Let $\rho : [0, \infty) \to Y$ be a geodesic ray which is **regular**, in the sense that its projection to each Y_i is a ray (in other words the end point $\rho(\infty)$ does not belong to the boundary of a subproduct).

Since G is cocompact, we can find a sequence (g_n) in G and a sequence (t_n) in $\mathbf{R}_+$ such that $g_n.\rho(t_n)$ converges to some point $y \in Y$ and $g_n.\rho$ converges uniformly on compacta to a geodesic line ℓ in Y. Set $h_{i,j} = g_i^{-1} g_j \in G$ and consider the angle

$$\theta = \angle_{\rho(t_i)}(h_{i,j}^{-1}.\rho(t_i), h_{i,j}.\rho(t_i)).$$

As in [Swe99, Theorem 11], observe that θ is arbitrarily close to π for $i < j$ large enough.

We shall prove that for all $i < j$ large enough, the isometry $h_{i,j}$ is regular hyperbolic, in the sense that its projection to each factor Y_k is hyperbolic. We argue by contradiction and assume that this is not the case. Notice that $\mathrm{Is}(Y_k)$ does not contain any parabolic isometry by [CM09a, Corollary 6.3(iii)]. Therefore, upon extracting and reordering the factors, we may then assume that there is some $s \leq q$ such that for all $i < j$, the projection of $h_{i,j}$ on $\mathrm{Is}(Y_1), \dots, \mathrm{Is}(Y_s)$ is elliptic, and the projection of $h_{i,j}$ on $\mathrm{Is}(Y_{s+1}), \dots, \mathrm{Is}(Y_q)$ is hyperbolic. We set $Y' = Y_1 \times \cdots \times Y_s$ and $Y'' = Y_{s+1} \times \cdots \times Y_q$. We shall prove that for $i < j$ large enough, the projections of $(h_{i,j})$ on $\mathrm{Is}(Y')$ forms a sequence of elliptic isometries which contradict Alexandrov angle rigidity.

Fix some small $\delta > 0$. Let x_i (resp. y_i) be the point at distance δ from $\rho(t_i)$ and lying on the geodesic segment $[h_{i,j}^{-1}.\rho(t_i), \rho(t_i)]$ (resp. $[\rho(t_i), h_{i,j}.\rho(t_i)]$). By construction, for $i < j$ large enough, the union of the two geodesic segments $[x_i, \rho(t_i)] \cup [\rho(t_i), y_i]$ lies in an arbitrary small tubular neighbourhood of the geodesic ray ρ. Since the projection $Y \to Y'$ is 1-Lipschitz, it follows that the Y'-component of $[x_i, \rho(t_i)] \cup [\rho(t_i), y_i]$, which we denote by $[x'_i, \rho'(t_i)] \cup [\rho'(t_i), y'_i]$, is uniformly close to the Y'-component of ρ, say ρ'. Since ρ is a regular ray, its projection ρ' is also a geodesic ray. Therefore, the angle

$$\theta' = \angle_{\rho'(t_i)}(x_i', y_i')$$

is arbitrarily close to π for $i < j$ large enough. Pick $i < j$ so large that $\theta' > \pi - \varepsilon$, where $\varepsilon > 0$ is the constant from Alexandrov angle rigidity for Y'. Set $h = h_{i,j}$ and let h' be the projection of h on $\mathrm{Is}(Y')$. By assumption h' is elliptic. Let c denote the projection of $\rho'(t_i)$ on the set of h'-fixed points. Then the isosceles triangles $\triangle\big(c, (h')^{-1}.\rho'(t_i), \rho'(t_i)\big)$ and $\triangle\big(c, \rho'(t_i), h'.\rho'(t_i)\big)$ are congruent, and we deduce

$$\begin{array}{rcl} \angle_c(\rho'(t_i), h'.\rho'(t_i)) & \leq & \pi - \angle_{\rho'(t_i)}(c, h'.\rho'(t_i)) - \angle_{\rho'(t_i)}(c, (h')^{-1}.\rho'(t_i)) \\ & \leq & \pi - \angle_{\rho'(t_i)}((h')^{-1}.\rho'(t_i), h'.\rho'(t_i)) \\ & = & \pi - \theta' \\ & < & \varepsilon. \end{array}$$

This contradicts Alexandrov angle rigidity. □

Proof of Proposition 2. Let Γ be a discrete group acting properly and cocompactly on X. First observe that (after passing to a finite index subgroup) we may assume that Γ preserves the given product decomposition of X, see see [CM09a, Corollary 5.3(ii)].

Let G be the closure of the projection of Γ to $\mathrm{Is}(Y_1) \times \cdots \times \mathrm{Is}(Y_q)$. Then G acts properly cocompactly on $Y = Y_1 \times \cdots \times Y_q$. Therefore it contains an element g acting as a hyperbolic isometry on Y_i for all i by Proposition 5. Since Γ maps densely to G and since the stabiliser of each point of Y in G is open by [CM09a, Theorem 1.2], it follows that Γ-orbits on $Y \times Y$ coincide with the G-orbits. In particular, given $y \in \mathrm{Min}(g)$, we can find $\gamma_Y \in \Gamma$ such that $\gamma_Y(y, g^{-1}y) = (gy, y)$. Since $\angle_y(\gamma_Y^{-1}y, \gamma_Y y) = \angle_y(g^{-1}y, gy) = \pi$, we infer that γ_Y is hyperbolic and has an axis containing the segment $[g^{-1}y, gy]$. In particular γ_Y acts as a hyperbolic isometry on Y_i for all i.

Let $\gamma_Y = (\alpha, h)$ be the decomposition of γ_Y along the splitting $\mathrm{Is}(X) = \mathrm{Is}(M) \times \mathrm{Is}(Y)$. By construction h acts as a hyperbolic isometry on Y_i for all i.

Let $U \leq \mathrm{Is}(Y)$ be the pointwise stabiliser of a ball containing $y, \gamma_Y y$ and $\gamma_Y^{-1}y$. Notice that every element of $\mathrm{Is}(Y)$ contained in the coset Uh maps y to $h.y$ and $h^{-1}y$ to y, and therefore acts also as a hyperbolic isometry on Y_i for all i.

On the other hand U is a compact open subgroup of $\mathrm{Is}(Y)$ by [CM09a, Theorem 1.2]. Set $\Gamma_U = \Gamma \cap (\mathrm{Is}(M) \times U)$. Notice that Γ_U acts properly and cocompactly on M by [CM09b, Lemma 3.2]. In other words the projection of Γ_U to $\mathrm{Is}(M)$ is a cocompact lattice. Abusing notation slightly, we shall denote this projection equally by Γ_U.

By the appendix from [BL93] (see also [Pra94] for an alternative argument), the group Γ_U contains an element γ_M acting as an $\mathbf{R}$-regular element on M. By [PR72, Lemma 3.5] there is a Zariski open set $V = V(\gamma_M)$ in $\mathrm{Is}(M)$ with the following property. For any $\delta \in V$ there exists n_δ such that an element $\gamma_M^n \delta$ is $\mathbf{R}$-regular for any $n \geq n_\delta$. By the Borel density theorem, the intersection $\Gamma_U \cap V\alpha^{-1}$ is nonempty. Pick an element $\delta \in \Gamma_U \cap V\alpha^{-1}$. Then $\delta\alpha \in V$ which means by definition that $\gamma_M^n \delta\alpha$ is $\mathbf{R}$-regular for all $n \geq n_0$ for some integer n_0.

Pick an element $\gamma_M' \in \Gamma$ (resp. $\delta' \in \Gamma$) which lifts γ_M (resp. δ). Set

$$\gamma = (\gamma_M')^{n_0} \delta' \gamma_Y \in \Gamma_U.$$

The projection of γ to $\mathrm{Is}(M)$ is $\gamma_M^{n_0}\delta\alpha$ and is thus $\mathbf{R}$-regular. The projection of γ to $\mathrm{Is}(Y)$ belongs to the coset Uh, and therefore acts as a hyperbolic isometry on Y_i for all i. □

References

[BS91] Victor Bangert and Viktor Schroeder, *Existence of flat tori in analytic manifolds of nonpositive curvature*, Ann. Sci. École Norm. Sup. (4) **24** (1991), no. 5, 605–634.

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[BL93] Yves Benoist and François Labourie, *Sur les difféomorphismes d'Anosov affines à feuilletages stable et instable différentiables*, Invent. Math. **111** (1993), no. 2, 285–308 (French, with French summary).

[CS11] Pierre-Emmanuel Caprace and Michah Sageev, *Rank rigidity for CAT(0) cube complexes*, Geom. Funct. Anal. **21** (2011), no. 4, 851–891.

[CM09a] Pierre-Emmanuel Caprace and Nicolas Monod, *Isometry groups of non-positively curved spaces: structure theory*, J. Topol. **2** (2009), no. 4, 661–700.

[CM09b] ———, *Isometry groups of non-positively curved spaces: discrete subgroups*, J. Topol. **2** (2009), no. 4, 701–746.

[Gro93] Mikhail Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.

[Kle99] Bruce Kleiner, *The local structure of length spaces with curvature bounded above*, Math. Z. **231** (1999), no. 3, 409–456.

[Pra94] Gopal Prasad, **R**-*regular elements in Zariski-dense subgroups*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 541–545. MR1315463 (96a:22022)

[PR72] Gopal Prasad and Madabusi Santanam Raghunathan, *Cartan subgroups and lattices in semi-simple groups*, Ann. of Math. (2) **96** (1972), 296–317.

[Rua01] Kim E. Ruane, *Dynamics of the action of a* CAT(0) *group on the boundary*, Geom. Dedicata **84** (2001), no. 1-3, 81–99.

[Sel60] Atle Selberg, *On discontinuous groups in higher-dimensional symmetric spaces*, Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960), Tata Institute of Fundamental Research, Bombay, 1960, pp. 147–164.

[Swe99] Eric L. Swenson, *A cut point theorem for* CAT(0) *groups*, J. Differential Geom. **53** (1999), no. 2, 327–358.

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