GROUPS WITH A ROOT GROUP DATUM

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Abstract. Root group data provide the abstract combinatorial framework common to all groups of Lie-type and of Kac-Moody-type. These notes intend to serve as a friendly introduction to their basic theory. We also survey some recent developments.

Introduction

Historical overview. Lie theory has a long and fascinating history. One of its most enthralling aspects is the gain in unity which has been acquired over the years through the contributions of many eminent figures. We try to roughly sum this up in the following paragraphs.

One of the foundational works of the theory has been the classification of simple Lie groups completed by W. Killing and É. Cartan in the first half of the 20th century: up to isomorphism, (center-free) complex simple Lie groups are in one-to-one correspondence with complex simple Lie algebras, which themselves are in one-to-one correspondence with the irreducible finite root systems. In particular, the Killing-Cartan classification highlighted five exceptional types of simple Lie groups besides the classical ones. Classical groups were then thoroughly studied and fairly well understood, mainly through case-by-case analysis [vdW35]. Still, some nice uniform constructions of them deserve to be mentioned: e.g., by means of algebras with involutions [Wei61], or constructions by means of automorphism groups of some linear structures defined over an arbitrary ground field [Die71]. In this respect, the simple Lie groups of exceptional type were much more mysterious; analogues of them had been defined over finite fields by L. Dickson for types E_6 and G_2. A wider range of concrete realizations of exceptional groups is provided by H. Freudenthal's work [Fre64].

From the 1950's on, the way was paved towards a theory which would eventually embody all these groups, regardless of their type or of the underlying ground field. Two foundational papers were those of C. Chevalley [Che55], who constructed analogues of simple Lie groups over arbitrary fields, and of A. Borel [Bor56], who began a systematic study of linear algebraic groups. For the sake of completeness and for the prehistory of buildings, see also [Tit57] for an approach from the geometer's viewpoint – where "geometer" has to be understood as in J. Tits' preface to [KMRT98]. A spectacular achievement consisted in the extension by C. Chevalley of É. Cartan's classification to all simple algebraic groups over arbitrary algebraically closed fields [Che05]. Remarkably surprising was the fact that, once the (algebraically closed) ground field is fixed, the classification is the same as for complex Lie groups: simple algebraic groups over the given field are again in one-to-one correspondence with irreducible finite root systems.

In order to extend this correspondence to all split reductive groups over arbitrary fields, M. Demazure [SGA70, Exp. XXI] introduced the notion of a *root datum* (in French: *donnée radicielle*), which is a refinement of the notion of root systems. These developments were especially exciting in view of the fact that most of the abstract simple groups known in the first half of the 20th century were actually related in some way to simple Lie groups.

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Another further step in the unification was made by J. Tits in his seminal paper [Tit64], where he proposed an axiomatic setting which allowed him to obtain a uniform proof of (projective) simplicity for all of these groups, as well as isotropic groups over arbitrary fields, at once. While reviewing the latter article, J. Dieudonné wrote: "*This paper goes a long way towards the realization of the hope expressed by the reviewer in 1951 that some general method be found which would give the structure of all "isotropic" classical groups without having to examine separately each type of group. It is well-known that the first breakthrough in that direction was made in the famous paper of Chevalley in 1955* [Che55]*, which bridged in a spectacular way the gap between Lie algebras and finite groups. The originality of the author has been to realize that the gist of Chevalley's arguments could be expressed in a purely group-theoretical way, namely, the existence in a group G of two subgroups B, N generating G, such that $H = B \cap N$ is normal in N, and that $W = N/H$ (the "Weyl group") is generated by a set S of involutory elements satisfying two simple conditions (and corresponding to the "roots" in Chevalley's case). This he calls a (BN)-pair (...).*"

This notion of a BN-pair was inspired to J. Tits by the decompositions in double cosets discovered by F. Bruhat [Bru54], which had then been extended and extensively used by C. Chevalley. What J. Dieudonné called "*purely group-theoretical*" in his review turned out to be the group-theoretic side of a unified geometrical approach to the whole theory, that J. Tits developed by creating the notion of *buildings* [Bou07b, IV §2 Exercice 15]. Exploiting beautifully the combinatorial and geometrical aspects of these objects, J. Tits was able to classify completely the irreducible buildings of rank $\geqslant 3$ with finite Weyl group [Tit74]. A key property of these buildings is that they happen to be all highly symmetric: they enjoy the so-called *Moufang property*. J. Tits' classification shows furthermore that they are all related to simple algebraic groups or to classical groups in some way. J. Tits also shows that a generalization of the fundamental theorem of projective geometry holds for buildings (seen as incidence structures). This result was used by G.D. Mostow to prove his famous strong rigidity theorem for finite volume locally symmetric spaces of rank $\geqslant 2$ [Mos73]; in this way the combinatorial aspects of Lie structures found a beautiful, deep and surprising application to differential geometry.

A few decades later, jointly with R. Weiss, J. Tits completed the extension of this classification to all irreducible Moufang buildings of rank $\geqslant 2$ with a finite Weyl group [TW02]. This result, combined with [BT73], yields a classification of all groups with an irreducible *split* BN-pair of rank $\geqslant 2$ with finite Weyl group. The condition that the BN-pair splits is the group-theoretic translation of the Moufang property (and has nothing to do with splitness in the sense of algebraic groups). Thus, every irreducible BN-pair of rank $\geqslant 3$ with a finite Weyl group splits. Concerning BN-pairs with finite Weyl groups, we finally note that what this group combinatorics does not cover in the theory of algebraic semisimple groups is the case of anisotropic groups. The structure of these groups is still mysterious and for more information about this, we refer to [Tit78], [Mar91, VIII.2.17] and [PR94].

A remarkable feature of the abstract notion of a BN-pair is that it does not require the Weyl group to be finite, even though J. Tits originally used them to study groups with a finite Weyl group in [Tit64] (the BN-pairs in these groups had been constructed in his joint work with A. Borel [BT65]). The possibility for the Weyl group to be infinite was called to play a crucial role in another breakthrough, initiated by the discovery of affine BN-pairs in p-adic semisimple groups by N. Iwahori and H. Mastumoto [IM65]. This was taken up by F. Bruhat and J. Tits in their celebrated theory of reductive groups over local fields [BT72]. In the latter, a refinement of the notion of split BN-pairs was introduced, namely *valuated root data* (in French: *données radicielles valuées*). These combine the information encoded in root data with extra information on the corresponding BN-pairs coming from the valuation of the ground field. Valuated root data turned out to be classifying data for *Bruhat-Tits buildings*, namely the buildings constructed from the aforementioned affine BN-pairs [Tit86b].

We note that in the case of Bruhat-Tits theory, the BN-pair structure (in fact the refined structure of valuated root datum) was not a way to encode *a posteriori* some previously known structure results proved by algebraic group tools (as in the case of Borel-Tits theory with spherical

BN-pairs and buildings). Indeed, the structure of valuated root datum, and its counterpart: the geometry of Euclidean buildings, is both the main tool and the goal of the structure theory. The existence of a valuated root datum structure on the group of rational points is proved by a very hard two-step descent argument, whose starting point is a split group. The argument involves both (singular) non-positive curvature arguments and the use of integral structures for the algebraic group under consideration. The final outcome can be nicely summed by the fact that the Bruhat-Tits building of the valuated root datum for the rational points is often the fixed point set of the natural Galois action in the building of the split group [BT84]. In fact, F. Bruhat and J. Tits formulate their results at such a level of generality (in particular with fields endowed with a possibly dense or even surjective valuation) that the structure of valuated root datum still makes sense while that of BN-pair doesn't in general (when the valuation is not discrete). At last, this study became complete after J. Tits' classification of affine buildings, regardless of any group action *a priori* [Tit86b]; roughly speaking, this classification reduces to the previous classification of spherical buildings after considering a suitably defined *building at infinity*. We refer to [Wei] for a detailed exposition of the classification in the discrete case.

At about the same time as Bruhat-Tits theory was developed, the first examples of groups with BN-pairs with infinite but non-affine Weyl groups were constructed by R. Moody and K. Teo [MT72] in the realm of Kac-Moody theory. The latter theory had been initiated by R. Moody and V. Kac independently a few years before in the context of classifying simple Lie algebras with growth conditions with respect to a grading. The corresponding groups (which were not so easily constructed) became known as Kac-Moody groups and were regarded as infinite-dimensional versions of the semisimple complex Lie groups. Several works in the 1980's, notably by V. Kac and D. Peterson, highlighted intriguing similarities between the finite-dimensional theory and the more recent Kac-Moody objects. Again, the notion of a BN-pair and its refinements played a crucial role in understanding these similarities, see e.g. [KP85]. We note that the present day situation is that there exist several versions of Kac-Moody groups, as explained for instance in [Tit89]. The biggest versions are often more relevant to representation theory (see [Mat88] or [Kum02]) than to group theory (see however [Moo82]). The relation between the complete and the minimal versions of these groups still needs to be elucidated precisely. As far as group theory and combinatorics are concerned, the theory gained once more in depth when J. Tits defined analogues of complex Kac-Moody groups over arbitrary fields in [Tit87], as C. Chevalley had done it for Lie groups some 30 years earlier. In [loc. cit.], some further refinements of the notion of BN-pairs had to be considered, the definitive formulation of which was settled in [Tit92] by the concept of *root group data*. This is the starting point of the present notes.

Content overview. The purpose of these notes is to highlight a series of structure properties shared by all groups endowed with a root group datum. One should view them as a guide through a collection of results spread over a number of different sources in the literature, which we have tried to present in a reasonably logical order. The proofs included here are often reduced to quotations of accurate references; however, we have chosen to develop more detailed arguments when we found it useful in grasping the flavour of the theory. The emphasis is placed on results of algebraic nature on the class of groups under consideration. Consequently, detailed discussions of the numerous aspects of the deep and beautiful theory of buildings are almost systematically avoided. Inevitably, the text is overlapping some parts of the second author's book [Rém02c], but the point of view adopted here is different and several themes discussed here (especially from Sect. 6 to 8) are absent from [loc. cit.].

The structure of the paper, divided into two parts, is the following.

Part I: survey of the theory and examples.— Sect. 1 collects some preliminaries on (usually infinite) root systems; it is the technical preparation required to state the definition of a root group datum. Sect. 2 is devoted to the latter definition and to some examples. The aim of Sect. 3 is to show that complex adjoint Kac-Moody groups provide a large family of groups endowed with a root group datum (with infinite Weyl group); the proof relies only on the very basics of the theory of Kac-Moody algebras (which are outlined as well). In Sect. 4, we first mention that any root group datum yields two BN-pairs, which in turn yield a pair of buildings acted upon by the ambient group G; this interplay between buildings and BN-pairs is then further described.

Part II: group actions on buildings and associated structure results.— The second part is devoted to the algebraic results that can be derived from the existence of a sufficiently transitive group action on a building. In Sect. 5, we first introduce a very important tool designed by J. Tits, namely the combinatorial analogue of techniques from algebraic topology for partially ordered sets; this is very useful for some amalgamation and intersection results. Subsequently we deduce a number of basic results on the structure of groups endowed with a root group datum. In Sect. 6, we explain that since the automorphism group of any building carries a canonical topology, these buildings may be used to endow G (admitting a root group datum) with two distinguished group topologies, with respect to which one may take metric completions; these yield two larger groups G_+ and G_- containing both G as a dense subgroup, and the diagonal embedding of G makes it a discrete subgroup in $G_+ \times G_-$. In Sect. 7, some simplicity results for $G_\pm$ and G are discussed. In Sect. 8 we show that, under some conditions, the group G admits certain nice presentations which can be used to describe classification results for root group data.

Notation. If G is a group, the order of an element $g \in G$ is denoted by $o(g)$. If moreover H is a subgroup of G, then ${}^gH$ denotes the conjugate gHg^{-1}.

What this article does not cover. The main aim of these notes is to highlight some algebraic properties common to all groups with a root group datum, with a special emphasis in those with an infinite Weyl group. However, root group data were initially designed to describe and study the combinatorial structure of rational points of isotropic simple algebraic groups, and it is far beyond the scope of this paper to describe the theory of algebraic groups. For a recent account of advanced problems in that area, we refer to [Gil07]. Another excellent reference on root group data with finite Weyl groups is the comprehensive book by J. Tits and R. Weiss [TW02], which is targeted at the classification in the rank two case. The case of rank one root group data, i.e. Moufang sets, is a subject in its own right: see [dMS07] in the same volume.

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CONTENTS

Part I. Survey of the theory and examples

1. Root data

Root data were first introduced by M. Demazure [SGA70, Exp. XXI] as data which classify, up to isomorphism, reductive group schemes over $\mathbb{Z}$ or split reductive algebraic groups over a given field [Spr98, Chapters 9-10]. Demazure's original definition can be viewed as a refinement of the notion of finite root systems, taking into account the possibility to have a non-trivial (connected) central torus. However, root systems encountered in Kac-Moody theory are mostly infinite, hence the definition of a root datum we give is not Demazure's (although it is closely related). The way towards a general theory of infinite root systems has been paved by R. Moody and A. Pianzola [MP89] (see also [MP95, Chapter 5] for a more comprehensive and self-contained treatment). However, this approach has two drawbacks that we want to avoid: it implicitly excludes non-reduced root systems and it requires a certain integrality condition. The axioms we propose here follow rather closely J.-Y. Hée's approach developed in [Hée91] (for a further comment on the comparison between these references, see Remark 1.1.1 below). We note that in another vein of generalization, N. Bardy has developed an abstract theory of root systems covering R. Borcherds' work using Lie algebras for number theory [Bar96]; this topic will not be covered here.

The content of this section is very simple: we first define root bases, which are designed to generate root systems, which themselves are the index sets of the combinatorics of root group data.

1.1. Root bases.

1.1.1. *Axioms of a root basis.* Let V be a real vector space. A **root basis** for V is a pair $\mathbf{B} = (\Pi, \Pi^\vee = \{\alpha^\vee\}_{\alpha\in\Pi})$ where Π is a (nonempty) subset of V and $\Pi^\vee$ is a set consisting of an element $\alpha^\vee \in V^*$ associated to each element $\alpha \in \Pi$, submitted to the following conditions:

- **(RB1):** For each $\alpha \in \Pi$, we have $\langle \alpha, \alpha^\vee \rangle = 2$.
- **(RB2):** For all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$, we have either $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 0$ or $\langle \alpha, \beta^\vee \rangle < 0$, $\langle \beta, \alpha^\vee \rangle < 0$ and $\langle \alpha, \beta^\vee \rangle\langle \beta, \alpha^\vee \rangle \in \{4\cos^2(\frac{\pi}{k}) \mid k \in \mathbb{Z}\} \cup \mathbb{R}_{\geqslant 4}$.
- **(RB3):** There exists $f \in V^*$ such that $\langle \alpha, f \rangle > 0$ for all $\alpha \in \Pi$.

Given a root basis $\mathbf{B} = (\Pi, \Pi^\vee = \{\alpha^\vee\}_{\alpha\in\Pi})$ as above, we make the following definitions:

- The matrix $A(\mathbf{B}) = (A_{\alpha,\beta})_{\alpha,\beta\in\Pi}$ defined by $A_{\alpha,\beta} = \langle \alpha, \beta^\vee \rangle$ is called the **Cartan matrix** of $\mathbf{B}$.
- The cardinality of Π is called the **rank** of $\mathbf{B}$.
- To each $\alpha \in \Pi$, we associate the involution $r_\alpha : V \to V; v \mapsto v - \langle v, \alpha^\vee \rangle \alpha$, which we call the **reflection** with respect to α.
- We set $S = S(\mathbf{B}) = \{r_\alpha \mid \alpha \in \Pi\}$.
- We define $W = W(\mathbf{B})$ to be the subgroup of $\mathrm{GL}(V)$ generated by $S(\mathbf{B})$; it is called the **Weyl group** of $\mathbf{B}$.
- We set

$$\Phi(\mathbf{B}) = \{w.\alpha \mid \alpha \in \Pi, w \in W\},\ \Phi(\mathbf{B})_+ = \Phi \cap (\textstyle\sum_{\alpha\in\Pi} \mathbb{R}_+\alpha) \text{ and } \Phi(\mathbf{B})_- = -\Phi(\mathbf{B})_+,$$

  and call $\Phi(\mathbf{B})$ the **root system** of $\mathbf{B}$.
- Given a subset $J \subset \Pi$, we set $\mathbf{B}_J = (J, J^\vee = \{\alpha^\vee\}_{\alpha\in J})$, $S_J = \{r_\alpha \mid \alpha \in J\}$ and $W_J = \langle S_J \rangle$. The tuple $\mathbf{B}_J$ is a root basis for V with Weyl group W_J.

We say that $\mathbf{B}$ is **integral** if each entry of the Cartan matrix is an integer. We say that $\mathbf{B}$ is **free** if Π is linearly independent in V.

Remarks.
1. The article [MP89] deals only with integral root bases, while [Hée91] considers only free root bases (note that under this assumption, axiom (RB3) is automatically satisfied).
2. The integrality condition is not appropriate when one wishes to study (non-algebraic) twisted forms of Chevalley groups or of Kac-Moody groups: the simplest illustration of this fact is provided by groups of type ${}^2F_4$.

3. The freeness condition is not appropriate to study root subbases and reflection subgroups of the Weyl group, see [MP89, Example 1]. Axiom (RB3), which was proposed by D. Krammer [Kra94], allows to combine naturally the approaches of both [Hée91] and [MP89]. In particular, most results from both [Hée91] (in the case $K = \mathbb{R}$) and [MP89] remain valid in the present context; the necessary modifications of arguments are mild and straightforward.

In the rest of this section, we collect only a few basic facts for later references. Concerning the comparison of the different notions of root bases, let us simply mention one useful construction adapted from [MP89]. Given a root basis $\mathbf{B} = (\Pi, \Pi^\vee)$ with Cartan matrix A, a free root basis $\widetilde{\mathbf{B}}$ may be constructed as follows. Define $\widetilde{V} = \bigoplus_{\alpha\in\Pi} \mathbb{R}\widetilde{\alpha}$. Moreover, for each $\beta \in \Pi$, define $\widetilde{\beta}^\vee \in \widetilde{V}^*$ by the assignment $\langle \widetilde{\alpha}, \widetilde{\beta}^\vee \rangle = A_{\alpha,\beta}$ for all $\alpha \in \Pi$. Then $\widetilde{\mathbf{B}} = (\widetilde{V}, \{\widetilde{\alpha}\}_{\alpha\in\Pi}, \{\widetilde{\alpha}^\vee\}_{\alpha\in\Pi})$ is a free root basis. Its Cartan matrix coincides with A. Furthermore, there is a canonical linear map $\pi : \widetilde{V} \to V$ induced by $\widetilde{\alpha} \mapsto \alpha$ which maps $\Phi(\widetilde{\mathbf{B}})$ to $\Phi(\mathbf{B})$. This restricted map turns out to be a bijection. Moreover π induces a canonical isomorphism $W(\widetilde{\mathbf{B}}) \to W(\mathbf{B})$: this follows from Theorem 1.1.4 below. Now all results of [Hée91] apply to the free root basis $\widetilde{\mathbf{B}}$ and then descend to $\mathbf{B}$ via π. In the rest of this section, we will often refer to [Hée91] to establish properties of $\mathbf{B}$; if we make no further comment on the lack of freeness of $\mathbf{B}$, it means that the desired property of $\mathbf{B}$ follows from the corresponding property of $\widetilde{\mathbf{B}}$ by the general principle we have just outlined.

1.1.2. *Products and irreducibility.* There is an obvious notion of a direct product of root bases: given root bases $\mathbf{B}_i = (V_i, \Pi_i, \Pi_i^\vee)$ for $i = 1, 2$, define $V = V_1 \oplus V_2$ and identify V_1 and V_2 with subspaces of V. We set $\Pi = \Pi_1 \cup \Pi_2$ and $\Pi^\vee = \Pi_1^\vee \cup \Pi_2^\vee$. It is straightforward to check that $(\Pi, \Pi^\vee)$ is a root basis, which is called the **direct product** of $\mathbf{B}_1$ and $\mathbf{B}_2$. Its Weyl group is the product $W(\mathbf{B}_1) \times W(\mathbf{B}_2)$. A root basis which does not split as a product is called **irreducible**.

1.1.3. *Example: the standard root basis of a Coxeter system.* The standard reference is [Bou07b, IV]. Let S be a set and $M = (m_{st})_{s,t\in S}$ be a **Coxeter matrix** over S. This means that

$$m_{st} \in \mathbb{Z} \cup \{\infty\}, \qquad m_{ss} = 1 \qquad \text{and} \qquad m_{st} = m_{ts} \geqslant 2$$

for all $s, t \in S$. The group W which is defined by the following presentation:

$$W = \langle S \mid \{(st)^{m_{st}} = 1 \mid s, t \in S,\ m_{st} < \infty\}\rangle$$

is called the **Coxeter group** of type M. The ordered pair (W, S) is called the **Coxeter system** of type M.

Given a Coxeter system (W, S) of type M, we set $V = \bigoplus_{s\in S} \mathbb{R}e_s$. Next we define a symmetric bilinear form $(\cdot, \cdot)$ on V by the formula

$$(e_s, e_t) = -\cos(\frac{\pi}{m_{st}})$$

for all $s, t \in S$. We also set $f_s = 2(\cdot, e_s) \in V^*$ for each $s \in S$. Then $\mathbf{B}(W, S) = (\{e_s\}_{s\in S}, \{f_s\}_{s\in S})$ is a free root basis. Note that (RB3) obviously holds here because the e_s's are linearly independent. This is called the **standard** root basis associated with (W, S).

Remark. It is well-known that the map $W \to \mathrm{GL}(V)$ attaching to each $s \in S$ the reflection $\sigma_s : v \mapsto v - 2(e_s, v)e_s$ is an injective group homomorphism [Bou07b, V.4].

1.1.4. *The Weyl group is a Coxeter group.* A basic result on root bases is the following:

Theorem. *Let $\mathbf{B} = (\Pi, \Pi^\vee)$ be a root basis. We have the following:*

(i) *The ordered pair (W, S) is a Coxeter system. Furthermore, for all distinct $\alpha, \beta \in \Pi$, the order $o(r_\alpha r_\beta)$ of $r_\alpha r_\beta$ is equal to k (resp. ∞) if $A_{\alpha,\beta}A_{\beta,\alpha} = 4\cos^2(\frac{\pi}{k})$ (resp. if $A_{\alpha,\beta}A_{\beta,\alpha} \geqslant 4$).*

(ii) *We have: $\Phi(\mathbf{B}) = \Phi(\mathbf{B})_+ \cup \Phi(\mathbf{B})_-$.*

Proof. The axioms (RB1)–(RB3) imply that any pair $\{\alpha, \beta\}$ of elements of Π is linearly independent. In other words $\mathbf{B}_{\{\alpha,\beta\}}$ is a free root basis. By [Hée91, (2.11)], it is thus a root basis in the sense of [loc. cit.]. Now the arguments of [Hée91, (2.10)] show that (ii) holds and allow moreover to apply *verbatim* the proof of [Bou07b, Ch. 5, §4, Th. 1], which yields (i). Finally, the rule that computes the order of $r_\alpha r_\beta$ is established in [Hée91, Prop. 1.23]. □

Remark. The result [Bou07b, Ch. 5, §4, Th. 1] quoted above is due to J. Tits; a more general version as the one in [loc. cit.] is stated in [Tit95, Lemme 1].

1.1.5. *The set* $\Phi(\mathbf{B})_w$. Let $\mathbf{B} = (\Pi, \Pi^\vee = \{\alpha^\vee\}_{\alpha\in\Pi})$ and let $W = W(\mathbf{B})$ be its Weyl group. For each $w \in W$, we set

$$\Phi(\mathbf{B})_w = \{\alpha \in \Phi(\mathbf{B})_+ \mid w.\alpha \in \Phi(\mathbf{B})_-\}.$$

Lemma. *Let ℓ denote the word length in W with respect to S, i.e., for any $w \in W$ we set: $\ell(w) = \min\{m \in \mathbb{N} : w = s_1 s_2 ... s_m$ with each s_i in $S\}$.*

(i) *For all $w \in W$ and $\alpha \in \Pi$, we have*

$$\ell(r_\alpha w) > \ell(w) \text{ if and only if } w^{-1}.\alpha \in \Phi(\mathbf{B})_+$$

and

$$\ell(r_\alpha w) < \ell(w) \text{ if and only if } w^{-1}.\alpha \in \Phi(\mathbf{B})_-.$$

(ii) *For each $\alpha \in \Pi$, we have $\Phi(\mathbf{B})_{r_\alpha} = \Phi(\mathbf{B}) \cap \mathbb{R}_+\alpha$.*

(iii) *For each $w = r_{\alpha_1} \dots r_{\alpha_n} \in W$ with $\alpha_i \in \Pi$ for each i and $\ell(w) = n$, we have*

$$\Phi(\mathbf{B})_w = \Phi(\mathbf{B})_{\alpha_n} \cup r_{\alpha_n}\Phi(\mathbf{B})_{\alpha_{n-1}} \cup \dots \cup r_{\alpha_n} \dots r_{\alpha_2}\Phi(\mathbf{B})_{\alpha_1}.$$

Proof. For (i), see [Hée91, (2.10)]. For (ii) and (iii), see [Hée91, (2.23)]. □

1.1.6. *Reflections and root subbases.* By [Hée91, (2.13)(d)], for all $\alpha, \beta \in \Pi$ and $w \in W$, we have $w.\alpha = \beta$ if and only if $w r_\alpha w^{-1} = r_\beta$. Therefore, given $\beta \in \Phi(\mathbf{B})$, we may write $\beta = w.\alpha$ for some $\alpha \in \Pi$ and $w \in W$, and the reflection $w r_\alpha w^{-1}$ depends only on β, but not on the specific choice of α and w. We denote this reflection by r_β. Note that for all $\lambda \in \mathbb{R}$ such that $\lambda\beta \in \Phi(\mathbf{B})$ we have $r_{\lambda\beta} = r_\beta$. In fact, it is convenient to define $r_{\lambda\beta} = r_\beta$ for all nonzero $\lambda \in \mathbb{R}$; in this way, we attach a reflection in W to every nonzero vector in V which is proportional to an element of Φ. Furthermore, given a nonzero vector $u \in V$ such that $u = \lambda\beta$ with $\beta \in \Phi$ and $\lambda \in \mathbb{R}$, we set $u^\vee = \lambda^{-1}\beta^\vee$. In this way, we have $r_u = r_{\lambda\beta} : v \mapsto v - \langle v, u^\vee\rangle u$.

The preceding discussion shows that the assignments $\alpha \mapsto \alpha^\vee$ with $\alpha \in \Pi$ extend uniquely to a map $\Phi(\mathbf{B}) \to V^* : \beta \mapsto \beta^\vee$ which is W–equivariant (V^* is endowed with the dual action of W). Indeed, since r_β is a reflection, it is of the form $r_\beta : v \mapsto v - \langle v, \beta^\vee\rangle\beta$ for a unique $\beta^\vee \in V^*$. Now, writing again $\beta = w.\alpha$ with $\alpha \in \Pi$ and $w \in W$, we have $r_\beta = w r_\alpha w^{-1}$ and it is straightforward to deduce that $\beta^\vee = w.\alpha^\vee$.

Let now Ψ be a subset of $\Phi(\mathbf{B})$. We set

$$W_\Psi = \langle r_\beta \mid \beta \in \Psi\rangle \quad \text{and} \quad \langle\Psi\rangle = \{w.\beta \mid \beta \in \Psi, w \in W_\Psi\}.$$

Note that $W_{\langle\Psi\rangle} = W_\Psi$ and that $\langle\Psi\rangle$ is W_Ψ–invariant. We set also:

$$C(\Psi) = \{f \in V^* \mid \langle\alpha, f\rangle > 0 \text{ for all } \alpha \in \langle\Psi\rangle \cap \Phi(\mathbf{B})_+\}$$

and

$$\Pi_\Psi = \textstyle\bigcap_\Delta \{\Delta \subset \langle\Psi\rangle \mid C(\Delta) = C(\Psi)\}.$$

We have the following:

Proposition. *The couple $\mathbf{B}_\Psi = (\Pi_\Psi, {\Pi_\Psi}^\vee = \{\alpha^\vee\}_{\alpha\in\Pi_\Psi})$ is a root basis which satisfies $\Phi(\mathbf{B}_\Psi) = \langle\Psi\rangle$ and $W(\mathbf{B}_\Psi) = W_\Psi$.*

Proof. Follows by arguments as in the proof of [MP89, Theorem 6]. □

The couple $\mathbf{B}_\Psi$ is called the **root subbasis** generated by Ψ. We say that $\mathbf{B}_\Psi$ is **parabolic** if $\Pi_\Psi \subset \Pi = \Pi_{\Phi(\mathbf{B})}$. This is the case whenever $\Psi \subset \Pi$. In that special case, we recover the root subbase which was considered in Sect. 1.1.1.

1.2. Root systems.

1.2.1. *Root systems with respect to a root basis.* Given a root basis $\mathbf{B} = (\Pi, \Pi^\vee = \{\alpha^\vee\}_{\alpha\in\Pi})$, a **B-root system** is a subset Φ of $V\backslash\{0\}$ which is $W(\mathbf{B})$–invariant, contained in $\{\lambda\alpha \mid \alpha \in \Phi(\mathbf{B}), \lambda \in \mathbb{R}\}$ and such that for each $\alpha \in \Pi$, the set $\Phi \cap \mathbb{R}\alpha$ is finite and non-empty. The set

$$\Pi_\Phi = \{\beta \in \Phi \mid \beta = \lambda\alpha \text{ for some } \alpha \in \Pi \text{ and } \lambda \in \mathbb{R}_+\}$$

is called the **basis** of Φ. The $\mathbf{B}$-root system Φ is called **reduced** if $\Phi \cap \mathbb{R}\alpha$ has cardinality 2 for each $\alpha \in \Pi$., i.e., if $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$. Given a $\mathbf{B}$-root system Φ, we set $\Phi_+ = \Phi \cap \mathbb{R}_+\Phi(\mathbf{B})_+$ and $\Phi_- = \Phi \cap \mathbb{R}_+\Phi(\mathbf{B})_-$. By Theorem 1.1.4(ii), we have $\Phi = \Phi_+ \cup \Phi_-$.

Note that by Sect. 1.1.6, there is a reflection $r_\beta \in W$ associated with every root β of a $\mathbf{B}$-root system Φ. A subset Ψ of Φ is called a **B-root subsystem** if Ψ is r_β–invariant for each $\beta \in \Psi$. Note that a root subsystem is a root system in a root subbase of $\mathbf{B}$, whose Weyl group is W_Ψ. We say that the $\mathbf{B}$-root subsystem Ψ is **parabolic** if W_Ψ is a parabolic subgroup of W, namely it is the Weyl group of a parabolic root subbasis.

Given any $\Psi \subset \Phi$, the set $\langle\Psi\rangle = \{w.\alpha \mid \alpha \in \Psi, w \in W_\Psi\}$, where $W_\Psi = \langle r_\beta \mid \beta \in \Psi\rangle$, is a root subsystem. It is called the **root subsystem generated by** Ψ.

For each $w \in W(\mathbf{B})$, we let

$$\Phi_w = \{\alpha \in \Phi_+ \mid w.\alpha \in \Phi_-\}.$$

Note that a decomposition similar to that of Lemma 1.1.5(iii) holds for Φ_w. In particular, this shows that the set Φ_w is finite.

Lemma. *Let* $\mathbf{B} = (\Pi, \Pi^\vee)$ *be a root basis. We have the following:*

(i) *Then* $\Phi(\mathbf{B})$ *is a reduced* $\mathbf{B}$*-root system if and only if for all* $\alpha, \beta \in \Pi$ *such that the order* $o(r_\alpha r_\beta)$ *is odd, one has* $A_{\alpha,\beta} = A_{\beta,\alpha}$.

(ii) *If there exists a* $\mathbf{B}$*-root system, then* $\Phi(\mathbf{B})$ *is a* $\mathbf{B}$*-root system.*

Proof. (i). By [Hée91, (2.17)], the set $\Phi(\mathbf{B})$ is a reduced root system if and only if $\Phi(\mathbf{B}_{\{\alpha,\beta\}})$ is a reduced root system for all distinct $\alpha, \beta \in \Pi$. Clearly, the subspace $V_{\alpha\beta}$ of V spanned by α and β is $W_{\{\alpha,\beta\}}$–invariant. Moreover, the $W_{\{\alpha,\beta\}}$–action on $V_{\alpha\beta}$ preserves the symmetric bilinear form $(\cdot,\cdot)$ defined by:

$$(\alpha, \alpha) = -A_{\alpha,\beta}, \qquad (\beta, \beta) = -A_{\beta,\alpha}, \qquad (\alpha, \beta) = -\frac{A_{\alpha,\beta}A_{\beta,\alpha}}{2}.$$

Therefore, in view of [Hée91, (2.16)], it follows that $\Phi(\mathbf{B}_{\{\alpha,\beta\}})$ is not a reduced root system if and only if $o(r_\alpha r_\beta)$ is finite and odd, and if moreover $A_{\alpha,\beta} \neq A_{\beta,\alpha}$.

(ii). Follows from the definitions. □

1.2.2. *Prenilpotent sets of roots.* Let $\mathbf{B} = (\Pi, \Pi^\vee)$ be a root basis and Φ be $\mathbf{B}$-root system. Given a set of roots $\Psi \subset \Phi$, we set

$$W_\varepsilon(\Psi) = \{w \in W \mid w.\alpha \in \Phi_\varepsilon \text{ for each } \alpha \in \Psi\}$$

for each sign $\varepsilon \in \{+, -\}$. Moreover, we set

$$\overline{\Psi} = \{\alpha \in \Phi \mid W_+(\Psi) \subset W_+(\alpha) \text{ and } W_-(\Psi) \subset W_-(\alpha)\}.$$

A subset $\Psi \subset \Phi$ is called **prenilpotent** if $W_+(\Psi)$ and $W_-(\Psi)$ are both nonempty. A prenilpotent set Ψ is called **nilpotent** if $\overline{\Psi} = \Psi$. Clearly for every set Ψ, the set $\overline{\Psi}$ is nilpotent.

Note that if Ψ is prenilpotent, there exist $v, w \in W$ such that $v.\Psi \subset \Phi_w$. Therefore, any prenilpotent set is finite (see Sect. 1.2.1). Furthermore, it is easy to verify that for each $w \in W$, the set Φ_w is nilpotent. Thus a set of positive roots is prenilpotent if and only if it is contained in Φ_w for some $w \in W$.

Given a pair $\{\alpha, \beta\} \subset \Phi$, we set

$$[\alpha, \beta] = \overline{\{\alpha, \beta\}}, \qquad]\alpha, \beta[= [\alpha, \beta]\backslash\{\lambda\alpha, \mu\beta \mid \lambda, \mu \in \mathbb{R}_+\}$$

and

$$[\alpha, \beta]_{\text{lin}} = \Phi \cap (\mathbb{R}_+\alpha + \mathbb{R}_+\beta), \qquad]\alpha, \beta[_{\text{lin}}= [\alpha, \beta]_{\text{lin}}\backslash\{\lambda\alpha, \mu\beta \mid \lambda, \mu \in \mathbb{R}_+\}.$$

Note that the set $[\alpha, \beta]_{\text{lin}}$ is contained in $[\alpha, \beta]$. However, the inclusion is proper in general, see [Rém02b, §5.4.2].

We record the following result for later references:

Lemma. *Let $\alpha, \beta \in \Phi$.*

(i) *If $\{\alpha, \beta\}$ generates a finite root subsystem, then $\{\alpha, \beta\}$ is prenilpotent if and only if $-\beta \notin \mathbb{R}_+\alpha$.*
(ii) *If $\{\alpha, \beta\}$ generates an infinite root subsystem, then $\{\alpha, \beta\}$ is prenilpotent if and only if $\langle \alpha, \beta^\vee \rangle > 0$.*
(iii) *If $\{\alpha, \beta\}$ is not prenilpotent, then $\{-\alpha, \beta\}$ is prenilpotent.*
(iv) *If $\{\alpha, \beta\}$ is prenilpotent with $\alpha \in \Pi_\Phi$ and $\beta \in \Phi_+$, then $]\alpha, \beta[\subset \Phi_+ \backslash \Phi_{r_\alpha}$.*
(v) *If $\{\alpha, \beta\}$ is prenilpotent, then for all $\gamma, \gamma' \in [\alpha, \beta]$, the pair $\{\gamma, \gamma'\}$ is prenilpotent and furthermore, we have $[\gamma, \gamma'] \subset [\alpha, \beta]$.*

Proof. (i). By Theorem 1.1.4(i), the Weyl group of a finite root system is a finite Coxeter group. The (unique) element of maximal length maps every positive root of this system to a negative one. The desired assertion follows easily.

(ii). By [MP95, Ch. 5, Prop. 8], we have $\langle \alpha, \beta^\vee \rangle > 0$ (resp. < 0) if and only if $\langle \beta, \alpha^\vee \rangle > 0$. Now if $\langle \alpha, \beta^\vee \rangle < 0$, then the set $[\alpha, \beta]_{\text{lin}}$ is infinite since the group $\langle r_\alpha, r_\beta \rangle$ is infinite. Therefore, the pair $\{\alpha, \beta\}$ cannot be prenilpotent, since the set $[\alpha, \beta]$, which contains $[\alpha, \beta]_{\text{lin}}$, is prenilpotent, hence finite. For the converse statement, see [Cap07, Lemma 2.3]

(iii). Follows from (i) and (ii).

(iv). Follows from Lemma 1.1.5(iii).

(v). We have mentioned above that a nilpotent set of roots is prenilpotent. Moreover, it is clear from the definition that any subset of a prenilpotent set of roots is prenilpotent. Thus $\{\gamma, \gamma'\}$ is prenilpotent. The inclusion $[\gamma, \gamma'] \subset [\alpha, \beta]$ follows from the definitions. □

1.3. **Root data.** A root datum consists in a root basis $\mathbf{B} = (\Pi, \Pi^\vee)$ such that $A_{\alpha,\beta} = A_{\beta,\alpha}$ for all $\alpha, \beta \in \Pi$ such that $o(r_\alpha r_\beta)$ is finite and odd, together with a $\mathbf{B}$-root system Φ. All the vocabulary used to qualify root bases (e.g. free, integral, irreducible, ...) will be used for root data as well, according as the property in question holds for the underlying root basis.

2. Root group data

2.1. **Axioms of a root group datum.** We are now ready to introduce the main object of study. Let G be a group and $E = (\mathbf{B}, \Phi)$ be a root datum. Thus $\mathbf{B} = (\Pi, \Pi^\vee)$ is a root basis in a real vector space V which will be held fixed throughout, and Φ is a $\mathbf{B}$-root system.

A **root group datum** of type E for G (formerly called a **twin root datum**) is a tuple $\{U_\alpha\}_{\alpha \in \Phi}$ of subgroups of G which, setting

$$U_+ = \langle U_\alpha \mid \alpha \in \Phi_+ \rangle \qquad \text{and} \qquad U_- = \langle U_\alpha \mid \alpha \in \Phi_- \rangle,$$

satisfies the following axioms.

(RGD0): For all $\alpha \in \Phi$, we have $U_\alpha \neq \{1\}$ and moreover $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$.
(RGD1): For each $\beta \in \Pi_\Phi$, we have $U_\beta \not\subset U_-$.
(RGD2): For each $\beta \in \Pi_\Phi$ and each $u \in U_\beta \backslash \{1\}$, there exists an element $\mu(u) \in U_{-\beta}.u.U_{-\beta}$ such that $\mu(u)\, U_\alpha\, \mu(u)^{-1} = U_{r_\beta.\alpha}$ for all $\alpha \in \Phi$.
(RGD3): For each prenilpotent pair $\{\alpha, \beta\} \subset \Phi$, we have $[U_\alpha, U_\beta] \subset \langle U_\gamma \mid \gamma \in]\alpha, \beta[\rangle$.
(RGD4): For each $\beta \in \Pi_\Phi$ there exists $\beta' \in \Phi_{r_\beta}$ such that $U_\alpha \subset U_{\beta'}$ for each $\alpha \in \Phi_{r_\beta}$.

The subgroups U_α of G are called **root subgroups**.

2.2. **Comments on the axioms of a root group datum.**

Remark 1.: Combining (RGD0) with (RGD2), it follows that G is generated by the set
$$\{U_\beta \mid \beta \in \Pi_\Phi\} \cup \{U_{-\beta} \mid \beta \in \Pi_\Phi\}.$$

Remark 2.: As it is the case for root bases, one obtains new systems of root subgroups from existing ones by taking products. We leave it to the reader to perform these constructions in details. In particular, if the root datum E is not irreducible, then G is a commuting product of subgroups, each one endowed with a root group datum indexed by a root subsystem of E.

Remark 3.: We will establish in Corollary 5.3(iii) below that $U_{-\beta} \not\subset U_+$ for each $\beta \in \Pi_\Phi$. Thus, the whole theory is 'symmetric in $+$ and $-$', although (RGD1) seems to break the symmetry at a first glance. In other words, if $\{U_\alpha\}_{\alpha\in\Phi}$ is a root group datum for G, then so is $\{U_{-\alpha}\}_{\alpha\in\Phi}$.

Remark 4.: A strengthened version of axiom (RGD3) is the following:

(RGD3)$_{\mathbf{lin}}$: For each prenilpotent pair $\{\alpha, \beta\} \subset \Phi$, we have
$$[U_\alpha, U_\beta] \subset \langle U_\gamma \mid \gamma \in]\alpha, \beta[_{\mathrm{lin}}\rangle.$$
This is indeed stronger than (RGD3), see Remark 1 of Sect. 1.2.2, and useful to prove Levi decompositions for parabolic subgroups. However, big parts of the theory can be developed using (RGD3) only.

Remark 5.: If $\widetilde{G}$ is an extension of G of the form $\widetilde{G} = T.G$, with G as above and T normalizing every root subgroup of G, then G is normal in $\widetilde{G}$ and it is common to view $\{U_\alpha\}_{\alpha\in\Phi}$ as a (non-generating) root group datum for $\widetilde{G}$. This is in fact the case in J. Tits original definition [Tit92]. In particular, the group $\widetilde{G}$ could be the direct product of G with *any* group. Thus most structure results on groups with a root group datum concern actually the subgroup $\widetilde{G}^\dagger = G$ generated by all root groups. That is why we have found natural to take the more restrictive condition that $G = \langle U_\alpha \mid \alpha \in \Phi\rangle$ as an axiom. It yields some technical simplifications and avoid to introduce a group T normalizing each root subgroup as part of the datum.

Remark 6.: Note that axiom (RGD4) is an empty condition if the $\mathbf{B}$-root system Φ is reduced. In fact, this axiom does not appear in [Tit92], but it does appear in the Bruhat-Tits' earlier definition of root group data [BT72, §6.1, (DR3)]. In fact, we will see in Lemma 2.4 that (RGD4) allows one to define a reduction of an arbitrary root group datum, which is a root group datum indexed by a reduced root system.

2.3. **Root group data for root subsystems.** Let $E = (\mathbf{B}, \Phi)$ be a root datum. Given a $\mathbf{B}$-root subsystem $\Psi \subset \Phi$ and a root group datum $\{U_\alpha\}_{\alpha\in\Phi}$ for a group G, we say that Ψ is **quasi-closed** if for each prenilpotent pair $\{\alpha, \beta\} \subset \Psi$, the group $[U_\alpha, U_\beta]$ is contained in the subgroup generated by root groups U_γ with $\gamma \in]\alpha, \beta[\cap \Psi$. The proof of the following statement is a straightforward verification:

Lemma. *Let G be a group endowed with a root group datum $(U_\alpha)_{\alpha\in\Phi}$ of type E. Given a $\mathbf{B}$-root subsystem $\Psi \subset \Phi$ which is quasi-closed, we set $G_\Psi = \langle U_\psi \mid \psi \in \Psi\rangle$. Then $\{U_\psi\}_{\psi\in\Psi}$ is a root group datum for G_Ψ.* □

Remark. An obvious sufficient condition for Ψ to be quasi-closed is that it is **closed**, that is to say: $[\alpha, \beta] \subset \Phi$ for each prenilpotent pair $\{\alpha, \beta\} \subset \Psi$. This is for example the case if Φ is a parabolic root subsystem. If the root group datum of G satisfies moreover the axiom (RGD3)$_{\mathrm{lin}}$, then Ψ is quasi-closed whenever it is **linearly closed**, namely $[\alpha, \beta]_{\mathrm{lin}} \subset \Psi$ for each prenilpotent pair $\{\alpha, \beta\} \subset \Psi$.

2.4. **A reduction.** Let $E = (\mathbf{B}, \Phi)$ be a root datum. By definition of root data, Lemma 1.2.1(i) shows that $\Phi(\mathbf{B})$ is a reduced $\mathbf{B}$-root system. For each $\alpha \in \Phi(\mathbf{B})$, we set
$$U_{(\alpha)} = \langle U_\beta \mid \beta \in \Phi,\ \beta = \lambda\alpha \text{ for some } \lambda > 0\rangle.$$

Lemma. *The system $\{U_{(\alpha)}\}_{\alpha\in\Phi(\mathbf{B})}$ is a root group datum of type $(\mathbf{B}, \Phi(\mathbf{B}))$ for G.*

Proof. It is clear from the definition that (RGD0) and (RGD1) hold. By (RGD4) for the original root group datum, we deduce that for each $\alpha \in \Pi$ there exists $\beta \in \Pi_\Phi$ such that $U_{(\alpha)} = U_\beta$. Therefore (RGD2) holds as well. The fact that (RGD3) holds follows easily by combining (RGD3) for the original root group datum with Lemma 1.2.2(v). Finally, since $\Phi(\mathbf{B})$ is reduced, the axiom (RGD4) is clearly satisfied. □

The lemma shows that any root group datum for a group G yields a root group datum for G indexed by a reduced root system. Most structure results on groups endowed with a root group datum assume that the underlying root system is reduced. In view of the reduction presented above, this assumption causes no loss of generality.

2.5. **Example: rank one groups.** The purpose of the present subsection and the following ones is to describe a first set of examples of groups admitting a root group datum.

A group G is called a **rank one group** if it admits a root group datum indexed by a root system of rank one, which can be assumed to be reduced in view of Sect. 2.4. Equivalently G possesses nontrivial subgroups U_+ and U_-, whose union generates G and such that for each $u \in U_+\backslash\{1\}$ there exists $\mu(u) \in U_-.u.U_-$ such that conjugation by $\mu(u)$ swaps U_+ and U_-. It is easy to see that the latter condition is equivalent to the following, where $A = U_+$ and $B = U_-$:

$$\text{for each } a \in A\backslash\{1\}, \text{ there exists } b \in B \text{ such that } {}^bA = {}^aB.$$

For instance, the group $G = \mathrm{SL}_2(k)$, where k is any field, is a rank one group with root subgroups

$$A = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in k \right\} \qquad \text{and} \qquad B = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in k \right\}.$$

Indeed, given $a = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $x \in k^\times$, one has ${}^bA = {}^aB$ with $b = \begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix}$.

It is common to consider a rank one group as a permutation group on the conjugacy class of its root subgroups (note that there is such a unique conjugacy class). This permutation action makes this conjugacy class a so called **Moufang set**; we refer to [dMS07] for a survey on this notion.

Finite rank one groups have been classified by C. Hering, W. Kantor and G. Seitz [HKS72] and this work is a fundamental step in the classification of finite simple groups. More precisely:

Theorem. *Let G be a finite 2-transitive group on a set Ω and suppose that, for $\alpha \in \Omega$, the stabilizer G_α has a normal subgroup regular on $\Omega \setminus \alpha$. Then G contains a normal subgroup M and M acts on Ω as one of the following groups in their usual 2-transitive representation: a sharply 2-transitive group,* $\mathrm{PSL}(2,q)$, $\mathrm{Sz}(q)$, $\mathrm{PSU}(3,q)$ *or a group of Ree type.*

Thus a finite rank-one group is either a sharply-2-transitive group or a finite group of Lie type and Lie rank one. No such classification is known in the infinite case, but this is an active area of research. Let us mention that very little is known about sharply-2-transitive infinite groups, and that the only known examples of infinite rank one groups which are not sharply-2-transitive are all of Lie type (in an appropriate sense). Furthermore, in these examples, the root groups are nilpotent of class at most 3. The case of abelian root subgroups seems to be intimately related to quadratic Jordan division algebras [dMW06] which paves the way towards a general theory of Moufang sets.

2.6. **Example: (isotropic) reductive algebraic groups.** Standard references are [Bor91] and [Spr98]. Let $\mathbf{G}$ be a reductive linear algebraic group defined over a field k. Assume that $\mathbf{G}$ is isotropic over k, namely that some proper parabolic subgroup of $\mathbf{G}$ is defined over k or, equivalently, that $\mathbf{G}(k)$ seen as a matrix group contains an infinite abelian subgroup of diagonal matrices. Let T be a maximal k-split k-torus. Borel-Tits theory [BT65] implies the existence of a root group datum $\{U_\alpha\}_{\alpha\in\Phi}$, indexed by the relative root system Φ of $(\mathbf{G}(k), T(k))$, for the group $\mathbf{G}(k)^\dagger$ which is generated by the k-points of unipotent radicals of parabolic k-subgroups of $\mathbf{G}$. This root group datum satisfies the extra condition $(\mathrm{RGD3})_{\mathrm{lin}}$.

A complementary fact is the following statement:

Theorem. *Groups endowed with a root group datum of rank $\geqslant 2$ and* ***finite*** *irreducible Weyl group are classified.*

This follows from the work of J. Tits [Tit74] for root data of rank $\geqslant 3$ and Tits-Weiss [TW02] for rank 2, all combined with [BT73]. The result can be loosely summarized by saying that all groups with such a root group datum are 'of Lie type' in an appropriate sense. In slightly more precise terms, these groups are classical groups over skew fields or reductive algebraic groups over fields, or twisted forms of them, which might not be algebraic in the strict sense (e.g. the Suzuki groups ${}^2B_2$, the Ree-Tits groups ${}^2F_4$ [Tit83] or the so-called "mixed groups" of Tits). An important step in the classification is that, denoting by $\{s, s'\}$ the canonical generating set of the finite Weyl group of a root group datum of rank 2, then $o(ss') \in \{2, 3, 4, 6, 8\}$. Therefore, it follows from Lemma 2.3 that for *any* root group datum indexed by a root system Φ, we have $o(r_\alpha r_\beta) \in \{1, 2, 3, 4, 6, 8, \infty\}$ for all $\alpha, \beta \in \Phi$.

2.7. **Example: some arithmetic groups.** Let k be any field and consider the (S-)arithmetic group $G = \mathrm{SL}_n(k[t, t^{-1}])$. Let

$$T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in k^\times \right\}$$

and $E = (\mathbf{B}, \Phi)$ be the classical root datum of $\mathrm{SL}_n(k)$ with respect to T, whose underlying vector space is $V \simeq \mathbb{R}^{n-1}$ endowed with the Killing form $(\cdot, \cdot)$. Note that $\Phi = \Phi(\mathbf{B})$ in this case. The basis $\Pi = \Pi_\Phi$ corresponds to the Borel subgroup of upper triangular matrices. Thus roots in Φ are in one-to-one correspondence with pairs (i, j) such that $i \neq j$ and $i, j \in \{1, \ldots, n\}$. If the root α corresponds to (i, j) one has a mapping (in fact: a morphism of functors) $u_\alpha : k \to \mathrm{SL}_n(k) : x \mapsto \mathbf{1}_{n\times n} + e_{ij}(x)$, where $e_{ij}(x)$ denotes the $n \times n$-matrix with x as the (i, j)-entry and 0 elsewhere. Furthermore, the tuple $\big\{\{u_\alpha(x) \mid x \in k\}\big\}_{\alpha\in\Phi}$ is a root group datum for $\mathrm{SL}_n(k)$.

Now we make the following definitions:

- $V^{\mathrm{aff}} = V \oplus \mathbb{R}e$;
- $\Phi^{\mathrm{aff}} = \{\alpha + n.e \mid \alpha \in \Phi, n \in \mathbb{Z}\}$;
- $\Pi^{\mathrm{aff}} = \Pi \cup \{-\alpha_0 + e\}$, where α_0 is the highest root of Φ;
- $(\cdot, \cdot)$ is the extension to V^{aff} of the Killing form, defined by the assignments
$$(\alpha, e) = (e, e) = 0$$
for all $\alpha \in \Phi$;
- ${}^\vee : \Phi^{\mathrm{aff}} \to (V^{\mathrm{aff}})^* : \alpha \to \alpha^\vee = 2(\cdot, \alpha)$.

One verifies that $\mathbf{B}^{\mathrm{aff}} = (\Pi^{\mathrm{aff}}, \{\beta^\vee\}_{\beta\in\Pi^{\mathrm{aff}}})$ is a root basis for V^{aff} with canonical root system Φ^{aff}. Its Weyl group is the so-called **affine Weyl group** of SL_n. It is isomorphic to the automorphism group of a tiling of Euclidean $(n-1)$-space by (hyper-)tetrahedra.

Note that a pair $\{\alpha+m.e, \beta+n.e\}$ of roots in Φ^{aff}, with $\alpha, \beta \in \Phi$ and $m, n \in \mathbb{Z}$, is prenilpotent if and only if $\alpha \neq -\beta$.

It is now an exercise to check that the system $\big\{\{u_\alpha(xt^n) \mid x \in k\}\big\}_{\alpha+n.e\in\Phi^{\mathrm{aff}}}$ is a root group datum of type $(B^{\mathrm{aff}}, \Phi^{\mathrm{aff}})$ for G.

2.8. **Example: a "free" construction.** Here we indicate how to construct a root group datum with infinite dihedral Weyl group starting from any two rank one groups. We first describe the underlying root datum.

Let $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ and $\Pi = \{e_1, -e_1 + e_2\}$. Let also $(\cdot, \cdot)$ by the symmetric bilinear form on V whose Gram matrix in the canonical basis $\{e_1, e_2\}$ is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and let ${}^\vee : \Phi \to V^* : \alpha \mapsto 2(\cdot, \alpha)$. Then $\mathbf{B} = (\Pi, \{\alpha^\vee\}_{\alpha\in\Pi})$ is a root basis. Its canonical root system is $\Phi(\mathbf{B}) = \Phi = \{\pm e_1 + n.e_2 \mid n \in \mathbb{Z}\}$ and its Weyl group W is infinite dihedral. We let $S = \{s_1, s_2\}$ be its canonical generating set, where $s_1 = r_{e_1}$ and $s_2 = r_{e_2 - e_1}$.

Let $\Phi_1 = \{\pm e_1\}$, $\Phi_2 = \{e_1 - e_2, -e_1 + e_2\}$, $\Pi_1 = \{e_1\}$ and $\Pi_2 = \{-e_1 + e_2\}$. Thus $E_i = \big((\Pi_i, \Pi_i^\vee), \Phi_i\big)$ is a rank one root datum for $i = 1, 2$. Let G_i be a group with a root datum $\{U_\alpha\}_{\alpha \in \Phi_i}$ of type E_i for $i = 1, 2$. Note that G_i may be any rank one group. Let

$$T_i = \langle \mu(u)\mu(v) \mid u, v \in U_\alpha \backslash \{1\}, \alpha \in \Pi_i \rangle$$

where $i = 1, 2$ and set $T = T_1 \times T_2$. We define

$$\widetilde{G} = (G_1 \times T_2) *_T (T_1 \times G_2)$$

and

$$N = T.\langle \mu(u) \mid u \in U_\alpha \backslash \{1\}, \alpha \in \Pi_1 \cup \Pi_2 \rangle < \widetilde{G}.$$

Note that T is normal in N. Furthermore, the unique homomorphism $W \to N/T$, defined by the assignments $s_1 \mapsto \mu(u_1).T$ and $s_2 \mapsto \mu(u_2).T$ where u_i is some fixed nontrivial element of U_α with $\alpha \in \Pi_i$, is in fact an isomorphism. Thus the quotient N/T is infinite dihedral. Therefore, there is a well-defined W–equivariant map $\Phi \to \{nU_\alpha n^{-1} \mid n \in T, \alpha \in \Pi_1 \cup \Pi_2\}$. In particular, we may use Φ as an index set for the family $\{nU_\alpha n^{-1} \mid n \in T, \alpha \in \Pi_1 \cup \Pi_2\}$. Now, one verifies that the system $\{U_\alpha\}_{\alpha \in \Phi}$ of subgroups of $\widetilde{G}$ satisfies (RGD0)–(RGD2). In order to make (RGD3) hold, one just add the necessary relations. More precisely, let H be the normal closure in $\widetilde{G}$ of the subset

$$\{[U_\alpha, U_\beta] \mid \alpha \neq \beta \text{ and } \{\alpha, \beta\} \subset \Phi \text{ is prenilpotent}\}.$$

We denote by G the quotient $\widetilde{G}/H$. The projection in G of the subgroup $U_\alpha < \widetilde{G}$ is again denoted by U_α. It turns out that the system $\{U_\alpha\}_{\alpha \in \Phi}$ is a root group system of type E for G.

This construction is due to J. Tits [Tit90, §9]. An alternative description, with detailed computations, and a generalization to other types of root data (with any right-angled Coxeter group as Weyl group), is carried out in [RR06].

3. Kac-Moody theory

The purpose of this section is to indicate that Kac-Moody theory provides a wide variety of examples of groups endowed with a root group datum with infinite Weyl groups. The origin of this theory lies in the classification of finite-dimensional simple Lie algebras over $\mathbb{C}$. A key tool in this classification is the existence of a Cartan decomposition, namely a root space decomposition with respect to a certain abelian subalgebra whose adjoint action is diagonalizable, and called a Cartan subalgebra. A basic idea in Kac-Moody theory is to construct a family of Lie algebras by generators and relations, where the relations impose the existence of a Cartan decomposition. Carrying out this idea, V. Kac was able to construct a *continuous* family of finitely generated simple Lie algebras. Our first goal is to explain this construction.

3.1. **Constructing Lie algebras with a Cartan decomposition.** We start with a matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$ of rank l and consider a triple $(\mathfrak{h}_\mathbb{R}, \Pi, \Pi^\vee)$ where $\mathfrak{h}_\mathbb{R}$ is a $\mathbb{R}$-vector space of dimension $2n - l$, $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is a linearly independent subset of $\mathfrak{h}_\mathbb{R}^*$, $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ is a linearly independent subset of $\mathfrak{h}_\mathbb{R}$ and the relation

$$\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$$

holds for all $i, j \in \{1, \dots, n\}$. Note that such a triple always exists and is unique up to isomorphism. Next we consider a Lie algebra $\tilde{\mathfrak{g}}(A)$ generated by $\{e_i, f_i \mid i = 1, \dots, n\}$ and a basis of $\mathfrak{h}_\mathbb{R}$, submitted to the following relations:

$$\begin{array}{rcll} [e_i, f_j] & = & \delta_{ij}\alpha_i^\vee & (i, j = 1, \dots n), \\ [h, h'] & = & 0 & (h, h' \in \mathfrak{h}_\mathbb{R}), \\ [h, e_i] & = & \langle \alpha_i, h \rangle.e_i & (i = 1, \dots, n; h \in \mathfrak{h}_\mathbb{R}), \\ [h, f_i] & = & -\langle \alpha_i, h \rangle.f_i & (i = 1, \dots, n; h \in \mathfrak{h}_\mathbb{R}). \end{array}$$

A fundamental result by V. Kac is the following:

Theorem. *Let $\tilde{n}_+$ (resp. $\tilde{n}_-$) be the subalgebra generated by $\{e_i \mid i = 1, \dots, n\}$ (resp. $\{f_i \mid i = 1, \dots, n\}$). Let also $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \otimes \mathbb{C}$, $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$ and $Q_+ = \sum_{i=1}^n \mathbb{Z}_+\alpha_i$. We have the following:*

(i) $\tilde{g}(A) = \tilde{n}_- \oplus \mathfrak{h} \oplus \tilde{n}_+$.
(ii) $\tilde{n}_+$ *(resp. $\tilde{n}_-$) is freely generated by $\{e_i \mid i = 1, \dots, n\}$ (resp. $\{f_i \mid i = 1, \dots, n\}$).*

(iii) *With respect to the adjoint* $\mathfrak{h}$*–action, one has a decomposition*

$$\tilde{\mathfrak{g}}(A) = \Big(\bigoplus_{\alpha\in Q_+\setminus\{0\}} \tilde{\mathfrak{g}}_\alpha\Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\alpha\in Q_+\setminus\{0\}} \tilde{\mathfrak{g}}_{-\alpha}\Big),$$

where $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}}(A) \mid [h, x] = \langle \alpha, h\rangle.x$ *for all* $h \in \mathfrak{h}\}$*.*

(iv) *The assignments* $e_i \mapsto -f_i$*,* $f_i \mapsto -e_i$ $(i = 1, \ldots, n)$*,* $h \mapsto -h$ $(h \in \mathfrak{h})$ *extend to an involutory automorphism* $\tilde{\omega} \in \operatorname{Aut} \tilde{\mathfrak{g}}(A)$*.*

(v) *Amongst all ideals intersecting* $\mathfrak{h}$ *trivially, there is a unique maximal one, say* $\mathfrak{r}$*.*

Proof. See [Kac90, Theorem 1.2]. Here, we merely note that (v) follows rather quickly from the root space decomposition (iii). Indeed, let U be any nontrivial ideal of $\tilde{\mathfrak{g}}(A)$ intersecting $\mathfrak{h}$ trivially and let $u \in U$ be a nonzero element. By (iii), we have $u = \sum_{i=1}^k u_i$, where $u_i \in \tilde{\mathfrak{g}}_{\alpha_i}$ and $\alpha_i \in \pm Q_+$ for each $i = 1, \ldots, k$. Since $\mathfrak{h}$ is not a finite union of hyperplanes, there exists $h \in \mathfrak{h}$ such that the scalars $\alpha_i(h)$ $(i = 1, \ldots, k)$ are all distinct. Now, for each $j \in \mathbb{N}$ we have

$$(\text{ad } h)^j(u) = \sum_{i=1}^k \langle \alpha_i, h\rangle^j.u_i \in U.$$

Since the matrix $(\langle \alpha_i, h\rangle^j)_{i,j=1}^k$ has nonzero determinant (it is a Vandermonde matrix), it follows that $u_i \in U$ for each $i = 1, \ldots, k$. In other words, the root space decomposition (iii) induces a similar decomposition of U. This shows that the sum of all ideals intersecting $\mathfrak{h}$ trivially is itself an ideal intersecting $\mathfrak{h}$ trivially. This is nothing else than the clever use of a classical trick to show that the restriction of a diagonalizable endomorphism is still diagonalizable. □

We define a Lie algebra $\mathfrak{g}(A)$ as the quotient $\tilde{\mathfrak{g}}(A)/\mathfrak{r}$, where $\mathfrak{r}$ is the maximal ideal of (v). As a consequence of the latter theorem, it is not difficult to establish the following (see [Kac90, Proposition 1.7]):

Corollary. *The Lie algebra* $\mathfrak{g}(A)$ *is simple if and only if* $\det A$ *is nonzero and for each* $i, j \in \{1, \ldots, n\}$ *there exists a sequence of indices* $i = i_0, i_1, \ldots, i_s = j$ *such that* $a_{i_{j-1} i_j}$ *is nonzero for each* $j = 1, \ldots, s$*.*

Note that it is an open problem to determine whether the matrix A (up to a permutation of the indices preserving A) is an invariant of the isomorphism class of the Lie algebra $\mathfrak{g}(A)$. This is only known for special classes of matrices, all of which are generalized Cartan matrices (see Sect. 3.2 below).

The root space decomposition (iii) above induces a decomposition $\mathfrak{g}(A) = \bigoplus_{\alpha\in Q} \mathfrak{g}_\alpha$. Note that by the definition of $\mathfrak{g}(A)$ we have $\mathfrak{g}_0 \simeq \mathfrak{h}$ and we will in fact identify the latter two algebras. Thus the decomposition of $\mathfrak{g}(A)$ is in fact a root space decomposition for the adjoint action of $\mathfrak{h}$. We define $\Phi = \{\alpha \in Q\setminus\{0\} \mid \mathfrak{g}_\alpha \neq 0\}$; elements of Φ are called **roots**. We also set $\Phi_\pm = \Phi \cap Q_\pm$, where $Q_- = -Q_+$.

The rule

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}, \tag{3.1}$$

valid for arbitrary $\alpha, \beta \in \mathfrak{h}^*$, shows that for each root $\alpha \in \Phi_+$, the root space $\mathfrak{g}_\alpha$ is the linear span of elements of the form

$$[\ldots[[e_{i_1}, e_{i_2}], e_{i_3}]\ldots, e_{i_s}]$$

such that $\alpha_{i_1} + \cdots + \alpha_{i_s} = \alpha$. Consequently, we obtain the obvious bound

$$\dim \mathfrak{g}_\alpha \leqslant n^{\text{height}\,\alpha} \tag{3.2}$$

for any $\alpha \in \Phi_+$, where by definition

$$\text{height}\Big(\sum_{i=1}^n \lambda_i \alpha_i\Big) = \Big|\sum_{i=1}^n \lambda_i\Big|$$

for any $\alpha = \sum_{i=1}^n \lambda_i \alpha_i \in Q$. The above description of $\mathfrak{g}_\alpha$ also shows that

$$\dim \mathfrak{g}_{\alpha_i} = 1 \qquad \text{and} \qquad \dim \mathfrak{g}_{\lambda\alpha_i} = 0 \tag{3.3}$$

for any $i = 1, \ldots, n$ and $\lambda \in \mathbb{Z}$, $\lambda > 1$. Similar statements hold for negative roots and f_i instead of e_i. A quick way to establish this is by applying the involution ω of $\mathfrak{g}(A)$, induced by the involution $\tilde{\omega} \in \mathrm{Aut}(\tilde{\mathfrak{g}}(A))$ mentioned in point (iv) of the theorem.

Finally, we remark that, in view of the root space decomposition of $\mathfrak{g}(A)$, the subalgebra $\mathfrak{r}_i$ generated by e_i and f_i is 3-dimensional. Now, if $a_{ii} = 0$, then $\mathfrak{r}_i$ is isomorphic to a Heisenberg Lie algebra. If $a_{ii} \neq 0$, then $\mathfrak{r}_i$ is not solvable and, hence, it must be isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

3.2. **Kac-Moody algebras.**

3.2.1. *The root basis and its canonical root system.* The Lie algebra $\mathfrak{g}(A)$ is called a **Kac-Moody algebra** if the matrix A is a **generalized Cartan matrix**, namely if $A \in \mathbb{Z}^{n\times n}$ and moreover $a_{ii} = 2$, $a_{ij} \leqslant 0$ and $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$ for all $i \neq j \in \{1, \ldots, n\}$. This is equivalent to the requirement that $\mathbf{B}(A) = (\Pi, \Pi^\vee)$ be an integral root basis. Note that $\mathbf{B}(A)$ is free by assumption. Let $S = S(\mathbf{B}(A))$ and $W = W(\mathbf{B}(A))$ be the Weyl group of $\mathbf{B}(A)$. By Theorem 1.1.4(i), for all distinct $\alpha, \beta \in \Pi$ we have $o(r_\alpha r_\beta) = 2, 3, 4, 6$ or ∞ according as $A_{\alpha,\beta}A_{\beta,\alpha} = 0, 1, 2, 3$ or $\geqslant 4$. In particular the set $\Phi(\mathbf{B}(A))$ is a reduced root system by Lemma 1.2.1(i). We will see in the next subsection that that the root system $\Phi(\mathbf{B}(A))$ has in fact a Lie theoretic interpretation in the present context.

3.2.2. *Lifting the Weyl group.* A basic fact on Kac-Moody algebras is that they satisfy Serre's relations:

$$(\mathrm{ad}\, e_i)^{1-a_{ij}}\, e_j = 0 \qquad \text{and} \qquad (\mathrm{ad}\, f_i)^{1-a_{ij}}\, f_j = 0$$

for all $i \neq j$. This follows from basic computations in $\mathfrak{sl}_2(\mathbb{C})$-modules, see [Kac90, §3.3]. An immediate consequence is the following:

Lemma. *The operators* $\mathrm{ad}\, e_i$ *and* $\mathrm{ad}\, f_i$ *are locally nilpotent on* $\mathfrak{g}(A)$ *for all* $i = 1, \ldots, n$.

Proof. Recall that a linear operator $A \in \mathrm{End}(V)$ of a vector space V is called locally nilpotent if every vector $v \in V$ is contained in a finite-dimension A-stable subspace U such that the restriction of A to U is nilpotent. In view of the definition of $\tilde{\mathfrak{g}}(A)$ and $\mathfrak{g}(A)$, we have $(\mathrm{ad}\, e_i)^2 h = 0$ for any $h \in \mathfrak{h}$. In view of Serre's relations, it follows that for any generator x of the Lie algebra $\mathfrak{g}(A)$ there is an integer N_x such that $(\mathrm{ad}\, e_i)^{N_x} x = 0$. Now, using Leibniz' rule (note that $\mathrm{ad}\, e_i$ is a derivation of $\mathfrak{g}(A)$ by Jacobi's identity), one deduces by a straightforward induction on iterated commutators of the generators of $\mathfrak{g}(A)$ that $\mathrm{ad}\, e_i$ is locally nilpotent. Similar discussions apply to $\mathrm{ad}\, f_i$. □

From the lemma it follows that

$$\exp \mathrm{ad}\, e_i = \sum_{m=0}^{\infty} \frac{1}{m!} (\mathrm{ad}\, e_i)^m$$

is a well-defined automorphism of $\mathfrak{g}(A)$.

Now, for each $i \in \{1, \ldots, n\}$, we consider the automorphism

$$r_i = \exp \mathrm{ad}\, e_i . \exp \mathrm{ad} - f_i . \exp \mathrm{ad}\, e_i \in \mathrm{Aut}(\mathfrak{g}(A)).$$

Note that r_i stabilizes the subalgebra $\mathfrak{r}_i \simeq \mathfrak{sl}_2(\mathbb{C})$ and acts on it as the involution $e_i \mapsto -f_i$, $f_i \mapsto -e_i$, $\alpha_i^\vee \mapsto -\alpha_i^\vee$. Furthermore, straightforward computations show that

$$r_i(h) = h - \langle \alpha_i, h \rangle \alpha_i^\vee$$

for all $h \in \mathfrak{h}$. In particular, the automorphism r_i preserves $\mathfrak{h}$ and, consequently, preserves the corresponding root space decomposition of $\mathfrak{g}(A)$. In other words r_i induces a permutation of Φ, which we denote by $r_i^\vee$. In fact, one can easily compute the action of $r_i^\vee$ on Φ by transforming the equation $[h, x] = \langle \alpha, h \rangle . x$ (satisfied by all $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha$ and $\alpha \in \Phi$) by r_i. Routine computations then show that

$$r_i^\vee(\alpha) = \alpha - \langle \alpha, \alpha_i^\vee \rangle . \alpha_i .$$

This extends to a linear action of $r_i^\vee$ on $\mathfrak{h}^*$ which is nothing but the dual action of r_i. The following result sums up the preceding discussion:

Proposition. *The canonical* $\mathbf{B}(A)$*–root system* $\Phi(\mathbf{B}(A))$ *identifies in a canonical way to a subset* ${}^{\mathrm{re}}\Phi$ *of the set of roots* Φ *of the Lie algebra* $\mathfrak{g}(A)$. $\square$

Note that we recover the fact that $\Phi(\mathbf{B}(A))$ is reduced thanks to Equation (3.3).

A remarkable feature of Kac-Moody theory is that Φ is real, i.e. $\Phi = {}^{\mathrm{re}}\Phi$, if and only if $\mathfrak{g}(A)$ is finite-dimensional, in which case it is a well understood semisimple Lie algebra, see [Kac90, Th. 5.6]. The elements of ${}^{\mathrm{im}}\Phi = \Phi \backslash {}^{\mathrm{re}}\Phi$ are called **imaginary roots**.

An important open problem of the theory is to compute the dimension of the root space $\mathfrak{g}_\alpha$ for α imaginary; recall that Equation (3.2) provides a rough upper-bound. In view of (3.3), we have $\dim \mathfrak{g}_\alpha = 1$ for any $\alpha \in {}^{\mathrm{re}}\Phi$. For such a root $\alpha \in {}^{\mathrm{re}}\Phi$, we set

$$U_\alpha = \langle \exp \operatorname{ad} x \mid x \in \mathfrak{g}_\alpha \rangle$$

which is a well-defined one-parameter subgroup of $\mathrm{Aut}(\mathfrak{g}(A))$ since $\operatorname{ad} x$ is locally nilpotent by the lemma above.

Note also that for each i, the reflection r_{α_i} of the root basis $\mathbf{B}(A)$ coincides with the restriction of $r_i^\vee$ to the $\mathbb{R}$-form $\mathfrak{h}_\mathbb{R}^*$ of $\mathfrak{h}^*$. The Weyl group $W < \mathrm{GL}(\mathfrak{h}_\mathbb{R}^*)$ is thus isomorphic to the subgroup of $\mathrm{GL}(\mathfrak{h})$ (resp. $\mathrm{GL}(\mathfrak{h}^*)$) generated by the corresponding restrictions of the r_i's (resp. $r_i^\vee$). Note however that the subgroup of $\mathrm{Aut}(\mathfrak{g}(A))$ generated by the r_i's is not isomorphic to W, but to an extension of W by an elementary abelian 2-group of rank n. This extended Weyl group is studied by J. Tits in [Tit66].

3.3. **Root group data for Kac-Moody groups.** Maintain the notation of the previous subsection. We let moreover G be the subgroup of $\mathrm{Aut}(\mathfrak{g}(A))$ generated by the U_α's. The group G is called the **adjoint Kac-Moody group** of type A over $\mathbb{C}$.

Theorem. *The tuple* $\{U_\alpha\}_{\alpha \in {}^{\mathrm{re}}\Phi}$ *is a root group datum for* G*, satisfying also* $(\mathrm{RGD3})_{\mathrm{lin}}$.

Proof. Condition (RGD0) holds by construction. For (RGD1), note that U_+ stabilizes the subalgebra $\mathfrak{n}_+$ generated by the e_i's. Moreover, the group $U_{-\alpha_i}$ stabilizes the subalgebra $\mathfrak{x}_i$. It follows that $U_{-\alpha_i} \not\subset U_+$, otherwise $U_{-\alpha_i}$ would stabilize $\mathfrak{x}_i \cap \mathfrak{n}_+ = \mathfrak{g}_{\alpha_i}$, which is absurd. A similar argument shows that $U_{\alpha_i} \not\subset U_-$, hence (RGD1) holds. Condition (RGD2) is satisfied as follows from the preceding discussion on the automorphisms $r_i \in \mathrm{Aut}(\mathfrak{g}(A))$. Moreover (RGD4) is empty since Φ is reduced. It remains to establish $(\mathrm{RGD3})_{\mathrm{lin}}$. To this end, for any prenilpotent pair $\{\alpha, \beta\} \subset {}^{\mathrm{re}}\Phi$ we let $\mathfrak{g}_{[\alpha,\beta]}$ be the vector space generated by all root spaces $\mathfrak{g}_\gamma$ with $\gamma \in [\alpha,\beta]_{\mathrm{lin}} = \Phi \cap (\mathbb{R}_+\alpha + \mathbb{R}_+\beta)$. Thus we have $\mathfrak{g}_{[\alpha,\beta]} = \bigoplus_{\gamma \in [\alpha,\beta]_{\mathrm{lin}}} \mathfrak{g}_\gamma$ and $\mathfrak{g}_{[\alpha,\beta]}$ is finite-dimensional since nilpotent sets of roots are necessarily finite by Sect. 1.2.2. Moreover, the rule (3.1) shows that $\mathfrak{g}_{[\alpha,\beta]}$ is in fact a nilpotent subalgebra.

Let now $\widetilde{U}_{[\alpha,\beta]}$ be the simply connected complex Lie group with Lie algebra $\mathfrak{g}_{[\alpha,\beta]}$. Thus $\widetilde{U}_{[\alpha,\beta]}$ is nothing but the set $\mathfrak{g}_{[\alpha,\beta]}$ endowed with a composition law $(u,v) \mapsto u * v$ given by the Baker-Campbell-Hausdorff formula. We also denote by $U_{[\alpha,\beta]}$ the subgroup of $\mathrm{Aut}(\mathfrak{g}(A))$ generated by $\exp \operatorname{ad} x$ for $x \in \mathfrak{g}_{[\alpha,\beta]}$. Now, it follows from the definitions that there is a canonical homomorphism

$$\varphi : \widetilde{U}_{[\alpha,\beta]} \to U_{[\alpha,\beta]}.$$

Furthermore, denoting by $\widetilde{U}_\gamma$ the one-parameter subgroup of $\widetilde{U}_{[\alpha,\beta]}$ with Lie algebra $\mathfrak{g}_\gamma$ for each $\gamma \in [\alpha,\beta]$, we have $\varphi(\widetilde{U}_\gamma) = U_\gamma$ and we obtain a product decomposition

$$\widetilde{U}_{[\alpha,\beta]} = \prod_{\gamma \in [\alpha,\beta]_{\mathrm{lin}}} \widetilde{U}_\gamma$$

induced by the decomposition of $\mathfrak{g}_{[\alpha,\beta]}$. Routine computations then show that the Lie algebra of the commutator group $[\widetilde{U}_\alpha, \widetilde{U}_\beta]$ is contained in $\sum_{\gamma \in]\alpha,\beta[_{\mathrm{lin}}} \mathfrak{g}_\gamma$, which yields $[\widetilde{U}_\alpha, \widetilde{U}_\beta] \subset \prod_{\gamma \in]\alpha,\beta[_{\mathrm{lin}}} \widetilde{U}_\gamma$. Transforming by φ, we deduce that axiom (RGD3) is satisfied. $\square$

3.4. **Generalizations to arbitrary fields and non-split groups.** In a similar way as complex semisimple Lie groups may be defined over arbitrary fields following Chevalley's construction, J. Tits [Tit87] has shown that similar constructions may be performed in the Kac-Moody context. A key point in this construction is to show that the simply connected nilpotent Lie groups $\widetilde{U}_{[\alpha,\beta]}$ that appeared in the proof of Theorem 3.3 are in fact the groups of $\mathbb{C}$-points of nilpotent group schemes defined over $\mathbb{Z}$ [Tit87, Prop. 1]. In somewhat less precise terms, this means that the commutation relations in $\widetilde{U}_{[\alpha,\beta]}$ may be written with integral coefficients in a similar way as in the classical case [Ste68, Lemma 15]. These integral coefficients may then be used to write down a Steinberg type presentation for a group over an arbitrary ground field, see [Tit87, §3.6].

In fact, Tits' construction associates a group functor

$$\mathcal{G}_{\mathbf{B}} : \mathrm{Rgs} \to \mathrm{Gps}$$

on the category of commutative unitary rings to every integral root basis $\mathbf{B} = (\Pi, \Pi^\vee)$ such that Π is finite. Given any field k, the group $\mathcal{G}_{\mathbf{B}}(k)$ is naturally endowed with a family of subgroups $\{U_\alpha\}_{\alpha\in\Phi(\mathbf{B})}$, all isomorphic to the additive group of k, which is a root group datum for a subgroup $\mathcal{G}_{\mathbf{B}}(k)^\dagger$ of $\mathcal{G}_{\mathbf{B}}(k)$ [Rém02c, Prop. 8.4.1]. This root group datum satisfies moreover $(\mathrm{RGD3})_{\mathrm{lin}}$. The functor $\mathcal{G}_{\mathbf{B}}$ is called a **Tits functor**. The value of a Tits functor on a field k is called a **split Kac-Moody group** over k.

An important feature of Tits functors is that their restriction to the category of fields is completely characterized by a short list of axioms inspired by the scheme-theoretic definition of linear algebraic groups [Tit87, Theorem 1]. One of these axioms is that the complex Kac-Moody group $\mathcal{G}_{\mathbf{B}}(\mathbb{C})$ has a natural adjoint action on the Lie algebra $\mathfrak{g}_A$, where $A = A(\mathbf{B})$ is the Cartan matrix of the root basis $\mathbf{B}$.

The analogy with the theory of reductive algebraic groups can be pushed one step further: Kac-Moody groups admit non-split forms which also possess naturally root group data. The non-split forms may be obtained by an algebraic process of Galois descent, which is defined and studied in [Rém02c, Chapters 11–13], or by using other twisting methods which do not fit into the context of Galois descent: see [Hée90] for Steinberg-Ree type constructions and [Müh99], [Müh02] for some others. In all cases, one obtains groups endowed with root group data; the Weyl group is generally infinite, and the underlying root basis might be of infinite rank as well.

We will not give more details about these constructions here. We merely mention that some of the groups they yield admit rather concise presentations, which allow to recover them in more direct manner, see Sect. 8.2 below.

4. Root group data, buildings and BN-pairs

There are several equivalent definitions of buildings which are all of different flavour and bring each a specific enlightenment to the theory. Here we present two of them and sketch some of their most basic features. Detailed accounts on the theory may be found in standard references: [Tit74] classifies the spherical buildings in connection with the theory of algebraic groups and their twisted analogues, [Wei03] takes into account simplifications made possible by the use of the Moufang property (as suggested by the addenda in [loc. cit.]), [Ron89] exploits the notion of a chamber system as introduced in [Tit81]. Finally, the book [AB] will present all the main viewpoints on buildings and a careful study of the relationships with combinatorial group theory, while [Dav08] provides a thorough treatment of the topological and metric viewpoints on Coxeter groups and buildings.

4.1. ***BN*-pairs from root group data.** Let us first introduce the definition and the basic properties of BN-pairs, another (less precise but of course more general) structure in group combinatorics.

4.1.1. *Axioms of a BN-pair.* Let G be a group. A ***BN*-pair** (or **Tits system**) [Bou07b, IV.2] for G is a pair B, N of subgroups of G, together with a set S of cosets of N modulo $B \cap N$, which satisfy the following axioms:

(BN1): $G = \langle B \cup N \rangle$ and $B \cap N \lhd N$.

(BN2): The elements of S have order 2 and generate the group $W := N/B \cap N$.
(BN3): For all $s \in S$ and $w \in W$, we have $sBw \subset BwB \cup BswB$.
(BN4): For each $s \in S$, we have $sBs \not\subset B$.

It follows from the axioms that the group W is a Coxeter group and that (W, S) is a Coxeter system [Bou07b, Ch. IV, §2, Th. 2]. Another important consequence is the following decomposition of G, called **Bruhat decomposition** [Bou07b, Ch. IV, §2, Th. 1]:

$$G = \bigsqcup_{w \in W} BwB.$$

In other words, the double cosets of B in G are in one-to-one correspondence with the elements of W.

An important concept associated with BN-pairs is that of a **parabolic subgroup**. Given any subset $J \subset S$, it follows from the axiom (BN3) that the set $P_J = \bigsqcup_{w \in W_J} BwB$ is a subgroup of G containing B, which is called a **standard parabolic subgroup of type** J. In fact, it follows from the Bruhat decomposition that *any subgroup of G containing B is obtained in this way* [Bou07b, Ch. IV, §2, Th. 3].

4.1.2. *BN-pairs from root group data.* As before, let now $\mathbf{B} = (\Pi, \Pi^\vee)$ be a root basis and $E = (\mathbf{B}, \Phi)$ be a root datum. Let also G be a group endowed with a root group datum $\{U_\alpha\}_{\alpha \in \Phi}$ of type E. We will also assume in this subsection that $\Phi = \Phi(\mathbf{B})$ is the canonical root system of $\mathbf{B}$; in particular it is reduced. This assumption causes no loss of generality in view of Lemma 2.4.

In order to construct BN-pairs for G, we introduce the following additional notation:

$$T = \langle \mu(u)\mu(v) \mid u, v \in U_\alpha \backslash \{1\},\ \alpha \in \Pi \rangle,$$

$$N = \langle \mu(u) \mid u \in U_\alpha \backslash \{1\},\ \alpha \in \Pi \rangle .T$$

and

$$B_\pm = T.U_\pm.$$

Clearly T normalizes each root group U_α, in particular B_+ and B_- are subgroups of G and we have $U_\pm \lhd B_\pm$. Given $\alpha \in \Pi$ and $u \in U_\alpha \backslash \{1\}$, we denote by r_α the coset $\mu(u).T \subset N/T$. Note that this is indeed independent of the choice of $u \in U_\alpha \backslash \{1\}$. Finally we set

$$S = \{r_\alpha \mid \alpha \in \Pi\}.$$

The expected relation between root group data and BN-pairs is the following statement:

Theorem. *The tuple $(B_\pm, N, S)$ is a BN-pair for G.*

The proof of this theorem is surprisingly difficult. The methods involved are completely elementary, but the complete proof is a very clever, quite technical, and fairly indirect one, due to J. Tits. The hardest point is to prove that for a root group datum as above, we have: $B_+ \cap U_- = \{1\}$. For this (and for other purposes among which are amalgamation theorems), J. Tits developed a combinatorial theory of coverings of partially ordered sets [Tit86a], which we sketch very briefly in 5.1. For a careful analysis of the proof, we recommend [AB, 8.6], which in fact contains the most detailed written treatment of this proof (to our knowledge); see also [Rém02b, §3] for reasonably detailed version suggested by [Abr96].

Corollary. *We have $B_- = N_G(U_-)$.*

Proof. Since U_- is normal in B_- by definition, we have $B_- \subset N_G(U_-)$. In view of the theorem, this implies that $N_G(U_-) = P_J^-$ for some $J \subset S$ since every subgroup containing B_- is a parabolic subgroup. Now if $J \neq \varnothing$, then $r_\alpha \in J$ for some $\alpha \in \Pi$ and hence $U_\alpha \in P_J^- = N_G(U_-)$. But we have just seen in the proof of (BN4) that $U_\alpha \not\subset N_G(U_-)$. Thus $J = \varnothing$ and $N_G(U_-) = B_-$. ☐

4.2. **Coset geometries.** The purpose of the next sections is to show that a group G endowed with a root group datum possesses two natural actions on two distinguished buildings, which are associated to G via BN-pairs constructed from the root group datum. Actions on buildings are very helpful in exploring the structure of the groups acting, as it will become clear in the subsequent study of G.

As we will see below, the construction of the building associated to a group with a BN-pair is a special example of a **coset geometry** associated to a group endowed with an inductive system of subgroups, and it is appropriate to start by defining the latter concept.

The coset geometry is obtained by the following construction. Let G be a group and let $\{G_a\}_{a\in F}$ be a system of subgroups indexed by some set F (in such a way that $G_a \neq G_b$ for $a \neq b$). The index set F is partially ordered by the inclusion of subgroups:

$$a \leqslant b \quad \Leftrightarrow \quad G_a \subset G_b.$$

We view $\{G_a\}_{a\in F}$ as an inductive system, all of whose morphisms are inclusion maps. The coset geometry of G with respect to $\{G_a\}_{a\in F}$ is the set

$$Y = \bigcup_{a\in F} G/G_a$$

which is partially ordered by the reverse inclusion:

$$gG_a \leqslant hG_b \quad \Leftrightarrow \quad gG_a \supset hG_b.$$

The poset $(F^{\mathrm{op}}, \leqslant) = (F, \geqslant)$, which is the dual of $(F, \leqslant)$, is thus isomorphic to a sub-poset of $(Y, \leqslant)$.

Recall that $(F^{\mathrm{op}}, \leqslant)$ has the structure of an **(abstract) simplicial complex** if any two elements of F^{op} have an infimum and if any nonmaximal element a of F^{op} coincides with the infimum of the set of elements strictly greater than a. In that case, the poset $(Y, \leqslant)$ also inherits of the structure of a simplicial complex, which is called the **simplicial coset geometry** associated with the system $\{G_a\}_{a\in F}$, and whose simplices are all the elements of Y, so that the order $\leq$ becomes the inclusion of simplices. The vertices of this complex are the minimal (nonempty) simplices, or equivalently, the cosets of the maximal subgroups in the system $\{G_a\}_{a\in F}$. The **diagram** of the poset Y (i.e. the graph with vertex set Y such that the vertices x, y form an edge if and only if $x \leqslant y$ or $y \leqslant x$) is nothing but the 1-skeleton of the first barycentric subdivision of the simplicial coset geometry.

A typical example is the case of an amalgam $G = A *_C B$ where $C = A \cap B$. In that case, the (simplicial complex associated to the) coset geometry is easily identified with the Bass-Serre tree associated to the amalgam.

Another example, important to us, is the **standard Coxeter complex** of a Coxeter system (W, S). This is defined as follows. Let F be the set of all proper subsets of S ordered by inclusion and consider the inductive system $\{W_J\}_{J\in F}$, where $W_J = \langle J\rangle$. We have $W_I \cap W_J = W_{I\cap J}$ for all $I, J \in S$, where $W_I = \langle I\rangle$. Moreover $W_J = \bigcap_{I\in F, I\supsetneq J} W_I$ for all nonmaximal $J \in F$. Thus $(F, \leqslant)$ is an abstract simplicial complex; in fact it is just a simplex. The standard Coxeter complex is the simplicial coset geometry associated with $\{W_J\}_{J\in F}$. Note that the maximal simplices in this complex are the cosets of $W_\varnothing = 1$, and are thus naturally in one-to-one correspondence with W.

For example, if W is infinite dihedral and $S = \{s_1, s_2\}$ is a Coxeter generating set, then the standard Coxeter complex is a simplicial line, which is simply the Bass-Serre tree of the amalgam $W = \langle s_1\rangle * \langle s_2\rangle$.

4.3. **Buildings as simplicial complexes.** Given a Coxeter system (W, S), a **building** of type (W, S) is a simplicial complex $\mathcal{X}$ together with a collection $\mathcal{A}$ of subcomplexes, all isomorphic to the standard Coxeter complex of (W, S), such that the following conditions are satisfied:

(Bu1): Any two simplices are contained in some $A \in \mathcal{A}$.
(Bu2): Given any two $A, B \in \mathcal{A}$, there is an isomorphism $A \to B$ fixing $A \cap B$ pointwise.

The maximal simplices of $\mathcal{X}$ are called **chambers**; the set of all chambers is denoted by $\mathrm{Ch}(\mathcal{X})$. The subcomplexes in $\mathcal{A}$ are called **apartments**. The Coxeter group W is called the **Weyl group** of $\mathcal{X}$.

A first basic property of buildings is the existence of a **type function** $\mathrm{typ} : \mathcal{X} \to \mathscr{P}(S)$ associating a subset of S to each simplex in $\mathcal{X}$ in such a way that each vertex is mapped to a maximal proper subset of S and for every simplex σ we have $\mathrm{typ}(\sigma) = \bigcap_{v\in\sigma} \mathrm{typ}(v)$. It is clear by construction that the standard Coxeter complex is endowed with such a type function: we can simply set $wW_J \mapsto J$ for every $w \in W$ and $J \subset S$. Now, transporting this type function to an apartment of $\mathcal{X}$, we can extend it in a coherent way to the whole of $\mathcal{X}$ using (Bu1) and (Bu2). Moreover, the isomorphisms in (Bu2) may always be assumed to be type-preserving [AB, Prop. 4.6]. The type of a chamber is the empty set.

The star of a simplex $\sigma \in \mathcal{X}$ is called a **residue**. It is itself a building whose apartments are the traces on $\mathrm{St}(\sigma)$ of apartments in $\mathcal{A}$. The type of this building is given by (W_J, J) where $J = \mathrm{typ}(\sigma)$.

4.4. **The Weyl distance.** An important feature about buildings is that the set of chambers is endowed with a so-called Weyl distance. Given a Coxeter system (W, S) and a set C, a map $\delta : C \times C \to W$ is called a **Weyl distance** if it satisfies the following conditions, where $x, y \in C$ and $w = \delta(x, y)$:

(WD1): $w = 1$ if and only if $x = y$.

(WD2): Given $z \in C$ such that $\delta(y, z) = s \in S$, we have $\delta(x, z) \in \{w, ws\}$; furthermore, if $\ell(ws) > \ell(w)$, then $\delta(x, z) = ws$.

(WD3): Given $s \in S$, there exists $z \in C$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

As we have seen above, the set $\mathrm{Ch}(A)$ of chambers in any apartment of a building $\mathcal{X}$ of type (W, S) can be identified with W. Consider the map

$$\delta_W : W \times W \to W : (x, y) \mapsto x^{-1}y.$$

It is immediate to check that δ_W is a Weyl distance. Note moreover that the composite map $\ell \circ \delta_W : W \times W \to W$ is nothing but the (combinatorial) distance in the Cayley graph of W with respect to S. Now one can transport the Weyl distance δ_W on $\mathrm{Ch}(A)$ for each apartment $A \in \mathcal{A}$ of $\mathcal{X}$. In view of the axioms (Bu1) and (Bu2), one verifies easily that this allows one to construct a well-defined Weyl distance $\delta : \mathrm{Ch}(\mathcal{X}) \times \mathrm{Ch}(\mathcal{X}) \to W$. One also checks that the composed map $d = \ell \circ \delta$ is a discrete metric in the usual sense, which is called the **numerical distance** on $\mathrm{Ch}(\mathcal{X})$.

The existence of a Weyl distance is in fact a characterizing property of buildings: any set endowed with a Weyl distance may be identified with the set of chambers of some building.

4.5. **Buildings from BN-pairs.** Given a group G with a BN-pair (B, N, S) and Weyl group $W = N/B \cap N$, let F be the set of proper subsets of S ordered by inclusion and consider the inductive system $\{P_J\}_{J\in S}$ consisting of the standard parabolic subgroups of G. We have $P_I \cap P_J = P_{I\cap J}$ and moreover $P_J = \bigcap_{I\in F, I\supsetneq J} P_I$ for all nonmaximal $J \in F$. Thus, as before, F is a simplicial complex. Let $\mathcal{X}$ be the simplicial coset geometry associated with $\{P_J\}_{J\in S}$. Let also A_0 be the simplicial coset geometry associated with the inductive system $\{N \cap P_J\}_{J\in F}$ of subgroups of N. Then A_0 is isomorphic to the Coxeter complex of type (W, S) and may be identified in a canonical way with a subcomplex of $\mathcal{X}$. Let $\mathcal{A} = \bigcup_{g\in G} g.A_0$. It turns out that $(\mathcal{X}, \mathcal{A})$ is a building of type (W, S); property (Bu1) is not difficult to deduce from the Bruhat decomposition.

The Weyl distance of $\mathcal{X}$ is also easy to identify: it is the map $\delta : \mathrm{Ch}(\mathcal{X}) \times \mathrm{Ch}(\mathcal{X}) \to W$ defined by

$$\delta(gB, hB) = w \Leftrightarrow Bh^{-1}gB = BwB.$$

This definition makes sense again thanks to the Bruhat decomposition of G.

Part II. Group actions on buildings and associated structure results

5. First structure results from actions on buildings

It is an old matter in group theory to try to obtain a presentation for a group that acts naturally on a space by preserving some structure, e.g. of topological or geometric nature. For example, in the case of a group Γ acting by homeomorphisms on an arcwise connected and simply connected topological space X possessing an arcwise connected open subset U such that $\Gamma.U = X$, a precise presentation for Γ is given in [Mac64]. An interesting special case is when the Γ-action is proper and totally discontinuous and $\overline{U}$ is compact: in that case (under some mild extra condition) the given presentation of Γ turns out to be finite. This is especially relevant to algebraic topology: the fundamental group of a topological space has a natural action on the universal cover, which is of course simply connected. Thus this method provides a way of obtaining presentations for fundamental groups.

This circle of ideas also lies behind Bass-Serre theory, which characterizes group amalgams in terms of actions on trees. Very early on, Tits realized that these ideas could be efficiently used in the context of buildings (recall that trees are indeed special examples of buildings!). This is what we want to explain in this section.

5.1. **Covering theory for partially ordered sets.** We now describe a very flexible method in the spirit of the ideas describe above, which applies in particular to all coset geometries. One of the goals is to make sense of a notion of simple-connectedness for posets in such a way that, under suitable hypotheses, the coset geometry $(Y, \leqslant)$ is simply connected if and only if $G = \varinjlim G_a$. We follow [Tit86a]; see also [Rém02b, Ch. 3].

We consider the category $\mathcal{O}$ whose objects are posets and whose morphisms are non-decreasing maps which are **descending bijections**. In other words, a non-decreasing map $f : (A, \leqslant) \to (B, \leqslant)$ is a morphism of $\mathcal{O}$ if and only if for any $a \in A$, the appropriate restriction of f induces a one-to-one map

$$\{x \in A \mid x \leqslant a\} \to \{y \in B \mid y \leqslant f(a)\}.$$

In the case of posets of simplices in simplicial complexes, this condition requires that the morphisms be simplicial maps.

A morphism $f : (E, \leqslant) \to (B, \leqslant)$ is called a **covering** if f is an **ascending bijection**. In other words f is a covering if and only if for any $a \in A$, the appropriate restriction of f induces a one-to-one map

$$\{x \in A \mid x \geqslant a\} \to \{y \in B \mid y \geqslant f(a)\}.$$

Again, in the language of simplicial complexes, this means that f induces a one-to-one map on the link of every simplex.

A covering $f : (E, \leqslant) \to (B, \leqslant)$ of a poset $(B, \leqslant)$ is called a **universal cover** if E is connected (i.e. the associated diagram is connected) and f factors through every other covering of $(B, \leqslant)$. A poset $(A, \leqslant)$ is called **simply connected** if the identity map defines a universal cover.

All basic properties of classical covering theory can be extended to the present context without difficulty, such as:

- existence and uniqueness of path-liftings (with a base-point);
- surjectivity of coverings whenever the base is connected;
- existence and uniqueness of universal covers (for based posets);
- a covering $f : (E, \leqslant) \to (B, \leqslant)$ such that E is connected and B is simply connected is automatically an isomorphism.

Let now $(A, \leqslant)$ be a poset and G be a group acting on A by automorphisms. A subset F of A is called a **fundamental domain** for the G-action on A if F contains exactly one elements of every G-orbit and if, moreover, one has $a \leqslant b \in F \Rightarrow a \in F$ for every $a \in A$. Given a fundamental domain F, let us consider the system $\{G_a\}_{a \in F}$ of stabilizers of points of F. By the definition of a fundamental domain, it is readily seen that $a \leqslant b \Rightarrow G_a \supset G_b$ for all $a, b \in F$.

We now consider the group $\widetilde{G}$ which is the direct limit of the system $\{G_a\}_{a\in F}$ and the associated coset geometry $(\widetilde{A}, \leqslant)$. In order to avoid confusion, we denote by $\widetilde{G}_a$ the canonical image of G_a in $\widetilde{G}$. Let also $\pi : \widetilde{G} \to G$ be the canonical map and define $\alpha : \widetilde{A} \to A$ by

$$\alpha(g\widetilde{G}_a) = \pi(g).a$$

for any $a \in F$. One verifies that α is a covering and that $\widetilde{A}$ is connected whenever F is connected. More importantly, we have the following [Tit86a, Prop. 1]:

Proposition. *The map $\alpha : \widetilde{A} \to A$ is a universal cover whenever F is simply connected. In particular, if F is simply connected, then the poset $\widetilde{A}$ is simply connected and $G \simeq \widetilde{G}$ if and only if A is simply connected.* □

5.2. **Buildings are simply connected.** Let now (W, S) be a Coxeter system and $(\mathcal{X}, \mathcal{A})$ be a building of type (W, S). Let $\mathcal{S}_2$ be the set of all subsets J (possibly empty) of S of cardinality at most 2 and such that $W_J = \langle J \rangle$ is finite. Let

$$|\mathcal{X}|_2 = \{\sigma \in \mathcal{X} \mid \sigma \text{ is a simplex of type } J \text{ for some } J \in \mathcal{S}_2\},$$

ordered by inclusion.

Since by definition, the group W is the inductive limit of the system $\{W_J\}_{J\in\mathcal{S}_2}$, it follows from Proposition 5.1 that the poset realization $|W|_2$ of an apartment of type (W, S) is simply connected. Consequently, we obtain:

Proposition. *The poset realization $|\mathcal{X}|_2$ is simply connected.*

Proof. Let $f : E \to |\mathcal{X}|_2$ be a covering. We must show that there exists a morphism $h : |\mathcal{X}|_2 \to E$ such that $f \cap h = \mathrm{id}$. Let σ_0 be a base chamber in $|\mathcal{X}|_2$ and choose $\sigma_1 \in f^{-1}(\sigma_0)$. Given any $\tau \in |\mathcal{X}|_2$, there exists by (Bu1) an apartment A containing both σ and τ. Since $|A|_2$ is simply connected, one deduces, by considering the restriction of f to the connected component of $f^{-1}(|A|_2)$ containing σ_1, that there exists a morphism $h_A : A \to f^{-1}(A)$ such that $f \circ h_A = \mathrm{id}_{|A|_2}$ and $h_A(\sigma_0) = \sigma_1$. In view of (Bu2) and the uniqueness of path-liftings, it follows that for any other apartment B containing σ_0, we have $h_A|_{A\cap B} = h_B|_{A\cap B}$. In particular $h_A(\tau)$ does not depend on the choice of the apartment A. Set $h(\tau) = h_A(\tau)$. Now one verifies easily that the map $h : |\mathcal{X}|_2 \to E$ is a morphism and the equality $f \circ h = \mathrm{id}$ follows by construction. □

One immediately deduces a decomposition as amalgamated sum for groups acting chamber-transitively on buildings. Indeed, a chamber is obvisouly simply connected and if the action is type-preserving and chamber-transitive, then any chamber is automatically a fundamental domain. Thus Propositions 5.1 and 5.2 apply. For example, if G is a group with a BN-pair (B, N, S), then G is the amalgamated sum of the standard parabolic subgroups of type J for $J \in \mathcal{S}_2$.

5.3. **Applications to root group data.** Let us now come back to a group G endowed with a root group datum $\{U_\alpha\}_{\alpha\in\Phi}$ of type $E = (\mathbf{B}, \Phi)$, whose Weyl group is denoted by W. We let (X_-, δ_-) be the building associated with the negative BN-pair (B_-, N, S) of G. Our present goal is to apply the technology we have just described to study the U_+-action on X_-.

We first recall the existence of an order $\leq$ on W defined as follows:

$$z \leqslant w \quad \Leftrightarrow \quad \ell(w) = \ell(z) + \ell(z^{-1}w).$$

This is called the **Bruhat ordering** of W. Using the solution of the word problem in Coxeter groups, this is seen to be equivalent to the existence of a reduced word $s_1 \dots s_n$ representing w as a product of elements of S, such that $z = s_1 \dots s_j$ for some $j \leqslant n$ (or $z = 1$).

Now, for each $w \in W$, we consider the following subgroup of U_+:

$$U_w = \langle U_\gamma \mid \gamma \in \Phi_{w^{-1}} \rangle.$$

Using Lemma 1.1.5, it is easily seen that if $z \leqslant w$, then $U_z \leqslant U_w$ for all $z, w \in W$. In other words, the system $\{U_w\}_{w\in W}$ is an inductive system of subgroups. As we will see in the sequel, the following result and its proof have many useful consequences concerning the structure of G:

Theorem. *The group U_+ is isomorphic to $\varinjlim U_w$.*

Proof. Let $\widetilde{U} = \varinjlim U_w$. Denote by $\widetilde{U}_w$ the canonical image of U_w in $\widetilde{U}$ and by $\pi : \widetilde{U} \to U_+$ the canonical homomorphism. Consider the set $\widetilde{X}$ consisting of all ordered pairs $(u\widetilde{U}_w, wW_J)$ such that $u \in \widetilde{U}$, $w \in W$, and $J \in \mathcal{S}_2$ is such that w is maximal in wW_J for the Bruhat ordering. Equivalently, the latter condition means that w is of maximal length in wW_J; it is a well known fact that there is such a unique element [Bou07b, Ch. IV, §1, Exerc. 3].

We define a partial order $\leqslant$ on $\widetilde{X}$ as follows:

$$(u\widetilde{U}_w, wW_I) \leqslant (v\widetilde{U}_z, zW_J) \quad \Leftrightarrow \quad wW_I \supset zW_J \text{ and } v^{-1}u \in \widetilde{U}_w.$$

The condition $wW_I \supset zW_J$ implies $z \in wW_I$ and hence $z \leqslant w$ and $U_z \subset U_w$. Thus the order $\leqslant$ is well-defined. Obviously there is an order-preserving action of $\widetilde{U}$ on $\widetilde{X}$ defined by $g : (u\widetilde{U}_w, wW_I) \mapsto (gu\widetilde{U}_w, wW_I)$.

Let now X_- be the negative building of G, namely the building associated with the BN-pair (G, B_-, S) as in Theorem 4.1.2. Consider the map

$$\nu : \widetilde{X} \to |X_-|_2 : (u\widetilde{U}_w, wW_I) \mapsto \pi(u)wP_I^-,$$

where P_I^- denotes the standard negative parabolic subgroup of type I.

The essential points are that $\widetilde{X}$ is connected and ν is a covering map. The verification of these points is slightly technical but straightforward; details may be found in [Rém02b, Th. 3.5.2]. Then it follows from the covering theory of posets (see Sect. 5.1) that ν is an isomorphism. Moreover ν is clearly π-equivariant by construction.

Let us now compare some point-stabilizers in $\widetilde{X}$ and $|X_-|_2$. For $w \in W$, we have clearly

$$\mathrm{Stab}_{\widetilde{U}}(\widetilde{U}_w, w) = \widetilde{U}_w.$$

On the other hand, we have $\nu(\widetilde{U}_w, w) = wB_-$ and

$$\begin{aligned}\mathrm{Stab}_{U_+}(wB_-) &= \{u \in U_+ \mid uwB_- = wB_-\}\\ &= \{u \in U_+ \mid w^{-1}uwB_- = B_-\}\\ &= U_+ \cap wB_-w^{-1}.\end{aligned}$$

From these facts, it follows clearly that

$$\pi^{-1}(U_+ \cap wB_-w^{-1}) = \widetilde{U}_w. \tag{5.1}$$

In particular, for $w = 1$ we get $\pi^{-1}(U_+ \cap wB_-w^{-1}) = \{1\}$ from which it follows that π is injective. □

Corollary. *We have the following:*

(i) *For each $w \in W$, we have $U_+ \cap wB_-w^{-1} = U_w$. In particular $U_+ \cap B_- = \{1\}$.*
(ii) *$B_+ \cap B_- = T$.*
(iii) *We have $U_{-\alpha} \not\subset U_+$ for each $\alpha \in \Pi$. In particular, the system $\{U_{-\alpha}\}_{\alpha\in\Phi}$ is a root group datum of type E for G and (B_+, N, S) is a BN-pair.*
(iv) *We have $T = \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$.*

Proof. (i). The first assertion follows by transforming (5.1) under π. The second assertion is the special case of the first one with $w = 1$.

(ii). Consider $g = tu \in B_+ = T.U_+$ and suppose that $g \in B_- = T.U_-$. Then $u \in t^{-1}B_- = B_-$ hence $u = 1$ by (i), whence $g \in T$ as desired.

(iii). The fact that $U_{-\alpha} \not\subset U_+$ follows from (i). The second assertion becomes then clear. In particular, we may apply Theorem 4.1.2 and its corollary. This shows that (B_+, N, S) is indeed a BN-pair for G and that $B_+ = N_G(U_+)$.

(iv). Let $\widetilde{T} = \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$. The inclusion $T \subset \widetilde{T}$ follows from the definitions. Note that $\widetilde{T} \subset N_G(U_+) \cap N_G(U_-)$. By Corollary 4.1.2 and (iii), we obtain $\widetilde{T} \subset B_+ \cap B_-$. Thus $\widetilde{T} \subset T$ by (ii). □

5.4. **Relationship between the positive and the negative BN-pairs.** The fact that the positive and negative BN-pairs of G have the subgroup N in common is not coincidental. In fact there is a tight relationship between these two BN-pairs, more precisely described by the following:

Proposition. *The following assertions, as well as similar assertions with + and − interchanged, hold:*

(i) *For all $w \in W$ and $s \in S$ such that $\ell(ws) < \ell(w)$, we have*

$$B_+wB_-sB_- = B_+wsB_-.$$

(ii) *For each $s \in S$, we have $B_+s \cap B_- = \varnothing$.*

(iii) *One has a **Birkhoff decomposition**, namely the map*

$$W \to B_+\backslash G/B_-; w \mapsto B_+wB_-$$

is bijective.

Proof. (i) is established by considerations similar to those used in the proof of Theorem 4.1.2.

For (ii), proceed as follows. Assume that $n = b.b'$ for some $b \in B_+$, $b' \in B_-$ and $n \in N$ such that $n.T = s \in S$. Let $\alpha \in \Pi$ such that $s = r_\alpha$. We have $U_\alpha = nU_{-\alpha}n^{-1}$ hence

$$U_\alpha{}^b = {}^{b'}U_{-\alpha}.$$

Since $b \in B_+$ normalizes U_+, the group $U_\alpha{}^b$ is contained in U_+. Similarly, we have ${}^{b'}U_{-\alpha} \subset U_-$ and the equality above shows that $U_\alpha \subset {}^b(U_+ \cap U_-)$. By Corollary 5.3, we have $U_+ \cap U_- = \{1\}$. This yields $U_\alpha = \{1\}$ which violates (RGD0). Hence (ii) is proven.

Assertion (iii) is deduced from (i) and (ii) in a similar way as the Bruhat decomposition is obtained from the axioms of BN-pairs. Details may be found in [Abr96, Lemma 1]. □

Using this result, we may know answer the question: when are B_+ and B_- conjugate in G?

Corollary. *The group B_+ and B_- are conjugate in G if and only if W is finite.*

Proof. Assume that W is finite and let w_0 be the longest element. It is well known that $\Phi_{w_0} = \Phi_+$ from which it follows that $U_+ = U_{w_0}$ and hence $w_0U_+w_0^{-1} = U_-$. Thus $w_0B_+w_0^{-1} = B_-$ as desired.

Assume now that $gB_+g^{-1} \subset B_-$ for some $g \in G$. Using the Birkhoff decomposition of G, it follows that $wB_+w^{-1} \subset B_-$ for some $w \in W$. Since $U_+ \cap B_- = \{1\}$ by Corollary 5.3, it follows that $w.\Phi_+ \subset \Phi_-$, that is to say, $\Phi_+ = \Phi_w$. By Lemma 1.1.5, the set Φ_w is finite. Thus Φ_+ is finite and so is $\Phi = \Phi_+ \cup -\Phi_+$. Consequently W is finite. □

Remark. When the group G is a Kac-Moody group, then G admits an (outer) automorphism which swaps B_+ and B_-. Such an automorphism can be constructed as a lift of the Cartan-Chevalley involution of the corresponding Lie algebra, see Theorem 3.1(iv). However, it is not clear that such an automorphism exists for any group endowed with a root group datum, although the whole theory is 'symmetric' under a sign change swapping + and −.

5.5. **More on the subgroup U_w.** We maintain the assumptions and notation of the preceding subsections (see 4.1.2).

Lemma. *Let $w \in W$ and write w as a reduced expression $w = r_{\alpha_1} \dots r_{\alpha_n}$ where $\alpha_i \in \Pi$ for each i. Let moreover $\beta_1 = \alpha_1$ and $\beta_i = r_{\alpha_1} \dots r_{\alpha_{i-1}}\alpha_i$ for each $i = 2, \dots, n$. Then the product set $U_{\beta_1}U_{\beta_2} \dots U_{\beta_n}$ coincides with the subgroup U_w and each element $u \in U_w$ has a unique writing as a product $u = u_1 \dots u_n$ with $u_i \in U_{\beta_i}$ for each $i = 1, \dots, n$.*

Furthermore, if U_α is nilpotent for each $\alpha \in \Pi$, then so is U_w for each $w \in W$.

Proof. Recall from Lemma 1.1.5(iii) that $\Phi_{w^{-1}} = \{\beta_1, \dots, \beta_n\}$, so the equality $U_w = U_{\beta_1}U_{\beta_2} \dots U_{\beta_n}$ follows by induction on $\ell(w)$ using (RGD3). Details may be found in [Rém02c, Lemma 1.5.2(iii)].

Now suppose that some $u \in U_w$ may be written in two different ways $u = u_1 \dots u_n = v_1 \dots v_n$. Note that $U_{\beta_1} \dots U_{\beta_{n-1}} = U_{wr_{\alpha_n}}$ is a subgroup of G. Thus, arguing by induction on $\ell(w)$, it suffices to show that $U_{\beta_n} \cap U_{wr_{\alpha_n}} = \{1\}$. Conjugating $U_{\beta_n} \cap U_{wr_{\alpha_n}}$ by (an element of G representing) the Weyl group element $r_{\alpha_n} w^{-1}$, we obtain the subgroup $U_{\alpha_n} \cap V$ where $V = r_{\alpha_n} w^{-1} U_{wr_{\alpha_n}} w r_{\alpha_n}$. By definition of U_w we have $V \subset U_-$, hence $U_{\alpha_n} \cap V \subset U_+ \cap U_-$, which is trivial in view of Corollary 5.3(i). The desired uniqueness result follows.

We have seen that the set $U_{\beta_1} \dots U_{\beta_{n-1}}$ is a subgroup of U_w which coincides with $U_{wr_{\alpha_n}}$. In fact, using (RGD3) one sees that $U_{\beta_1} \dots U_{\beta_{n-1}}$ is normal in U_w. Similarly $U_{\beta_2} \dots U_{\beta_n}$ is a normal subgroup of U_w. Therefore, assuming the nilpotency of each root group, the nilpotency of U_w follows by induction on $\ell(w)$, using a standard criterion for nilpotency [Hal76, Th. 10.3.2]. □

Remark. When G is a split Kac-Moody group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}_A$ (see Sect. 3.3), then U_w is a complex nilpotent Lie group of dimension $\ell(w)$. Its Lie algebra is the subalgebra $\mathfrak{g}_w = \sum_{\alpha \in \Phi_w} \mathfrak{g}_\alpha$ of $\mathfrak{g}_A$. It turns out that in this case, the nilpotency degree of U_w is bounded above by a constant depending only on G (in fact: on the generalized Cartan matrix A), but not on w: this is the main result of [Cap07]. It implies that a similar bound exists for all split or almost split Kac-Moody groups over arbitrary fields.

Here is another characterization of root group data (of finite rank) with finite Weyl group:

Proposition. *Assume that root groups are nilpotent and that the boot basis* $\mathbf{B}$ *is of finite rank. Then W is finite if and only if U_+ is nilpotent.*

Proof. We have seen in the proof of Corollary 5.4 that if W is finite, then U_+ coincides with U_w for some $w \in W$. Thus the 'only if' part is clear in view of the proposition.

Suppose now that the Weyl group W is infinite. Let α be a simple root. Then, since W is an infinite Coxeter group, there exists a positive root, say β, such that the associated reflections r_α and r_β generate an infinite dihedral group: this is well known, a proof may be found e.g. in [NV02]. Up to replacing β by $r_\alpha(\beta)$, we may – and shall – assume that $\{\alpha; \beta\}$ is a non-prenilpotent pair of positive roots. In order to prove that U_+ is not nilpotent, it is enough to show that F is isomorphic to the (centre-free, hence non nilpotent) free product $U_\alpha * U_\beta$. This follows from the general fact, stated in [Tit90, §4, Proposition 5], that if $\{\gamma; \delta\}$ is a non-prenilpotent pair of roots, then the canonical map $U_\gamma * U_\delta \to G$ is injective. The proof follows closely the idea of the proof of Theorem 5.3: the group F is analyzed by means of its (discrete) action on the negative building. More precisely, as suggested by [loc. cit., comment after Lemme 3], it is not difficult to construct an F-invariant subset of that building which features a treelike structure. This tree is in fact isomorphic to the Bass-Serre tree of the amalgam F, which shows the desired injectivity. □

5.6. **The Weyl codistance.** In the same way as positive and negative Bruhat decompositions of G allow to define the Weyl distance on $\mathrm{Ch}(X_+) \times \mathrm{Ch}(X_+)$ and $\mathrm{Ch}(X_-) \times \mathrm{Ch}(X_-)$ respectively, the Birkhoff decomposition allows to define a map

$$\delta^* : \mathrm{Ch}(X_+) \times \mathrm{Ch}(X_-) \cup \mathrm{Ch}(X_-) \times \mathrm{Ch}(X_+) \to W$$

by

$$\delta^*(gB_+, hB_-) = w \quad \Leftrightarrow \quad B_- h^{-1} g B_+ = B_- w B_+$$

and similarly for $+$ and $-$ interchanged. Using Proposition 5.4, one sees that the mapping δ^* is a **Weyl codistance**, which means that it enjoys the following properties, as well as similar properties obtained by swapping $+$ and $-$, where $x \in \mathrm{Ch}(X_+)$ and $y \in \mathrm{Ch}(X_-)$:

(WCod1): $\delta^*(x, y) = \delta^*(y, x)^{-1}$.
(WCod2): If $\delta^*(x, y) = w$ and $\delta_-(y, z) = s \in S$ with $\ell(ws) < \ell(w)$ for some $z \in \mathrm{Ch}(X_-)$, then $\delta^*(x, z) = ws$.
(WCod3): If $\delta^*(x, y) = w$, then for each $s \in S$, there exists $z \in \mathrm{Ch}(X_-)$ such that $\delta_-(y, z) = s$ and $\delta^*(x, z) = ws$.

A Weyl codistance defined on a pair of buildings of the same type is also-called a **twinning** between these buildings. Two chambers are called **opposite** if their Weyl codistance is 1. More generally, simplices of the same type are called opposite if they are contained in opposite chambers. Since the parabolic subgroups of G (i.e. subgroups containing some conjugate of B_+ or B_-) are the simplex-stabilizers, the opposition relation may also be defined between parabolic subgroups of G. Roughly speaking, two parabolic subgroups are opposite if their intersection is as small as possible.

Here is an example of the usefulness of the Weyl codistance:

Proposition. *We have:* $\bigcap_{w\in W} wB_+w^{-1} \subset B_-$.

Proof. By definition of δ^*, we have

$$\delta^*(wB_+, B_-) = w \quad \text{for all } w \in W. \tag{5.2}$$

We claim that the latter property characterizes the chamber $B_- \in \mathrm{Ch}(X_-)$. Suppose indeed that an element $g \in G$ is such that $\delta^*(wB_+, gB_-) = w$ for all $w \in W$. Let $z = \delta_-(gB_-, B_-)$, where δ_- is the Weyl distance of X_-. Let $z = s_n \dots s_1$ be a reduced decomposition of z in elements s_i of S. It follows from (WD2), (WD3) that there exist elements $g_0, g_1, \dots, g_n \in G$, with $g_0 = 1$ and $g_n = g$, such that $\delta_-(g_{i-1}B_-, g_i) = s_i$. By (5.2) we have $\delta^*(zB_+, B_-) = z$ and a straightforward induction on i using (WCod2) shows that $\delta^*(zB_+, g_iB_-) = zs_1 \dots s_i$ for each $i = 1, \dots, n$. In particular $\delta^*(zB_+, gB_-) = 1$. By our assumption on g, we have also $\delta^*(zB_+, gB_-) = z$, whence $z = 1$. In view of (WD1) this implies that $g \in B_-$ and the claim is proven.

Now, since $H = \bigcap_{w\in W} wB_+w^{-1}$ fixes the chamber wB_- for each $w \in W$ and since δ^* is clearly G-invariant, it follows that H fixes B_-. Equivalently, we get $H \subset B_-$ as desired. □

Corollary. *The kernel of the action of G on X_+ (resp. X_-) is the center of G and we have $Z(G) \cap U_+ = Z(G) \cap U_- = \{1\}$ and $Z(G/Z(G)) = \{1\}$.*

Proof. Let $K = \bigcap_{g\in G} gB_+g^{-1}$ be the kernel of the action of G on X_+ and let Z be the center of G.

Clearly $Z \subset \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$, hence $Z \subset T \subset B_+$ by Corollary 5.3(iv). Since Z is normal in G we deduce $Z \subset K$.

Conversely, by the lemma we have $K \subset B_-$ hence $K \subset T$ by Corollary 5.3(ii). In particular K normalizes U_α for each $\alpha \in \Phi$. Conversely, each U_α clearly normalizes K, from which we deduce $[K, U_\alpha] \subset K \cap U_\alpha \subset T \cap U_\alpha = \{1\}$, where the latter equality follows again from Corollary 5.3(i). Thus $K \subset \bigcap_{\alpha\in\Phi} C_G(U_\alpha) = Z$ by (RGD0).

Note that since $K \subset T$ and $T \cap U_+ = T \cap U_- = \{1\}$, it follows that the canonical projection $\pi : G \to G/Z$ maps the system $\{U_\alpha\}_{\alpha\in\Phi}$ to a root group datum for G/Z. By construction the buildings associated with G and G/Z coincide and G/Z acts faithfully. Thus G/Z is center-free by the above. □

6. Group topology

6.1. **Topological completions.** The existence of BN-pairs and, hence, of building-actions, for a group G endowed with a root group datum allows one to construct other groups obtained by some simple process of topological completion. The idea behind this is the following: the isometry group of a metric space is naturally endowed with a structure of topological group, the topology being that of uniform convergence on *bounded* subsets. Since buildings are in particular discrete metric spaces (the metric is given by the numerical distance), this provides a topology for any group acting on a building or, more precisely, for the quotient of the group by the kernel of the action. Here, in order to avoid the necessity of replacing G by a quotient, we proceed as follows.

Let X_+ be the building associated with the positive BN-pair (B_+, N, S) of G. Let $c_+ = B_+$ be the chamber fixed by B_+. For each $n \in \mathbb{N}$, we define

$$U_{+,n} = \{g \in U_+ \mid g.c = c \text{ for each chamber } c \text{ such that } d_+(c, c_+) \leqslant n\}.$$

Thus $U_{+,n}$ is the kernel of the action of U_+ on the ball of radius n centered at c_+ in $\mathrm{Ch}(X_+)$. Consider now the map $\mathrm{dist}_+ : G \times G \to \mathbb{R}_+$ defined by

$$\mathrm{dist}_+(g,h) = \begin{cases} 2 & \text{if } \ h^{-1}g \notin U_+ \\ 2^{-n} & \text{if } \ h^{-1}g \in U_+ \text{ and } n = \max\{k \in \mathbb{N} \mid h^{-1}g \in U_k\}. \end{cases}$$

By definition, for all $g \in G$ we have $\mathrm{dist}_+(1, g) = 0$ only if g belongs to U_+ and acts trivially on X_+. By Corollary 5.6, this implies that $g = 1$. Moreover, it is straightforward to check that dist_+ satisfies the triangle inequality. Therefore dist_+ is a left-invariant metric on G. Let G_+ denote the completion of the metric space (G, dist_+) and let $\varphi_+ : G \to G_+$ be the inclusion map. The extension of dist_+ to G_+ is again denoted by dist_+. Clearly the space G_+ is discrete whenever X_+ is of finite diameter, which happens if and only if W is finite.

As usual, the preceding discussion may be done with the sign $-$ instead of $+$, thereby providing a complete metric space (G_-, dist_-) and an inclusion map $\varphi_- : G \to G_-$.

Proposition. *Let $\varepsilon \in \{+, -\}$. The following assertions hold:*

(i) *The topology defined by the metric $\mathrm{dist}_\varepsilon$ makes G into a topological group. In particular G_ε is a topological group which is totally disconnected.*
(ii) *Let $\widehat{B}_\varepsilon$ (resp. $\widehat{U}_\varepsilon$) be the closure of B_ε (resp. U_ε) in G_ε. Then $\widehat{B}_\varepsilon \simeq T \ltimes \widehat{U}_\varepsilon$.*
(iii) *The system $(\widehat{B}_\varepsilon, N, S)$ is a BN-pair of G_ε. The corresponding building is canonically isomorphic to X_ε. The kernel of the action of G_ε on X_ε is the center $Z(G_\varepsilon)$ and $Z(G_\varepsilon) = Z(G)$ is a discrete subgroup of G_ε.*
(iv) *The homomorphism $\varphi_\varepsilon : G_\varepsilon \to \mathrm{Aut}(X_\varepsilon)$ is continuous and open, where $\mathrm{Aut}(X_\varepsilon)$ is endowed with the topology of uniform convergence on* bounded *subsets (i.e. the bounded-open topology). Moreover φ_ε is proper if and only if $Z(G)$ is finite.*
(v) *The subgroup $U_{-\varepsilon}$ is discrete in G_ε.*
(vi) *The subgroup $(\varphi_\varepsilon \times \varphi_{-\varepsilon})(G)$ is discrete in $G_\varepsilon \times G_{-\varepsilon}$.*

Proof. (i). It is immediate to check that $\{U_{\varepsilon,n}\}_{n\in\mathbb{N}}$ satisfy the standard axioms of a system of neighborhoods of the identity in G, see [HR79, Th. 4.5]. Thus G is indeed a topological group and so is G_ε; moreover, the map φ_ε is obviously an injective homomorphism.

For $n \in \mathbb{N}$, denote by $\widehat{U}_{\varepsilon,n}$ the closure of $U_{\varepsilon,n}$ in G_ε. It follows easily from the definitions that

$$\widehat{U}_{\varepsilon,n} = \{g \in G_\varepsilon \mid \mathrm{dist}_\varepsilon(1, g) \leqslant n\} = \{g \in \widehat{U}_\varepsilon \mid g.c = c \text{ for each chamber } c \text{ such that } d_\varepsilon(c, c_\varepsilon) \leqslant n\}. \tag{6.1}$$

Since any open subgroup of a topological group contains the identity component, we have $(G_\varepsilon)^\circ \subset \widehat{U}_{\varepsilon,n}$ for each $n \in \mathbb{N}$. By (6.1), the subgroups $\widehat{U}_{\varepsilon,n}$ intersect trivially, whence $(G_\varepsilon)^\circ = \{1\}$. Thus G_ε is totally disconnected.

(ii). Since T normalizes U_ε, it also normalizes $\widehat{U}_\varepsilon$. Moreover T is a discrete subgroup of G_ε by Corollary 5.3(i). Thus $T.\widehat{U}_\varepsilon$ is a closed subgroup containing B_ε, whence $\widehat{B}_\varepsilon \subset T.\widehat{U}_\varepsilon$. Since the reverse inclusion obviously holds, we obtain $\widehat{B}_\varepsilon = T.\widehat{U}_\varepsilon$. It remains to show that $T \cap \widehat{U}_\varepsilon = \{1\}$. Note that for any nontrivial $t \in T$, we have $\mathrm{dist}_\varepsilon(1, t) = 2$ by Corollary 5.3(i). Hence the desired result follows from (6.1).

(iii). The subgroup of G_ε generated by $\widehat{B}_\epsilon \cup N$ contains $\widehat{U}_\epsilon$, hence it is open. Therefore it is closed. But clearly it contains G, whence $G_\varepsilon = \langle \widehat{B}_\varepsilon \cup N \rangle$. Moreover, it follows from (ii) and (6.1) that $G \cap \widehat{B}_\varepsilon = B_\varepsilon$. Therefore, we have $T \subset \widehat{B}_\varepsilon \cap N \subset B_\varepsilon \cap N \subset T$. Thus (BN1) holds. Now axioms (BN2) and (BN4) are immediate and (BN3) follows from the corresponding property of G by taking closures.

Consider the map

$$f_\varepsilon : G/B_\varepsilon \to G_\varepsilon/\widehat{B}_\varepsilon : gB_\varepsilon \mapsto g\widehat{B}_\varepsilon.$$

Since $G \cap \widehat{B}_\varepsilon = B_\varepsilon$, it follows that f_ε is injective. On the other hand, for any $g \in G_\varepsilon$, there exists $g' \in G$ such that $g^{-1}g' \in \widehat{U}_\varepsilon$ by the definition of G_ε. This shows that f_ε is surjective. Since the BN-pairs of G and G_ε have the same Weyl group (more precisely: the same N and S), it follows that f_ε is a canonical isomorphism between the corresponding buildings.

Let $K = \bigcap_{g\in G_\varepsilon} g\widehat{B}_\varepsilon g^{-1}$ be the kernel of the action of G_ε on X_ε. Note that $\widehat{U}_\varepsilon$ acts faithfully on X_ε by (6.1), hence $K \cap \widehat{U}_\varepsilon = \{1\}$. Since $K \subset \widehat{B}_\varepsilon$, it follows that K normalizes $\widehat{U}_\varepsilon$. Conversely $\widehat{U}_\varepsilon$ obviously normalizes K, so we deduce $[K, \widehat{U}_\varepsilon] \subset K \cap \widehat{U}_\varepsilon = \{1\}$. Since G is generated by conjugates of $\widehat{U}_\varepsilon$ as follows easily from (BN1) and (ii), we deduce that $K \subset Z(G_\varepsilon)$. Since $Z(G_\varepsilon)$ normalizes $\widehat{B}_\varepsilon$ we obtain $Z(G_\varepsilon) \subset \widehat{B}_\varepsilon$ because $(\widehat{B}, N, S)$ is a BN-pair. Hence $Z(G_\varepsilon) \subset K$.

Let now $k \in K$ and write $k = t.u$ according to (ii). Since G_ε/K is nothing but the completion of $G/Z(G)$, we deduce by applying (ii) to G_ε/K that t and u both belong to K. We have seen above that $K \cap \widehat{U}_\varepsilon$ is trivial. This shows that $K \subset T \subset G$. Therefore Corollary 5.6 yields $K = Z(G)$.

(iv). It suffices to check the continuity of φ_ε at 1. This property is an obvious consequence of the definition of the topology on G_ε. The fact that φ_ε is open essentially follows because the restriction of φ_ε to the open subgroup $\widehat{U}_\varepsilon$ is injective and maps it to an open subgroup of $\mathrm{Aut}(X_\varepsilon)$. Since $Z(G)$ is a discrete subgroup of G_ε by (iii), it is clear that φ_ε can be proper only if $Z(G)$ is finite. Assume conversely that $Z(G)$ is finite and let $C \subset \mathrm{Aut}(X_\varepsilon)$ be a compact subset. Let $B = \varphi_\varepsilon^{-1}(C)$ and let $(x_n)_{n\geqslant 0}$ be any sequence of points in B. Up to extracting, we may assume that the sequence $\big(\varphi_\varepsilon(x_n)\big)_{n\geqslant 0}$ converges to some $c \in C$. Since φ_ε has finite fibers, there are finitely many points $b_1, \dots, b_k$ such that $\varphi_\varepsilon(b_i) = c$. Now, it is clear by the pigeonhole principle that $(x_n)_{n\geqslant 0}$ has a subsequence converging to b_i for some $i \in \{1, \dots, k\}$.

(v). We have $U_{-\varepsilon} \cap \widehat{U}_\varepsilon \subseteq U_{-\varepsilon} \cap U_\varepsilon = \{1\}$ by Corollary 5.3(i). Since $\widehat{U}_\varepsilon$ is an open subgroup of G_ε, it follows that $U_{-\varepsilon}$ is discrete.

(vi). Similarly $\widehat{U}_\varepsilon \times \widehat{U}_{-\varepsilon}$ is an open subgroup of $G_\varepsilon \times G_{-\varepsilon}$. On the other hand we have

$$(\varphi_\varepsilon \times \varphi_{-\varepsilon})(G) \cap (\widehat{U}_\varepsilon \times \widehat{U}_{-\varepsilon}) \subseteq U_\varepsilon \cap U_{-\varepsilon} = \{1\}.$$

The proof is complete. □

The example to keep in mind here is the group $G = \mathrm{SL}_n\big(k[t, t^{-1}]\big)$, where k is an arbitrary field, see Sect. 2.7. The completions G_+ and G_- are then respectively $\mathrm{SL}_n\big(k(\!(t)\!)\big)$ and $\mathrm{SL}_n\big(k(\!(t^{-1})\!)\big)$. Note also that if the Weyl group W is finite, then the buildings X_+ and X_- have finite diameter, hence are bounded. Therefore, in that case the topologies defined by dist_+ and dist_- are discrete and we have $G_+ = G = G_-$. It is only for an infinite Weyl group that the completions G_+ and G_- are potentially bigger than G.

Remark. It is known that the completed group $\mathrm{SL}_n\big(k(\!(t)\!)\big)$ has the property of being transitive on the *complete* system of apartments in the positive building X_+ associated to $G = \mathrm{SL}_n\big(k[t, t^{-1}]\big)$. The complete system of apartments consists of all subsets A of $\mathrm{Ch}(X_+)$ such that the restriction of the Weyl distance to A is Weyl–isometric to (W, δ_W), where W is the Weyl group of X_+. It is however *not* clear in general that the analogue of this property holds for the action of the completion G_+ on the positive building X_+ associated to any group G endowed with a root group datum. Nevertheless, in the special case when G is a split or almost split Kac-Moody group, it is indeed true that G_+ acts transitively on the complete system of apartments of X_+: this property may be deduced from [CR06, Prop. 4].

6.2. **Levi decompositions.** At this point, it is appropriate to make a digression concerning the structure of parabolic subgroups of G and its topological completions. The decompositions $B_+ = T \ltimes U_+$ and $\widehat{B}_+ = T \ltimes \widehat{U}_+$ (see Corollary 5.3 and Proposition 6.1 respectively) are special cases of semi-direct decompositions which apply to all parabolic subgroups of spherical type of G and its completions.

Let $J \subset S$ be such that $W_J = \langle J \rangle$ is finite. Let $\Phi_J = \{\alpha \in \Phi \mid r_\alpha \in W_J\}$ be the associated finite root subsystem. We define

$$L_J = T.\langle U_\alpha \mid \alpha \in \Phi_J \rangle,$$

and, for $\varepsilon \in \{+, -\}$,

$$U_{\varepsilon,J} = U_\varepsilon \cap w_J U_\varepsilon w_J^{-1} \quad \text{and} \quad \widehat{U}_{\varepsilon,J} = \widehat{U}_\varepsilon \cap w_J \widehat{U}_\varepsilon w_J^{-1},$$

where w_J denotes the unique element of maximal length in W_J (which is an involution). Let also $P_{\varepsilon,J}$ be the parabolic subgroup of type J and sign ε in G and let $\widehat{P}_{\varepsilon,J}$ be the parabolic subgroup of type J in G_ε.

Theorem. *For any $\varepsilon \in \{+,-\}$, the following assertions hold:*

(i) *Parabolic subgroups of type J admit a **Levi decomposition**:*

$$P_{\varepsilon,J} = L_J \ltimes U_{\varepsilon,J} \quad \textit{and} \quad \widehat{P}_{\varepsilon,J} = L_J \ltimes \widehat{U}_{\varepsilon,J}.$$

(ii) *The group $\widehat{U}_{\varepsilon,J}$ is the closure of $U_{\varepsilon,J}$ in G_ε and $\widehat{P}_{\varepsilon,J}$ is the closure of $P_{\varepsilon,J}$.*

(iii) *We have:*

$$U_{\varepsilon,J} = \bigcap_{g \in P_{\varepsilon,J}} gU_\varepsilon g^{-1} \quad \textit{and} \quad \widehat{U}_{\varepsilon,J} = \bigcap_{g \in \widehat{P}_{\varepsilon,J}} g\widehat{U}_\varepsilon g^{-1}.$$

Proof. We refer to [Rém02b, Th. 6.2.2] for the statements in G and [RR06, Lemma 1.C.2] for the corresponding extensions to G_ε. □

The group L_J is called a **Levi subgroup** of $P_{\varepsilon,J}$ (resp. $\widehat{P}_{\varepsilon,J}$). The group $U_{\varepsilon,J}$ (resp. $\widehat{U}_{\varepsilon,J}$) is called the **unipotent radical** of $P_{\varepsilon,J}$ (resp. $\widehat{P}_{\varepsilon,J}$). Note that the group L_J is discrete in G_ε by (i), since $\widehat{U}_{\varepsilon,J}$ is open by definition. Moreover, assertion (iii) shows that $\widehat{U}_{\varepsilon,J}$ acts trivially on $\mathrm{Res}_J(B_\varepsilon)$.

Remark. We emphasize the importance of the assumption that the type J of the parabolic subgroups to which the Levi decomposition applies be such that W_J is finite. It is to be expected that such a decomposition fails in general for other types of parabolic subgroups. However, if the strengthened commutation relation axiom (RGD3)$_{\mathrm{lin}}$ holds (see Remark 4 in Sect. 2.1), then parabolic subgroups of all types admit a Levi decomposition by [Rém02b, Th. 6.2.2].

6.3. **The group $\widehat{U}_+$ and other projective limits.** Given a collection $\mathscr{V}$ of groups (e.g. finite groups, nilpotent groups, solvable groups), we say that a totally disconnected group G is **pro-$\mathscr{V}$** if every continuous discrete quotient of G is in $\mathscr{V}$. We also define p**-groups** to be groups all of whose elements have order a power of p; in particular, p-groups need not be finite.

Proposition. *Suppose that for each $\alpha \in \Pi$, the root group U_α is finite (resp. solvable, a p-group). Then $\widehat{U}_+$ is profinite (resp. pro-solvable, pro-p).*

Proof. We give only a sketch. Supplementary details may be found in [RR06, Th. 1.C(ii)] and [CR06, Prop. 3]. The family $\{\widehat{U}_{+,n}\}_{n\geqslant 0}$, as defined in (6.1), is a basis of open neighborhoods of the identity in $\widehat{U}_+$ consisting of normal subgroups. Furthermore, by definition of the topology, the quotient $\widehat{U}_+/\widehat{U}_{+,n}$ is isomorphic to $U_+/U_{+,n}$ for each n. Hence it suffices to show that the successive quotients $U_{+,n}/U_{+,n+1}$ have the desired property (i.e. are finite, solvable, p-groups). This is done by induction on n.

Let $c \in \mathrm{Ch}(X_+)$ be a chamber at numerical distance n from B_+ and let $g \in G$ be such that $g.B_+ = c$. We have $gB_+g^{-1} = \mathrm{Stab}_G(c) \supset U_{+,n}$. Hence, for each $s \in S$, the group $U_{+,n}$ is contained in the parabolic subgroup $P_s(c) := gP_{+,\{s\}}g^{-1}$ of type $\{s\}$. The latter group admits a Levi decomposition, so we get a homomorphism $\varphi_{s,c} : U_{+,n} \to L_s(c)$, where $L_s(c)$ is a Levi subgroup of the parabolic subgroup $P_s(c)$. An induction on n using Theorem 6.2 shows that $U_+ \cap \mathrm{Stab}_G(c)$ is actually contained in the unipotent radical of $\mathrm{Stab}_G(c)$.

Under the canonical projection of $P_s(c)$ onto $L_s(c)$, the latter group is mapped onto a "unipotent" subgroup $U_s(c)$ which turns out to coincide with $gU_s(B_+)g^{-1}$, where $U_s(B_+) = g\langle U_\beta \mid \beta \in \Phi_s\rangle g^{-1}$. By (RGD3), the root groups U_β for $\beta \in \Phi_s$ are mutually centralizing. To summarize, we obtain a homomorphism:

$$\varphi_{s,c} : U_{+,n} \to U_s(c),$$

where $U_s(c)$ is isomorphic to a quotient of the direct product $\prod_{\beta\in\Phi_s} U_\beta$. In view of Theorem 6.2(iii), the kernel of $\varphi_{s,c}$ acts trivially on $\mathrm{Res}_{\{s\}}(c)$. Therefore, the intersection $\bigcap_{s\in S}\mathrm{Ker}(\varphi_{s,c})$ acts trivially on the ball of numerical radius 1 centered at c. Therefore, it follows that

$$\bigcap_{s,c} \mathrm{Ker}(\varphi_{s,c}) = U_{+,n+1},$$

where the intersection is taken over all $s \in S$ and all $c \in \mathrm{Ch}(X_+)$ such that $d_+(c, B_+) = n$. Hence the product homomorphism (defined componentwise)

$$\prod_{s,c} \varphi_{s,c} : U_{+,n} \to \prod_{s,c} U_s(c)$$

induces an injection of the quotient $U_{+,n}/U_{+,n+1}$ into the product $\prod_{s,c} U_s(c)$. All the desired assertions follow, modulo the fact that if each U_α is finite, then the ball of radius n centered at B_+ is finite. The latter fact is clear since the assumption implies that the ball of numerical radius 1 is finite and since G is transitive on $\mathrm{Ch}(X_+)$. □

The above proposition shows that, thanks to root group data with finite root groups, we can obtain (most presumably) interesting families of profinite groups. In the case when root groups are moreover p-groups, the corresponding group $\widehat{U}_+$ is pro-p and a natural question is to compare such a group with well-known examples, e.g., analytic groups over local fields. This is a subtle question because the local fields for which the question is relevant are of positive characteristic (the group U_+, hence $\widehat{U}_+$, contains a lot of torsion elements). Here is a first result showing that Kac-Moody theory shall provide new interesting examples of pro-p groups.

Theorem. *For any sufficiently large prime number p, there exist Kac-Moody groups G over the field $\mathbb{Z}/p\mathbb{Z}$ such that:*
- (i) *each root group is isomorphic to the additive group $(\mathbb{Z}/p\mathbb{Z}, +)$;*
- (ii) *the group U_+ enjoys Kazhdan's property* (T) *– in particular, it is finitely generated;*
- (iii) *the completion $\widehat{U}_+$ is a Golod-Shafarevich pro-p group.*

Part (ii) is due to J. Dymara and T. Januszkiewicz [DJ02]; Part (iii) is due to M. Ershov [Ers06, Theorem 1.6]. On the one hand, a finitely generated group Γ which is *Golod-Shafarevich* at p has the property that its pro-p completion $\widehat{\Gamma}_p$ admits a presentation with remarkably few relations with respect to the number of generators [loc. cit., Introduction]. The reason why it is connected to the previous discussion is that in this case the group $\widehat{\Gamma}_p$ contains a non-abelian free pro-p group, which cannot be analytic. On the other hand, Kazhdan's property (T) is a property with many equivalent characterizations (in terms of isometric actions on separable Hilbert spaces, of unitary representations etc) [dlHV89]; it is satisfied by most lattices in semisimple Lie groups and can be used to prove that for most of these (center-free) lattices any proper quotient has to be finite. Therefore the existence, observed by M. Ershov, of groups combining Kazhdan's property (T) and a Golod-Shafarevich presentation is rather surprising. We refer to [Ers06, Sect. 8] for a deeper discussion on the usefulness in discrete group theory of the Kac-Moody groups in the above theorem.

To sum up, at this stage we already know that the completion procedure described in 6.1 provides totally disconnected locally pro-p groups which look like simple algebraic matrix groups over local fields (at least from a combinatorial viewpoint), but are new in general since their pro-p Sylow subgroups, which are their maximal compact subgroups up to finite index, are not analytic groups over local fields.

6.4. **Lattices.** When the root groups U_α ($\alpha \in \Pi$) are finite, the group $\widehat{U}_+$ is compact open by Proposition 6.3 and, hence, G_+ is locally compact. Therefore G_+ admits a Haar measure denoted Vol [Bou07a, Ch. VII, §1, Th. 1] and it makes sense to talk about **lattices**, i.e. discrete subgroups Γ such that $\mathrm{Vol}(G_+/\Gamma)$ is finite [Bou07a, Ch. VII, §2, n°5].

We already know some discrete subgroups of G_+ and $G_+ \times G_-$ by Proposition 6.1. In order to check whether the covolume of these is finite, the following simple criterion is useful:

Proposition. *Let G be a locally compact group acting continuously and properly by automorphisms on a locally finite building X with finitely many orbits on* $\mathrm{Ch}(X)$ *(as before, the group* $\mathrm{Aut}(X)$ *is endowed with the bounded-open topology). Let $\mathcal{C}$ be a set of representatives of the Γ-orbits in* $\mathrm{Ch}(X)$. *A discrete subgroup $\Gamma \subset G$ is a lattice in G if and only if the series*

$$\sum_{c \in \mathcal{C}} \frac{1}{|\,\mathrm{Stab}_\Gamma(c)|}$$

converges.

Proof. We refer to [Ser77, p. 116] and note that the idea to apply Serre's argument to automorphism groups of buildings already appeared in [Bou00, Prop. 1.4.2]. □

The following result is due to the second author in [Rém99]; similar results were obtained by Carbone-Garland [CG99].

Theorem. *Let G be a group with finite center, endowed with a root group datum $\{U_\alpha\}_{\alpha\in\Phi}$ with Φ reduced such that U_α is finite for each $\alpha \in \Pi$ and that*

$$\sum_{w\in W} (1/q)^{\ell(w)} < \infty,$$

where $q = \min_{\alpha\in\Pi} |U_\alpha|$ (a sufficient condition is: $q > |S|$). Then U_+ is a lattice in G_- and G is a lattice in $G_+ \times G_-$.

Proof. Let us first consider the group U_+. By the Birkhoff decomposition G is the disjoint union of subsets of the form $U_+.w.B_-$ where w runs over W. This means that the set $\mathcal{C} = \{w.B_- \mid w \in W\}$ is a set of representatives of the U_+-orbits in $\mathrm{Ch}(X_-)$. By Corollary 5.3(i), we have $\mathrm{Stab}_{U_+}(w.B_-) = U_+ \cap wB_-w^{-1} = U_w$. By the proposition (see also Proposition 6.1(iv)), the group U_+ is a lattice in G_- if and only if $\sum_{w\in W} \frac{1}{|U_w|} < \infty$.

Let us now consider the action of G on $X = X_+ \times X_-$. The product X is a building of type (W, S). Its chamber set $\mathrm{Ch}(X)$ is $\mathrm{Ch}(X_+) \times \mathrm{Ch}(X_-)$. The G-action on X preserves the Weyl codistance. Moreover, by the Birkhoff decomposition, it is **Weyl co-transitive** in the following sense: for any $x, x' \in \mathrm{Ch}(X_+)$ and $y, y' \in \mathrm{Ch}(X_-)$ such that $\delta^*(x, y) = \delta^*(x', y')$ there exists $g \in G$ such that $(g.x, g.y) = (x', y')$. Therefore, it follows that the set $\{(B_+, w.B_-) \mid w \in W\}$ is a set of representatives for the G-orbits in $\mathrm{Ch}(X)$. Moreover we have $\mathrm{Stab}_G(B_+, w.B_-) = B_+ \cap wB_-w^{-1} = T.U_w$ by Corollary 5.3. Combining Corollary 5.6 with [Tit90, Th. 1], we see that the quotient $T/Z(G)$ is finite, hence so is T because $Z(G)$ is finite by hypothesis. It follows again from the proposition that G is a lattice in $G_+ \times G_-$ if and only if $\sum_{w\in W} \frac{1}{|U_w|} < \infty$.

It remains to evaluate the sum $z = \sum_{w\in W} \frac{1}{|U_w|}$. In view of Lemma 5.5, we have $|U_w| \geqslant q^{\ell(w)}$, where $q = \min_{\alpha\in\Pi} |U_\alpha|$. Therefore $z \leqslant \sum_{w\in W} (1/q)^{\ell(w)}$ as desired.

Note that $\sum_{w\in W} x^{\ell(w)} = \sum_{n\geqslant 0} |W(n)|x^n$, where $W(n) = \{w \in W \mid \ell(w) = n\}$. Since we have $|W(n)| \leqslant |S|^n$, the condition $q > |S|$ is clearly sufficient for $\sum_{w\in W} (1/q)^{\ell(w)}$ to converge. □

For the theory of lattices in Lie groups we refer to [Rag72], and for the more advanced and specific theory of lattices in semisimple Lie groups we refer to [Mar91]. These references are the guidelines for the study of lattices arising from the theory of root data with finite root groups as below, at least for the part of the study which relies on analogies with arithmetic groups [Rém02a].

7. Simplicity results

7.1. **Tits' transitivity lemma.** It is an elementary fact on permutation groups that if a group G acts transitively and primitively on a set X (e.g. G is 2-transitive), then any normal subgroup of G acts either trivially or transitively. If a group G has a BN-pair, it is not quite true that its action on the chambers of the corresponding building is primitive, but it is indeed true that a chamber-stabilizer has very few over-groups: as mentioned in Sect. 4.1.1, any subgroup containing B is a standard parabolic subgroup. This should shed some light upon the following:

Lemma. ***(Tits' transitivity lemma)*** *Let G be a group with a BN-pair (B, N, S) and X be the associated building and W be the Weyl group. If the Coxeter system (W, S) is irreducible, then any normal subgroup of G acts either trivially or transitively on $\mathrm{Ch}(X)$.*

Proof. See [Tit64, Prop. 2.5] or [Bou07b, Ch. IV, §2, Lemma 2]. □

In fact, this very useful result might be seen as a variant of a previously known and quite classical theme, according to which groups admitting a sufficiently transitive action on a set shall be submitted to strong restrictions concerning their normal subgroups. To be more precise, we need to introduce some further notions (they will be useful – and still relevant to simplicity – when discussing some local actions on trees): the action of a group G on a set X is called *quasi-primitive* if G, as well as any non-trivial normal subgroup of G, acts transitively on X. This is the case if the action is *primitive*, namely if the only equivalence relations on X which are compatible with the G-action are the trivial ones; note that primitivity itself is implied by 2-transitivity. In fact, denoting by $G^\dagger$ the subgroup of G (acting on X) generated by the stabilizers of the various elements x in X, we have the following implications:

$$G \text{ acts 2-transitively on } X \Rightarrow G \text{ acts primitively on } X \Rightarrow G \text{ acts quasi-primitively on } X,$$

which finally implies that $G = G^\dagger$ or that G acts simply transitively on X. A variant of this is the well-known Iwasawa's lemma: let G act quasi-primitively on X such that there exists a G-equivariant map $T : X \to \{\text{abelian subgroups of } G\}$; $x \mapsto T_x$ with $G = \langle T_x : x \in X \rangle$. Then for any normal subgroup $N \lhd G$ acting non-trivially on X, we have: $N \supseteq [G, G]$. This is a well-known elegant way to prove the projective simplicity of linear groups (using unipotent subgroups).

In Tits' specific lemma, some combinatorial structure (namely, the building structure) is needed on the set on which the group acts, but the transitivity condition is not as strong as it is for classical simplicity criteria.

As a final remark concerning 2-transitive (or slightly less transitive) group actions, we note that one of J. Tits' earliest works is the generalization of projective groups by means of multiple transitivity properties [Tit52]. J. Tits proves in this work that if a group G acts sharply n-transitively on an arbitrary set X, with $n \geqslant 4$, then the set is finite and either the group is a symmetric or alternating group with its standard action, or the set has at most 12 points and there are very few examples, only with $n = 4$ or 5. The example of Moufang sets [dMS07], as defined by him in 1992, therefore provides a nice way to see that J. Tits' latest subjects of interest are in close connection with the very earliest ones.

7.2. **Topological simplicity of topological completions.** Tits' original use of his transitivity lemma was to obtain a proof of abstract simplicity applying uniformly to all isotropic simple algebraic groups. The notion of a BN-pair (and later that of a root group datum) was created by him precisely in order to obtain such a uniform theory. Recall that in the context of algebraic groups, the Weyl group W is finite (see Sect. 2.6), the group U_+ is nilpotent (see Lemma 5.5) and G coincides with the completions G_+ and G_-. However, letting Tits' arguments work in the more general context of arbitrary root group data, one obtains the following statement:

Theorem. *Let G be a group endowed with a root group datum $\{U_\alpha\}_{\alpha\in\Phi}$ of irreducible type. Assume that the completion G_+ is topologically perfect (i.e. $[G_+, G_+]$ is dense in G_+) and that U_α is solvable for each $\alpha \in \Pi$. Then $G_+/Z(G_+)$ is topologically simple (i.e. any closed normal subgroup is trivial).*

Proof. Let H be a normal subgroup of G_+ not contained in $Z(G_+)$. In view of Proposition 6.1(iii) and Tits' transitivity lemma, we have $G_+ = H.\widehat{B}_+$. Since $\widehat{B}_+$ normalizes $\widehat{U}_+$ it follows that every conjugate of $\widehat{U}_+$ in G_+ is of the form $h\widehat{U}_+h^{-1}$ for some $h \in H$. By Proposition 6.1(ii) and (iii), the group G_+ is clearly generated by all these conjugates, hence we obtain $G_+ = H.\widehat{U}_+$. It follows that

$$G_+/H = H.\widehat{U}_+/H \simeq \widehat{U}_+/H \cap \widehat{U}_+. \tag{7.1}$$

Assume now that H is closed. Since G_+ is topologically perfect, so is the continuous quotient G_+/H. On the other hand, the group $\widehat{U}_+$ is pro-solvable by Proposition 6.3, hence the derived series of $\widehat{U}_+$ penetrates every open neighborhood of the identity in $\widehat{U}_+$. Clearly this property is inherited by any continuous quotient. Therefore, the only continuous quotient of $\widehat{U}_+$ which is topologically perfect is the trivial one. Now, it follows from (7.1) that $H = G_+$. $\square$

Remark. It is in good order to wonder when the condition that G_+ be topologically perfect is fulfilled. If G itself is abstractly perfect, then G_+ is clearly topologically perfect since G is dense in G_+ by definition. Now for G to be perfect, it suffices that each rank one group $X_\alpha = \langle U_\alpha \cup U_{-\alpha}\rangle$ be perfect since G is generated by those. This happens for example when G is any split Kac-Moody group over a field k of order > 3, since then $X_\alpha \simeq \mathrm{SL}_2(k)$. However, G_+ turns out to be topologically perfect in many circumstances, even when G is not abstractly perfect. We refer to [CR06, Sect. 2.2] for sufficient conditions ensuring that G_+ is topologically perfect. These conditions are fulfilled by all split or almost split Kac-Moody groups over *arbitrary* fields (as long as the Weyl group is infinite), as well as by most root group data obtained by exotic constructions, such as those mentioned in Sect. 2.8, see [CR06, Sect. 2.1].

7.3. **Abstract simplicity of topological completions.** As demonstrated by L. Carbone, M. Ershov and G. Ritter [CER06], in the case when $\widehat{U}_+$ is a profinite group, the arguments of the proof of Theorem 7.2 may be pushed further in order to obtain *abstract* simplicity of the completion G_+. In fact, the latter reference deals primarily with the case when $\widehat{U}_+$ is pro-p. Using some results of Dan Segal's [Seg00], this can be extended to the more general case when $\widehat{U}_+$ is pro-solvable:

Theorem. *Maintain the assumptions of Theorem 7.2 and assume moreover that U_α is finite for each $\alpha \in \Pi$ and that $\widehat{U}_+$ is topologically finitely generated (i.e. $\widehat{U}_+$ possesses a finitely generated dense subgroup). Then $G_+/Z(G_+)$ is abstractly simple.*

Proof. By Proposition 6.3, the group $\widehat{U}_+$ is profinite. By [Seg00, Corollary 1], the group $[\widehat{U}_+, \widehat{U}_+]$ is closed, hence $\widehat{U}_+/[\widehat{U}_+, \widehat{U}_+]$ is a finitely generated abelian profinite group. Moreover the group $\widehat{U}_+$ is topologically generated by U_+, which is itself generated by $\{U_\alpha \mid \alpha \in \Phi\}$. Since all root groups are finite and since there is finitely many of them up to conjugacy, it follows that $\widehat{U}_+/[\widehat{U}_+, \widehat{U}_+]$ is of finite exponent. It must therefore be finite since it is finitely generated. Thus $[\widehat{U}_+, \widehat{U}_+]$ is of finite index in $\widehat{U}_+$, hence open by [Seg00, Theorem 1], since $\widehat{U}_+$ is itself open in G_+. Now it follows that the derived group $[G_+, G_+]$, which contains $[\widehat{U}_+, \widehat{U}_+]$ is open, hence closed. By assumption, this implies that G_+ is abstractly perfect, namely $G_+ = [G_+, G_+]$.

The arguments of the proof of Theorem 7.2 may now be repeated, thereby establishing (7.1). In order to conclude, it remains to prove that a finitely generated pro-solvable (profinite) group has no nontrivial perfect quotient, which is indeed true by the proposition below. □

Remark. Again, one should ask when it actually happens that $\widehat{U}_+$ is topologically finitely generated. This is discussed in [CER06, Sect. 6 and 7], where some sufficient conditions are given in the case when G is a split Kac-Moody group over a field. Here we merely mention that the case when (W, S) is **2-spherical** (i.e. $o(st) < \infty$ for all $s, t \in S$) is especially favourable, because then the group U_+ is (mostly) *abstractly* finitely generated, see Theorem 8.1(i) below, and hence its closure $\widehat{U}_+$ is of course topologically finitely generated.

The following statement is a consequence of Dan Segal's results proven in [Seg00]. Since it is of independent interest but not explicitly stated in [loc. cit.], we include it here:

Proposition. *Let G be topologically finitely generated pro-solvable (profinite) group. Then G has no nontrivial perfect quotient.*

Proof. Let H be a normal subgroup of G such that G/H is perfect. Thus we have $G = H.[G, G]$. Since G is topologically finitely generated, the derived group $[G, G]$ is closed by [Seg00, Corollary 1] and, hence, the quotient $G/[G, G]$ is a topologically finitely generated abelian profinite group. Since it is generated by the projection of H, it follows right away that there exist finitely many elements $h_1, \dots, h_d \in H$ such that $G = \overline{\langle h_1, \dots, h_d\rangle}.[G, G]$.

Let now N be an open normal subgroup of G. Thus G/N is a finite solvable group generated by the projections of $h_1, \dots, h_d$. Using the last equation in [Seg00, p. 52], we obtain that

$$[G, G] = \Big(\prod_{i=1}^{d} [h_i, G]\Big)^{*f(d)}.N$$

for some $f(d) \in \mathbb{N}$, where the notation $*f(d)$ is used to denote the image of the $f(d)^{\text{th}}$ Cartesian power under the product map. Since the latter equation holds for any open normal subgroup N, we deduce from [DdSMS99, Prop. 1.2(iii)] that

$$[G,G] = \overline{\Big(\prod_{i=1}^{d}[h_i,G]\Big)^{*f(d)}}.$$

Since the map $G \to G : g \mapsto [h_i, g]$ is continuous and G is compact, the set $[h_i, G]$ is closed in G. Hence the big product in the right-hand side of the latter equation is closed and we obtain

$$[G,G] = \Big(\prod_{i=1}^{d}[h_i,G]\Big)^{*f(d)}.$$

Since H is normal, we have $[h_i, G] \subset H$ for each i, from which we finally deduce that $[G,G] \subset H$. Since we have $G = H.[G,G]$ by assumption, it finally follows that $G = H$ as desired. □

At this stage, we note that Kac-Moody groups over finite fields provide, through their geometric completions, intriguing topological groups. Indeed, they are often abstractly simple, locally pro-p and share further (combinatorial) properties with adjoint simple algebraic groups over local fields of positive characteristic. This is a probably non-exhaustive list of arguments supporting the analogy with classical matrix groups, but we also saw that the maximal compact subgroups of some of them contain finite index subgroups which are Golod-Shafarevich and hence contain free pro-p subgroups. It would be interesting to provide further arguments supporting and/or disproving this analogy, from the point of view of representation theory for instance.

7.4. **Weyl transitivity of normal subgroups.** The previous simplicity results deal only with the topological completions. No such general simplicity results should be expected for the uncomplete group G. Indeed, recall from Sect. 2.7 that the group $G = \mathrm{SL}_n(k[t, t^{-1}])$ possesses a root group datum, but it is far from simple in view of the existence of evaluation homomorphisms. However, in the context of root group data, Theorem 7.2 may be used to obtain a strengthening of Tits' transitivity lemma.

Before stating it, we introduce the following definition: a group G, acting on a building X with Weyl distance δ, is called **Weyl transitive** if for any $x, y, x', y' \in \mathrm{Ch}(X)$ such that $\delta(x, y) = \delta(x', y')$, there exists $g \in G$ such that $(g.x, g.y) = (x', y')$. It is an immediate consequence of the Bruhat decomposition that if G has a BN-pair, then G is Weyl transitive on the associated building. The following result is a straightforward consequence of Theorem 7.2:

Corollary. *Let G be a group endowed with a root group datum and assume that the hypotheses of Theorem 7.2 hold. Then any normal subgroup of G is either central or Weyl transitive on X_+.*

Proof. Let H be a normal subgroup of G which is not contained in $Z(G)$. Let $\overline{H}$ denote the closure of H in G_+. By Proposition 6.1(iii) and Theorem 7.2, we have $\overline{H} = G_+$, hence H is dense. The point-stabilizers of G_+ for its diagonal action on $\mathrm{Ch}(X_+) \times \mathrm{Ch}(X_+)$ are open in G_+. Since H is dense, it follows immediately that H and G_+ have the same orbits in $\mathrm{Ch}(X_+) \times \mathrm{Ch}(X_+)$. The result follows, since G_+ is Weyl transitive on X_+ by Proposition 6.1. □

Coming back again to the group $\mathrm{SL}_n\big(\mathbb{F}_q[t, t^{-1}]\big)$, it follows from the corollary that it contains Weyl transitive subgroups of arbitrarily large finite index, since it is residually finite. More information on Weyl transitivity and other families of examples may be found in [AB06] and [AB, Sect. 6].

7.5. **Simplicity of lattices.** As mentioned in the previous section, the discrete group G should not be expected to be simple in general. It was shown in [CR06] that the existence of finite quotients for G is related to the geometry of its Weyl group. In fact, building upon earlier work of Y. Shalom [Sha00], Bader-Shalom [BS06] and B. Rémy [Rém05], the following result was proven in [CR06, Theorem 19]:

Theorem. *Let G be a group with a root group datum $\{U_\alpha\}_{\alpha \in \Phi}$ with Φ reduced of finite rank such that:*

- *U_α is finite and nilpotent for each $\alpha \in \Pi$,*
- *$\sum_{w\in W}(1/q)^{\ell(w)} < \infty$, where $q = \min_{\alpha\in\Pi}|U_\alpha|$,*
- *(W, S) is irreducible,*
- *W is not virtually abelian (i.e. W is not of spherical or of affine type).*

Then $G/Z(G)$ is infinite, finitely generated and virtually simple. All of its finite quotients are nilpotent and factor through (i.e. are quotients of) the direct product $\prod_{\alpha\in\Pi} U_\alpha$. □

The fact that G embeds as an irreducible lattice in $G_+ \times G_-$ (see Theorem 6.4) enables one to appeal to the results of Y. Shalom [Sha00], Bader-Shalom [BS06] and B. Rémy [Rém05]. Combining them all, it follows that any noncentral normal subgroup of G is of finite index. On the other hand, if W is not virtually abelian then the geometry of the associated Coxeter complex enjoys some form of combinatorial hyperbolicity which may be exploited to obtain strong obstructions to the existence of finite quotients of G, see [CR06]. All together, these arguments yield the theorem above.

It is the right place to mention that a construction of finitely presented torsion free groups as lattices in product of buildings (in fact, trees), standing by the (rich, but fortunately not complete!) analogy with irreducible lattices in products of simple Lie groups, was first due to M. Burger and Sh. Mozes [BM00b]. The groups they construct are fundamental groups of finite square complexes; in fact, they are uniform lattices for products of two trees. An important tool in the study of these simple lattices is the projections on factors. This amounts to investigating the closures of the projections of these lattices in the full automorphism group of a single tree [BM00a]. For this, a general structure theory is developped for closed non-discrete groups acting on trees: if the local actions (i.e. the actions of vertex stabilizers on the spheres around the vertices) are sufficiently transitive on large enough spheres, then a strong dichotomy holds for closed normal subgroups [loc. cit., lines 20-22 of the introduction]. This is where transitivity properties for group action are back as one of the main conditions: (quasi-)primitivity, 2-transitivity appear in the above theory at local level, but also as a condition on the action on the asymptotic boundaries of the trees under consideration [loc. cit., §3]. We finally note that these groups cannot have property (T) since they act nontrivially on trees, as opposed to many simple Kac-Moody lattices who often do enjoy property (T), and are finitely presented.

For the general problem of constructing infinite finitely generated groups, we recommend the concise but instructive historical note in [Rat04].

8. Curtis-Tits type presentations and existence results

We have already encountered presentations of groups with BN-pairs as a corollary of Proposition 5.2. It turns out that for groups with a root group datum, there often exist much more economical presentations, called **Curtis-Tits type** presentations.

For groups with a finite Weyl group, these were first obtained by R. Curtis and J. Tits. This was extended to the case of certain infinite Weyl groups by P. Abramenko and B. Mühlherr [AM97]. When all root groups are finite, this presentation happens to be finite. Homological finiteness properties of groups with a root group datum were extensively studied by P. Abramenko; we refer to [Abr04] for a survey of some known results. In this section we focus on the Curtis-Tits type presentations. We mention in passing some facts on Steinberg-type presentations for the universal central extensions, and conclude with some remarks on existence of root group data for groups given by a Curtis-Tits type presentation.

8.1. Curtis-Tits and Steinberg type presentations of the universal central extension.

The set-up is the following. As before, we let G be a group with a root group datum $\{U_\alpha\}_{\alpha\in\Phi}$ of type $E = (\mathbf{B}, \Phi)$ and assume that $\Phi = \Phi(\mathbf{B})$ is the canonical root system of $\mathbf{B}$.

We will assume moreover that the Coxeter system (W, S) is **2-spherical**, i.e. $o(st) < \infty$ for all $s, t \in S$. As a justification for this assumption, let us just mention the fact that the group $SL_2(\mathbb{F}_q[t, t^{-1}])$ is finitely generated but *not* finitely presented, see [Beh98]. As we know from Sect. 2.7, this group is endowed with a root group datum with infinite dihedral Weyl group.

Another condition that we will take as a hypothesis is the following:

$$(*) \qquad X_{\alpha,\beta}/Z(X_{\alpha,\beta}) \not\in \{B_2(2), G_2(2), G_2(3), {}^2F_4(2)\} \text{ for all } \alpha, \beta \in \Pi.$$

The importance of this condition comes from the following:

Lemma. *Suppose that (W,S) is 2-spherical. Then Condition $(*)$ holds if and only if for all $\alpha, \beta \in \Pi$, $\alpha \neq \beta$, we have*

$$[U_\alpha, U_\beta] = \langle U_\gamma \mid \gamma \in]\alpha, \beta[\rangle.$$

Proof. See [Abr96, Prop. 7]. □

Note that the inclusion $\subset$ in the previous lemma is covered by axiom (RGD3); the essential point is that $(*)$ allows to express root subgroups as commutators.

In order to simplify notation, we make the following convention: given a set of roots $\Psi \subset \Phi$, we denote by U_Ψ the group generated by all U_γ with $\gamma \in \Psi$.

Theorem. *Suppose that (W,S) is 2-spherical, that S is finite and that $(*)$ holds. Then we have the following:*

(i) *$U_+ = \langle U_\alpha \mid \alpha \in \Pi\rangle$; in particular U_+ is finitely generated if all root groups are finite.*
(ii) *If (W,S) is 3-spherical (i.e. any triple of elements of S generates a finite subgroup of W) and if $|U_\alpha| \geqslant 16$ for each $\alpha \in \Pi_\Phi$ then U_+ is the direct limit of the inductive system formed by the U_α and $U_{[\alpha,\beta]}$, where $\alpha, \beta \in \Pi$; in particular U_+ is finitely presented if all root groups are finite.*
(iii) *Let $\widetilde{G}$ be the direct limit of the inductive system formed by the X_α and $X_{\alpha,\beta}$ in G, where $\alpha, \beta \in \Pi$. Then $\widetilde{G}$ is endowed with a root group datum and the kernel of the canonical homomorphism $\widetilde{G} \to G$ is central. In particular, G is finitely presented if all root groups are finite.*
(iv) *Let $\mathrm{St}(G)$ be the direct limit of the inductive system formed by the U_α and $U_{[\alpha,\beta]}$, where $\{\alpha, \beta\} \subset \Phi$ is a prenilpotent pair such that $o(r_\alpha r_\beta)$ is finite. If (W,S) is irreducible and $|S| \geqslant 3$ and if $|U_\alpha| \geqslant 5$ for all $\alpha \in \Pi$, then $\mathrm{St}(G) \to G$ is a universal central extension of G. In particular, the center $Z(\mathrm{St}(G))$ (and hence $Z(G)$) is finite if all root groups are finite.*

Proof. For (i), one shows by induction on $\ell(w)$ that $U_{w.\alpha} \subset \langle U_\beta \mid \beta \in \Pi\rangle$ for all $\alpha \in \Pi$ such that $w.\alpha > 0$. The point is to view $U_{w.\alpha}$ as a subgroup of a commutator of root subgroups which are already known to be contained in $\langle U_\beta \mid \beta \in \Pi\rangle$ by induction. This uses the lemma and the 2-sphericity of (W,S). For (ii), we refer to [DM06, Cor. 1.2]. A statement similar to (iii) was first obtained in [AM97]. For a complete proof of the above, see [Cap05, Theorem 3.7]. Statement (iv) follows from a combination of [Cap05, Theorem 3.11] and the results of [dMT06]. □

Remark 1.: P. Abramenko has proved that, provided $(*)$ holds, the group U_+ is finitely presented if and only if (W,S) is 3-spherical [Abr04]. Thus the presentation in (ii) should not be expected to hold when (W,S) is not 3-spherical. More information on the (homological) finiteness properties of G may be found in Abramenko's book [Abr96] or in the survey paper [Abr04].

Remark 2.: We emphasize that the relations which present the Steinberg group are not *all* commutation relations of G but only those commutation relations which appear in rank two Levi subgroups of spherical type (i.e. with finite Weyl group).

8.2. **Existence and classification results.** One way of interpreting Theorem 8.1(iii) is by saying that the group G is completely determined by triple the $(E, \mathscr{X}, K)$ consisting of the root datum E, the inductive system $\mathscr{X} = \{X_\alpha, X_{\alpha,\beta} \mid \alpha, \beta \in \Pi\}$ and the (central) kernel K of the homomorphism $\widetilde{G} \to G$. This motivates the following definition.

A **local datum** is a triple $\mathscr{D} = (E, \mathscr{X}, K)$ consisting of the following data:

- a root datum $E = (\mathbf{B}, \Phi)$ of 2-spherical type and of finite rank;

- an inductive system $\mathscr{X}$ of groups parameterized as follows: for each $\gamma \in \bigcup_{\alpha,\beta\in\Pi} \Phi_{\{\alpha,\beta\}}$, a group U_γ and for all distinct $\alpha, \beta \in \Pi$, a group $X_{\alpha,\beta}$ such that $\{U_\gamma\}_{\gamma\in\Phi_{\{\alpha,\beta\}}}$ is a root group datum of type $E_{\{\alpha,\beta\}}$ for $X_{\alpha,\beta}$; all morphisms of the inductive system $\mathscr{X}$ are inclusions;
- a subgroup K of the center $Z(\widetilde{G})$, where $\widetilde{G}$ is defined as the direct limit of the inductive system $\mathscr{X}$.

Given a local datum $\mathscr{D}$, the inductive limit $\widetilde{G} = \widetilde{G}(\mathscr{D})$ is called its **universal enveloping group** and the quotient $G(\mathscr{D}) = \widetilde{G}/K$ is called the **enveloping group**. The subgroup $K = K(\mathscr{D})$ is called the **kernel** of $\mathscr{D}$.

Thus local data provide excellent candidates for being classifying data of all groups G with a twin root datum with 2-spherical Weyl group satisfying the condition $(*)$. In order to make this correspondence a genuine classification of the isomorphism classes of groups endowed with such a root group datum, there are two questions to answer:

Question 1.: Given a local datum, is its enveloping group endowed with a root group datum?

Question 2.: Given two non-isomorphic local data, are their respective enveloping groups non-isomorphic?

Both problems are still incompletely solved. In order to make a precise statement of some of the known information, let us make some additional definitions. The local datum $\mathscr{D}$ is called **locally finite** if U_γ is finite for each $\gamma \in \Pi$. It is called **locally split** if $X_{\alpha,\beta}$ is a split Chevalley group of rank 2 for all distinct $\alpha, \beta \in \Pi$. Furthermore, we let $\mathcal{LS}$ be the collection of all local data which are locally finite or locally split and which satisfy condition $(*)$.

Theorem. *We have the following:*

(i) *For each $\mathscr{D} \in \mathcal{LS}$, the enveloping group $G(\mathscr{D})$ is endowed with a twin root datum of type E such that the associated local datum coincides with $\mathscr{D}$.*

(ii) *Let $\mathscr{D}_1, \mathscr{D}_2 \in \mathcal{LS}$ be such that $G(\mathscr{D}_1)$ is infinite and the root datum of $\mathscr{D}_1$ is of irreducible type. Let also $\varphi : G(\mathscr{D}_1) \to G(\mathscr{D}_2)$ be an isomorphism. Then there exists a bijection $\sigma : \Pi_1 \to \Pi_2$, a sign $\varepsilon \in \{+,-\}$, an inner automorphism* $\mathrm{Ad}\ g$ *of $G(\mathscr{D}_2)$ and for each root α with $\pm\alpha \in \Pi$, an isomorphism $\varphi_\alpha : U_\alpha \to U_{\varepsilon\sigma(\alpha)}$ such that the diagram*

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\varphi_\alpha} & U_{\varepsilon\sigma(\alpha)} \\ \downarrow & & \downarrow \\ G(\mathscr{D}_1) & \xrightarrow{\mathrm{Ad}\ g\circ\varphi} & G(\mathscr{D}_2) \end{array}$$

commutes for each root α with $\pm\alpha \in \Pi$, where the vertical arrows are the canonical inclusions. In particular, for all distinct $\alpha, \beta \in \Pi$, the restriction of $\mathrm{Ad}\ g \circ \varphi$ *to $X_{\alpha,\beta}$ is an isomorphism onto $X_{\varepsilon\sigma(\alpha),\varepsilon\sigma(\beta)}$. Moreover, the isomorphism* $\mathrm{Ad}\ g \circ \varphi$ *induces an isomorphism between the universal enveloping groups of $\mathscr{D}_1$ and $\mathscr{D}_2$ which maps the kernel $K(\mathscr{D}_1)$ to the kernel $K(\mathscr{D}_2)$.*

Proof. Statement (i) is a reformulation of the main result of [Müh99]. Once (i) is known to hold, part (ii) follows from the results of [CM06] and [Cap06] and the fact that $G(\mathscr{D}_2)$ can be embedded in a Kac-Moody group by [Müh99]. More precisely, the statement above is obtained by an argument which goes along the following lines.

First, it follows from (i) that the group $G(\mathscr{D})$ is finitely generated if and only if $\mathscr{D}$ is locally finite. Therefore, we may assume that $\mathscr{D}_1$ and $\mathscr{D}_2$ are either both locally finite or both locally split (and infinite). The case of locally finite ones is covered by [CM06, Theorem 5.1 and Corollary 3.8]. In fact, technically speaking the latter reference requires all root groups to be of order at least 4, but this assumption can be bypassed by taking advantage of the fact that Weyl groups are assumed to be 2-spherical in the present context.

Now, for locally split $\mathscr{D}$, it essentially follows from (i) (see [Müh99]) that $G(\mathscr{D})/Z(G(\mathscr{D}))$ is in fact a split adjoint Kac-Moody group. The desired statement then follows from [Cap06, Theorem A]. □

Remark 1.: The results of B. Mühlherr [Müh99] quoted above are originally stated in the setting of twin buildings, but they can be easily reformulated in the setting of root group data. We refer to [Tit92] for details on the translation from one context to the other. B. Mühlherr has designed a program to extend the results of [Müh99] to all local data satisfying $(*)$ and has successfully carried out large parts of this program, see [Müh02]. Let us mention here that, in order to check that the enveloping group $G(\mathscr{D})$ is endowed with a root group datum, the main difficulty is to prove that axiom (RGD1) is fulfilled. Indeed (RGD0) trivially holds, (RGD2) is satisfied by construction, and arguments similar to those of the proof of Theorem 8.1(iii) show that (RGD3) holds as well. Now, in order to prove that (RGD1) is also satisfied, it suffices to show the subgroup U_+ of $G(\mathscr{D})$ is residually nilpotent. This is because the rank one group $X_\alpha = \langle U_\alpha \cup U_{-\alpha} \rangle$ is never nilpotent (see Corollary 5.6); in fact X_α is quasisimple unless it is finite of very small order. However, the residual nilpotency of U_+ is delicate to establish. The way it is done in [Müh99] is by realizing the inductive system of rank two groups of $\mathscr{D}$ in a certain large group which is known to possess a root group datum (mostly the latter group is a split Kac-Moody group). This allows to embed U_+ in some unipotent radical of this larger group. Now the latter group is residually nilpotent as a consequence of Proposition 6.3, hence so is U_+.

Remark 2.: The article [CM06] quoted above is concerned with the isomorphism problem for groups endowed with locally finite root group data, while [Cap06] deals with the case of split Kac-Moody groups. None of these references makes the assumption that the Weyl group is 2-spherical.

Remark 3.: The only reason for the assumption that $G(\mathscr{D}_1)$ is infinite and $\mathscr{D}_1$ of irreducible type in Theorem 8.2(ii) is to avoid the exceptional isomorphisms between small finite Chevalley groups. Of course, the conclusions of that theorem are known to hold for all sufficiently large finite Chevalley groups: this is all classical, see [Ste68].

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