“Abstract” homomorphisms of split Kac-Moody groups

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Abstract

This work is devoted to the isomorphism problem for split Kac-Moody groups over arbitrary fields. This problem turns out to be a special case of a more general problem, which consists in determining homomorphisms of isotropic semisimple algebraic groups to Kac-Moody groups, whose image is bounded. Since Kac-Moody groups possess natural actions on twin buildings, and since their bounded subgroups can be characterized by fixed point properties for these actions, the latter is actually a rigidity problem for algebraic group actions on twin buildings. We establish some partial rigidity results, which we use to prove an isomorphism theorem for Kac-Moody groups over arbitrarily fields of cardinality at least 4. In particular, we obtain a detailed description of automorphisms of Kac-Moody groups. This provides a complete understanding of the structure of the automorphism group of Kac-Moody groups over ground fields of characteristic 0.

The same arguments allow to treat unitary forms of complex Kac-Moody groups. In particular, we show that the Hausdorff topology that these groups carry is an invariant of the abstract group structure.

Finally, we prove the non-existence of cocentral homomorphisms of Kac-Moody groups of indefinite type over infinite fields with finite-dimensional target. This provides a partial solution to the linearity problem for Kac-Moody groups.

Contents

Introduction

The starting point of this work is the study of the automorphism group of a split Kac-Moody group over an arbitrary field.

In a general sense, a Kac-Moody group is a group attached to a Kac-Moody Lie algebra. The initial motivation of the construction of these algebras by V. Kac and R. Moody has been largely surpassed nowadays by the spectacular developments and ramified applications that their theory has known since the origin. As the complex semisimple Lie algebras coincide with the finite-dimensional Kac-Moody algebras, it is natural and useful to ask whether one can obtain interesting groups by 'integrating' these algebras. Although it became quickly clear that this question had an affirmative answer, the actual construction of the corresponding groups turned out to be a delicate problem, whose definitive solution was given by J. Tits [Tit87b]. We refer to [Tit89] for a thorough historical and comparative introduction to the different ways of constructing a Kac-Moody group.

Given a **generalized Cartan matrix** $A = (A_{ij})_{i,j \in I}$, i.e. a matrix with integral coefficients such that $A_{ii} = 2$, $A_{ij} \leq 0$ and $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$ for all $i \neq j \in I$, Tits [Tit87b] constructs a group functor[1] $\mathcal{G}$ on the category of rings, together with a family $(\varphi_i)_{i \in I}$ of morphisms of functors $\mathrm{SL}_2 \to \mathcal{G}$, and shows that the restriction of $\mathcal{G}$ to the category of fields is completely characterized by a short list of simple properties, one of which being the existence of a natural adjoint action of $\mathcal{G}(\mathbb{C})$ on the Kac-Moody algebra $\mathfrak{g}(A)$ of type A. These functors will be called **Tits functors** in the sequel. By definition, a split Kac-Moody group over a field $\mathbb{K}$ is a group obtained by evaluating a Tits functor on $\mathbb{K}$.

The aforementioned characterization of Tits functors is inspired by the scheme-theoretic definition of algebraic groups. It paves thereby the way for a development of the structure theory of Kac-Moody groups which draws naturally parallels to the rich and well known theory of algebraic groups. This program has been carried out to a certain extent by several mathematicians among whom V. Kac, D. Peterson, G. Rousseau and B. Rémy (see [Rém02b] and references therein). In this respect, the study of automorphisms of split Kac-Moody groups, which is the central theme of this work, is aimed to parallel the celebrated theory of "abstract" homomorphisms of algebraic groups by A. Borel and J. Tits [BT73].

Recall that a *diagonal* automorphism of $\mathrm{SL}_2(\mathbb{K})$ is an automorphism of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & xb \\ x^{-1}c & d \end{pmatrix}$$

[1] Actually, the parameter system of a Tits functor consists of a 'Kac-Moody root datum', which is a richer structure than just a generalized Cartan matrix and generalizes in an appropriate way the data which classify reductive groups (see §1.1.3). In order to avoid irrelevant technicalities in this introduction, we only emphasize the dependence on a generalized Cartan matrix. The results we state hold for all types of Tits functors.

for some $x \in \mathbb{K}^\times$. An automorphism of $\mathrm{SL}_2(\mathbb{K})$ is called *diagonal-by-sign* if it is either diagonal or the composite of a diagonal automorphism with a transpose-inverse.

Our main result is the following.

Theorem A. *Let $(A, \mathcal{G}, (\varphi_i)_{i\in I})$ and $(A', \mathcal{G}', (\varphi'_i)_{i\in I'})$ be systems as above and let $\mathbb{K}, \mathbb{K}'$ be fields. Let $\varphi : \mathcal{G}(\mathbb{K}) \to \mathcal{G}'(\mathbb{K}')$ be an isomorphism.*

Suppose that $|\mathbb{K}| \geq 4$ and $\mathcal{G}(\mathbb{K})$ is infinite. Then there exist an inner automorphism α of $\mathcal{G}(\mathbb{K})$, a bijection $\pi : I \to I'$ and, for each $i \in I$, a field isomorphism $\zeta_i : \mathbb{K} \to \mathbb{K}'$, a diagonal-by-sign automorphism δ_i of $SL_2(\mathbb{K})$ such that the diagram

$$\begin{array}{ccc} \mathbf{SL_2}(\mathbb{K}) & \xrightarrow{SL_2(\zeta_i)} & \mathbf{SL_2}(\mathbb{K}') \\ {\scriptstyle \varphi_i\circ\delta_i}\big\downarrow & & \big\downarrow{\scriptstyle \varphi'_{\pi(i)}} \\ \mathcal{G}(\mathbb{K}) & \xrightarrow{\varphi\circ\alpha} & \mathcal{G}'(\mathbb{K}') \end{array}$$

commutes for every $i \in I$. Furthermore, if $\mathbb{K}$ is infinite then $A_{ij}A_{ji} = A'_{\pi(i)\pi(j)}A'_{\pi(j)\pi(i)}$ for all $i, j \in I$ and if $\mathrm{char}(\mathbb{K}) = 0$ then $A_{ij} = A'_{\pi(i)\pi(j)}$ for all $i, j \in I$.

It follows in particular that the isomorphism φ induces an isomorphism of the respective Weyl groups of $\mathcal{G}(\mathbb{K})$ and $\mathcal{G}'(\mathbb{K}')$ which preserves the set of canonical generators.

Theorem A can be used to characterize automorphisms of the Kac-Moody group $\mathcal{G}(\mathbb{K})$. Denoting by $(U_\alpha)_{\alpha\in\Phi}$ the system of root groups of $\mathcal{G}(\mathbb{K})$, it follows that, under the hypotheses of the theorem, every automorphism of $\mathcal{G}(\mathbb{K})$ leaves the union of the conjugacy classes of U_+ and U_- invariant, where $U_+ := \langle U_\alpha | \ \alpha \in \Phi_+\rangle$ and $U_- := \langle U_\alpha | -\alpha \in \Phi_+\rangle$. This fact in turn yields naturally a decomposition of any automorphism of $\mathcal{G}(\mathbb{K})$ as a product of an inner automorphism, a sign automorphism, a diagonal automorphism, a field automorphism and a graph automorphism (see Theorem 4.2 below). In the case where $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K})$ is prime to every off-diagonal entry of the generalized Cartan matrix A, this provides a complete description of the group $\mathrm{Aut}(\mathcal{G}(\mathbb{K}))$.

The proof of Theorem A combines the use of the two main available tools to explore the structure of a Kac-Moody group $\mathcal{G}(\mathbb{K})$. The first one is the strongly transitive action of $\mathcal{G}(\mathbb{K})$ on a twin building $\mathcal{B}$, constructed by M. Ronan and J. Tits and described in [Tit90], [Tit92]. Such a twin building $\mathcal{B}$ consists of the product of two thick buildings, say $\mathcal{B}_+ \times \mathcal{B}_-$, each corresponding to a BN-pair of $\mathcal{G}(\mathbb{K})$. Both BN-pairs have the same Weyl group; actually, the strong link which relates these two BN-pairs is an opposition relation between their respective Borel groups. This opposition relation yields an opposition relation between the chambers of $\mathcal{B}_+$ and $\mathcal{B}_-$, which is called a **twinning**. The existence of such a twinning invariant under the diagonal $\mathcal{G}(\mathbb{K})$-action makes these structures rather rigid, as we will see in the sequel.

The second tool is of more algebraic nature: It is the **adjoint representation** of the Kac-Moody group $\mathcal{G}(\mathbb{K})$ on a $\mathbb{K}$-vector space obtained by tensoring up a $\mathbb{Z}$-form[2] of the universal enveloping algebra of the Kac-Moody algebra $\mathfrak{g}_A$ of type A. This adjoint representation, constructed by B. Rémy [Rém02b, Chapter 9], is functorial and should be compared to the adjoint representation of a group scheme on its algebra of distributions.

A striking feature of these two actions is that they are strongly related. The main relationship to keep in mind is the following: *The adjoint action of a subgroup of $\mathcal{G}(\mathbb{K})$*

[2]This $\mathbb{Z}$-form was constructed by Tits [Tit87b] and plays a fundamental role in the construction of Tits functors.

is locally finite if and only if this subgroup has fixed points in both halves $\mathcal{B}_+$, $\mathcal{B}_-$ of the twin building $\mathcal{B}$. A subgroup satisfying one of these equivalent conditions is called a **bounded subgroup**. The adjoint representation can be used to endow certain bounded subgroups with a structure of algebraic groups and serves in this way as a substitute for an algebro-geometric structure for $\mathcal{G}(\mathbb{K})$.

A key observation made in [CM05b] and inspired by [KW92], is that the conclusions of Theorem A would follow for a given Kac-Moody group isomorphism whenever one shows that this isomorphism maps bounded subgroups to bounded subgroups. This observation relies on the understanding of the structure of maximal bounded subgroups, which allows to reduce the Kac-Moody group isomorphism problem to the finite-dimensional case, for which a complete solution is available in [BT73]. In this way, the isomorphism problem for Kac-Moody groups becomes a special case of the following.

Problem. *Let $\mathcal{K}$ be a Kac-Moody group, $\mathbf{G}$ be a connected reductive $\mathbb{F}$-isotropic $\mathbb{F}$-group and $\varphi : \mathbf{G}(\mathbb{F}) \to \mathcal{K}$ be a homomorphism. Find conditions under which φ has bounded image.*

In view of the above description of bounded subgroups of Kac-Moody groups, this problem may be viewed as a rigidity problem for reductive group actions on twin buildings. It turns out that for split Kac-Moody groups acting on one-dimensional buildings, this problem can be completely solved by means of the following result.

Theorem. (J. Tits [Tit77]) *Let $\mathbf{G}(\mathbb{F})$ act on a tree T, where $\mathbf{G}$ is a semisimple algebraic $\mathbb{F}$-group of positive $\mathbb{F}$-rank. Then one of the following holds, where $\mathbf{G}^\dagger(\mathbb{F})$ denotes the subgroup of $\mathbf{G}(\mathbb{F})$ generated by the $\mathbb{F}$-points of the unipotent radicals of the Borel subgroups of $\mathbf{G}$ defined over $\mathbb{F}$.*

(i) $\mathbf{G}^\dagger(\mathbb{F})$ *has a global fixed point.*

(ii) $\mathbf{G}^\dagger(\mathbb{F})$ *has no global fixed point but a unique fixed end.*

(iii) $\mathbb{F}\text{-rank}(\mathbf{G}) = 1$ *and the root datum of* $\mathbf{G}(\mathbb{F})$ *has a valuation such that the corresponding Bruhat-Tits tree has* $\mathbf{G}^\dagger(\mathbb{F})$*-equivariant embedding in T.*

Specializing this result to $\mathbf{G}(\mathbb{F})$-actions on one-dimensional twin buildings, one obtains the following.

Corollary B. *Let $\mathcal{K}$ be a split Kac-Moody group whose twin building is one-dimensional and $\mathbf{G}$ be a connected reductive $\mathbb{F}$-group of positive $\mathbb{F}$-rank. Then every homomorphism of $\mathbf{G}^\dagger(\mathbb{F})$ to $\mathcal{K}$ has bounded image.*

This motivates the search for rigidity results analogous to Tits' theorem but for higher dimensional buildings. An example of such an analogue is provided by the following.

Theorem C. *Let $\Gamma := SL_2(\mathbb{Q})$ act by cellular isometries on a $\mathrm{CAT}(0)$ polyhedral complex X. Then one of the following holds:*

(i) Every finitely generated subgroup of Γ has a fixed point in X.

(ii) There exists finitely many primes $p_1, \dots, p_n$ such that for each $i = 1, \dots, n$, there exists a Γ_{p_i}-equivariant embedding of the vertices of the Bruhat-Tits p_i-adic tree T_i in X, where $\Gamma_{p_i} = \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p_i}])$. Moreover, for each integer m prime to all p_i's, the group $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{m}])$ has fixed points in X.

The basic ingredient of the proof of this theorem is a result of M. Bridson [Bri99] which describes arbitrary abelian group actions on CAT(0) polyhedral complexes (see Proposition 2.8 below). One also needs the fact that the group $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{m}])$ has bounded generation [Mor05].

Since no assumption on the local compactness of X is made in Theorem C, this result applies in particular to all buildings of finite rank (see [Dav98]). In the special case of Kac-Moody buildings, one obtains the following.

Corollary D. *Let $\mathcal{K}$ be a Kac-Moody group and $\mathbf{G}$ be a $\mathbb{Q}$-split semisimple algebraic $\mathbb{Q}$-group. Then every homomorphism of $\mathbf{G}(\mathbb{Q})$ to $\mathcal{K}$ has bounded image.*

This corollary is the key ingredient of the proof of Theorem A over fields of characteristic 0. Though similar in spirit, the proof of Theorem A in positive characteristic follows a slightly different strategy. In the latter, one considers homomorphisms of a $\mathbb{F}$-isotropic reductive $\mathbb{F}$ group to a split Kac-Moody group whose restriction to the center of the reductive group is injective. The main idea, which was at the heart of [CM05a], is to study the action of the semisimple part on the fixed point set of the abelian part in the twin building. Combining the aforementioned result of M. Bridson, a fixed point theorem for automorphism groups of twin buildings by B. Mühlherr [Müh94] and Borel-Tits' description of centralizers of tori in reductive groups [BT65], one shows essentially that the image of the center centralizes a subgroup of $\mathcal{G}(\mathbb{K})$ which is of Kac-Moody type but not necessarily split. If the dimension of the center is large enough, the twin building of this Kac-Moody group becomes one-dimensional, which makes Tits' theorem available again. However, the presence of a possibly non-trivial anisotropic kernel creates some technical difficulties coming from the existence of anisotropic elements in algebraic groups (see Theorem 6.6 below for a precise statement).

It is rather natural to consider the 'dual' of the problem addressed above and study homomorphisms of Kac-Moody groups to reductive groups. The question of the existence of injective such homomorphisms is known as the *linearity problem* for Kac-Moody groups, to which the following result gives a partial answer.

Theorem E. *Let A be a generalized Cartan matrix, $\mathcal{G}$ be a Tits functor of type A and $\mathbb{K}$ be an infinite field. Let $\mathbb{F}$ be a field, n be an integer and $\varphi : \mathcal{G}(\mathbb{K}) \to GL_n(\mathbb{F})$ be a homomorphism with central kernel. Then every indecomposable component of the generalized Cartan matrix A is of finite or affine type.*

This shows in particular that there does not exist any cocentral homomorphism of an indefinite type Kac-Moody group over an infinite field to a reductive group. Note that modulo the conjectural simplicity of indefinite type Kac-Moody groups, this shows the nonexistence of any nontrivial homomorphism of these groups with finite-dimensional target.

The linearity problem has been considered for Kac-Moody groups over finite fields by B. Rémy, who proved that Kac-Moody groups of certain hyperbolic types over sufficiently big finite fields are non-linear (see [Rém02a] and [Rém04]). It is expected that the conclusions of Theorem E actually hold without any restriction on the cardinality of the field[3]; however, non-linearity results for Kac-Moody groups over finite fields seem to

[3] More precisely, one expects that Kac-Moody groups of type A over finite fields are non-linear as soon as the Coxeter diagram $M(A)$ associated with A is non-spherical and non-affine. There are however generalized Cartan matrices A of indefinite type such that $M(A)$ is a Coxeter diagram of type $\tilde{A}_1$. Over finite fields, the corresponding Kac-Moody groups should be considered with special care, see §4.1.2.

be much harder to prove. In particular, the techniques developed by B. Rémy to tackle this problem are considerably more elaborated than the ones we use to prove Theorem E. Actually, according to B. Rémy's work, the algebraic group point of view on Kac-Moody groups should be replaced by a discrete group point in the case of finite ground fields[4]; this allows to combine classical arguments from the theory of algebraic groups with tools from dynamics and ergodic theory (see [Rém03] for a survey).

Although the Kac-Moody groups considered in Theorem A are split, it is probable that some of the ideas developed in this context can also be used to study the isomorphism problem in the non-split case (see [Rém02b] for the relative theory of Kac-Moody groups). As an illustration of this possibility, we include the solution of the isomorphism problem for unitary forms of complex Kac-Moody groups (see Theorem 8.2). These unitary forms were defined and studied by V. Kac and D. Peterson (see [KP87] and references therein). In the finite-dimensional case, they coincide with the compact semisimple Lie groups. In the affine case, they coincide (up to a central extension by a copy of S^1) with the so-called 'algebraic' loop groups of the compact semisimple Lie groups. In the indefinite type case, no such convenient description is known. However, in all cases, unitary forms of complex Kac-Moody groups carry a natural structure of connected Hausdorff topological group, and it follows from our result that any epimorphism with central kernel between two such unitary forms is continuous. This is of course well known in the finite-dimensional case.

We now come to the organization of the text. The first chapter collects standard prerequisites on Kac-Moody groups, their root data and twin buildings. The second chapter is devoted to CAT(0) geometry. After reviewing some standard definitions, we recall Bruhat-Tits fixed point theorem and mention some consequences for groups with bounded generation. The third chapter reviews B. Rémy's construction of the adjoint representation of Kac-Moody groups, as well as the relationship between the adjoint action and the action on the twin building. These first three chapters are essentially preliminary and contain nothing new (although the Jordan decomposition of bounded elements of Kac-Moody groups over field of positive characteristic does not seem to appear in the literature). The next three chapters are devoted to the isomorphism problem for Kac-Moody groups; they constitute the heart of this work. Chapter 4 contains a statement of the isomorphism theorem, and proceeds next to a detailed study of diagonalizable and completely reducible subgroups of Kac-Moody groups. The rest of the proof of the isomorphism theorem is divided up among Chapter 5 and Chapter 6, which correspond respectively to the case of characteristic 0 and positive characteristic. Chapter 5 also contains the proof of Theorem C. The proof of Theorem E is given in Chapter 7. Finally, Chapter 8 is devoted to unitary forms of complex Kac-Moody groups. It contains the corresponding versions of the isomorphism theorem and the non-linearity theorem.

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[4]A Kac-Moody group over a finite field is finitely generated and embeds as a lattice in the product of the automorphism groups of its two buildings; this automorphism group is canonically endowed with a locally compact totally disconnected topology. These facts fail for Kac-Moody groups over infinite fields.

Chapter 1

The objects: Kac-Moody groups, root data and Tits buildings

1.1 Kac-Moody groups and Tits functors

1.1.1 Parameters of the construction

Let I be a finite set. A **generalized Cartan matrix** over I is a matrix $A = (A_{ij})_{i,j\in I}$ with integral coefficients such that

$$A_{ii} = 2,\quad A_{ij} \le 0 \text{ if } i \neq j,\quad A_{ij} = 0 \Leftrightarrow A_{ji} = 0$$

for all $i, j \in I$. A **(classical) Cartan matrix** over I is a generalized Cartan matrix over I which can be decomposed as a product of an invertible diagonal matrix and a positive definite matrix. A generalized Cartan matrix is called **symmetrizable** if it is the product of an invertible diagonal matrix and a symmetric matrix.

A **Kac-Moody root datum** is a system $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ where I is a finite set, A is a generalized Cartan matrix over I, Λ is a free $\mathbb{Z}$-module, c_i is an element of Λ for each $i \in I$, h_i is an element of the $\mathbb{Z}$-dual $\Lambda^\vee$ of Λ and the relation

$$\langle c_i | h_j \rangle = A_{ji}$$

holds for all $i, j \in I$. The Kac-Moody root datum $\mathcal{D}$ is called **simply connected** if the h_i's form a basis of $\Lambda^\vee$.

1.1.2 Kac-Moody algebras

Let $A = (A_{ij})_{i,j\in I}$ be a generalized Cartan matrix. The **Kac-Moody algebra** of type A over $\mathbb{C}$ is the complex Lie algebra, noted $\mathfrak{g}_A$, generated by the elements e_i, f_i and h_i ($i \in I$) with the following presentation ($i, j \in I$):

$$\begin{array}{rcl}
[h_i, e_j] &=& A_{ij} e_j, \\
[h_i, f_j] &=& -A_{ij} f_j, \\
[h_i, h_j] &=& 0, \\
[e_i, f_i] &=& -h_i, \\
[e_i, f_j] &=& 0 \text{ for } i \neq j, \\
(\text{ad } e_i)^{-A_{ij}+1}(e_j) &=& \text{ad } (f_i)^{-A_{ij}+1}(f_j) = 0.
\end{array}$$

The Lie algebra $\mathfrak{g}_A$ is the derived algebra of the "classical" Kac-Moody algebra $\mathfrak{g}(A)$ considered by Kac in [Kac90]. This follows from Gabber-Kac' theorem (see [Kac90, Theorem 9.11]).

1.1.3 Existence and uniqueness of Tits functors

Throughout, we make the convention that a *ring* is a commutative $\mathbb{Z}$-algebra with a unit.

Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum. Let $\mathcal{T}_\Lambda$ is the **split torus scheme**, i.e. $\mathcal{T}_\Lambda$ is the group functor on the category of rings defined by $\mathcal{T}_\Lambda(R) = \mathrm{Hom}_{\mathrm{gr}}(\Lambda, R^\times)$.

With the datum $\mathcal{D}$, J. Tits [Tit87b, §3.6] associates a system $\mathcal{F} = (\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$ consisting of:

- a group functor $\mathcal{G}$ on the category of rings,
- a collection $(\varphi_i)_{i\in I}$ of morphisms of functors $\varphi_i : SL_2 \to \mathcal{G}$,
- a morphism of functors $\eta : \mathcal{T}_\Lambda \to \mathcal{G}$,

which satisfies the following conditions, where r^{h_i} denotes the element $\lambda \mapsto r^{\langle\lambda,h_i\rangle}$ of $\mathcal{T}_\Lambda$:

(KMG1) if $\mathbb{K}$ is a field, $\mathcal{G}(\mathbb{K})$ is generated by the images of $\varphi_i(\mathbb{K})$ and $\eta(\mathbb{K})$;

(KMG2) for every ring R, the homomorphism $\eta(R) : \mathcal{T}_\Lambda(R) \to \mathcal{G}(R)$ is injective;

(KMG3) for $i \in I$ and $r \in R^\times$, one has $\varphi_i \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} = \eta(r^{h_i})$;

(KMG4) if ι is an injective homomorphism of a ring R in a field $\mathbb{K}$, then $\mathcal{G}(\iota) : \mathcal{G}(R) \to \mathcal{G}(\mathbb{K})$ is injective;

(KMG5) there is a homomorphism $\mathbf{Ad} : \mathcal{G}(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g}_A)$ whose kernel is contained in $\eta(\mathcal{T}_\Lambda(\mathbb{C}))$, such that, for $c \in \mathbb{C}$,

$$\mathbf{Ad}\left(\varphi_i \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\right) = \exp \mathrm{ad}\ ce_i,$$

$$\mathbf{Ad}\left(\varphi_i \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right) = \exp \mathrm{ad}\ (-cf_i),$$

and, for $t \in \mathcal{T}_\Lambda(\mathbb{C})$,

$$\mathbf{Ad}(\eta(t))(e_i) = t(c_i) \cdot e_i, \qquad \mathbf{Ad}(\eta(t))(f_i) = t(-c_i) \cdot f_i.$$

The group functor $\mathcal{G}$ as above is called a **Tits functor of type $\mathcal{D}$ and of basis $\mathcal{F}$**. By definition, a **(split) Kac-Moody group** of type $\mathcal{D}$ over a field $\mathbb{K}$ is the value on $\mathbb{K}$ of a Tits functor of type $\mathcal{D}$. The main result of [Tit87b] asserts that the restriction of $\mathcal{G}$ to the category of fields is completely characterized by the conditions (KMG1)–(KMG5) modulo some additional non-degeneracy condition on the images of the φ_i's (see [Tit87b, Theorem 1] for a precise statement).

1.1.4 An alternative construction in the 2-spherical case

Let A be a generalized Cartan matrix over a (finite) set I. For each subset $J \subset I$, we set $A_J := (A_{ij})_{i,j \in J}$. The matrix A is called **2-spherical** if for every 2-subset J of I the matrix A_J is a (classical) Cartan matrix. Equivalently, A is 2-spherical if and only if $A_{ij}A_{ji} \leq 3$ for all $i \neq j \in I$.

In this section we present an explicit construction of Kac-Moody groups of type $\mathcal{D}$, where $\mathcal{D} = (I, A, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ is the simply connected Kac-Moody root datum associated with a 2-spherical generalized Cartan matrix A.

Let $\mathbb{K}$ be a field and assume that $\mathbb{K}$ is of cardinality at least 3 (resp. at least 4) if $A_{ij} = -2$ (resp. $A_{ij} = -3$) for some $i, j \in I$. For each $i \in I$, let X_i be a copy of $SL_2(\mathbb{K})$ and for each 2-subset $J = \{i, j\}$ of I, let $X_{i,j}$ be a copy of the universal Chevalley group of type A_J over $\mathbb{K}$. Let also $\varphi_{i,j} : X_i \hookrightarrow X_{i,j}$ be the canonical monomorphism corresponding to the inclusion of Cartan matrices $A_{\{i\}} \hookrightarrow A_{\{i,j\}}$. The direct limit of the inductive system formed by the groups X_i and $X_{i,j}$ along with the monomorphisms $\varphi_{i,j}$ $(i, j \in I)$ coincides with the simply connected Kac-Moody group $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ of type $\mathcal{D}$ over $\mathbb{K}$ (see Theorem A and its application in [Cap05a]).

1.2 Root data

1.2.1 Definition

Let (W, S) be a Coxeter system, let Φ be the associated root system (viewed either as a set of half-spaces in the chamber system associated with (W, S) or as a subset of the real vector space $\mathbb{R}^S$) and let Π be a basis of Φ. Let Φ_+ (resp. Φ_-) be the subset of positive (resp. negative) roots. We refer to [Wei03, Chapter 3] (resp. [Bou81]) for general facts on root systems from the combinatorial (resp. algebraic) viewpoint.

A pair of roots $\{\alpha, \beta\} \subset \Phi$ is called **prenilpotent** if there exist $w, w' \in W$ such that $\{w(\alpha), w(\beta)\} \subset \Phi_+$ and $\{w'(\alpha), w'(\beta)\} \subset \Phi_-$. In that case, we set

$$[\alpha, \beta] := \bigcap_{\substack{w \in W \\ \epsilon \in \{+,-\}}} \{\gamma \in \Phi |\ \{w(\alpha), w(\beta)\} \subset \Phi_\epsilon \Rightarrow w(\gamma) \in \Phi_\epsilon\}$$

and

$$]\alpha, \beta[:= [\alpha, \beta] \backslash \{\alpha, \beta\}.$$

A **twin root datum** of type (W, S) is a system $\mathcal{Z} = (G, (U_\alpha)_{\alpha \in \Phi})$ consisting of a group G together with a family of subgroups U_α indexed by the root system Φ, which satisfy the following axioms, where $H := \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$, $U_+ := \langle U_\alpha |\ \alpha \in \Phi_+ \rangle$ and $U_- := \langle U_\alpha |\ \alpha \in \Phi_- \rangle$:

(TRD0) For each $\alpha \in \Phi$, we have $U_\alpha \neq \{1\}$.

(TRD1) For each prenilpotent pair $\{\alpha, \beta\} \subset \Phi$, the commutator group $[U_\alpha, U_\beta]$ is contained in the group $U_{]\alpha,\beta[} := \langle U_\gamma |\ \gamma \in]\alpha, \beta[\rangle$.

(TRD2) For each $\alpha \in \Pi$ and each $u \in U_\alpha \backslash \{1\}$, there exists elements $u', u'' \in U_{-\alpha}$ such that the product $\mu(u) := u'uu''$ conjugates U_β onto $U_{s_\alpha(\beta)}$ for each $\beta \in \Phi$.

(TRD3) For each $\alpha \in \Pi$, the group $U_{-\alpha}$ is not contained in U_+ and the group U_α is not contained in U_-.

(TRD4) $G = H\langle U_\alpha | \alpha \in \Phi \rangle$.

This definition was first given in [Tit92]; more details can be found in [Rém02b, Chapter 1] (see also [Abr96, § I.1]). The following two lemmas are well known. The first one shows in particular that the product $u'uu''$ in (TRD2) is uniquely determined by the element u, as suggested by the notation $\mu(u)$.

Lemma 1.1. *Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha \in \Phi})$ be a twin root datum and set $H := \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$. Let $\alpha \in \Phi$ and set $X_\alpha := \langle U_\alpha \cup U_{-\alpha} \rangle$. We have the following.*

(i) There are unique elements $v', v'' \in U_{-\alpha}$ such that conjugation by $v'uv''$ swaps U_α and $U_{-\alpha}$.

(ii) An element $h \in H$ centralizes X_α if and only if it centralizes U_α.

Proof. The existence part of Assertion (i) follows from (TRD2). The uniqueness part is well known and follows from the fact, easy to deduce from (TRD2), that X_α has a split BN-pair of rank one. Since H normalizes X_α, Assertion (ii) follows from (i). □

Lemma 1.2. *Let $N := H\langle \mu(u) | u \in U_\alpha \backslash \{1\}, \alpha \in \Pi \rangle$. Then H is normal in N and N/H is isomorphic to W.*

Proof. This follows (for example) from [Tit92, Proposition 4]. □

The Coxeter group W is called the **Weyl group** of $\mathcal{Z}$.

Groups endowed with a twin root datum include isotropic semisimple algebraic groups, split and quasi-split Kac-Moody groups, as well as some other more exotic families of groups, including those constructed in [RR06]. Here, we will focus on split Kac-Moody groups, but a great deal of our discussion will apply to the slightly more general class of groups endowed with a locally split twin root datum. We now recall this notion.

1.2.2 Locally split twin root data

A twin root datum $(G, (U_\alpha)_{\alpha \in \Phi})$ of type (W, S) is called **locally split over** $(\mathbb{K}_\alpha)_{\alpha \in \Phi}$ (or over $\mathbb{K}$ if $\mathbb{K}_\alpha \simeq \mathbb{K}$ for all α) if the following conditions are satisfied:

(LS1) The group $T := \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$ is abelian.

(LS2) For each $\alpha \in \Phi^+$, there is a homomorphism

$$\varphi_\alpha : SL_2(\mathbb{K}_\alpha) \to \langle U_\alpha \cup U_{-\alpha} \rangle$$

which maps the subgroup of upper (resp. lower) triangular unipotent matrices onto U_α (resp. $U_{-\alpha}$).

Notice that the second condition holds for all $\alpha \in \Phi$ as soon as it holds for all roots α in some basis of Φ (this is a direct consequence of (TRD2) and the fact that $W.\Pi = \Phi$ for each basis Π of Φ).

Lemma 1.3. *Let $(G, (U_\alpha)_{\alpha \in \Phi})$ be a twin root datum which is locally split over $(\mathbb{K}_\alpha)_{\alpha \in \Phi}$, let $T := \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$ and let $\alpha \in \Phi$. Set $X_\alpha := \langle U_\alpha \cup U_{-\alpha} \rangle$. We have the following.*

(i) If $|\mathbb{K}_\alpha| \geq 4$, then the derived subgroup of $\langle T \cup X_\alpha \rangle$ coincides with X_α.

(ii) The normalizer of U_α in $\langle T \cup X_\alpha \rangle$ is solvable.

Proof. Note that if $|\mathbb{K}_\alpha| \geq 4$ then $SL_2(\mathbb{K}_\alpha)$ is perfect. Thus (i) and (ii) follow from (LS1) and (LS2) and the fact that T normalizes U_α and $U_{-\alpha}$. □

1.2.3 Twin root data for Kac-Moody groups

Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum and let $M(A) = (m_{ij})_{i,j\in I}$ be the Coxeter matrix over I defined as follows: $m_{ii} = 1$ and for $i \neq j$, $m_{ij} = 2, 3, 4, 6$ or ∞ according as the product $A_{ij}A_{ji}$ is equal to $0, 1, 2, 3$ or ≥ 4. Let (W, S) be a Coxeter system of type $M(A)$ with $S = \{s_i | \ i \in I\}$, Φ be its root system and $\Pi = \{\alpha_i | \ i \in I\}$ be a basis of Φ such that for each $i \in I$, the reflection associated with α_i is s_i.

Let $\mathcal{F} = (\mathcal{G}, (\phi_i)_{i\in I}, \eta)$ be the basis of a Tits functor of type $\mathcal{D}$ and $\mathbb{K}$ be a field. Let $G := \mathcal{G}(\mathbb{K})$, $T := \eta(\mathcal{T}_\Lambda(\mathbb{K}))$ and for each $i \in I$ let $\bar{s}_i := \phi_i(\mathbb{K}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and U_{α_i} (resp. $U_{-\alpha_i}$) be the image under $\phi_i(\mathbb{K})$ of the subgroup of upper (resp. lower) triangular unipotent matrices.

Lemma 1.4. *One has the following:*

(i) The assignments $\bar{s}_i \mapsto s_i$ extend to a surjective homomorphism onto W, denoted ζ, whose kernel is contained in T.

(ii) For each $w \in W$ and $i \in I$, the group $U_{w(\alpha_i)} := \zeta(w)U_{\alpha_i}\zeta(w)^{-1}$ depends only on the element $\alpha := w(\alpha_i)$ of Φ (and not on α_i or on w).

(iii) The group T coincides with $\bigcap_{\alpha\in\Phi} N_G(U_\alpha)$.

(iv) The system $\mathcal{Z}_{\mathcal{D}}(\mathbb{K}) := (G, (U_\alpha)_{\alpha\in\Phi})$ is a twin root datum of type (W, S) which is locally split over $\mathbb{K}$.

(v) Given $\alpha \in \Phi$ and a subgroup H of T, if H normalizes a conjugate of U_α contained in $X_\alpha := \langle U_\alpha \cup U_{-\alpha}\rangle$ and different from U_α and $U_{-\alpha}$, then H centralizes X_α.

Proof. This statement is implicitly contained in [Tit87b]. See also [Tit92, §3.3] and [Rém02b, Proposition 8.4.1]. The technical assertion (v) follows from the fact that T acts on X_α by diagonal automorphisms (and never by field automorphisms). $\square$

The twin root datum $\mathcal{Z}_{\mathcal{D}}(\mathbb{K})$ is called the **standard** twin root datum associated with $\mathcal{F}$ and $\mathbb{K}$. The basis $\Pi = \{\alpha_i | \ i \in I\}$ of Φ is called **standard** (with respect to $\mathcal{F}$).

1.2.4 Isomorphisms of root data

Let $\mathcal{Z} := (G, (U_\alpha)_{\alpha\in\Phi})$ and $\mathcal{Z}' := (G', (U'_\alpha)_{\alpha\in\Phi'})$ be twin root data of type (W, S) and (W', S') respectively.

Let $S_1, S_2, \ldots, S_n$ be the irreducible subsets of S. In other words, $S = S_1 \cup \cdots \cup S_n$ is the finest partition of S such that $[S_i, S_j] = 1$ whenever $1 \leq i < j \leq n$.

An ordered pair (φ, π) consisting of an isomorphism $\varphi : G \to G'$ and an isomorphism $\pi : W \to W'$ is called an **isomorphism of $\mathcal{Z}$ to $\mathcal{Z}'$** if the following condition hold:

(ITRD1) $\pi(S) = S'$ and, hence, π induces an equivariant bijection $\Phi \to \Phi'$ again denoted π.

(ITRD2) There exists $x \in G'$ and a sign ϵ_i for each $1 \leq i \leq n$ such that

$$x\varphi(U_\alpha)x^{-1} = U'_{\epsilon_i\pi(\alpha)}$$

for every $\alpha \in \Phi$ such that $s_\alpha \in W_{S_i}$.

Thus, if (W,S) is irreducible, then either $x\varphi(U_\alpha)x^{-1} = U'_{\pi(\alpha)}$ or $x\varphi(U_\alpha)x^{-1} = U'_{-\pi(\alpha)}$ for all $\alpha \in \Phi(W,S)$. In particular, this means that φ maps the union of conjugacy classes

$$\{gU_+g^{-1} | g \in G\} \cup \{gU_-g^{-1} | g \in G\}$$

to

$$\{gU'_+g^{-1} | g \in G'\} \cup \{gU'_-g^{-1} | g \in G'\},$$

with the notation of §1.2.1.

A crucial fact on isomorphisms between twin root data we will need later is the following.

Theorem 1.5. *Let* $\mathcal{Z} := (G, (U_\alpha)_{\alpha\in\Phi(W,S)})$ *and* $\mathcal{Z}' := (G', (U'_\alpha)_{\alpha\in\Phi(W',S')})$ *be twin root data with* S *and* S' *finite and let* $\varphi : G \to G'$ *be an isomorphism. Assume that*

$$\{\varphi(U_\alpha) | \alpha \in \Phi(W,S)\} = \{xU'_\alpha x^{-1} | \alpha \in \Phi(W',S')\} \tag{$*$}$$

for some $x \in G'$*. Then there exists an isomorphism* $\pi : W \to W'$ *such that* (φ, π) *is a twin root data isomorphism of* $\mathcal{Z}$ *to* $\mathcal{Z}'$*.*

Proof. See [CM05a, Theorem 2.2]. □

1.3 Tits buildings

1.3.1 Buildings

Let (W,S) be a Coxeter system. A **building** of type (W,S) is a pair $\mathcal{B} = (\mathcal{C}, \delta)$ where $\mathcal{C}$ is a set and $\delta : \mathcal{C} \times \mathcal{C} \to W$ is a **distance function** satisfying the following axioms where $x, y \in \mathcal{C}$ and $w = \delta(x,y)$:

(Bu1) $w = 1$ if and only if $x = y$;

(Bu2) if $z \in \mathcal{C}$ is such that $\delta(y,z) = s \in S$, then $\delta(x,z) = w$ or ws, and if, furthermore, $l(ws) = l(w) + 1$, then $\delta(x,z) = ws$;

(Bu3) if $s \in S$, there exists $z \in \mathcal{C}$ such that $\delta(y,z) = s$ and $\delta(x,z) = ws$.

The Coxeter group W is called the **Weyl group** of $\mathcal{B}$.

Given $s \in S$, chambers $x, y \in \mathcal{C}$ are called **s-adjacent** if $\delta(x,y) \in \{1, s\}$. Two chambers are called **adjacent** if they are s-adjacent for some $s \in S$. A building of type (W,S) is called **thick** if for every chamber x and every $s \in S$, there exist at least three chambers s-adjacent to x.

1.3.2 Twin buildings

Let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+), \mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be two buildings of the same type (W,S). A **codistance** (or a **twinning**) between $\mathcal{B}_+$ and $\mathcal{B}_-$ is a mapping $\delta_* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$ satisfying the following axioms, where $\epsilon \in \{+,-\}, x \in \mathcal{C}_\epsilon, y \in \mathcal{C}_{-\epsilon}$ and $w = \delta_*(x,y)$:

(Tw1) $\delta_*(y,x) = w^{-1}$;

(Tw2) if $z \in \mathcal{C}_{-\epsilon}$ is such that $\delta_{-\epsilon}(y,z) = s \in S$ and $l(ws) = l(w) - 1$, then $\delta_*(x,z) = ws$;

(Tw3) if $s \in S$, there exists $z \in \mathcal{C}_{-\epsilon}$ such that $\delta_{-\epsilon}(y, z) = s$ and $\delta_*(x, z) = ws$.

A **twin building of type** (W, S) is a triple $(\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ where $\mathcal{B}_+, \mathcal{B}_-$ are buildings of type (W, S) and δ_* is a twinning between $\mathcal{B}_+$ and $\mathcal{B}_-$. Two chambers c, d of $\mathcal{B}$ are called **opposite** if $\delta^*(x, y) = 1$.

A crucial feature of twin buildings is that they constitute rather rigid structures. This is made more precise by the following basic but extremely important result, due to J. Tits.

Theorem 1.6. *Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a thick twin building and let c, d be opposite chambers of $\mathcal{B}$. Let $E_1(c)$ denote the set of all chambers of $\mathcal{B}$ adjacent to c. Any two automorphisms of $\mathcal{B}$ coincide if and only if their restrictions to $E_1(c) \cup \{d\}$ coincide.*

Proof. See [Tit90, Théorème 1]. □

1.3.3 Building combinatorics: apartments, residues and projections

Although we won't need much from the theory of buildings, there are a few crucial fundamental concepts which we review all together in the present section. The main references are [Tit74], [Tit90], [Ron89], [Abr96] and [Wei03].

Throughout this section, we let (W, S) be a Coxeter system.

Let $\delta : W \times W \to W : (x, y) \mapsto x^{-1}y$. Then δ is a distance function and $\mathcal{A}(W, S) := (W, \delta)$ is a building of type (W, S). Given any building $\mathcal{B}$ of type (W, S), an **apartment** of $\mathcal{B}$ is a set of chambers isometric to $\mathcal{A}(W, S)$. A fundamental property of buildings is that any pair of chambers is contained in some apartment.

Given a subset J of S, we write W_J for the subgroup of W generated by J. We say that J is **spherical** if W_J is finite.

Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S). Given a subset $J \subset S$ and a chamber $c \in \mathcal{C}$, the **residue** of type J (or the J-residue) containing c is the set

$$\mathrm{Res}_J(c) := \{x \in \mathcal{C} |\ \delta(c, x) \in W_J\}.$$

The **rank** of a J-residue is the cardinality of J. A residue of rank 0 is a chamber. Note that, given $s \in S$, the $\{s\}$-residue containing c is nothing but the set of all chambers which are s-adjacent to c. This residue is also called the s-**panel** containing c. Crucial to the theory of buildings is the fact that a residue ρ of type J, endowed with the appropriate restriction of the W-distance, is itself a building of type (W_J, J). If $\mathcal{A}$ is an apartment of $\mathcal{B}$ then $\rho \cap \mathcal{A}$ is either empty or an apartment of ρ. All apartments of ρ can be obtained in this way.

Let ℓ be the length function of (W, S). Given a building $\mathcal{B} = (\mathcal{C}, \delta)$ of type (W, S), the composite $d := \ell \circ \delta : \mathcal{C} \times \mathcal{C} \to \mathbb{Z}_{\geq 0}$ is called the **numerical distance** of $\mathcal{B}$. Note that $(\mathcal{C}, d)$ is a (discrete) metric space in the usual sense.

Let ρ be a residue of spherical type J. Two chambers $c, d \in \rho$ are called **opposite** (in ρ) if $d(c, d) = \max\{d(x, y) |\ x, y \in \rho\}$. This definition makes sense because W_J is a finite Coxeter group.

Given two residues ρ, σ of $\mathcal{B}$, the set

$$\mathrm{proj}_\rho(\sigma) := \{x \in \rho |\ d(x, \sigma) = d(\rho, \sigma)\}$$

consisting of all chambers of ρ at minimal numerical distance from σ is called the **projection** of σ to ρ. It is itself a residue, whose rank is bounded from above by the ranks of ρ and σ. In particular, if c is a chamber then $\mathrm{proj}_\rho(c)$ is a chamber of ρ.

Given a panel σ and an apartment $\mathcal{A}$ of $\mathcal{B}$ such that $\sigma \cap \mathcal{A}$ is nonempty, the intersection $\sigma \cap \mathcal{A}$ consists of exactly two chambers, say c and d, and $\mathcal{A}$ is the disjoint union of

$$\phi_{\mathcal{A}}(\sigma, c) := \{x \in \mathcal{A} |\ \mathrm{proj}_\sigma(x) = c\}$$

and

$$\phi_{\mathcal{A}}(\sigma, d) := \{x \in \mathcal{A} |\ \mathrm{proj}_\sigma(x) = d\}.$$

The sets $\phi_{\mathcal{A}}(\sigma, c)$ and $\phi_{\mathcal{A}}(\sigma, d)$ are called **roots** (or **half-apartments**) of $\mathcal{B} = (\mathcal{C}, \delta)$. Given a root ϕ of $\mathcal{A}$, the set of all panels meeting both ϕ and its complement in $\mathcal{A}$ is denoted by $\partial\phi$; it is called the **wall** determined by ϕ.

We now turn to twin buildings.

Let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+)$, $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be buildings of type (W, S) and let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be a twinning. Let $\mathcal{A}_+$ (resp. $\mathcal{A}_-$) be an apartment of $\mathcal{B}_+$ (resp. $\mathcal{B}_-$). We say that $\mathcal{A}_+$ and $\mathcal{A}_-$ are **twins** if every chamber of $\mathcal{A}_+$ has a unique opposite in $\mathcal{A}_-$ (or vice-versa). In that case $(\mathcal{A}_+, \mathcal{A}_-)$ is called a **twin apartment** of $\mathcal{B}$. Not all apartments of $\mathcal{B}_+$ have a twin in $\mathcal{B}_-$ but if it exists, the twin is unique. Furthermore, any given pair x_+, x_- of opposite chambers is contained in a unique twin apartment $(\mathcal{A}_+, \mathcal{A}_-)$, which is obtained as follows ($\epsilon \in \{+, -\}$):

$$\mathcal{A}_\epsilon := \{c \in \mathcal{C}_\epsilon |\ \delta_\epsilon(x_\epsilon, c) = \delta_*(x_{-\epsilon}, c)\}.$$

A **residue** ρ of $\mathcal{B}$ is a residue of $\mathcal{B}_+$ or of $\mathcal{B}_-$. The **sign** of the residue ρ is defined to be $+$ or $-$ accordingly. Two residues (of opposite signs) are called **opposite** if they are of the same type and contain opposite chambers.

Given a twin building $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ of type (W, S), the composite $d^* := \ell \circ \delta^*$ is called the **numerical codistance** of $\mathcal{B}$. Thus, two chambers are at numerical codistance 0 if and only if they are opposite.

Given residues ρ, σ of $\mathcal{B}$ of spherical type and opposite signs, the set

$$\mathrm{proj}_\rho(\sigma) := \{x \in \rho |\ \exists y \in \sigma : d^*(x, y) \geq d^*(x', y')\ \forall x' \in \rho, y' \in \sigma\}$$

consisting of all chambers of ρ at maximal numerical codistance from σ is called the **projection** of σ to ρ. It is itself a residue, whose rank is bounded from above by the ranks of ρ and σ. In particular, if c is a chamber then $\mathrm{proj}_\rho(c)$ is a chamber of ρ.

1.3.4 The Moufang property

Let $\mathcal{B} = ((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta_*)$ be a thick twin building of type (W, S). Let $\mathcal{A} = (\mathcal{A}_+, \mathcal{A}_-)$ be a twin apartment. A subset $\phi \subset \mathcal{A}_+ \cup \mathcal{A}_-$ is called a **twin root** if $\phi_\epsilon := \phi \cap \mathcal{A}_\epsilon$ is a root of $\mathcal{A}_\epsilon$ and if the set of chambers of $\mathcal{A}_{-\epsilon}$ opposite a chamber of ϕ_ϵ coincides with the complement of $\phi_{-\epsilon}$ in $\mathcal{A}_{-\epsilon}$ ($\epsilon \in \{+, -\}$). Any two chambers $x_+ \in \mathcal{C}_+$ and $c_- \in \mathcal{C}_-$ such that $\delta_*(x_+, x_-) = s \in S$ are contained in a unique twin root, noted $\phi(x_+, x_-)$, which is the union of $\phi_+(x_+, x_-)$ and $\phi_-(x_+, x_-)$ where $\phi_\epsilon(x_+, x_-)$ consists of all chambers $c \in \mathcal{C}_\epsilon$ such that

$$\delta_\epsilon(x_\epsilon, c) = s\delta_*(x_{-\epsilon}, c)$$

and

$$\ell(\delta_\epsilon(x_\epsilon, c)) < \ell(s\delta_\epsilon(x_\epsilon, c)).$$

Given ϕ a twin root of $\mathcal{B}$, the group U_ϕ consisting of all $g \in \mathrm{Aut}(\mathcal{B})$ which fix pointwise each panel ρ such that $|\rho \cap \phi| = 2$ is called the **root group** associated with ϕ.

We now assume that (W, S) has no direct factor of type A_1 (i.e. no element of S is central in W). It is then a consequence of Theorem 1.6 that for each twin root ϕ and each panel ρ such that $|\rho \cap \phi| = 1$, the root group U_ϕ acts freely on the set $\rho \backslash \phi$. We say that the twin building $\mathcal{B}$ has the **Moufang property** (or simply is **Moufang**) if U_ϕ is transitive on $\rho \backslash \phi$ for each twin root ϕ and each panel ρ such that $|\rho \cap \phi| = 1$.

Moufang twin buildings provide the geometric counterpart of the algebraic notion of twin root data, as we will see in §1.4.2.

1.4 Twin root data and twin buildings: a short dictionary

This section aims to recall the correspondence between twin root data and Moufang twin buildings (see [Tit90, Théorème 3] and [Tit92, Proposition 7]).

1.4.1 Twin buildings from twin root data

Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha \in \Phi})$ be a twin root datum of type (W, S). Let H be the intersection of the normalizers of all U_ϕ's, let N be the subgroup of G generated by H together with all $\mu(u)$ such that $u \in U_\phi \backslash \{1\}$, where $\mu(u)$ is as in (TRD2), and for each $\epsilon \in \{+, -\}$, let $U_\epsilon := \langle U_\alpha |\ \alpha \in \Phi_\epsilon \rangle$ and $B_\epsilon := H.U_\epsilon$. We recall from [Tit92, Proposition 4], that (G, B_+, N) and (G, B_-, N) are both BN-pairs of type (W, S). Thus, we have corresponding Bruhat decompositions of G:

$$G = \coprod_{w \in W} B_+ w B_+ \qquad \text{and} \qquad G = \coprod_{w \in W} B_- w B_-.$$

For each $\epsilon \in \{+, -\}$, the set $\mathcal{C}_\epsilon := G/B_\epsilon$ endowed with the map $\delta_\epsilon : \mathcal{C}_\epsilon \times \mathcal{C}_\epsilon \to W$ by

$$\delta_\epsilon(gB_\epsilon, hB_\epsilon) = w \Leftrightarrow B_\epsilon g^{-1} h B_\epsilon = B_\epsilon w B_\epsilon,$$

has a canonical structure of a thick building of type (W, S).

The twin root datum axioms imply that G also admits Birkhoff decompositions (this statement appears in [Tit83, §6.3]; see [Abr96, Lemma 1] for a proof):

$$G = \coprod_{w \in W} B_\epsilon w B_{-\epsilon}$$

for each $\epsilon \in \{+, -\}$. The pair $((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-))$ of buildings admits a natural twinning by means of the W-codistance δ^* defined by

$$\delta^*(gB_\epsilon, hB_{-\epsilon}) = w \Leftrightarrow B_\epsilon g^{-1} h B_{-\epsilon} = B_\epsilon w B_{-\epsilon}$$

for each $\epsilon \in \{+, -\}$. The triple $\mathcal{B} := ((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta^*)$ is a twin building of type (W, S). The diagonal action of G on $\mathcal{C}_+ \times \mathcal{C}_-$ by left multiplication is transitive on pairs of opposite chambers.

1.4.2 Parabolic subgroups and root subgroups

Let (W, S), $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$, H, N, U_+, U_-, B_+, B_- be as in §1.4.1.

We recall from the theory of BN-pairs (see [Bou81, Chapter IV]) that a subgroup P of G containing B_ϵ is called a **standard parabolic subgroup** of sign ϵ, where $\epsilon \in \{+, -\}$. The conjugates of P are called **parabolic subgroups** of sign ϵ. If P contains B_ϵ, then there exists $J \subseteq S$ such that P has a Bruhat decomposition

$$P = \coprod_{w\in W_J} B_\epsilon w B_\epsilon;$$

the set J is called the **type** of the parabolic subgroup P. If J is spherical, then P is said to be **of finite type** (or of spherical type). Two parabolic subgroups P_+ and P_- of type J are called **opposite** if there exists $g \in G$ such that gP_+g^{-1} and gP_-g^{-1} are the two standard parabolic subgroup of type J of opposite signs. A minimal parabolic subgroup (i.e. a parabolic subgroup of type $\emptyset$) is called a **Borel subgroup**; so are e.g. B_+ and B_-.

Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be the twin building associated with $\mathcal{Z}$. The group B_+ (resp. B_-) fixes a unique chamber c_+ of $\mathcal{B}_+$ (resp. c_- of $\mathcal{B}_-$) which is called **standard** (with respect to $\mathcal{Z}$) and we have $B_+ = \mathrm{Stab}_G(c_+)$ (resp. $B_- = \mathrm{Stab}_G(c_-)$). Moreover c_+ and c_- are opposite. The unique twin apartment $\mathcal{A}$ containing them is called **standard** (with respect to $\mathcal{Z}$). We have the following:

$$N = \mathrm{Stab}_G(\mathcal{A}), \qquad H = B_+ \cap N = B_- \cap N = \mathrm{Fix}_G(\mathcal{A})$$

and for $J \subset S$,

$$P_+^J = \mathrm{Stab}_G(\mathrm{Res}_J(c_+)), \qquad \text{and} \qquad P_-^J = \mathrm{Stab}_G(\mathrm{Res}_J(c_-)),$$

where P_ϵ^J denotes the parabolic subgroup of type J containing B_ϵ. A subgroup of G which stabilizes a pair of opposite chambers of $\mathcal{B}$ is called **diagonalizable**. Thus H is a maximal diagonalizable subgroup of G and all such subgroups are conjugate (because G is transitive on pairs of opposite chambers).

Let $\Phi(\mathcal{A})$ be the set of all twin roots contained in $\mathcal{A}$. There is a canonical one-to-one correspondence $\zeta : \Phi \to \Phi(\mathcal{A})$ such that for all $\phi \in \Phi$, the group U_ϕ fixes pointwise each panel σ with $|\sigma\cap\zeta(\phi)| = 2$ and acts regularly on $\sigma\backslash\zeta(\phi)$ for each panel with $|\sigma\cap\zeta(\phi)| = 1$. Thus U_ϕ coincides with the root group $U_{\zeta(\phi)}$ and if (W, S) has no direct factor of type A_1 then the twin building $\mathcal{B}$ is Moufang.

Conversely, let $\mathcal{B}$ be a Moufang twin building of type (W, S). Let $x_+ \in \mathcal{C}_+$ and $x_- \in \mathcal{C}_-$ be opposite chambers and $\mathcal{A}$ be the unique twin apartment containing them. Let $\Phi(\mathcal{A})$ be the set of all twin roots ϕ of $\mathcal{B}$ such that $\phi \subset \mathcal{A}$. Let $G(\mathcal{B})$ be the group generated by all root groups U_ϕ such that $\phi \in \Phi(\mathcal{A})$. Then $(G(\mathcal{B}), (U_\phi)_{\phi\in\Phi(\mathcal{A})})$ is a twin root datum of type (W, S) and the associated twin building is isomorphic to $\mathcal{B}$.

1.4.3 Kernel of the action on the twin building

Lemma 1.7. *Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum and H be the intersection of the normalizers of all U_α's in G. The kernel of the action of G on the twin building associated with $\mathcal{Z}$ coincides with $\bigcap_{\alpha\in\Phi} C_G(U_\alpha)$. In particular, if G is generated by the U_α's or if H is abelian, then this kernel is the center of G.*

Proof. Let K be the kernel of the action of G on the twin building associated with $\mathcal{Z}$. The fact that K normalizes every root group is a consequence of [Rém02b, Corollary 3.5.4]. This shows that $K \subset H$. Since K normalizes U_α and U_α normalizes K for each α, the commutator $[K, U_\alpha]$ is contained in $K \cap U_\alpha \subset H \cap U_\alpha = \{1\}$, where the last equality follows from [Rém02b, Theorem 3.5.4]. Hence we have $K \subset \bigcap_{\alpha \in \Phi} C_G(U_\alpha)$. The reverse inclusion is a consequence of Theorem 1.6. $\square$

Chapter 2

Basic tools from geometric group theory

2.1 CAT(0) geometry

The standard reference is the book [BH99].

2.1.1 CAT(0) metric spaces and their isometries

Let (X, d) be a metric space. A **geodesic path** is a map c from a closed interval of $\mathbb{R}$, endowed with the natural metric, to X which is an isometry onto its image. A **geodesic segment** is the image of a geodesic path; a **geodesic line** is the image of an isometry of $\mathbb{R}$ to X. The metric space (X, d) is called **geodesic** if every two points are joined by a geodesic path.

A **geodesic triangle** of a metric space (X, d) consists of three points of X, called the **vertices** of the triangle, and three geodesic paths joining the vertices pairwise, called the **edges**.

Let Δ be a geodesic triangle with vertices a, b, c and let $\alpha : I_\alpha \to X$, $\beta : I_\beta \to X$, $\gamma : I_\gamma \to X$ be the edges of Δ joining respectively b to c, c to a and a to b. If a point x lies on the union of the images of the edges of Δ, we write $x \in \Delta$.

Let $(\mathbb{E}^2, d_\mathbb{E})$ be the Euclidean plane and let $\bar{\Delta}$ be a geodesic triangle of $(\mathbb{E}^2, d_\mathbb{E})$ with vertices $\bar{a}$, $\bar{b}$, $\bar{c}$ such that $d(a, b) = d_\mathbb{E}(\bar{a}, \bar{b})$, $d(b, c) = d_\mathbb{E}(\bar{b}, \bar{c})$ and $d(a, c) = d_\mathbb{E}(\bar{a}, \bar{c})$. Let $\bar{\alpha} : I_\alpha \to \mathbb{E}^2$ be the geodesic joining $\bar{b}$ to $\bar{c}$ and define $\bar{\beta}$ and $\bar{\gamma}$ similarly. Then Δ is said to satisfy the **CAT(0) inequality** if for all $\phi, \psi \in \{\alpha, \beta, \gamma\}$ and all $x \in I_\phi$, $y \in I_\psi$, one has

$$d(\phi(x), \psi(y)) \leq d_\mathbb{E}(\bar{\phi}(x), \bar{\psi}(y)).$$

A **CAT(0) space** is a geodesic metric space all of whose geodesic triangles satisfy the CAT(0) inequality. In such a space, every two points are joined by a unique geodesic.

Given an isometry γ of a metric space X, let

$$|\gamma| = \inf\{d(x, \gamma.x) |\ x \in X\}.$$

The isometry γ is called **semisimple** if there exists $x \in X$ such that $d(x, \gamma.x) = |\gamma|$. It is called **elliptic** (resp. **hyperbolic**) if it is semisimple and if $|\gamma| = 0$ (resp. $|\gamma| > 0$).

The following lemma collects a few standard facts about isometries of complete CAT(0) spaces.

Lemma 2.1. *Let X be a complete CAT(0) space and γ be an isometry of X. Let $\operatorname{Min}(\gamma) := \{x \in X | \, d(x, \gamma.x) = |\gamma|\}$. We have the following.*

(i) If γ^n is elliptic (resp. hyperbolic) for some $n \in \mathbb{Z}_{>0}$, then so is γ.

(ii) Given a closed convex γ-invariant subset C of X, then γ is elliptic (resp. hyperbolic) if and only if $\gamma|_C$ is elliptic (resp. hyperbolic) and one has $\operatorname{Min}(\gamma|_C) = \operatorname{Min}(\gamma) \cap C$.

(iii) γ is hyperbolic if and only if $\operatorname{Min}(\gamma)$ is a disjoint union of geodesic lines along each of which γ acts by translation.

Proof. See [BH99, Propositions II.6.2, II.6.7 and Theorem II.6.8]. □

A fundamental feature of CAT(0) metric spaces on which Assertion (ii) of this lemma rests is the following.

Lemma 2.2. *Let (X, d) be a complete CAT(0) space, C be a closed convex subset of X and $x \in X$ be any point. Then there is a unique point c of C such that $d(x, c) = d(x, C) := \inf_{y \in C} d(c, y)$.*

Proof. See [BH99, Proposition II.2.4]. □

2.1.2 Bruhat-Tits' fixed point theorem and some consequences

The celebrated fixed point theorem of F. Bruhat and J. Tits holds in the general framework of CAT(0) spaces.

Theorem 2.3. *Let X be a complete CAT(0) space and let Γ be a group acting on X by isometries. If Γ has a bounded orbit, then Γ has a global fixed point.*

Proof. See [BH99, Corollary II.2.8]. □

Also relevant to us are the following corollaries.

A group G is said to be **boundedly generated** by a family of subgroups $(U_i)_{i \in I}$ if there exists a constant $\nu \in \mathbb{Z}_{>0}$ such that every $g \in G$ may be written as a product $g = g_1 g_2 \dots g_\nu$ with $g_j \in \bigcup_{i \in I} U_i$.

Corollary 2.4. *Let X be a complete CAT(0) space, let Γ be a group acting on X by isometries. Suppose that Γ is boundedly generated by a (possibly infinite) family of subgroups $(U_i)_{i \in I}$ and that there exists a bounded subset $B \subset X$ such that $B \cap \operatorname{Fix}(U_i)$ is nonempty for each $i \in I$. Then Γ has a global fixed point.*

Proof. Let $x_0 \in X$ be a base point and let $g \in \Gamma$. For each $i \in I$ let $x_i \in B \cap \operatorname{Fix}(U_i)$. Let us write $g = g_1 g_2 \dots g_\nu$ with $g_j \in U_{i(j)}$ and define $x_{i(0)} := x_0$. Hence for each $j = 0, 1, \dots, \nu$, the point $x_{i(j)}$ is defined and it is fixed by $U_{i(j)}$ for $j > 0$. We have

$$\begin{array}{rcl} d(x_0, g.x_0) & \leq & d(x_0, x_{i(1)}) + d(x_{i(1)}, g_1 g_2 \dots g_\nu.x_0) \\ & = & d(x_0, x_{i(1)}) + d(x_{i(1)}, g_2 g_3 \dots g_\nu.x_0) \\ & \leq & d(x_0, x_{i(1)}) + d(x_{i(1)}, x_{i(2)}) + d(x_{i(2)}, g_2 g_3 \dots g_\nu.x_0) \\ & \leq & \dots \\ & \leq & \sum_{j=1}^{\nu} d(x_{i(j-1)}, x_{i(j)}) \\ & \leq & \nu. \max\{d(x_k, x_l) | \, k, l \in I \cup \{0\}\}. \end{array}$$

Hence Γ has a bounded orbit and we may apply Theorem 2.3. □

Corollary 2.5. *Let X be a complete CAT(0) space and let Γ be a group acting on X by isometries. Suppose Γ is boundedly generated by a finite family of subgroups $U_1, \ldots, U_n$, each of which has a fixed point in X. Then Γ has a global fixed point.*

Proof. Let $B \subset X$ be a bounded subset which contains an element of $\mathrm{Fix}(U_i)$ for each $i = 1, 2, \ldots, n$. It suffices to apply Corollary 2.4. □

2.1.3 CAT(0) polyhedral complexes

Let X be a M_κ-polyhedral complex. Such a complex is obtained by taking a disjoint union of possibly infinitely many copies of convex polyhedral cells in M_κ^n (where M_κ^n denotes the complete 1-connected n-dimensional manifold of constant sectional curvature κ) and gluing them along isometric faces. Following [BH99], we denote by $Shapes(X)$ the set of isometry classes of cells of X. A celebrated theorem of M. Bridson ensures that a locally CAT(0), 1-connected M_κ-polyhedral complex X with $Shapes(X)$ finite, endowed with the intrinsic metric, is complete and CAT(0) (see [BH99, §II.5]). For the sake of brevity, such a complex will be called a **CAT(0) polyhedral complex**.

A group action on a metric space X is called **semisimple** if every element of the group acts as a semisimple isometry. We need the following result of M. Bridson.

Proposition 2.6. *Let X be a connected M_κ-polyhedral complex with $Shapes(X)$ finite and let Γ be a group acting on X by cellular isometries. Then the action of Γ is semisimple and the set $\{|\gamma| : \gamma \in \Gamma\}$ is a discrete subset of $\mathbb{R}$.*

Proof. See Theorem A and the proposition of [Bri99]. □

Recall from [BH99] that a semisimple isometry γ of a metric space X is called **hyperbolic** if $|\gamma| > 0$, or equivalently if $\langle \gamma \rangle$ has no fixed point. In that case, assuming that X is complete and CAT(0), there exists a geodesic line ℓ stabilized by $\langle \gamma \rangle$ and along which γ acts by translation of length $|\gamma|$.

Corollary 2.7. *Let X be a CAT(0) polyhedral complex and let Γ be a group acting on X by cellular isometries. If Γ is abelian and m-divisible for some integer $m \in \mathbb{Z}_{>0}$, then every element of Γ has a fixed point in X.*

Proof. Suppose $\gamma \in \Gamma$ acts without fixed point. In view of Proposition 2.6, this implies that γ is hyperbolic. For each $r \in \mathbb{Z}_{>0}$, let $\gamma_r \in \Gamma$ be such that $(\gamma_r)^{m^r} = \gamma$. Each γ_r is hyperbolic and, hence, one has $|\gamma_r| = |\gamma|/m^r$. Since r is arbitrary, this contradicts Proposition 2.6. □

The last general result about CAT(0) polyhedral complexes we need can also be deduced from a theorem of M. Bridson.

Proposition 2.8. *Let X be a CAT(0) polyhedral complex and let Γ be an abelian group acting on X by cellular isometries. Then the subset $\Gamma_0 \subset \Gamma$ consisting of those elements of Γ which have fixed points in X constitutes a subgroup of Γ and the quotient Γ/Γ_0 is a free abelian group of finite rank bounded above by* $\dim(X)$.

Proof. Since Γ is abelian, it follows from Corollary 2.5 that the subset Γ_0 of Γ consisting of all elements which have fixed points in X is actually a subgroup of Γ. By Lemma 2.1(i), if $\gamma \in \Gamma$ does not belong to Γ_0, then $\gamma^n \notin \Gamma_0$ for all $n \in \mathbb{Z}_{\neq 0}$. In particular, the group Γ/Γ_0 is torsion free. Let $\gamma_1, \ldots, \gamma_n \in \Gamma$ be such that their images in Γ/Γ_0 generate a free abelian group of rank n. Clearly the group $\Gamma_n := \langle \gamma_1, \dots, \gamma_n \rangle$ is free abelian of rank n, and by definition, the intersection $\Gamma_n \cap \Gamma_0$ is trivial (otherwise the image of Γ_n in the quotient Γ/Γ_0 could not be of rank n). In other words, the group Γ_n acts freely on X. Now it follows from [Bri99, Theorem B] that n is bounded by the dimension of X. It follows that the rank of Γ/Γ_0 is also bounded by $\dim(X)$. □

2.1.4 Buildings are CAT(0)

The only aim of this section is to recall that any building of finite rank has a geometric realization which is a CAT(0) polyhedral complex. This result is due to F. Bruhat and J. Tits for affine buildings and to M. Davis and G. Moussong in the general case. The construction of this realization is described in [Dav98] and is extremely useful; it allows in particular to apply the results of the preceding subsections to buildings.

Following a standard convention, we denote by $|\mathcal{B}|$ the CAT(0) geometric realization of a building $\mathcal{B}$. Any group which acts on $\mathcal{B}$ by automorphisms acts on $|\mathcal{B}|$ by cellular isometries. Some important points to remember are the following:

- The facets of the polyhedral complex $|\mathcal{B}|$ correspond to the residues of $\mathcal{B}$ of spherical type. This leads us to adopt the convention that expressions like: "The group Γ fixes a point of $\mathcal{B}$" or "The group Γ fixes a point of $|\mathcal{B}|$" or else "The group Γ stabilizes a spherical residue of $\mathcal{B}$" are synonyms.

- If C is a set of spherical residues of $\mathcal{B}$ which is **combinatorially convex** (i.e. for all $x, y \in C$ and every spherical residue ρ containing y, one has $\mathrm{proj}_\rho(x) \in C$), then the subset $\bigcup_{c\in C} |c|$ of $|\mathcal{B}|$ is closed and convex. In particular, if $\mathcal{A}$ is an apartment of $\mathcal{B}$ then $|\mathcal{A}|$ is a closed convex subset of $|\mathcal{B}|$.

2.2 Rigidity of algebraic-group-actions on trees

Besides the generalities on geometric group theory collected in the preceding sections, we will need Tits' theorem on rigidity of algebraic group actions on tree. The statement of this result requires the following additional preparation.

Let $\mathcal{Z} = (G, (U_\epsilon)_{\epsilon\in\{+,-\}})$ be a twin root datum of rank 1 and suppose that $G = \langle U_+ \cup U_- \rangle$. Let also

$$T := N_G(U_+) \cap N_G(U_-)$$

and

$$M := \{m \in G|\ {}^mU_+ = U_- \text{ and } {}^mU_- = U_+\}.$$

A nontrivial homomorphism $\eta : T \to \mathbb{R}$ is called a **valuation** of $\mathcal{Z}$ if the following conditions hold:

(V1) $\eta(mtm) = -\eta(t)$ for all $t \in T$ and $m \in M$.

(V2) For every real number r, the set $\varphi^{-1}([r, \infty])$ is a group, where φ is the function $U_+ \to \mathbb{R} \cup \{\infty\}$ defined by

$$\varphi(u) = \begin{cases} \frac{1}{2}\eta(\mu(u) \cdot m_0) & \text{if } u \neq 1, \\ \infty & \text{if } u = 1, \end{cases}$$

for a suitable $m_0 \in M$.

Recall from [BT72, §§7, 8] that with a valuation η of $\mathcal{Z}$, one associates a $\mathbb{R}$-tree T_η, called **Bruhat-Tits tree**, on which G operates by isometries. This tree is constructed as follows.

Given $r \in \mathbb{R}$, let G_r be the subgroup of G generated by $\varphi^{-1}([r, \infty])$, $m_0 \cdot \varphi^{-1}([-r, \infty]) \cdot m_0^{-1}$ and $\mathrm{Ker}(\eta)$. One then defines an equivalence relation $\sim$ on $G \times \mathbb{R}$ by

$$(g, r) \sim (g', r') \qquad \Leftrightarrow \qquad r = r' \text{ and } g^{-1}g' \in G_r.$$

The group G acts on the quotient $T_\eta^0 := (G \times \mathbb{R})/\sim$ from the left, and there is a unique metric in T_η^0 which is invariant under G and such that $r \mapsto (1, r) \mod \sim$ is an isometric embedding of $\mathbb{R}$ into T_η^0. The tree T_η is defined as the completion of the tree T_η^0.

Theorem 2.9. *Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum of spherical irreducible type such that $G = \langle U_\alpha | \ \alpha \in \Phi\rangle$. Let G act on an $\mathbb{R}$-tree T by isometries. If U_α is nilpotent for each $\alpha \in \Phi$, the one of the following holds:*

(i) G fixes a point of T.

(ii) G fixes a unique end but no point of T.

(iii) $\mathcal{Z}$ is of rank one (i.e. $|\Phi| = 2$) and admits a valuation η such that the Bruhat-Tits tree T_η embeds G-equivariantly in T. In particular, if T is simplicial then η is discrete.

Proof. This follows from [Tit77, Corollary 4 and Proposition 4]. □

Chapter 3

Kac-Moody groups and algebraic groups

The main purpose of this chapter is to introduce the adjoint representation of Tits functors and bring its relationship with the action on the twin building into focus. One of the determining byproducts of the adjoint representation is the existence of an intrinsic Jordan decomposition of bounded elements of Kac-Moody groups.

Some basics from the theory of algebraic groups, which become applicable in the Kac-Moody context thanks to the adjoint action, are collected in the last section of the chapter.

3.1 Bounded subgroups

Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum of type (W, S) and $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be the associated twin building.

By definition, a **bounded subgroup** of G is a subgroup which is contained in the intersection of two finite type parabolic subgroups of opposite signs. In other words, a subgroup of G is bounded if and only if it stabilizes a residue of spherical type in $\mathcal{B}_+$ and in $\mathcal{B}_-$. This notion will play a fundamental role in this work, mainly because in a Kac-Moody group, which is not an algebraic group in general, the intersection of two finite type parabolic subgroups of opposite signs is an algebraic group (see Proposition 3.6 below). Thus every bounded subgroup embeds in a (in general non-reductive) algebraic group, and this makes available a series of tools from the theory of algebraic groups.

First of all, we need to review the Levi decomposition of parabolic subgroups. Actually, we will only need the Levi decomposition of parabolic subgroups of finite type, and it turns out that this decomposition can be obtained in the general framework of twin root data (and not only for split Kac-Moody groups). This is the point of view we adopt here.

3.1.1 Levi decomposition of parabolic subgroups

Throughout this section and the following one, we let (W, S), Φ, $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$, H, N, U_+, U_-, B_+, B_- be as in §1.4.1. Moreover we identify the Coxeter group W with the Weyl group N/H of $\mathcal{Z}$. Finally, let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be the twin building associated with $\mathcal{Z}$.

Given $J \subset S$, we set $\Phi_J := \{\phi \in \Phi |\ s_\phi \in W_J\}$ and

$$L^J := H.\langle U_\phi |\ \phi \in \Phi_J\rangle.$$

For $\epsilon \in \{+,-\}$, let also P_ϵ^J denote the parabolic subgroup of type J containing B_ϵ and

$$U_\epsilon^J := \bigcap_{w \in W_J} wU_\epsilon w^{-1}.$$

Proposition 3.1. *Suppose that J is spherical. Then, for $\epsilon \in \{+,-\}$, we have*

$$P_\epsilon^J = L^J \ltimes U_\epsilon^J.$$

Denoting by R_ϵ^J the unique J-residue of $\mathcal{B}_\epsilon$ stabilized by P_ϵ^J, we have

$$L^J = P_+^J \cap P_-^J = \mathrm{Stab}_G(R_+^J) \cap \mathrm{Stab}_G(R_-^J)$$

and the group U_ϵ^J acts regularly on the J-residues opposite R_ϵ^J in $\mathcal{B}$.

Proof. This follows from [Rém02b, Théorème 6.2.2]. □

The group U_ϵ^J is called the **unipotent radical** of P_ϵ^J and L^J is called a **Levi factor** (or a **Levi subgroup**).

3.1.2 Levi decomposition of bounded subgroups

In this section we describe a Levi decomposition for bounded subgroups. A result of this type was obtained in [Rém02b, § 6.3] under a technical assumption, called (NILP). It was later shown in [CM05b, Proposition 3.6] that (NILP) is not a necessary condition: Levi decompositions of bounded subgroups exist for arbitrary groups endowed with a twin root datum. We refer to [CM05b] for more details concerning the geometric interpretations of these decompositions.

Given $J, K \subset S$, $w \in W$ and $\epsilon \in \{+,-\}$, let

$$\Psi_\epsilon^{J,K,w} := (\Phi_\epsilon \cap w^{-1}.\Phi_{-\epsilon}) \backslash \Phi_{J \cap wKw^{-1}}$$

and

$$U_\epsilon^{J,K,w} := \langle U_\phi |\ \phi \in \Psi_\epsilon^{J,K,w} \rangle.$$

Proposition 3.2. *Suppose that J and K are spherical. Then, for $\epsilon \in \{+,-\}$, we have*

$$P_\epsilon^J \cap wP_{-\epsilon}^K w^{-1} = L^{J \cap wKw^{-1}} \ltimes U_\epsilon^{J,K,w}$$

and

$$P_\epsilon^J \cap wU_{-\epsilon}^K w^{-1} = U_\epsilon^{J,K,w}.$$

Moreover, there exists a prenilpotent pair of roots $\{\alpha, \beta\} \subset \Psi_\epsilon^{J,K,w}$ such that $\Psi_\epsilon^{J,K,w} = [\alpha, \beta]$ (in particular $\Psi_\epsilon^{J,K,w}$ is finite). Denoting by ρ (resp. σ) the unique J-residue (resp. K-residue) stabilized by P_ϵ^J (resp. $wP_{-\epsilon}^K w^{-1}$), we have

$$L^{J \cap wKw^{-1}} = \mathrm{Stab}_G(\mathrm{proj}_\rho(\sigma)) \cap \mathrm{Stab}_G(\mathrm{proj}_\sigma(\rho))$$

and $\Psi_\epsilon^{J,K,w}$ coincides with the set of twin roots of the standard twin apartment which contain both $\mathrm{proj}_\rho(\sigma)$ and $\mathrm{proj}_\sigma(\rho)$.

Proof. This follows from [CM05b, Lemma 3.3 and Proposition 3.6]. □

3.1.3 Bounded unipotent subgroups

Let $(G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum. A subgroup of G is called **bounded unipotent** if it is contained in the unipotent radical of the intersection of two Borel subgroups of opposite signs.

Lemma 3.3. *Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum and U be a bounded unipotent subgroup of G. Given a Borel subgroup B of G, the intersection $B \cap U$ is contained in the unipotent radical of B.*

Proof. Let B_+ and B_- be Borel subgroups of respective signs $+$ and $-$ such that U is contained in the unipotent radical of B_+ and B_-. Thus $U \leq U_+ \cap U_-$ where U_+ and U_- denote the unipotent radicals of B_+ and B_- respectively. Let $\epsilon \in \{+,-\}$ be the sign of B. Then $B \cap U \leq B \cap U_{-\epsilon}$ and Proposition 3.2 implies that $B \cap U_{-\epsilon}$ is contained in the unipotent radical of B. □

3.2 Adjoint representation of Tits functors

This section aims to recall the construction of an adjoint representation of Tits functors, due to B. Rémy [Rém02b, Chapter 9], which can be used as a substitute for an algebro-geometric structure on Kac-Moody groups. Exactly as in loc. cit., the adjoint representation will provide bounded subgroups with a structure of algebraic groups, thereby allowing to use arguments from the theory of algebraic groups (e.g. results of §3.3).

3.2.1 Kac-Moody algebras of type $\mathcal{D}$ and their universal enveloping algebras

Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum. The **Kac-Moody algebra of type $\mathcal{D}$** is the complex Lie algebra, noted $\mathfrak{g}_\mathcal{D}$, generated by $\mathfrak{g}_0 := \Lambda^\vee \otimes_\mathbb{Z} \mathbb{K}$ and the sets $\{e_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ with the following presentation ($i, j \in I$ and $h, h' \in \mathfrak{g}_0$):

$$\begin{array}{rcl} [h, e_i] &=& \langle c_i, h\rangle e_i, \\ [h, f_i] &=& -\langle c_i, h\rangle f_i, \\ [h, h'] &=& 0, \\ [e_i, f_i] &=& -h_i \otimes 1, \\ [e_i, f_j] &=& 0 \text{ for } i \neq j, \\ (\text{ad } e_i)^{-A_{ij}+1}(e_j) &=& \text{ad } (f_i)^{-A_{ij}+1}(f_j) = 0. \end{array}$$

The algebra $\mathfrak{g}_A$ considered in §1.1.2 is nothing but the Kac-Moody algebra of simply connected type (i.e. when $\Lambda^\vee = \bigoplus_{i\in I} \mathbb{Z}h_i$). Note that, given a Kac-Moody algebra of type $\mathcal{D}$, there is a canonical homomorphism $\mathfrak{g}_A \to \mathfrak{g}_\mathcal{D}$.

The **universal enveloping algebra** of $\mathfrak{g}_\mathcal{D}$ is denoted by $\mathcal{U}\mathfrak{g}_\mathcal{D}$. Following J. Tits [Tit87b], for $u \in \mathcal{U}\mathfrak{g}_\mathcal{D}$ and $n \in \mathbb{Z}$ we set

$$u^{[n]} := (n!)^{-1}u^n \qquad \text{and} \qquad \binom{u}{n} := (n!)^{-1}u(u-1)\dots(u-n+1).$$

For $i \in I$ let $\mathcal{U}_i$ (resp. $\mathcal{U}_{-i}$) be the subring $\sum_{n\in\mathbb{Z}_{\geq 0}} \mathbb{Z}e_i^{[n]}$ (resp. $\sum_{n\in\mathbb{Z}_{\geq 0}} \mathbb{Z}f_i^{[n]}$) of $\mathcal{U}\mathfrak{g}_\mathcal{D}$ and let $\mathcal{U}_0$ be the subring generated by all $\binom{u}{n}$ for $\lambda \in \Lambda^\vee$ and $n \in \mathbb{Z}_{\geq 0}$. Finally, let $\mathcal{U}_\mathcal{D}$ be the subring of $\mathcal{U}\mathfrak{g}_{\mathcal{D}}$ generated by $\mathcal{U}_0$ and all $\mathcal{U}_i$ and $\mathcal{U}_{-i}$ for $i \in I$. The ring $\mathcal{U}_{\mathcal{D}}$ is a $\mathbb{Z}$-form of the enveloping algebra $\mathcal{U}\mathfrak{g}_{\mathcal{D}}$, in the sense that the canonical map $\mathcal{U}_{\mathcal{D}} \otimes_{\mathbb{Z}} \mathbb{C} \to \mathcal{U}\mathfrak{g}_{\mathcal{D}}$ is one-one (see [Tit87b, §4.4]).

3.2.2 Definition of the adjoint representation

We keep the notation of the previous subsection.

Given a ring R (i.e. a commutative ring with a unit) and a subring $\mathcal{A}$ of $\mathcal{U}_{\mathcal{D}}$, we set $(\mathcal{A})_R := \mathcal{A} \otimes_{\mathbb{Z}} R$. The assignment $R \mapsto \mathrm{Aut}(\mathcal{U}_{\mathcal{D}})_R$ is functorial (see [Rém02b, §9.5.1]), and the corresponding functor is abusively denoted by $\mathrm{Aut}(\mathcal{U}_{\mathcal{D}})$.

Let now $\mathcal{F} = (\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$ be the basis of a Tits functor $\mathcal{G}$ of type $\mathcal{D}$. Given $i \in I$, let $\mathfrak{u}_i$ be the natural transformations defined by

$$\mathfrak{u}_i : R \to \mathcal{G}(R) : r \mapsto \phi_i \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}.$$

Proposition 3.4. *Let $\mathbb{K}$ be a field. Then the assignments*

$$\mathfrak{u}_i(x) \mapsto \exp(\mathrm{ad}\, e_i \otimes x) = \sum_{n\geq 0} \frac{(\mathrm{ad}\, e_i)^n}{n!} \otimes x^n$$

and

$$h \mapsto \begin{cases} h' \mapsto h' \\ e_i \otimes y \mapsto \langle c_i, h\rangle (e_i \otimes y) \end{cases}$$

where $x, y \in \mathbb{K}$ and $h, h' \in \mathcal{T}_\Lambda(\mathbb{K})$, extend to a unique morphism $\mathbf{Ad}_{\mathbb{K}} : \mathcal{G}(\mathbb{K}) \to \mathrm{Aut}(\mathcal{U}_{\mathcal{D}})_{\mathbb{K}}$, which defines a morphism of functors $\mathbf{Ad} : \mathcal{G} \to \mathrm{Aut}(\mathcal{U}_{\mathcal{D}})$ (where $\mathcal{G}$ and $\mathrm{Aut}(\mathcal{U}_{\mathcal{D}})$ are restricted to the category of fields).

Proof. See [Rém02b, Théorème 9.5.3]. □

The homomorphism $\mathbf{Ad}_{\mathbb{K}}$ is called the **adjoint representation** of the Kac-Moody group $\mathcal{G}(\mathbb{K})$.

Lemma 3.5. *The kernel of the adjoint representation of $\mathcal{G}(\mathbb{K})$ coincides with the center of $\mathcal{G}(\mathbb{K})$.*

Proof. See [Rém02b, Proposition 9.6.2]. □

3.2.3 Bounded subgroups as algebraic groups

Let $\mathcal{D}$ be a Kac-Moody root datum, $\mathcal{G}$ be a Tits functor of type $\mathcal{D}$ and $\mathbb{K}$ be a field.

A subgroup H of $G := \mathcal{G}(\mathbb{K})$ is called $\mathrm{Ad}_{\mathbb{K}}$**-locally finite** (resp. $\mathrm{Ad}_{\mathbb{K}}$**-locally unipotent**) if every vector of the $\mathbb{K}$-vector space $(\mathcal{U}_{\mathcal{D}})_{\mathbb{K}}$ is contained in a finite-dimensional subspace V invariant under $\mathbf{Ad}_{\mathbb{K}}(H)$ (resp. and such that $\mathbf{Ad}_{\mathbb{K}}(H)|_V$ is a unipotent subgroup of $GL(V)$). A subgroup H of G is called $\mathrm{Ad}_{\mathbb{K}}$**-diagonalizable** (resp. $\mathrm{Ad}_{\mathbb{K}}$**-semisimple**) if $(\mathcal{U}_{\mathcal{D}})_{\mathbb{K}}$ decomposes into a direct sum of one-dimensional subspaces invariant under $\mathbf{Ad}_{\mathbb{K}}(H)$ (resp. a direct sum of finite-dimensional irreducible $\mathbf{Ad}_{\mathbb{K}}(H)$-modules). An element $g \in G := \mathcal{G}(\mathbb{K})$ is called $\mathrm{Ad}_{\mathbb{K}}$**-locally finite** (resp. $\mathrm{Ad}_{\mathbb{K}}$**-locally unipotent**, $\mathrm{Ad}_{\mathbb{K}}$**-diagonalizable**, $\mathrm{Ad}_{\mathbb{K}}$**-semisimple**) if $\langle g\rangle$ is so.

Let now P_+ and P_- be finite type parabolic subgroups of G of opposite signs and let $A := P_+ \cap P_-$. We know from Proposition 3.2 that A has an abstract Levi decomposition

$A = L \ltimes U$, where L is a Levi factor of finite type and U is a bounded unipotent subgroup of G. It follows implicitly from Tits' construction of G that A is abstractly isomorphic to the $\mathbb{K}$-points of a connected algebraic $\mathbb{K}$-group, and that the abstract Levi decomposition above corresponds to a Levi decomposition in the sense of algebraic groups. The following result makes this algebraic structure more intrinsic by means of the adjoint representation.

We assume here that the field $\mathbb{K}$ is infinite.

Proposition 3.6. *Given $A \leq G$ as above, let T be a maximal diagonalizable subgroup of G contained in A and let $A = L \ltimes U$ be a Levi decomposition such that $T \leq L$. Then there exists a finite-dimensional subspace W of $(\mathcal{U}_{\mathcal{D}})_{\mathbb{K}}$ such that $A = \mathrm{Stab}_G(W)$ and, moreover, the following assertions hold:*

(i) The Zariski closure $\bar{A}$ (resp. $\bar{L}$, $\bar{U}$, $\bar{T}$) of $\mathbf{Ad}_{\mathbb{K}}(A)|_W$ (resp. $\mathbf{Ad}_{\mathbb{K}}(L)|_W$, $\mathbf{Ad}_{\mathbb{K}}(U)|_W$, $\mathbf{Ad}_{\mathbb{K}}(T)|_W$) in $GL(W_{\bar{\mathbb{K}}})$ is a connected $\mathbb{K}$-subgroup, where $W_{\bar{\mathbb{K}}} := W \otimes_{\mathbb{K}} \bar{\mathbb{K}} \subset (\mathcal{U}_{\mathcal{D}})_{\bar{\mathbb{K}}}$. Moreover, one has $\bar{A} = \mathbf{Ad}_{\mathbb{K}}(A)|_W$ if $\mathbb{K} = \bar{\mathbb{K}}$ and similarly for L, U and T.

(ii) $\bar{L}$ is reductive, $\bar{T}$ is a maximal torus of $\bar{L}$, $\bar{U}$ is unipotent, and $\bar{A} = \bar{L} \ltimes \bar{U}$ is a Levi decomposition. Moreover $\mathbf{Ad}_{\mathbb{K}}$ maps root subgroups of L (in the sense of §1.4.2) to root subgroups of $\bar{L}$ (in the algebraic sense).

(iii) The kernel of the restriction $\mathbf{Ad}_{\mathbb{K}} : A \to GL(W)$ is the center of A and is contained in the center of L, which is $\mathrm{Ad}_{\mathbb{K}}$*-diagonalizable.*

(iv) An element s of A is $\mathrm{Ad}_{\mathbb{K}}$*-semisimple if and only if $\mathbf{Ad}_{\mathbb{K}}(s)|_W$ is a semisimple element of $\bar{A}$.*

Proof. See [Rém02b, §10.3] for (i), (ii) and (iii).

For Assertion (iv), note that the 'only if' part is clear. Let now $\bar{s} := \mathbf{Ad}_{\mathbb{K}}(s)|_W$ and suppose $\bar{s}$ semisimple. Then $\bar{s}$ is contained in a maximal torus of $\bar{L}$. In view of (i), (ii) and using Theorem 3.7(ii) and the functoriality of the adjoint representation, one sees that there exists an element $s' \in \mathcal{G}(\bar{\mathbb{K}})$ which is $\mathbf{Ad}_{\bar{\mathbb{K}}}$-diagonalizable and such that $\mathbf{Ad}_{\bar{\mathbb{K}}}(s')|_{W_{\bar{\mathbb{K}}}} = \bar{s}$. It follows from (iii) that $(s')^{-1}s$ is $\mathbf{Ad}_{\bar{\mathbb{K}}}$-diagonalizable and centralizes s'. Therefore s is an $\mathbf{Ad}_{\bar{\mathbb{K}}}$-diagonalizable element of $\mathcal{G}(\bar{\mathbb{K}})$ and is thus $\mathbf{Ad}_{\mathbb{K}}$-semisimple. □

3.2.4 Link between the adjoint representation and the action on the twin building

By Lemmas 1.7 and 3.5, the kernel of the adjoint action of $\mathcal{G}(\mathbb{K})$ and its action on the associated twin building coincide. Actually, the relationship between these actions is very sharp. The present subsection aims to bring this relationship into focus.

Theorem 3.7. *Let $\mathbb{K}$ be an infinite field, $\mathcal{G}$ be a Tits functor and H be a subgroup of $G := \mathcal{G}(\mathbb{K})$. We have the following:*

(i) H is bounded if and only if it is $\mathbf{Ad}_{\mathbb{K}}$-locally finite.

(ii) H is diagonalizable (in the sense of §1.4.2) if and only if it is $\mathbf{Ad}_{\mathbb{K}}$-diagonalizable.

(iii) If H is bounded unipotent then it is $\mathbf{Ad}_{\mathbb{K}}$-locally unipotent. If $Z(G) = \{1\}$, $\mathbb{K}$ is perfect and H is $\mathbf{Ad}_{\mathbb{K}}$-locally unipotent, then H is bounded unipotent.

Proof. Assertions (i) and (ii) can be found in [Rém02b, Théorème 10.2.2 and Lemme 10.4.1]. Assertion (i) was proved by V. Kac and D. Peterson in characteristic 0 (see [KP87, §2.4, Theorem 1]).

Assertion (iii) can be proved as follows. The first statement is a consequence of Propositions 3.2 and 3.4. Suppose now that $Z(G) = \{1\}$, $\mathbb{K}$ is perfect and H is a $\mathbf{Ad}_{\mathbb{K}}$-locally unipotent subgroup of G. By (i), a $\mathbf{Ad}_{\mathbb{K}}$-locally unipotent subgroup of G is bounded. Furthermore, Proposition 3.6(ii) allows to apply [BT71, Corollaire 3.7] because $\mathbb{K}$ is perfect. This implies that H is bounded unipotent in G modulo some $\mathbf{Ad}_{\mathbb{K}}$-diagonalizable subgroup. In other words H is contained in a subgroup of G of the form $T \ltimes V$ where T is diagonalizable and V is bounded unipotent. Thus every element $h \in H$ can written as a product of the form $h = tv$ where $t \in T$ and $v \in V$. By (ii), $\mathbf{Ad}_{\mathbb{K}}(t)$ must be trivial, otherwise u would not be $\mathbf{Ad}_{\mathbb{K}}$-locally unipotent. This means that $t \in Z(G)$ (see Lemma 3.5), whence $H \subset V$. □

3.2.5 Jordan decomposition

In order to illustrate the intrinsic character of the constructions of the preceding paragraphs, we show in this section the existence of a Jordan decomposition of bounded elements of Kac-Moody groups. Such a Jordan decomposition has been obtained by V. Kac and S.P. Wang [KW92] over ground fields of characteristic 0, but the result over fields of positive characteristic does not seem to appear in the literature.

Note that a bounded element of a Kac-Moody group is contained in the intersection of a pair P_+, P_- of finite type parabolic subgroups of opposite signs. Thus it follows from Proposition 3.6 that a bounded element has an "abstract" Jordan decomposition. However, this decomposition depends a priori on the choice of P_+ and P_-. This inconvenience is lifted by the following.

Proposition 3.8. *Let $\mathcal{G}$ be a Tits functor, $\mathbb{K}$ be an infinite perfect field and set $G := \mathcal{G}(\mathbb{K})$. Let $g \in G$ be an $\mathbf{Ad}_{\mathbb{K}}$-locally finite element. Then there exist unique $\mathbf{Ad}_{\mathbb{K}}$-semisimple $g_s \in G$ and bounded unipotent $g_u \in G$ such that $g = g_s g_u = g_u g_s$. Moreover, given a finite-dimensional $\mathbf{Ad}_{\mathbb{K}}(g)$-stable subspace V of $(\mathcal{U}_{\mathcal{D}})_{\mathbb{K}}$, then V is $\mathbf{Ad}_{\mathbb{K}}(g_s)$-stable and $\mathbf{Ad}_{\mathbb{K}}(g_u)$-stable; moreover, $\mathbf{Ad}_{\mathbb{K}}(g_s)|_V$ and $\mathbf{Ad}_{\mathbb{K}}(g_u)|_V$ are respectively the semisimple and unipotent parts of $\mathbf{Ad}_{\mathbb{K}}|_V(g)$ in $GL(V)$.*

Proof. Existence. Let P_+ (resp. P_-) be a finite type parabolic subgroup of positive (resp. negative) sign containing g and set $A := P_+ \cap P_-$. Let W be the finite-dimensional $\mathbf{Ad}_{\mathbb{K}}(A)$-stable subspace of $(\mathcal{U}_{\mathcal{D}})_{\mathbb{K}}$ provided by Proposition 3.6. It follows from this proposition that one has a Jordan decomposition $\mathbf{Ad}_{\mathbb{K}}(g)|_W = s.u = u.s$, where s and u are respectively semisimple and unipotent parts of $\mathbf{Ad}_{\mathbb{K}}(g)|_W$ in $GL(W)$. By Proposition 3.6(iii), the restriction of $\mathbf{Ad}_{\mathbb{K}}$ to the set of bounded unipotent elements of A is injective. Therefore, Proposition 3.6(ii) and [BT71, Corollaire 3.7] imply that there exists a unique bounded unipotent element $g_u \in A$ such that $\mathbf{Ad}_{\mathbb{K}}(g_u)|_W = u$. Let $g_s := g(g_u)^{-1}$. It follows from the uniqueness of the Jordan decomposition in $GL(W)$ that $\mathbf{Ad}_{\mathbb{K}}(g_s)|_W = s$. Therefore, in view of Proposition 3.6(iii), $[g_s, g_u]$ is central in A and $\mathbf{Ad}_{\mathbb{K}}$-diagonalizable, and g_s is $\mathbf{Ad}_{\mathbb{K}}$-semisimple. Since g_s and g_u are contained in a common finite type parabolic subgroup of G by construction, it follows that the commutator $[g_s, g_u]$ is contained in the unipotent radical of a Borel subgroup of G. Moreover, it follows from Proposition 3.2 that any bounded subgroup contained in the unipotent radical of a Borel subgroup is bounded unipotent. Thus $[g_s, g_u]$ is both $\mathbf{Ad}_{\mathbb{K}}$-diagonalizable and contained in a bounded unipotent subgroup of G. Therefore $[g_s, g_u]$ is trivial and g_s commutes with g_u.

Set $\bar{g} := \mathbf{Ad}_{\mathbb{K}}(g)$, $\bar{g}_s := \mathbf{Ad}_{\mathbb{K}}(g_s)$ and $\bar{g}_u := \mathbf{Ad}_{\mathbb{K}}(g_u)$.

Let V be any $\bar{g}$-stable finite-dimensional subspace of $(\mathcal{U}_{\mathcal{D}})_{\mathbb{K}}$. Since $\bar{g}_s$ commutes with $\bar{g}$, it follows that $\bar{g}_s^n.V$ is $\bar{g}$-stable for each $n \in \mathbb{Z}$. Since $\bar{g}_s$ is locally finite, there exists an $n \in \mathbb{Z}_{\geq 0}$ such that $V' := \sum_{i=0}^{n} \bar{g}_s^i.V$ is $\bar{g}_s$-stable. Thus V' is a finite-dimensional subspace which is $\bar{g}$-stable and $\bar{g}_s$-stable subspace, hence it is also $\bar{g}_u$-stable. By Theorem 3.7, $\bar{g}_s|_{V'}$ is semisimple and $\bar{g}_u|_{V'}$ is unipotent. It follows from the uniqueness of the Jordan decomposition in $GL(V')$ that $\bar{g}|_{V'} = \bar{g}_s|_{V'}\bar{g}_u|_{V'}$ is a Jordan decomposition. In particular, $V \subset V'$ is stable under $\bar{g}_s$ and $\bar{g}_u$ and $\bar{g}_s|_V$ and $\bar{g}_u|_V$ are the semisimple and unipotent parts of $\bar{g}|_V$ in $GL(V)$.

Uniqueness. Suppose that g'_s and g'_u are respectively an $\mathbf{Ad}_{\mathbb{K}}$-semisimple element and a bounded unipotent element of G such that $g = g'_s g'_u = g'_u g'_s$. By the arguments of the preceding paragraph, it follows that every finite-dimensional $\mathbf{Ad}_{\mathbb{K}}(g)$-stable subspace V of $(\mathcal{U}_{\mathcal{D}})_{\mathbb{K}}$ is $\mathbf{Ad}_{\mathbb{K}}(g'_s)$-stable and $\mathbf{Ad}_{\mathbb{K}}(g'_u)$-stable and that, moreover, $\mathbf{Ad}_{\mathbb{K}}(g'_s)|_V$ and $\mathbf{Ad}_{\mathbb{K}}(g'_u)|_V$ are respectively the semisimple and unipotent parts of $\mathbf{Ad}_{\mathbb{K}}(g)$ in $GL(V)$. This is in particular true for $V = W$. Now it follows from the definition of g_u that $g'_u = g_u$, which establishes the claim. □

3.3 A few facts from the theory of algebraic groups

3.3.1 Some basics about Borel-Tits' theory

In this section, we collect a few basic lemmas extracted from Borel-Tits' theory of abstract homomorphisms of algebraic groups [BT73]. Since the part of this theory we need is extremely tiny compared to the monumental paper [BT73], we provide it with full proofs.

Lemma 3.9. *Let $\mathbb{K}$ be an infinite field and $\mathbf{G}$ be an algebraic group defined over $\mathbb{K}$, of positive dimension. Let X be an abstract group and $\varphi : X \to \mathbf{G}(\mathbb{K})$ be a homomorphism. Suppose that X is solvable. We have the following:*

(i) X has a finite index subgroup X^0 such that the Zariski closure of the image of $[X^0, [X^0, X]]$ under φ is a connected unipotent $\mathbb{K}$-group.

(ii) If for every finite index subgroup D of X, one has $[D, [X, X]] = [X, X]$, then the Zariski closure of the image under φ of $[X, X]$ is a connected unipotent $\mathbb{K}$-group.

Proof. The reference is [BT73, Proposition 7.1].

Given a subgroup Y of X, we denote by $\bar{\varphi}(Y)$ the Zariski closure of $\varphi(Y)$ in $\mathbf{G}$.

Since X is soluble, so is $L := \bar{\varphi}(X)$ (see [Bor91, §2.4, Corollary 2]). Moreover L is defined over $\mathbb{K}$ (see [Bor91, AG.14.4]).

Let $X^0 := \varphi^{-1}(L^\circ)$. Thus X^0 is a finite index subgroup of X and $\bar{\varphi}(X^0) = L^\circ$ is connected and defined over $\mathbb{K}$ (see [Spr98, Proposition 12.1.1]).

Next one obtains successively

$$\bar{\varphi}([X^0, X]) = [L^\circ, L] \subset L^\circ$$

(see [Spr98, Corollary 2.2.8]) and then

$$\bar{\varphi}([X^0, [X^0, X]]) \subset [L^\circ, L^\circ].$$

In view of [Spr98, Corollary 2.2.8], this proves Assertion (i).

Notice that

$$[X^0, [X^0, X]] \subset [X^0, [X, X]] \subset [X^0, X] \subset [X, X].$$

Thus (ii) is a straightforward consequence of (i). $\square$

Recall that a **diagonal automorphism** of the group $SL_2(\mathbb{K})$ is an automorphism of the form $x \mapsto dxd^{-1}$ where d is a diagonal matrix of $GL_2(\mathbb{K})$.

Lemma 3.10. *Let $\mathbb{F}$ and $\mathbb{K}$ be fields and suppose that $|\mathbb{F}| \geq 4$. If $\mathbb{F}$ is finite, suppose also* $\operatorname{char}(\mathbb{F}) = \operatorname{char}(\mathbb{K})$. *Let $\pi_{\mathbb{F}} : SL_2(\mathbb{F}) \to \Gamma_{\mathbb{F}}$ and $\pi_{\mathbb{K}} : SL_2(\mathbb{K}) \to \Gamma_{\mathbb{K}}$ be nontrivial surjective homomorphisms. Given a nontrivial group homomorphism $\varphi : \Gamma_{\mathbb{F}} \to \Gamma_{\mathbb{K}}$ there exists a field homomorphism $\zeta : \mathbb{F} \to \mathbb{K}$, an inner automorphism ι and a diagonal automorphism δ of $SL_2(\mathbb{K})$ such that the diagram:*

$$\begin{tikzcd} SL_2(\mathbb{F}) \arrow[r, "SL_2(\zeta)"] \arrow[d, "\pi_{\mathbb{F}}"'] & SL_2(\mathbb{K}) \arrow[d, "\pi_{\mathbb{K}}\circ\delta\circ\iota"] \\ \Gamma_{\mathbb{F}} \arrow[r, "\varphi"] & \Gamma_{\mathbb{K}} \end{tikzcd}$$

commutes.

Proof. The hypotheses imply that $\mathbb{K}$ has cardinality ≥ 4.

Let $U_+^{\mathbb{F}}$ (resp. $U_-^{\mathbb{F}}$, $D^{\mathbb{F}}$, $N^{\mathbb{F}}$) be the subgroup of $SL_2(\mathbb{F})$ consisting of non trivial upper triangular unipotent (resp. lower triangular unipotent, diagonal, monomial) matrices, and define $U_+^{\mathbb{K}}$, $U_-^{\mathbb{K}}$, $D^{\mathbb{K}}$ and $N^{\mathbb{K}}$ similarly.

Suppose first that $\mathbb{F}$ is finite. Then we have $\operatorname{char}(\mathbb{F}) = \operatorname{char}(\mathbb{K})$ by assumption. Let u be a nontrivial element of $U_+^{\mathbb{F}}$. Using Jordan decomposition and the fact that the order of u coincides with $\operatorname{char}(\mathbb{F}) = \operatorname{char}(\mathbb{K})$, we see that $\varphi(\pi_{\mathbb{F}}(u)) = \pi_{\mathbb{K}}(u')$ for some unipotent matrix $u' \in SL_2(\mathbb{K})$. An easy computation shows that the centralizer of a nontrivial unipotent matrix of $SL_2(\mathbb{K})$ consists of unipotent matrices (modulo the center). It follows that $\varphi(\pi_{\mathbb{F}}(U_+))$ and $\varphi(\pi_{\mathbb{F}}(U_-))$ are contained in (distinct) unipotent subgroups of $\Gamma_{\mathbb{K}}$.

If $\mathbb{F}$ is infinite, the same results hold in view of Lemma 3.9. This implies in particular that $\operatorname{char}(\mathbb{F}) = \operatorname{char}(\mathbb{K})$ in this case as well.

Thus, there exists an inner automorphism ι of $SL_2(\mathbb{K})$ such that $\varphi\circ\pi_{\mathbb{F}}(U_+^{\mathbb{F}}) \subset \pi_{\mathbb{K}}\circ\iota(U_+^{\mathbb{K}})$ and $\varphi \circ \pi_{\mathbb{F}}(U_-^{\mathbb{F}}) \subset \pi_{\mathbb{K}} \circ \iota(U_-^{\mathbb{K}})$. An easy computation in $SL_2(\mathbb{K})$ shows that a matrix which normalizes any nonempty subset of $U_+^{\mathbb{K}}\backslash\{1\}$ (resp. $U_-^{\mathbb{K}}\backslash\{1\}$) must be upper (resp. lower) triangular. Therefore, the normalizer of $\varphi \circ \pi_{\mathbb{F}}(U_+^{\mathbb{F}})$ (resp. $\varphi \circ \pi_{\mathbb{F}}(U_-^{\mathbb{F}})$) is contained in the image under $\pi_{\mathbb{K}} \circ \iota$ of the subgroup of upper triangular (resp. lower triangular) matrices of $SL_2(\mathbb{K})$. In particular, we have also $\varphi \circ \pi_{\mathbb{F}}(D^{\mathbb{F}}) \subset \pi_{\mathbb{K}} \circ \iota(D^{\mathbb{K}})$. Since $|\mathbb{F}| \geq 4$, the normalizer of $D^{\mathbb{F}}$ in $SL_2(\mathbb{F})$ is $D^{\mathbb{F}} \cup N^{\mathbb{F}}$, and we deduce by similar arguments as above that $\varphi \circ \pi_{\mathbb{F}}(N^{\mathbb{F}}) \subset \pi_{\mathbb{K}} \circ \iota(N^{\mathbb{K}})$.

Let now δ be a diagonal automorphism of $SL_2(\mathbb{K})$ such that

$$\varphi \circ \pi_{\mathbb{F}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \pi_{\mathbb{K}} \circ \delta \circ \iota \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and let $\zeta : \mathbb{F} \to \mathbb{K}$ be the unique mapping such that

$$\varphi \circ \pi_{\mathbb{F}} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \pi_{\mathbb{K}} \circ \delta \circ \iota \begin{pmatrix} 1 & \zeta(t) \\ 0 & 1 \end{pmatrix}$$

for all $t \in \mathbb{F}$. Given $t \in \mathbb{F}^\times$, the unique element x of $\mathbb{F}$ such that the product

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

belongs to $N^{\mathbb{F}}$ is $x = -t^{-1}$. We deduce that

$$\varphi \circ \pi_{\mathbb{F}} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \pi_{\mathbb{K}} \circ \delta \circ \iota \begin{pmatrix} 1 & 0 \\ \zeta(t^{-1})^{-1} & 1 \end{pmatrix} \tag{3.1}$$

for all $t \in \mathbb{F}$ and that $\varphi \circ \pi_{\mathbb{F}}(\mu^{\mathbb{F}}) = \pi_{\mathbb{K}} \circ \delta \circ \iota(\mu^{\mathbb{K}})$, where $\mu^{\mathbb{F}} = \mu^{\mathbb{K}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Using conjugation by $\mu^{\mathbb{F}}$ in the equality (3.1), we obtain $\zeta(t^{-1}) = \zeta(t)^{-1}$ for all $t \in \mathbb{F}^\times$. Since on the other hand, the mapping ζ is a homomorphism of additive groups $(\mathbb{F}, +) \to (\mathbb{K}, +)$, we may apply Hua's theorem (see for example [Coh89, §6.11, Theorem 6.11]), which insures that ζ is a field homomorphism. This completes the proof. □

3.3.2 Homomorphisms of affine Kac-Moody groups to $SL_2(\mathbb{K})$

In the following lemma, we consider homomorphisms of Chevalley groups over rings of Laurent polynomials to $SL_2(\mathbb{K})$, where $\mathbb{K}$ is a field. Such homomorphisms can be described completely, in the same vein as Lemma 3.10. However, we won't need such a general description in the sequel: only the following specific piece of information will be relevant to us.

Lemma 3.11. *Let $\mathcal{G}_0$ be a Chevalley group scheme, $\mathbb{F}$ and $\mathbb{K}$ be fields, t be an indeterminate, $G := \mathcal{G}_0(\mathbb{F}[t, t^{-1}])$ and $\varphi : G \to SL_2(\mathbb{K})$ be a homomorphism with non-abelian image. Then $\mathcal{G}_0$ is of rank one.*

Proof. Let $\bar{G} := \mathcal{G}_0(\mathbb{F}(t))$ and let $(\bar{G}, (\bar{U}_\alpha)_{\alpha \in \Phi})$ be the twin root datum of spherical type which is associated with $\bar{G}$ (see Lemma 1.4). The functoriality of $\mathcal{G}_0$ allows to identify G to a subgroup of $\bar{G}$. Modulo this identification, we set $U_\alpha := G \cap \bar{U}_\alpha$ for each $\alpha \in \Phi$ and $T := \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$.

(Although we will not need this fact, we mention in passing that $(G, (U_\alpha)_{\alpha \in \Phi})$ is *not* a twin root datum. This can be explained geometrically by the fact that G does not act strongly transitively on the spherical buildings at infinity of the affine buildings associated to G.)

For each $\alpha \in \Phi$, the group $B_\alpha = T.U_\alpha$ is solvable. Moreover, for every finite index subgroup T^0 of T, one has $[T^0, U_\alpha] = U_\alpha = [B_\alpha, B_\alpha]$ if t is algebraic over $\mathbb{F}$. If else t is transcendental over $\mathbb{F}$ then $[T^0, [T^0, U_\alpha]]$ is an infinite subgroup of U_α. In particular, Lemma 3.9 implies that $\varphi(U_\alpha)$ contains an infinite unipotent subgroup of $SL_2(\mathbb{K})$. Since U_α is abelian, an easy matrix computation shows, as in the proof of Lemma 3.10, that $\varphi(U_\alpha)$ is a unipotent subgroup of $SL_2(\mathbb{K})$ modulo the center. Let V_α denote the Zariski closure of $\varphi(U_\alpha)$ in $SL_2(\bar{\mathbb{K}})$.

Let now $\alpha, \beta \in \Phi$ be such that $\alpha \neq \pm\beta$. Then the group $\langle U_\alpha \cup U_\beta \rangle$ is nilpotent, and arguments as above show that the Zariski closure of its image in $SL_2(\bar{\mathbb{K}})$, say $V_{\alpha\beta}$, is a unipotent subgroup modulo the center. But $SL_2(\bar{\mathbb{K}})$ has only one conjugacy class of nontrivial unipotent closed subgroups and any two such subgroups generate the whole of $SL_2(\bar{\mathbb{K}})$. This shows that if $\alpha \neq \pm\beta$ then $V_\alpha = V_{\alpha\beta} = V_\beta$ modulo the center. In particular, if the cardinality of Φ is greater than 2, then, for all $\alpha, \beta \in \Phi$, one has $V_\alpha = V_\beta$ modulo the center.

Since G is generated by the U_α's, we finally deduce that the image of φ is abelian. This contradicts our hypotheses. □

3.3.3 Commuting sets of semisimple elements

The following result is a variation on a classical theme.

Proposition 3.12. *Let $\mathbb{K}$ be a field and $\mathbf{G}$ be a reductive algebraic $\mathbb{K}$-group which is $\mathbb{K}$-split. Let H be a commutative subgroup of $\mathbf{G}(\mathbb{K})$ and suppose that each element of H is contained in a $\mathbb{K}$-split torus. Then $\mathbf{G}$ contains a maximal $\mathbb{K}$-split torus $\mathbf{T}$ normalized by H. In particular $\mathbf{T}(\mathbb{K})$ contains a finite index subgroup of H. Moreover, if H is finite of order prime to the order of the Weyl group of $\mathbf{G}$, then H is contained in $\mathbf{T}(\mathbb{K})$.*

Proof. A slightly different version of this result appears in [SS70, §II.5] (see especially Corollaries II.5.17 and II.5.19).

The version stated above can be obtained as follows.

We work by induction on the dimension n of $\mathbf{G}$. For $n = 1$, the identity component of $\mathbf{G}$ is a maximal $\mathbb{K}$-split torus and the result is clear. Suppose now $n > 1$. If H is contained in the center of the identity component of $\mathbf{G}$, then H is central in $\mathbf{G}(\mathbb{K})$ and the result is equally clear. Assume thus that H is not and choose an element $t \in H$ which is non-central in $\mathbf{G}^0$. Let $\mathbf{G}^t$ be the centralizer of t in $\mathbf{G}$. Then $\mathbf{G}^t$ is reductive (because $\mathbf{G}$ is), defined over $\mathbb{K}$ (because t is semisimple, see [Spr98, Corollary 12.1.4(i)]) and $\mathbb{K}$-split (because any maximal $\mathbb{K}$-split torus of $\mathbf{G}$ containing t is contained in $\mathbf{G}^t$). Moreover $(\mathbf{G}^t)^0$ is a proper subgroup of $\mathbf{G}^0$ by the definition of t; therefore, one has $\dim \mathbf{G}^t < n$. Clearly H is contained in $\mathbf{G}^t(\mathbb{K})$. By induction, H normalizes a maximal $\mathbb{K}$-split torus $\mathbf{T}$ of $\mathbf{G}^t$, and it follows from the definition of $\mathbf{G}^t$ that $\mathbf{T}$ is also a maximal torus of $\mathbf{G}$. The result follows, since $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ is finite. □

Chapter 4

Isomorphisms of Kac-Moody groups: an overview

The proof of the isomorphism theorem for Kac-Moody groups, which is stated below, requires rather different arguments according as the characteristic of the ground field is positive or not. However, the proofs in both cases follow a similar global strategy, and the purpose of the present chapter is to collect all the technical preparations that these two proofs have in common. An important idea, which was actually at the basis of [CM05a], is that the structure of a Kac-Moody group is roughly 'controlled' by its rank one Levi subgroups. One is thus interested in obtaining an abstract characterization of these Levi subgroups; such a characterization is provided by Proposition 4.17 below and rests heavily on Tits' rigidity theorem for actions on trees. However, this proposition requires in turn to get a sharp control on diagonalizable subgroups of Kac-Moody groups, which is a rather delicate problem. A whole section of this chapter is devoted to diagonalizable subgroups; another one is concerned with a slightly larger class of subgroups called completely reducible. The chapter ends with the main technical auxiliary to the proof of the isomorphism theorem.

4.1 The isomorphism theorem

4.1.1 The statement

The setting is the following:

- $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ is a Kac-Moody root datum,
- $\mathcal{F} = (\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$ is the basis of a Tits functor $\mathcal{G}$ of type $\mathcal{D}$,
- $\mathbb{K}$ is a field,
- $G := \mathcal{G}(\mathbb{K})$ is the corresponding Kac-Moody group.

Let also $\mathcal{F}' = (\mathcal{G}', (\varphi'_i)_{i\in I'}, \eta')$ be the basis of a Tits functor $\mathcal{G}'$ of type $\mathcal{D}' = (I', A', \Lambda', (c'_i)_{i\in I'}, (h'_i)_{i\in I'})$, let $\mathbb{K}'$ be a field and set $G' := \mathcal{G}'(\mathbb{K}')$.

Theorem 4.1. *Let $\varphi : G \to G'$ be an isomorphism. Suppose that G is infinite and $|\mathbb{K}| \geq 4$. Then there exist an inner automorphism α of G', a bijection $\pi : I \to I'$ and, for each $i \in I$, a field isomorphism $\zeta_i : \mathbb{K} \to \mathbb{K}'$, a diagonal automorphism δ_i of $SL_2(\mathbb{K}')$ and* *an automorphism ι_i of $SL_2(\mathbb{K}')$ which is either trivial or transpose-inverse, such that the diagram*

$$\begin{tikzcd} SL_2(\mathbb{K}) \arrow[r, "SL_2(\zeta_i)"] \arrow[d, "\varphi_i"'] & SL_2(\mathbb{K}') \arrow[d, "\varphi'_{\pi(i)}\circ\delta_i\circ\iota_i"] \\ G \arrow[r, "\alpha\circ\varphi"] & G' \end{tikzcd}$$

commutes for every $i \in I$. In particular, the bijection π extends to an isomorphism, also noted π, of the Weyl groups of G and G' such that (φ, π) is an isomorphism of the twin root data associated with G and G'.

The dependence of ζ_i, δ_i and ι_i in i in the preceding statement is of technical nature. By adding an irreducibility assumption on G, one obtains the following more homogeneous reformulation of the isomorphism theorem.

Theorem 4.2. *Suppose additionally that $\mathcal{G}$ is of irreducible type (i.e. A is indecomposable). Then there exist a bijection $\pi : I \to I'$, an inner automorphism α, a diagonal automorphism δ and a sign automorphisms σ of G' and for each $i \in I$, a field isomorphism $\zeta_i : \mathbb{K} \to \mathbb{K}'$, such that the diagram*

$$\begin{tikzcd} SL_2(\mathbb{K}) \arrow[r, "SL_2(\zeta_i)"] \arrow[d, "\varphi_i"'] & SL_2(\mathbb{K}') \arrow[d, "\varphi_{\pi(i)}"] \\ G \arrow[r, "\sigma\circ\delta\circ\alpha\circ\varphi"] & G' \end{tikzcd}$$

commutes for every $i \in I$. Furthermore, if $\mathbb{K}$ is infinite then $A_{ij}A_{ji} = A'_{\pi(i)\pi(j)}A'_{\pi(j)\pi(i)}$ for all $i, j \in I$. Moreover, if $\mathrm{char}(\mathbb{K}) = 0$ or if $\mathrm{char}(\mathbb{K}) = p > 0$, $\mathbb{K}$ infinite and A_{ij} prime to p for all $i \neq j \in I$, then $A_{ij} = A'_{\pi(i)\pi(j)}$ and $\zeta_i = \zeta_j$ for all $i, j \in I$.

Diagonal automorphisms of Kac-Moody groups are straightforward generalizations of diagonal automorphisms of Chevalley groups. More precisely, given a Tits functor $\mathcal{G}$ with basis $\mathcal{F} = (\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$, a field $\mathbb{K}$ and a diagonal automorphism δ_i of $SL_2(\mathbb{K})$ for each $i \in I$, then there exists an automorphism δ of $\mathcal{G}(\mathbb{K})$ such that $\delta \circ \varphi_i = \varphi_i \circ \delta_i$ for each $i \in I$ (see [CM05a, §8.2]; actually, Lemma 58 of [Ste68] is valid for Kac-Moody groups of all types). Automorphisms of this form are called **diagonal**. Similarly, the group $\mathcal{G}(\mathbb{K})$ possesses an automorphism σ whose restriction to $\varphi_i(\mathrm{SL}_2(\mathbb{K}))$ is a transpose-inverse for each $i \in I$; note that σ is involutory and swaps the two standard Borel subgroups of opposite signs of $\mathcal{G}(\mathbb{K})$. This automorphism, together with the identity, are the **sign** automorphisms of $\mathcal{G}(\mathbb{K})$.

The proof of Theorem 4.1 occupies the major part of the present chapter and the following two ones. Assuming that Theorem 4.1 holds, Theorem 4.2 can be obtained as follows.

Proof of Theorem 4.2. The existence of an inner automorphism α, a diagonal automorphism δ and a sign automorphism σ of $\mathcal{G}'(\mathbb{K}')$ such that the diagram above commutes directly follows from Theorem 4.1 and the assumption on G.

In order to compare the different field isomorphisms ζ_i provided by Theorem 4.1, one proceeds as follows. Given $i, j \in I$, one transforms by $\varphi' := \sigma\circ\delta\circ\iota\circ\varphi$ the defining relation of $\mathcal{G}(\mathbb{K})$ which prescribes how the diagonal elements of $\varphi_i(SL_2(\mathbb{K}))$ act on $\varphi_j(SL_2(\mathbb{K}))$, and similarly with i and j interchanged (see [Tit87b, §3.6, Relation (4)]). One obtains

$$\zeta_i(x)^{A_{\pi(i)\pi(j)}} = \zeta_j(x)^{A_{ij}} \qquad \text{and} \qquad \zeta_i(x)^{A_{ji}} = \zeta_j(x)^{A_{\pi(j)\pi(i)}} \tag{4.1}$$

for all $x \in \mathbb{K}^\times$. For $f := \zeta_j^{-1}\zeta_i$, this can be rewritten

$$f(x)^{A_{\pi(i)\pi(j)}} = x^{A_{ij}} \qquad \text{and} \qquad f(x)^{A_{ji}} = x^{A_{\pi(j)\pi(i)}}. \tag{4.2}$$

By definition f is an automorphism of $\mathbb{K}$. By computing $f(x^{A_{\pi(i)\pi(j)}A_{ji}})$ in two different ways using (4.2), one deduces that

$$x^{A_{ij}A_{ji}} = x^{A_{\pi(i)\pi(j)}A_{\pi(j)\pi(i)}}$$

for all $x \in \mathbb{K}^\times$. If $\mathbb{K}$ is infinite, one obtains $A_{ij}A_{ji} = A_{\pi(i)\pi(j)}A_{\pi(j)\pi(i)}$ for all $i, j \in I$.

Similarly, by computing $f((x^{A_{ji}} + 1)^{A_{ji}})$ using (4.2), one obtains

$$\sum_{k=0}^{A_{\pi(j)\pi(i)}} \binom{A_{\pi(j)\pi(i)}}{k} x^{k.A_{ji}} = \sum_{m=0}^{A_{ji}} \binom{A_{ji}}{m} x^{m.A_{\pi(j)\pi(i)}} \tag{4.3}$$

for all $x \in \mathbb{K}^\times$. In particular, if $\mathbb{K}$ is infinite and A_{ji} or $A_{\pi(j)\pi(i)}$ is prime to char($\mathbb{K}$), then $A_{ji} = A_{\pi(j)\pi(i)}$.

An easy argument shows that the only automorphism of an infinite field which acts trivially on all n^{th} powers for some positive integer n is the identity. Therefore, if $A_{ji} = A_{\pi(j)\pi(i)}$ then (4.2) implies that $f = \text{id}$ whence $\zeta_i = \zeta_j$. The desired result follows easily. □

4.1.2 Coxeter diagrams vs. generalized Cartan matrices

If follows from Theorem 4.1 that the isomorphism $\varphi : G \to G'$ induces an isomorphism of the respective Weyl groups of G and G' which preserves the canonical generators. However, it is not immediate that φ induces an isomorphism of the generalized Cartan matrices of G and G'; the equality $A_{ij} = A_{\pi(i)\pi(j)}$ was obtained in Theorem 4.2 under some additional assumptions on the characteristic of the ground fields. This kind of condition is not surprising: It is well known that Chevalley groups of type B_n and C_n over perfect fields of characteristic 2 are abstractly isomorphic. Similar phenomena occur for all types of Kac-Moody groups. We do not want to go into details on this topic; relevant related results may be found in [Hée90] and [Cho00]. In the present section, we merely illustrate by an example that over finite fields, the abstract structure of Kac-Moody groups might contain only very poor information on their defining generalized Cartan matrices.

Given integers $m, n \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{D}_m^n$ the simply connected Kac-Moody root datum associated with the generalized Cartan matrix $\begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}$ over $I = \{1, 2\}$. Let $\mathcal{F}_m^n = (\mathcal{G}_m^n, ((\varphi_m^n)_i)_{i=1,2}, \eta)$ be the basis of a Tits functor $\mathcal{G}_m^n$ of type $\mathcal{D}_m^n$. We have the following.

Lemma 4.3. *Let $\mathbb{K}$ be a finite field of cardinality at least 3. Given integers m, m', n, n' which are all multiples of $|\mathbb{K}^\times|$, then the Kac-Moody groups $\mathcal{G}_m^n(\mathbb{K})$ and $\mathcal{G}_{m'}^{n'}(\mathbb{K})$ are isomorphic. Moreover, given any two automorphisms ζ_1, ζ_2 of $\mathbb{K}$, then there exists an automorphism φ of $G := \mathcal{G}_m^n(\mathbb{K})$ such that the diagram*

$$\begin{array}{ccc} SL_2(\mathbb{K}) & \xrightarrow{SL_2(\zeta_i)} & SL_2(\mathbb{K}) \\ {\scriptstyle (\varphi_m^n)_i}\downarrow & & \downarrow{\scriptstyle (\varphi_m^n)_i} \\ G & \xrightarrow{\varphi} & G \end{array}$$

commutes for $i = 1, 2$.

Proof. Since $m, n \geq 2$, it follows that all commutation relations between root groups corresponding to prenilpotent pairs of roots are trivial in $\mathcal{G}_m^n(\mathbb{K})$; this is true for all fields, see [Mor88, §3.6]. Since $x^m = x^n = 1$ for all $x \in \mathbb{K}$, it follows that the diagonal elements of $(\varphi_m^n)_1(SL_2(\mathbb{K}))$ centralize $(\varphi_m^n)_2(SL_2(\mathbb{K}))$ and similarly with 1 and 2 interchanged. Therefore, it follows that the groups $\mathcal{G}_m^n(\mathbb{K})$ and $\mathcal{G}_{m'}^{n'}(\mathbb{K})$ have the same Steinberg type presentation (see [Tit87b, §3.6]). The other assertion of the lemma follows for the same reasons. $\square$

This lemma could be easily generalized, e.g. for Kac-Moody groups with a right-angled Weyl group. This shows that in general, *over finite fields*, the field isomorphisms $(\zeta_i)_{i \in I}$ provided by Theorems 4.1 and 4.2 are independent of one another.

4.2 Diagonalizable subgroups and their centralizers

4.2.1 Commuting sets of diagonalizable subgroups

The following result is a version of Proposition 3.12 in the Kac-Moody context.

Proposition 4.4. *Let $\mathcal{G}$ be a Tits functor, $\mathbb{K}$ be a field and H be a commutative subgroup of $G := \mathcal{G}(\mathbb{K})$. Suppose that H is generated by finitely many subgroups $H_1, \dots, H_n$, each of which is diagonalizable in G. Then H normalizes a maximal diagonalizable subgroup of G. Moreover, if each H_i has finite order prime to the order of every finite subgroup of the Weyl group of $\mathcal{G}(\mathbb{K})$, then H itself is diagonalizable.*

Proof. Each H_i is bounded and, hence, so is H by Corollary 2.5. Let $\mathcal{B}$ the twin building associated to $\mathcal{G}(\mathbb{K})$ and ρ_+, ρ_- be spherical residues of opposite signs which are stabilized by H.

If ρ_+ and ρ_- are opposite, then the result is a direct consequence of Proposition 3.12, which we may apply in view of Proposition 3.6.

Suppose now that ρ_+ and ρ_- are not opposite. Then there exists a spherical residue σ containing ρ_- properly and such that $\mathrm{proj}_\sigma(\rho_+) = \rho_-$ (this follows from [Abr96, Proposition 4] together with [CM05b, Proposition 2.7]). By Proposition 3.2, the double stabilizer $\mathrm{Stab}_G(\rho_+) \cap \mathrm{Stab}_G(\sigma)$ has a Levi decomposition of the form $L \ltimes U$, and the action of H on σ coincides with the action of its image under the canonical projection $L \ltimes U \to L$. Therefore, in view of Proposition 3.6, it follows again from Proposition 3.12 that H stabilizes an apartment of σ containing ρ_-. In particular, it stabilizes a residue ρ'_- of σ such that ρ_- and ρ'_- are opposite in σ. By the definition of σ, the numerical codistance between ρ_+ and ρ_- is strictly greater than the numerical codistance between ρ_+ and ρ'_-. Now, a straightforward induction shows that H stabilizes a residue of $\mathcal{B}$ opposite ρ_+, and the proposition has already been proven in this case. $\square$

4.2.2 Fixed points of diagonalizable subgroups

Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha \in \Phi})$ be a twin root datum of type (W, S). Recall from §1.4.2 that a subgroup of G is called **diagonalizable** if it is contained in the intersection of two opposite Borel subgroups or, equivalently, if it fixes a pair of opposite chambers in the twin building associated with G. In particular, a diagonalizable subgroup is bounded.

The following lemma, well known to the experts, is a basic exercise in the theory of twin buildings. By lack of an appropriate reference, we include a proof.

Lemma 4.5. *Let H be a diagonalizable subgroup of G and $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be the twin building associated with $\mathcal{Z}$. Let $\mathcal{B}_+^H$ (resp. $\mathcal{B}_-^H$) be the set of chambers of $\mathcal{B}_+$ (resp. $\mathcal{B}_-$) fixed by H. Then $\mathcal{B}^H := (\mathcal{B}_+^H, \mathcal{B}_-^H)$ has a canonical structure of twin building of type (W, S); two chambers of $\mathcal{B}^H$ are opposite if and only if they are opposite as chambers of $\mathcal{B}$.*

Proof. It suffices to prove the following claim: If two chambers x, y are fixed by H, then there exists a twin apartment all of whose chambers are fixed by H.

By hypothesis, H fixes some twin apartment $\mathcal{A}$ of $\mathcal{B}$. Let $f(x)$ (resp. $f(y)$) be the numerical distance between x (resp. y) and $\mathcal{A}$. The proof of the claim above is by induction on $f(x) + f(y)$. The result is clear for $f(x) + f(y) = 0$. Now assume that $f(x) + f(y) > 0$. Without loss of generality we have $f(y) > 0$. Considering a gallery of minimal possible length joining y to a chamber of $\mathcal{A}$, we obtain a chamber y' adjacent to y such that $f(x)+f(y') < f(x)+f(y)$. By induction, there exists a twin apartment $\mathcal{A}'$ all of whose chambers are fixed by H, which contains both x and y'. Thus y is at numerical distance at most 1 of $\mathcal{A}'$. Let π be a panel intersecting $\mathcal{A}'$ and containing y, ϕ be the unique twin root of $\mathcal{A}'$ which contains x and such that $\pi \in \partial\phi$ and π' be the unique panel opposite π which meets $\mathcal{A}'$. Since H fixes $\mathcal{A}$, it stabilizes π' and hence fixes $\mathrm{proj}_{\pi'}(y)$. Moreover $\mathrm{proj}_{\pi'}(y)$ consists of a unique chamber y' which is at numerical codistance 1 from y. Hence y and y' are contained in a unique twin root ϕ' of $\mathcal{B}$, all of whose chambers must therefore be fixed by H. Now $\mathcal{A}'' := \phi \cup \phi'$ is a twin apartment of $\mathcal{B}$, all of whose chambers are fixed by H. The claim is proven and the lemma follows. □

4.2.3 Centralizers of diagonalizable subgroups

Although the twin building $\mathcal{B}^H$ of Lemma 4.5 is not thick in general (it might be reduced to a twin apartment), there is a canonical way of attaching a thick twin building to it (see [Cap05b]). This plays a crucial role in proving that centralizers of diagonalizable subgroups of Kac-Moody groups are endowed with twin root data. This is the purpose of the present section.

Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum and $\mathcal{B}$ be the associated twin building. A diagonalizable subgroup of G is called **regular** (with respect to $\mathcal{Z}$) if $\mathcal{B}^H$ is reduced to a single twin apartment. Thus, roughly speaking a diagonalizable subgroup is regular if its fixed point set is as small as possible. We will come back to this notion a little further. At this point, we just record the following.

Proposition 4.6. *Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ be the twin root datum associated to a Kac-Moody group G over the field $\mathbb{K}$. Let $T := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$. Given a subgroup H of T, we set*

$$\Phi^H := \{\alpha \in \Phi |\ [U_\alpha, H] = 1\} \qquad \text{and} \qquad G^H := T.\langle U_\alpha |\ \alpha \in \Phi^H\rangle.$$

Then we have the following:

(i) Φ^H is empty if and only if H is regular.

(ii) If Φ^H is nonempty, then $\mathcal{Z}^H := (G^H, (U_\alpha)_{\alpha\in\Phi^H})$ is a twin root datum which is locally $\mathbb{K}$-split, and whose Weyl group W^H is generated by $\{s_\alpha |\ \alpha \in \Phi^H\}$. Furthermore T is a maximal diagonalizable subgroup of G^H.

(iii) Given a subgroup $H' \leq T$, then H' is regular with respect to $\mathcal{Z}^H$ if and only if $\langle H \cup H'\rangle$ is regular with respect to $\mathcal{Z}$.

(iv) A subgroup of G^H is bounded with respect to $\mathcal{Z}^H$ if and only if it is bounded with respect to $\mathcal{Z}$.

(v) $G^H = C_G(H)$.

Proof. Assertion (i) follows from Lemma 1.4(v).

One knows from Lemma 4.5 that the set $\mathcal{B}^H$ of all fixed points of H constitutes a sub-building of $\mathcal{B}$. This building is not necessarily thick, but the main result of [Cap05b] describes a canonical equivalence relation on the chambers of $\mathcal{B}^H$ such that the corresponding quotient is a thick twin building, say $\bar{\mathcal{B}}^H$. Using the action of G^H on $\mathcal{B}^H$ and on $\bar{\mathcal{B}}^H$ and the correspondence between the structures of $\mathcal{B}^H$ and $\bar{\mathcal{B}}^H$ described in [Cap05b], one shows that the building $\bar{\mathcal{B}}^H$ is Moufang and that the system $(U_\alpha)_{\alpha\in\Phi^H}$ is precisely the set of root groups of some twin apartment of $\bar{\mathcal{B}}^H$. This implies that $\mathcal{Z}^H$ is a twin root datum whose associated twin building is isomorphic to $\bar{\mathcal{B}}^H$ (see §1.4.2). Clearly $\mathcal{Z}^H$ satisfies (LS2).

Let $\mathcal{A}$ be the standard twin apartment of $\mathcal{B}$. Recall that $T = \mathrm{Fix}_G(\mathcal{A})$. Since $T \leq G^H$, we have also $T = \mathrm{Fix}_{G^H}(\mathcal{A})$ from which is follows that T is maximal diagonalizable with respect to $\mathcal{Z}^H$. In particular $\mathcal{Z}^H$ verifies (LS1) and is thus locally split over $\mathbb{K}$. This proves (ii).

Assertion (iii) follows from the way the building $\bar{\mathcal{B}}^H$ is related to $\mathcal{B}^H$ (see [Cap05b]).

Every spherical residue of $\mathcal{B}^H$ projects to a spherical residue of $\bar{\mathcal{B}}^H$. Moreover, the building $\mathcal{B}^H$ is embedded in $\mathcal{B}$ as a combinatorially convex subset. This, together with Lemma 2.2, implies that a bounded subgroup of G which is contained in G^H is also bounded with respect to $\mathcal{Z}^H$. This shows the 'if' part of (iv).

A Borel subgroup of G^H fixes a chamber of $\bar{\mathcal{B}}^H$ and hence a chamber of $\mathcal{B}^H$. This shows that a Borel subgroup of G^H is contained in a Borel subgroup of G. In view of (ii) and of Proposition 3.1, this implies that a bounded unipotent subgroup of G^H is contained in a bounded unipotent subgroup of G. On the other hand, since every finite root subsystem of Φ^H is also a finite root subsystem of Φ, we deduce that finite type Levi factors of G^H are contained in finite type Levi factors of G. Now Proposition 3.2, together with Corollary 2.5, implies that a bounded subgroup of G^H is also bounded with respect to $\mathcal{Z}$. This shows the 'only if' part of (iv).

For (v), notice first that by definition, G^H centralizes H. Let $g \in C_G(H)$. Then g acts on $\mathcal{B}^H$ and on $\bar{\mathcal{B}}^H$. By (ii), G^H acts transitively on pairs of opposite chambers of $\bar{\mathcal{B}}^H$. Hence there exists $g' \in G^H$ such that $g'g$ fixes a pair of opposite chambers of $\bar{\mathcal{B}}^H$. This implies that $g'g$ fixes a pair of opposite chambers of $\mathcal{B}^H$ (see [Cap05b]). Thus, by Lemma 4.5, we may assume that $g'g$ fixes the standard twin apartment $\mathcal{A}$. Since $\mathrm{Fix}_G(\mathcal{A}) = T$, we obtain $g'g \in T$ and hence $g \in (g')^{-1}T \subset G^H$. □

Proposition 4.6(ii) can be interpreted as a combinatorial version of the fact that the centralizer of a diagonalizable subgroup of a reductive algebraic group is itself reductive.

4.2.4 Regular diagonalizable subgroups

Recall that a diagonalizable subgroup of a group G endowed with a twin root datum is called **regular** if its fixed point set in the twin building associated with G is thin. The term *regular* is inspired by [Bor91, §13.1]. If G is a Kac-Moody group over an infinite ground field, then one can show that a diagonalizable subgroup H of G is regular if and only if the following condition holds: Given a finite type parabolic subgroup P of G

containing H, then H is contained in finitely many Borel subgroups of P. Equivalently, a diagonalizable subgroup of G is regular if and only if it is contained in a unique maximal diagonalizable subgroup of G. This justifies the choice of terminology

Lemma 4.7. *Let $\mathcal{G}$ be a Tits functor, $\mathbb{K}$ be a field and T be a maximal diagonalizable subgroup of $\mathcal{G}(\mathbb{K})$. One has the following:*

(i) If $\mathbb{K} = \mathbb{F}_2$ then $T = \{1\}$.

(ii) If $|\mathbb{K}| \geq 4$ then T is regular and $N_{\mathcal{G}(\mathbb{K})}(T) = N$, where N is as in Lemma 1.2.

Proof. If $\mathbb{K} = \mathbb{F}_2$ then $\mathbb{K}^\times$ is trivial, whence (i). Assertion (ii) follows from [Rém02b, Lemma 8.4.1 and Proposition 10.1.3]; it can also be viewed as a consequence of Lemma 4.8 below. □

For $\mathbb{K} = \mathbb{F}_3$ the regularity of H depends on the type of $\mathcal{G}$. The absence of regular diagonalizable subgroups over the smallest fields is responsible for the hypothesis $|\mathbb{K}| \geq 4$ in the isomorphism theorem. On the other hand, an important fact which was used in [CM05a] and will be exploited again in the sequel is the existence of many non-maximal diagonalizable subgroups which are regular, which follows from the following result.

Lemma 4.8. *Suppose that $\mathcal{Z}$ is locally split over fields $(\mathbb{K}_\alpha)_{\alpha\in\Phi}$ and let $T := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$. Given a subgroup H of T, if one of the following conditions hold, then H is regular:*

(i) For each $\alpha \in \Phi$, the intersection $H \cap X_\alpha$ has at least 3 elements, where $X_\alpha := \langle U_\alpha \cup U_{-\alpha}\rangle$.

(ii) H is invariant under the Weyl group action on T, and there exists a basis Π of Φ such that for each $\alpha \in \Pi$, the intersection $H \cap X_\alpha$ has at least 3 elements.

Proof. Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be the twin building associated with G, and let $\mathcal{A}$ be the standard twin apartment. Note that H fixes every chamber of $\mathcal{A}$ because $H < T$. If x is any chamber of $\mathcal{B}$ fixed by H, then H fixes all minimal galleries joining x to a chamber of $\mathcal{A}$. This shows that H is regular if and only if for every panel π intersecting $\mathcal{A}$, the only chambers of π fixed by H are the two chambers contained in $\mathcal{A}$. If H is invariant under the Weyl group, which is the stabilizer of $\mathcal{A}$ in G, this in turn is equivalent to the fact that for a given chamber c of $\mathcal{A}$, the fixed chambers of H in any panel π containing c are the two chambers of π contained in $\mathcal{A}$.

On the other hand, given $\alpha \in \Phi$, any subgroup of $T \cap X_\alpha$ having at least 3 elements fixes exactly two chambers in every panel of $\mathcal{A}$ belonging to $\partial\alpha$. Therefore, if (i) or (ii) holds then H is regular. □

4.2.5 Coregular diagonalizable subgroups

Regular diagonalizable subgroups are diagonalizable subgroups whose centralizers are 'minimal'. In the finite-dimensional case, i.e. for reductive groups, this can be formalized by means of the well defined notion of dimension. One would now like to define a class of diagonalizable subgroups whose centralizers are not minimal but 'cominimal' in a sense to be defined. In a reductive group, these diagonalizable subgroups would be those whose centralizer is of semisimple rank 1. This leads us to make the following definition.

Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum. A diagonalizable subgroup H of G is called **coregular** if it is not regular and if the Weyl group of the twin root datum $\mathcal{Z}^H$

(see Proposition 4.6) is a free product of groups of order 2. (A free product of groups of order 2 is a Coxeter group which is sometimes called **universal**.)

In other words, a diagonalizable subgroup is coregular if its centralizer, which is endowed with a twin root datum, acts on a 1-dimensional building. In the finite-dimensional case, this is equivalent to say that the centralizer is of semisimple rank 1, as desired.

The notion of coregular diagonalizable subgroups plays a crucial role in the sequel (see notably Propositions 4.17 and 4.19). The following technical lemma provides useful sufficient conditions for a diagonalizable subgroup of a Kac-Moody group to be coregular.

Lemma 4.9. *Let $\mathcal{G}$ be a Tits functor with basis $\mathcal{F} = (\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$, let $\mathbb{K}$ be a field and $\mathcal{Z} = (\mathcal{G}(\mathbb{K}), (U_\alpha)_{\alpha\in\Phi})$ be the standard twin root datum associated with G. Let X be a subgroup of $\mathbb{K}^\times$ of order at least 10 and let $H := \langle \varphi_i(\mathrm{diag}(x, x^{-1}))|\ i \in I, x \in X\rangle$. Then we have the following:*

(i) H is invariant under the action of Weyl group of $G := \mathcal{G}(\mathbb{K})$ on $T := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$.

(ii) H is regular.

(iii) For each $\alpha \in \Phi$, the group $H^\vee_\alpha := C_H(X_\alpha)$ is coregular, where $X_\alpha := \langle U_\alpha \cup U_{-\alpha}\rangle$.

Proof. For each $\alpha \in \Phi$, let $T_\alpha := T \cap X_\alpha$. By definition, one has $H < T$, hence H is diagonalizable. Furthermore, one knows from [Tit87b] that $\mathcal{G}(\mathbb{K})$ has a Steinberg-type presentation and, using the defining relations of $\mathcal{G}(\mathbb{K})$ in such a presentation, one verifies that H is invariant under the action of the Weyl group on T. In particular, it follows from Lemma 4.8 that H is regular. Thus (i) and (ii) hold.

We also recall from [Tit87b] that, given $\alpha \in \Phi$, there exists a group isomorphism $u_\alpha : \mathbb{K} \to U_\alpha$ such that, for each $i \in I$, there exists $n_i \in \mathbb{Z}$ for which one has

$$\varphi_i(\mathrm{diag}(x, x^{-1}))u_\alpha(y)\varphi_i(\mathrm{diag}(x, x^{-1}))^{-1} = u_\alpha(x^{n_i} \cdot y) \tag{$*$}$$

for all $x, y \in \mathbb{K}^\times$.

Let now $\alpha \in \Phi$. Since $H^\vee_\alpha$ centralizes X_α, it cannot be regular.

Suppose by contradiction that $H^\vee_\alpha$ is not coregular. Then, by definition, there exist roots $\phi, \psi \in \Phi$ such that $\phi \neq \pm\psi$, the product $s_\phi s_\psi$ is of finite order and $X_{\phi,\psi} := \langle X_\phi \cup X_\psi\rangle$ centralizes $H^\vee_\alpha$. Since H is normalized by the Weyl group, we may and shall assume that there exist $i(\phi), i(\psi) \in I$ such that $i(\phi) \neq i(\psi)$ and

$$\langle \varphi_{i(\phi)}(\mathrm{diag}(x, x^{-1})), \varphi_{i(\psi)}(\mathrm{diag}(y, y^{-1}))|\ x, y \in \mathbb{K}^\times\rangle = \langle T_\phi \cup T_\psi\rangle.$$

In particular, one has

$$\langle \varphi_{i(\phi)}(\mathrm{diag}(x, x^{-1})), \varphi_{i(\psi)}(\mathrm{diag}(y, y^{-1}))|\ x, y \in X\rangle = H \cap X_{\phi,\psi}.$$

On the other hand, it follows from $(*)$ that for each $x \in X$, the element

$$h(x) := \varphi_{i(\phi)}(\mathrm{diag}(x, x^{-1}))^{n_{i(\psi)}}.\varphi_{i(\psi)}(\mathrm{diag}(x, x^{-1}))^{-n_{i(\phi)}}$$

centralizes U_α. By Lemma 1.1(ii), such an element centralizes X_α and is thus contained in $H \cap X_{\phi,\psi}$.

Since $i(\phi) \neq i(\psi)$, it follows that if $x \in X$ is of infinite order, then $h(x) \in H^\vee_\alpha \cap X_{\phi,\psi}$ is of infinite order. Moreover, by the definition of ϕ and ψ, the intersection $H^\vee_\alpha \cap X_{\phi,\psi}$ is central in $X_{\phi,\psi}$. Since the center of $X_{\phi,\psi}$ is finite order ≤ 4, this yields a contradiction in the case where X contains an element of infinite order.

Suppose now that all elements of X are of finite order. Then X possesses an element of order $n \geq 10$. It follows from the definitions that H contains all n-torsion elements of $T \cap G^\dagger$, where $G^\dagger := \langle U_\alpha | \ \alpha \in \Phi \rangle$. In particular, since $n \geq 10$ and $X_{\phi,\psi}$ is a Chevalley group of rank 2 over $\mathbb{K}$, the group $H \cap X_{\phi,\psi}$ contains a subgroup Y, direct product of two cyclic subgroups of order n (or $n/2$). On the other hand, $(*)$ implies that any finite subgroup of T of given exponent m acts on U_α as a cyclic group of order at most m. Therefore, the action of Y on U_α by conjugation has a kernel of order at least $n/2$. This kernel is contained in $H_\alpha^\vee \cap X_{\phi,\psi}$ (see Lemma 1.1(ii)), whose order is bounded from above by 4, a contradiction. □

4.3 Completely reducible subgroups and their centralizers

The notion of diagonalizable subgroups of Kac-Moody groups is of central importance in this work; it is an abstract counterpart of the notion of split tori in the theory of algebraic groups. However, it should be mentioned that it is often a difficult problem to show that a given subgroup of a Kac-Moody group is diagonalizable. This is the reason why we will be led to consider a slightly more general class of subgroups, called completely reducible subgroups, which we now introduce. They should be viewed as analogues of (possibly) non-split tori in algebraic groups.

4.3.1 Definition

Let G be a group endowed with a twin BN-pair and $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be the associated twin building. A subgroup H of G is called **completely reducible** if the following conditions hold:

(CR1) H is bounded;

(CR2) Given a finite type parabolic subgroup P containing H, there exists a finite type parabolic subgroup opposite P which contains H as well.

This condition is equivalent to the following one:

(CR) H stabilizes a pair of opposite spherical residues of rank r in $\mathcal{B}$, but no residue of rank $< r$.

Note that diagonalizable subgroups of G are completely reducible.

Automorphism groups of twin buildings satisfying (CR) were considered by B. Mühlherr [Müh94, p. 80], where the above condition is named Condition (O). The terminology of *complete reducibility* for group actions on spherical buildings is due to J. P. Serre (see [Ser04] and references therein). Note that Condition (CR) is more general than Conditions (CR1)-(CR2), in the sense that (CR) does not refer to the existence of a larger group which is strongly transitive; in particular a completely reducible group action in the sense of (CR) need not be type-preserving. A typical highly interesting case is provided by Galois group actions; this case was actually an important motivation for this definition (see [Müh94, §3.1]).

For our purposes, all completely reducible groups we will consider will arise via the following lemma.

Lemma 4.10. *Let $\mathcal{G}$ be a Tits functor and $\mathbb{K}$ be a field. Let H be a subgroup of $G := \mathcal{G}(\mathbb{K})$. If H is $\mathbf{Ad}_{\mathbb{F}}$-diagonalizable for some field extension $\mathbb{F}/\mathbb{K}$, then it is completely reducible.*

Proof. Let $\mathbb{F}/\mathbb{K}$ be a field extension such that H is $\mathbf{Ad}_{\mathbb{F}}$-diagonalizable. Thus H is a bounded subgroup of $\mathcal{G}(\mathbb{F})$. Since the twin building of $\mathcal{G}(\mathbb{K})$ embeds as a closed and convex sub-building of the twin building associated with $\mathcal{G}(\mathbb{F})$ (this is essentially a consequence of (KMG4)), it follows from Lemma 2.2 that H is bounded as a subgroup of $G = \mathcal{G}(\mathbb{K})$. Let $P_+ \leq G$ (resp. $P_- \leq G$) be a finite type parabolic subgroup of positive (resp. negative) sign which contains H and is minimal with respect to this property.

Applying Proposition 3.6 to $A := P_+ \cap P_-$, one can appeal to algebraic group arguments, as follows. Since H is $\mathbf{Ad}_{\mathbb{F}}$-diagonalizable, the group $\mathbf{Ad}_{\mathbb{K}}(H)|_W$ is contained in a maximal $\mathbb{K}$-torus of the Zariski closure of $\mathbf{Ad}_{\mathbb{K}}(A)|_W$ (notation of Proposition 3.6). Using the conjugacy theorem for maximal $\mathbb{K}$-tori (see [Spr98, Theorem 14.4.3]), we deduce that H is contained in a common Levi factor of P_+ and P_-, whence the result in view of Proposition 3.1. □

4.3.2 Algebraic group background

Before proceeding to a description of centralizers of completely reducible subgroups, we briefly review classical results on centralizers of tori in algebraic groups. Getting acquainted with the rather technical descriptions of the next subsection is made easier if one keeps these classical facts in mind.

Let $\mathbb{K}$ be a field and $\mathbf{G}$ be a connected reductive algebraic $\mathbb{K}$-group. Let $\mathbf{S}$ be a $\mathbb{K}$-subtorus of $\mathbf{G}$ and $\mathbf{Z}$ be the centralizer of $\mathbf{S}$ in $\mathbf{G}$. The following facts are classical; the reference is [BT65]:

(i) $\mathbf{Z}$ is a connected reductive $\mathbb{K}$-group; in fact it is a Levi factor.

(ii) If the derived group of $\mathbf{Z}$ is isotropic over $\mathbb{K}$, then $\mathbf{Z}(\mathbb{K})$ is naturally endowed with a twin root datum of spherical type. Borel subgroups of this twin root datum are $\mathbb{K}$-points of minimal parabolic $\mathbb{K}$-subgroups of $\mathbf{Z}$.

(iii) The intersection of two opposite minimal parabolic $\mathbb{K}$-subgroups of $\mathbf{Z}$ coincides with the centralizer of a maximal $\mathbb{K}$-split torus of $\mathbf{Z}$.

(iv) The derived group of the centralizer of a maximal $\mathbb{K}$-split torus of $\mathbf{Z}$ is anisotropic over $\mathbb{K}$.

4.3.3 Fixed points and centralizers of completely reducible subgroups

We need analogues of Lemma 4.5 and Proposition 4.6 for completely reducible subgroups of Kac-Moody groups. Such analogues have been obtained by B. Mühlherr [Müh94] in the case of completely reducible group actions on spherical buildings, but all arguments developed in that context can be generalized to twin buildings (this was pointed out in loc. cit., §3.1).

We place ourself in the following setting.

- $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ is a twin root datum of type (W, S) and $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ is the associated twin building.

- H is a completely reducible subgroup of G.

- Given a spherical residue ρ of $\mathcal{B}$, we denote by U^H_ρ the centralizer of H in the unipotent radical of $\mathrm{Stab}_G(\rho)$.

- For $\epsilon \in \{+, -\}$, $\mathcal{B}^H_\epsilon$ denotes the set of all H-stable spherical residues of $\mathcal{B}_\epsilon$ of minimal rank. Elements of $\mathcal{B}^H_\epsilon$ are called **H-chambers**.

- $r \in \mathbb{Z}_{\geq 0}$ is the rank of H-chambers.

- Two H-chambers are called **adjacent** if they are contained in a common spherical residue of rank $r + 1$.

Recall a set X of spherical residues of a building to be **combinatorially convex** if the following condition holds: Given two elements $\rho, \rho' \in X$, then $\mathrm{proj}_\tau(\rho)$ belongs to X for every spherical residue τ containing ρ'. The **combinatorial convex closure** of a set of spherical residues is defined accordingly.

Lemma 4.11. *The set $\mathcal{B}^H_+ \cup \mathcal{B}^H_-$ of all H-chambers is combinatorially convex. Let A be the combinatorial convex closure of a pair of opposite H-chambers. One has the following.*

(i) Given any H-chamber $C \in A$, there is a unique H-chamber of A opposite C.

(ii) A coincides with the convex closure of any pair of opposite H-chambers belonging to A.

Proof. In the case where H-chambers are genuine chambers of $\mathcal{B}$ (i.e. when $r = 0$), these are basic properties of twin apartments in twin buildings (a detailed proof of them can be found in [Abr96, Lemma 2]). In the general case, the same arguments can be adapted with the help of [CM05b, Proposition 2.7]. □

Before stating the main result of this section, we introduce the following useful definitions, which are taken from [HP98].

Let X be a set. A **wall** of X is a partition of X into two nonempty subsets called **half-spaces**. A wall is said to **separate** two given points x, y of X if x belongs to one of the half-spaces corresponding to that wall and y belongs to the other. A **wall system** on X is a collection $\mathcal{M}$ of walls such that for all pairs of points x, y of X, the collection $\mathcal{M}(x, y)$ of walls separating x from y is finite. A **wall space** is a pair $(X, \mathcal{M})$ consisting of a set X and a wall system $\mathcal{M}$ on X; it is called **reduced** if $\mathcal{M}(x, y)$ is nonempty whenever x and y are distinct.

Obviously, every wall space $(X, \mathcal{M})$ has a canonical **reduction**, obtained by identifying a point x of X with the points contained in the intersection of all half-spaces containing x. Two wall spaces are called **equivalent** if their reductions are isomorphic (in an obvious sense).

Given a Coxeter system (W, S) and the corresponding thin building $\mathcal{A} = \mathcal{A}(W, S)$, the collection of all pairs of opposite half-apartments of $\mathcal{A}$ form a canonical wall system on $\mathcal{A}$, and the corresponding wall space is reduced.

Theorem 4.12. *Let A be the combinatorial convex closure of a pair of opposite H-chambers. Let $\Psi \subset A \times A$ denote the set of all pairs (ρ, ρ') of elements of A such that ρ is of positive sign and ρ' is adjacent to the unique H-chamber of A opposite ρ (see Lemma 4.11(i)). Two elements (ρ, ρ') and (σ, σ') of Ψ are called* ***opposite*** *if ρ (resp. ρ') is adjacent to σ (resp. σ'). Given $(\rho, \rho') \in \Psi$, we set $U_{(\rho,\rho')} := U_\rho^H \cap U_{\rho'}^H$. Let Ψ_0 denote the set of all $\psi \in \Psi$ such that U_ψ is nontrivial.*

(i) Given a pair $\{(\rho, \rho'), (\sigma, \sigma')\}$ of opposite elements of Ψ_0, the respective combinatorial convex closures of $\{\rho, \rho'\}$ and $\{\sigma, \sigma'\}$ constitute a wall of A. The collection of all walls obtained in this way is a wall system on A. The corresponding wall space is equivalent to the canonical wall space of a Coxeter system, say (W^H, S^H).

(ii) The respective combinatorial convex closures of two given pairs $(\rho_1, \rho_1'), (\rho_2, \rho_2') \in \Psi_0$ coincide if and only if $U_{(\rho_1,\rho_1')} = U_{(\rho_2,\rho_2')}$.

(iii) Let Ψ^H be the quotient of Ψ_0 by the equivalence relation which identifies elements $\psi, \psi' \in \Psi_0$ such that $U_\psi = U_{\psi'}$. If Ψ^H is nonempty, then the tuple $\mathcal{Z}^H = (G^H, (U_\psi)_{\psi \in \Psi^H})$ is a twin root datum of type (W^H, S^H), where $G^H := C_G(H)$.

Proof. This is a twin version of Theorem 3.4.8 of [Müh94]. □

Note that Proposition 4.6 may be viewed as a special case of Theorem 4.12.

4.3.4 Link with the algebraic group setting

We have mentioned above that we will consider only certain completely reducible subgroups of Kac-Moody groups in the sequel, namely those which arise via Lemma 4.10. The aim of the present section is to specialize Theorem 4.12 to this situation, and to relate this result to the classical algebraic setting recalled in §4.3.2.

Let $\mathcal{G}$ be a Tits functor, $\mathbb{K}$ be a field and $\mathcal{Z} = (G, (U_\alpha)_{\alpha \in \Phi})$ be the twin root datum associated with $G := \mathcal{G}(\mathbb{K})$.

Lemma 4.13. *Let H be a subgroup of G which is $\mathbf{Ad}_{\mathbb{F}}$-diagonalizable for some field extension $\mathbb{F}/\mathbb{K}$ and $\mathcal{Z}^H = (G^H, (U_\psi)_{\psi \in \Psi^H})$ be the twin root datum provided by Theorem 4.12(iii). We have the following.*

(i) If G is of finite type, then $\mathcal{Z}^H$ coincides with the twin root datum associated with $C_G(H)$ as in §4.3.2(ii).

(ii) Let Γ be a subgroup of G^H. Then Γ is bounded with respect to $\mathcal{Z}^H$ if and only if it is bounded with respect to $\mathcal{Z}$.

(iii) Given a finite type Levi subgroup L of G containing H, then $C_L(H)$ is a finite type Levi subgroup of G^H (with respect to $\mathcal{Z}^H$).

(iv) Conversely, if L^H is a finite type Levi subgroup of G^H, then there exists a finite type Levi subgroup L of G such that $L^H = L \cap G^H$.

Proof. (i) may be viewed as a consequence of Proposition 3.6; we omit the technical details.

(iii) can be deduced from (i).

(ii) is obtained as follows. Let $\mathcal{B}$ be the twin building associated with $\mathcal{Z}$ and let $|\mathcal{B}|^H$ be the set of fixed points of H in the geometric realization of $\mathcal{B}$. Note that $|\mathcal{B}|^H$ is a closed convex subset of $|\mathcal{B}|$ which is Γ-invariant. Hence, if Γ has fixed points in $|\mathcal{B}|$, it has fixed points in $|\mathcal{B}|^H$ (see Lemma 2.2). By (iii), this shows that if Γ is bounded with respect to $\mathcal{Z}$ then it is bounded with respect to $\mathcal{Z}^H$.

Conversely, suppose that Γ is bounded with respect to $\mathcal{Z}^H$. Without loss of generality, we may assume that Γ is the intersection of a pair of finite type parabolic subgroups of G^H of opposite signs. It then follows from Proposition 3.2 and from the Bruhat decomposition of the Levi factor that Γ is boundedly generated by a finite family of root subgroups of G^H. Moreover, Theorem 4.12 implies that root subgroups of G^H (with respect to $\mathcal{Z}^{\mathcal{H}}$) are bounded (unipotent) subgroups of G (with respect to $\mathcal{Z}$). Therefore, the fact that Γ is bounded with respect to $\mathcal{Z}$ follows from Corollary 2.5.

(iv) is a consequence of (i) and (ii). □

The derived group of a maximal diagonalizable subgroup of G^H (with respect to $\mathcal{Z}^H$) is called the **anisotropic kernel** of G^H. It follows from Lemma 4.13 that the anisotropic kernel is isomorphic to the $\mathbb{K}$-points of a $\mathbb{K}$-anisotropic semisimple $\mathbb{K}$-group. We remark that the twin root datum $\mathcal{Z}^H$ might be degenerate, in the sense that Ψ^H is empty. In that case, a 'maximal diagonalizable' subgroup of G^H should be understood as the stabilizer in G^H of a pair of opposite H-chambers, and it still makes sense to consider the anisotropic kernel.

4.3.5 Regular and coregular completely reducible subgroups

As before, let $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum.

A completely reducible subgroup $H \leq G$ is called **regular** if for every finite type parabolic subgroup $P \leq G$ containing H, there is a unique parabolic subgroup P' opposite P and containing H. This property has several possible reformulations. Some of them are collected in the following.

Lemma 4.14. *Let H be a completely reducible subgroup of G. The following assertions are equivalent:*

(i) H is regular.

(ii) The set of H-chambers coincides with the combinatorial convex closure of two opposite H-chambers.

(iii) The set Ψ_0 of Theorem 4.12 is empty. □

The proof, which is an easy exercise, is left to the reader.

A completely reducible subgroup $H \leq G$ is called **coregular** if it is not regular and if the Coxeter system (W^H, S^H) provided by Theorem 4.12(i) is of universal type.

Note that in the case where H is diagonalizable, the definition of (co)regularity given in the present section coincides with the definition of §4.2.4.

4.4 Basic recognition of the ground field

4.4.1 The cardinality of the ground field

Proposition 4.15. *Let $\mathcal{G}$ be a Tits functor and $\mathbb{K}$ be a field. The Kac-Moody group $\mathcal{G}(\mathbb{K})$ is finitely generated if and only if $\mathbb{K}$ is finite.*

Proof. This is well known and follows from the fact that the defining relations of $\mathcal{G}(\mathbb{K})$ use only the ring structure of $\mathbb{K}$, and that a field which is finitely generated as a ring is finite (see for example [CM05b, Proposition 6.1]). $\square$

4.4.2 The characteristic

Proposition 4.16. *Let $\mathcal{G}$ be a Tits functor, $\mathbb{K}$ be an infinite field and $p \in \mathbb{Z}_{>0}$ be a prime. Suppose that $G := \mathcal{G}(\mathbb{K})$ is infinite. Then $p = \mathrm{char}(\mathbb{K})$ if and only if one of the following two conditions holds:*

(i) $\mathbb{K}$ is finite and the set of orders of finite p-subgroups of G has no upper-bound.

(ii) $\mathbb{K}$ is infinite and the set of ranks of elementary abelian p-subgroups of G has no upper-bound.

Proof. If $\mathbb{K}$ is finite, the equivalence between $p = \mathrm{char}(\mathbb{K})$ and (i) is proven in [CM05b, Proposition 6.2]. We assume now that $\mathbb{K}$ is infinite.

Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ be the twin root associated with G. By Lemma 1.4(iv), each U_α is isomorphic to the additive group of $\mathbb{K}$. Thus $p = \mathrm{char}(\mathbb{K})$ implies (ii). Suppose now that $p \neq \mathrm{char}(\mathbb{K})$ and let P be an elementary abelian p-subgroup of G. By Theorem 2.3, every finite subgroup of G is bounded. In view of (KMG4) we may identify G with a subgroup of $\bar{G} := \mathcal{G}(\bar{\mathbb{K}})$. Given an element $g \in P$, let $g = g_s g_u$ be its Jordan decomposition (see Proposition 3.8). Since g is of order p, we deduce that g_s and g_u are both of finite order. Therefore, g_u must be trivial, otherwise its order would be a power of a prime different from p, which contradicts the fact that the order of g is a p-power. Hence $g = g_s$ is $\mathbf{Ad}_{\bar{\mathbb{K}}}$-diagonalizable. In particular, it is diagonalizable in $\bar{G}$ (see Theorem 3.7(ii)). Therefore, we may apply Proposition 4.4, from which we deduce that the group P normalizes some maximal diagonalizable subgroup T of $\bar{G}$. On the other hand, since T is isomorphic to a direct product of finitely many copies of $\bar{\mathbb{K}}^\times$ (see Lemma 1.4(iii)) and since the Weyl group $N_{\bar{G}}(T)/T$ is a Coxeter group of finite rank, it follows that there is an upper-bound on the set of ranks of elementary abelian p-subgroups of $N_{\bar{G}}(T)$. Thus (ii) fails because all maximal diagonalizable subgroups of $\bar{G}$ are conjugate. $\square$

4.5 Detecting rank one subgroups of Kac-Moody groups

One of the most important tools in the proof of the isomorphism theorem is the identification of rank one subgroups of Kac-Moody groups (more precisely: Levi subgroups of rank one). The derived group of such a Levi is isomorphic to a copy of $SL_2(\mathbb{K})$. Proposition 4.17 below aims to characterize Levi factors of rank one among all subgroups of a given Kac-Moody groups which are isomorphic to $SL_2(\mathbb{K})$. This characterization is then used to prove an important technical auxiliary to the isomorphism theorem (Proposition 4.19).

Let $\mathcal{F} = (\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$ be the basis of a Tits functor $\mathcal{G}$. Let $\mathbb{K}$ be a field, let $G := \mathcal{G}(\mathbb{K})$, $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ be the standard twin root datum associated with $\mathcal{F}$ and $\mathbb{K}$, $\Pi = \{\alpha_i \mid i \in I\}$ be the basis of Φ which is standard with respect to $\mathcal{F}$ and $T := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$.

A version of the following result appears as Proposition 2.6 in [CM05a], where all fields were supposed algebraically closed.

Proposition 4.17. *Let $\mathbb{F}$ be a field of cardinality ≥ 4, $\varphi : SL_2(\mathbb{F}) \to G$ be a nontrivial homomorphism and H be a coregular diagonalizable subgroup of G. Suppose that:*

(1) $\Gamma := \varphi(SL_2(\mathbb{F}))$ is contained in $G^H := C_G(H)$.

(2) Γ has a subgroup $H_\Gamma \leq \varphi(\{\mathrm{diag}(x, x^{-1}) \mid x \in \mathbb{F}^\times\})$ such that $H_r := \langle H \cup H_\Gamma\rangle$ is diagonalizable and regular.

Then there exist an automorphism $\nu \in \mathrm{Inn}(G)$, an element $i \in I$, a field morphism $\zeta : \mathbb{F} \to \mathbb{K}$, a diagonal-by-sign automorphism δ of $SL_2(\mathbb{K})$ such that $\nu(\Gamma)$ is contained in $\langle U_{\alpha_i} \cup U_{-\alpha_i}\rangle$ and that the diagram

$$\begin{array}{ccc} SL_2(\mathbb{F}) & \xrightarrow{SL_2(\zeta)} & SL_2(\mathbb{K}) \\ \varphi\big\downarrow & & \big\downarrow\varphi_i\circ\delta \\ \Gamma & \xrightarrow{\nu} & \langle U_{\alpha_i} \cup U_{-\alpha_i}\rangle \end{array}$$

commutes.

Proof. Since H_r is diagonalizable and regular, there exists $\nu \in \mathrm{Inn}(G)$, unique modulo $N_G(T)$, such that $\nu(H_r) \leq T$.

Since H is coregular, the group G^H is endowed with a twin root datum $\mathcal{Z}^H = (G^H, (U_\alpha)_{\alpha\in\Phi^H})$ of universal type (see Proposition 4.6). Note that for each $\alpha \in \Phi^H$ there is a unique $\alpha' \in \Phi$ such that $\nu(U_\alpha) = U_{\alpha'}$.

Let $\mathcal{B}^{\mathcal{Z}^H} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be the twin building associated with $\mathcal{Z}^H$. By (1), the group $\Gamma \leq G^H$ acts on $\mathcal{B}^{\mathcal{Z}^H}$. The rest of the proof is divided into several steps.

Step 1. *For $\epsilon \in \{+, -\}$, the geometric realization $|\mathcal{B}_\epsilon|$ is a tree.*

Immediate because the Weyl group of $\mathcal{B}_\epsilon$ is of universal type and, hence, the geometric realization of $\mathcal{B}_\epsilon$ is one-dimensional.

Step 2. *The set of all points of $|\mathcal{B}^{\mathcal{Z}^H}|$ fixed by H_Γ coincides with $|\mathcal{A}|$ for some twin apartment $\mathcal{A}$ of $\mathcal{B}^{\mathcal{Z}^H}$.*

Indeed, by Proposition 4.6(iii), the group H_r is a regular diagonalizable subgroup of G^H, and fixes therefore a unique twin apartment $\mathcal{A}$ of $\mathcal{B}^{\mathcal{Z}^H}$ and no chamber outside $\mathcal{A}$. But $H_r = H_\Gamma.H$ and by Lemma 1.7, H acts trivially on $\mathcal{B}^{\mathcal{Z}^H}$. Thus H_Γ fixes all chambers of $\mathcal{A}$ and no chamber outside $\mathcal{A}$. Therefore H_Γ fixes a chamber of every residue it stabilizes and, hence, every such residue is a residue of $\mathcal{A}$. The claim follows.

Step 3. *The group Γ fixes no end of $|\mathcal{B}_\epsilon|$, for $\epsilon \in \{+, -\}$.*

Suppose by contradiction that Γ fixes an end ξ of $|\mathcal{B}_\epsilon|$. Then ξ is fixed by H_Γ and, hence, ξ is an end of $|\mathcal{A}|$ by Step 2.

Let us now consider $\mu := \varphi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$. For each $h \in H_\Gamma$ one has $\mu h \mu^{-1} = h^{-1}$. In particular μ normalizes H_Γ and, hence, stabilizes $\mathcal{A}$. It follows from Proposition 4.6(ii) that the pointwise stabilizer $\mathrm{Fix}_{G^H}(\mathcal{A})$ is abelian. In particular μ acts nontrivially on $\mathcal{A}$ because the group $\langle H_\Gamma \cup \{\mu\}\rangle$ is non-abelian. Since μ is of finite order, we deduce that μ acts as a reflection on $\mathcal{A}$. By Step 1, this means that μ has a unique fixed point in $|\mathcal{A}|$, which contradicts the fact that μ fixes an end of $|\mathcal{A}|$.

Step 4. *Given a chamber x of $\mathcal{B}_\epsilon$, where $\epsilon \in \{+,-\}$, if Γ fixes x then Γ acts trivially on every panel containing x.*

Let σ be a panel containing x. By Proposition 3.1, one has a Levi decomposition $\mathrm{Stab}_{G^H}(\sigma) = L \ltimes U(\sigma)$ where the unipotent radical $U(\sigma)$ acts trivially on σ. Since $\mathcal{Z}^H$ is locally split over $\mathbb{K}$, it follows that $\mathrm{Stab}_L(x)$ is solvable (see Lemma 1.3(ii)).

Suppose now that Γ fixes x. Then Γ is contained in $\mathrm{Stab}_{G^H}(\sigma)$ and the image of Γ under the canonical projection $\mathrm{Stab}_{G^H}(\sigma) \to \mathrm{Stab}_{G^H}(\sigma)/U(\sigma) \simeq L$ is contained in $\mathrm{Stab}_L(x)$. Since Γ is quasisimple, it follows that $\Gamma \leq U(\sigma)$. In particular it acts trivially on σ.

Step 5. *The group Γ fixes no chamber of $\mathcal{B}_\epsilon$, for $\epsilon \in \{+,-\}$.*

By Step 4, if Γ fixes a chamber of $\mathcal{B}_\epsilon$ then it acts trivially on $\mathcal{B}_\epsilon$. On the other hand, by Lemma 1.7 and Proposition 4.6, the kernel of the action of G^H on $\mathcal{B}_\epsilon$ is central in G^H and, hence, abelian.

Step 6. *Given $\epsilon \in \{+,-\}$, if Γ fixes a panel of $\mathcal{B}_\epsilon$, then it fixes a panel of $\mathcal{B}_{-\epsilon}$.*

Assume by contradiction that Γ fixes a panel σ of $\mathcal{B}_\epsilon$ but no panel of $\mathcal{B}_{-\epsilon}$.

By Step 2, σ is a panel of $\mathcal{A}$.

By Steps 1, 3, 5 and Theorem 2.9, the field $\mathbb{F}$ admits a discrete valuation η such that the Bruhat-Tits tree T_η associated with $(SL_2(\mathbb{F}), \eta)$ has a Γ-equivariant embedding in $|\mathcal{B}_{-\epsilon}|$. The group $D := \{\mathrm{diag}(x, x^{-1}) |\ x \in \mathbb{F}^\times\}$ stabilizes a unique line λ of T_η. Since $\varphi(D)$ contains H_Γ it follows from Step 2 that λ is contained in $|\mathcal{A}|$.

Let x be a point of $\lambda \subset |\mathcal{A}|$ which corresponds to a panel σ_x of $\mathcal{A}$. Since both σ and σ_x are panels of $\mathcal{A}$ it follows that $\mathrm{proj}_{\sigma_x}(\sigma)$ either coincides with σ or is a chamber of $\mathcal{A}$. But $\mathrm{Stab}_\Gamma(x)$ fixes no neighbor of x in T_η and, in particular, it fixes no chamber of σ_x contained in $\mathcal{A}$. It follows that $\mathrm{proj}_{\sigma_x}(\sigma) = \sigma_x$. Since $\mathcal{B}^{\mathcal{Z}^H}$ is of universal type, this implies that σ and σ_x are opposite.

We have proven that for each point x of $\lambda \subset |\mathcal{A}|$ which corresponds to a panel σ_x of $\mathcal{A}$, the panels σ and σ_x are opposite. This is impossible, because there are infinitely many such x's, while there is a unique panel of $\mathcal{A}$ opposite σ.

Step 7. *The group Γ fixes a unique pair of opposite panels σ_+, σ_- of $\mathcal{A}$.*

In view of Steps 2 and 6 and the fact that $\mathcal{B}^{\mathcal{Z}^H}$ is of universal type, it suffices to prove that Γ fixes a panel of $\mathcal{B}_+$.

Suppose the contrary. By Steps 1, 3, 5, 6 and Theorem 2.9, the field $\mathbb{F}$ admits a discrete valuation η_+ (resp. η_-) such that the Bruhat-Tits tree T_+ (resp. T_-) associated with $(SL_2(\mathbb{F}), \eta_+)$ (resp. $(SL_2(\mathbb{F}), \eta_-)$) has a Γ-equivariant embedding in $|\mathcal{B}_+|$ (resp. $|\mathcal{B}_-|$).

Let $\epsilon \in \{+,-\}$. Given a vertex x of $T_\epsilon \subset |\mathcal{B}_\epsilon|$, then x corresponds to a (uniquely defined) panel σ_x of $\mathcal{B}_\epsilon$ (because if x corresponds to a chamber, then $\mathrm{Stab}_{G^H}(x)$ acts trivially on the neighbors of x in $\mathcal{B}_\epsilon$).

The group $D := \{\mathrm{diag}(x, x^{-1}) |\ x \in \mathbb{F}^\times\}$ fixes a unique line λ_+ of T_+ (resp. λ_- of T_-). Since $\varphi(D)$ contains H_Γ it follows from Step 2 that λ_+ (resp. λ_-) is contained in $|\mathcal{A}|$. Let $\mu(t) := \varphi\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \in \Gamma$, where $t \in \mathbb{F}^\times$. Then $\mu(t)$ fixes a unique point x_ϵ^t of λ_ϵ, where $\epsilon \in \{+,-\}$. Since $\mathcal{B}^{\mathcal{Z}^H}$ is of universal type, the panels $\sigma_{x_+^t}$ and $\sigma_{x_-^t}$ are opposite. Let $\pi_\epsilon \in \mathbb{F}^\times$ be a uniformizer for η_ϵ ($\epsilon \in \{+,-\}$), i.e. an element π_ϵ of $\mathbb{F}^\times$ such that $\eta_\epsilon(\pi_\epsilon) = 1$. For all $t \in \mathbb{F}^\times$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned} \eta_+(t) = n &\Leftrightarrow \mu(t) \text{ fixes } x_+^{\pi_+^n} \\ &\Leftrightarrow \mu(t) \text{ fixes } x_-^{\pi_-^n} \\ &\Leftrightarrow \eta_-(t) = n. \end{aligned}$$

Hence the valuations η_+ and η_- coincide. In particular, it follows that given a pair x_+, x_- of vertices of λ_+, λ_- respectively such that σ_{x_+} and σ_{x_-} are opposite, we have $\mathrm{Stab}_\Gamma(x_+) = \mathrm{Stab}_\Gamma(x_-)$.

Let now $e = \{x, y\}$ be an edge of λ_+ and let $e' = \{x', y'\}$ be the edge of λ_- such that σ_x and $\sigma_{x'}$ (resp. σ_y and $\sigma_{y'}$) are opposite. Let c_e (resp. $c_{e'}$) be the unique chamber of σ_x (resp. σ'_x) such that $|c_e| \cap e$ (resp. $|c_{e'}| \cap e'$) is not reduced to x (resp. x') in the geometric realization $|\mathcal{B}_+|$ (resp. $|\mathcal{B}_-|$). Since σ_x and $\sigma_{x'}$ (resp. σ_y and $\sigma_{y'}$) are opposite, it follows that c_e and $c_{e'}$ are opposite. In particular $\mathrm{proj}_{\sigma_x}(c_{e'}) \neq c_e$. Since $\mathrm{Stab}_\Gamma(e) = \mathrm{Stab}_\Gamma(e')$ fixes $\mathrm{proj}_{\sigma_x}(c_{e'})$, we deduce that $\mathrm{Stab}_\Gamma(e)$ fixes both of the edges of λ_+ containing x. This contradicts the fact that e is the unique fixed edge of $\mathrm{Stab}_\Gamma(e)$ in the Bruhat-Tits tree T_+.

Step 8.

Let α be a twin root of $\mathcal{A}$ such that $\sigma_+, \sigma_- \in \partial\alpha$. Note that α is unique up to a sign. Up to modifying ν if necessary, we may and shall assume that α belongs to Π. Thus $\alpha = \alpha_i$ for some $i \in I$. By Step 7, we know that $\varphi(\Gamma)$ is contained in $L_\alpha := \mathrm{Stab}_{G^H}(\sigma_+) \cap \mathrm{Stab}_{G^H}(\sigma_-)$. Moreover, since Γ is quasisimple, we have $\varphi(\Gamma) \leq X_\alpha := [L_\alpha, L_\alpha]$. By Lemma 1.3(i), we have $X_\alpha = \langle U_\alpha \cup U_{-\alpha} \rangle$. Up to transforming by ν, we may and shall assume that $\Phi^H \subset \Phi$. Recall that the image of $\varphi_i : SL_2(\mathbb{K}) \to G$ coincides with X_α (see Lemma 1.4). We may apply Lemma 3.10. Note that, by Step 2, the only root subgroup of X_α normalized by H_Γ are U_α and $U_{-\alpha}$. In particular, the inner automorphism ι of $SL_2(\mathbb{K})$ provided by Lemma 3.10 may be chosen to be either trivial or conjugation by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The conclusion follows. $\square$

4.6 Images of diagonalizable subgroups under Kac-Moody group isomorphisms

When applying Proposition 4.17 to concrete situations, the main difficulty is to have the diagonalizable subgroup H required by the statement at one's disposal. In the present section, we provide a tool which rules out this difficulty in the situations considered in this work.

We place ourself in the setting of the isomorphism theorem (see §4.1.1).

Proposition 4.18. *Let $\varphi : G \to G'$ be an isomorphism. Suppose that the field $\mathbb{K}$ is infinite. Then there exists a subset $X \subset \mathbb{K}^\times$ of cardinality ≥ 10 such that φ maps the group $H := \langle \varphi_i(\mathrm{diag}(x, x^{-1})) | i \in I, x \in X \rangle$ to a diagonalizable subgroup of G'.*

This proposition constitutes a decisive step in the proof of the isomorphism theorem. Its proof will be given in the next two chapters, for fields of characteristic zero and of positive characteristic respectively.

4.7 A technical auxiliary to the isomorphism theorem

Let $\mathcal{F} = (\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$, $\mathcal{F}' = (\mathcal{G}', (\varphi'_i)_{i\in I'}, \eta')$ be the respective bases of Tits functors $\mathcal{G}$, $\mathcal{G}'$. Let $\mathbb{K}$ and $\mathbb{K}'$ be fields and set $G := \mathcal{G}(\mathbb{K})$, $G' := \mathcal{G}'(\mathbb{K}')$.

Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$, $\mathcal{Z}' = (G', (U'_\alpha)_{\alpha\in\Phi'})$ be the standard twin root data associated with G and G' respectively. Let Π and Π' be the standard bases of Φ and Φ' (with respect to $\mathcal{F}$ and $\mathcal{F}'$).

For each $\alpha \in \Phi^+$ (resp. $\alpha \in (\Phi')^+$), let $X_\alpha := \langle U_\alpha \cup U_{-\alpha}\rangle$ (resp. $X'_\alpha := \langle U'_\alpha \cup U'_{-\alpha}\rangle$).

Proposition 4.19. *Let $\varphi : G \to G'$ be an isomorphism. Suppose that $T := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$ has a subgroup H such that the following conditions hold:*

(1) $\varphi(H)$ is regular diagonalizable in G'.

(2) Given $\alpha \in \Pi$, there exists a group $H^\vee_\alpha \leq C_H(X_\alpha)$ such that $\varphi(H^\vee_\alpha)$ is non-regular and coregular.

(3) Given $\alpha \in \Pi$, the group $\langle \varphi(H_\alpha) \cup \varphi(H^\vee_\alpha)\rangle$ is regular in G', where $H_\alpha := H \cap X_\alpha$.

(4) $\langle H_\alpha \cup H^\vee_\alpha\rangle$ is regular in G.

(5) $H^\vee_\alpha$ is coregular.

Then there exists a bijection $\pi : I \to I'$, an element $\nu \in \mathrm{Inn}(G')$, and, for each $i \in I$, a field isomorphism $\zeta_i : \mathbb{K} \to \mathbb{K}'$, a diagonal-by-sign automorphism δ_i of $SL_2(\mathbb{K}')$ such that the diagram

$$\begin{array}{ccc} SL_2(\mathbb{K}) & \xrightarrow{SL_2(\zeta_i)} & SL_2(\mathbb{K}') \\ {\scriptstyle\varphi_i}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_{\pi(i)}\circ\delta_i} \\ G & \xrightarrow{\nu\circ\varphi} & G' \end{array}$$

commutes for each $i \in I$. In particular, π induces an isomorphism ϱ between the respective Weyl groups of G and G', such that (φ, ϱ) is an isomorphism of $\mathcal{Z}$ to $\mathcal{Z}'$.

Proof. Since $\varphi(H)$ is diagonalizable, there exists $\nu \in \mathrm{Inn}(G')$ such that $\nu \circ \varphi(H)$ is contained in $T' := \bigcap_{\alpha\in\Phi'} N_{G'}(U'_\alpha)$. Moreover ν is unique modulo $N_{G'}(T')$ because $\varphi(H)$ is regular and by (3), for every $\alpha \in \Pi$ the group $\nu(T')$ is the unique maximal diagonalizable subgroup of G' containing $\langle\varphi(H_\alpha)\cup\varphi(H^\vee_\alpha)\rangle$. Therefore, we may apply Proposition 4.17 to the restriction of φ to X_α for every $\alpha \in \Pi$. In particular we obtain a map $f : \Pi \to (\Phi')^+$ such that $\varphi(X_\alpha)$ is contained in $X'_{f(\alpha)}$ for each $\alpha \in \Pi$.

Let $\varphi' := \nu \circ \varphi$, $H' := \varphi'(H)$ and for each $\alpha \in \Phi'$, let $H'_\alpha := H' \cap X'_\alpha$ and $(H'_\alpha)^\vee := \varphi'(H^\vee_\alpha)$. As a consequence of Proposition 4.17, we have $\varphi'(H_\alpha) \leq H'_{f(\alpha)}$. Moreover, it follows from assumptions (4) and (5) that $(\varphi')^{-1}((H'_\alpha)^\vee)$ is coregular in G and that $\langle(\varphi')^{-1}(H'_\alpha) \cup (\varphi')^{-1}((H'_\alpha)^\vee)\rangle$ is regular. Therefore, we may apply Proposition 4.17 to the restriction of $(\varphi')^{-1}$ to $X'_{f(\alpha)}$, for $\alpha \in \Pi$. In particular, it follows that φ induces an isomorphism of X_α onto $X'_{f(\alpha)}$ for every $\alpha \in \Pi$.

Using conjugation under the respective Weyl groups of G and G', one next shows that the map $f : \Pi \to (\Phi')^+$ extends to a bijection $f : \Phi^+ \to (\Phi')^+$. In particular, the hypotheses of Theorem 1.5 are satisfied. Now all conclusions of the proposition follow, using again Proposition 4.17. $\square$

Remark 4.20. Note that the conclusions of Proposition 4.19 immediately imply that the isomorphism theorem (Theorem 4.1) holds for the given isomorphism $\varphi : G \to G'$. Therefore, in order to prove that the isomorphism theorem holds for a given an isomorphism φ between two Kac-Moody groups G and G', it suffices to exhibit a system $(H, (H_\alpha^\vee)_{\alpha\in\Pi})$ of subgroups of G which satisfy conditions (1)–(5) of Proposition 4.19. The construction of such a system is the main purpose of the next two chapters.

Chapter 5

Isomorphisms of Kac-Moody groups in characteristic zero

The main purpose of this chapter is to prove the isomorphism theorem for Kac-Moody groups over arbitrary fields of characteristic 0. This will be done by proving that a Kac-Moody group isomorphism in characteristic zero satisfies the hypotheses of Proposition 4.19, which requires first to prove Proposition 4.18. This in turn will be obtained as a consequence of Theorem C of the introduction. The proof of the latter is rather elementary and strongly inspired by Tits' beautiful proof of Theorem 2.9. Actually, one can view Theorem C of the introduction as a kind of weak version of Tits' theorem on trees which remains valid in higher dimensional CAT(0) spaces. However, while Tits' result applies to arbitrary $\mathbb{R}$-trees, we consider only discrete (i.e. polyhedral) CAT(0) spaces. This restriction allows to appeal to Bridson's results recalled in §2.1.3. In particular, Proposition 2.8 applied respectively to the abelian subgroups of $SL_2(\mathbb{Q})$ consisting of the diagonal and the upper triangular unipotent matrices happens to yield the key to the proof of Theorem C.

Let us finish by mentioning that Theorem C is probably far from optimal. However, specialized to the Kac-Moody setting, it is sufficient to describe homomorphisms from Chevalley groups over $\mathbb{Q}$ to arbitrary Kac-Moody groups, as stated in Corollary D.

5.1 Rigidity of $SL_2(\mathbb{Q})$-actions on CAT(0) polyhedral complexes

5.1.1 Upper diagonal matrices

Proposition 5.1. *Let X be a CAT(0) polyhedral complex and let $\Gamma = SL_2(\mathbb{Q})$ act on X by cellular isometries. Let $a, b \in \mathbb{Z}$ be relatively prime, and let*

$$h := \begin{pmatrix} \frac{a}{b} & 0 \\ 0 & \frac{b}{a} \end{pmatrix}, \quad D := \langle h \rangle \quad \textit{and} \quad U := \langle \begin{pmatrix} 1 & (\frac{1}{ab})^n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \rangle.$$

The following assertions are equivalent.

(i) D has global fixed points;

(ii) U has global fixed points;

(iii) $D \ltimes U$ has global fixed points.

Proof. Clearly (iii) implies (i) and (ii) and conversely, if (i) and (ii) both hold, then so does (iii) by Corollary 2.5. We have to show that (i) and (ii) are equivalent. Notice that U is abelian and a-divisible. It follows from Corollary 2.7 that every element of U has a fixed point.

(i) $\Rightarrow$ (ii) For each $n \in \mathbb{Z}$ let $U_n := \langle h^n u h^{-n} \rangle$, where $u := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in U$. Let d be the distance function on X. Each U_n has fixed points in X and, since D has fixed points by hypothesis, we deduce from the equality $d(\mathrm{Fix}(D), \mathrm{Fix}(U_n)) = d(\mathrm{Fix}(D), \mathrm{Fix}(U_0))$ valid for all $n \in \mathbb{Z}$ that there exists a bounded subset $B \subset X$ which intersects $\mathrm{Fix}(U_n)$ non-trivially for all n. By Corollary 2.4, it suffices to show that the group U is boundedly generated by the family $(U_n)_{n\in\mathbb{Z}}$.

Given $z \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$, there exist $x, y \in \mathbb{Z}$ such that

$$x \cdot a^{4k} + y \cdot b^{4k} = z \cdot (ab)^k$$

because a and b are relatively prime. We have then

$$\begin{pmatrix} 1 & \frac{z}{(ab)^k} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{a^{2k}}{b^{2k}} \cdot x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{b^{2k}}{a^{2k}} \cdot y \\ 0 & 1 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 & \frac{a^{2k}}{b^{2k}} \cdot x \\ 0 & 1 \end{pmatrix} \in U_k$ and $\begin{pmatrix} 1 & \frac{b^{2k}}{a^{2k}} \cdot y \\ 0 & 1 \end{pmatrix} \in U_{-k}$, this shows that U is boundedly generated by the family $(U_n)_{n\in\mathbb{Z}}$ as desired.

(ii) $\Rightarrow$ (i) Suppose by contradiction that D has no fixed point but $\mathrm{Fix}(U)$ is nonempty. Then h is hyperbolic. Since D normalizes U, it stabilizes $\mathrm{Fix}(U)$ and we may assume that D stabilizes a geodesic line $\ell \subset \mathrm{Fix}(U)$ (see Lemma 2.1). Let $U_- := \mu U \mu^{-1}$, where $\mu := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$. Since μ normalizes D, the geodesic line $\ell_- := \mu(\ell) \subset \mathrm{Fix}(U_-)$ is also stabilized by D. Hence the geodesic lines ℓ and ℓ_- are parallel. Let $\xi_1, \xi_2 \in \partial X$ be the points of the geometric boundary of X which are the common ends of ℓ and ℓ_-. Since $\mu h \mu^{-1} = h^{-1}$, it follows that ξ_1 and ξ_2 are swapped by μ. On the other hand, we have $\mu = u.u_-.u$ where $u := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in U$ and $u_- := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in U_-$. Since U and U_- act both trivially on ξ_1 and ξ_2 because $\ell \subset \mathrm{Fix}(U)$ and $\ell_- \subset \mathrm{Fix}(U_-)$, we deduce that μ fixes ξ_1 and ξ_2. This is a contradiction.

□

Corollary 5.2. *Let X be a CAT(0) polyhedral complex and let $\Gamma = SL_2(\mathbb{Q})$ act on X by cellular isometries. Then there exist at most finitely many primes $p_1, p_2, \dots, p_n$ such that $\langle h_i \rangle$ has no fixed point in X, where $h_i := \mathrm{diag}(p_i, p_i^{-1}) \in \Gamma$.*

Proof. The subgroup $H < \Gamma$ of diagonal matrices is a free abelian group of infinite rank. By Proposition 2.8, there exists a subgroup $H_0 \leq H$ such that H/H_0 is of finite rank and each element of H_0 has fixed points.

Let $h := \mathrm{diag}(\frac{a}{b}, \frac{b}{a})$ be an element of H_0, where a and b are relatively prime. It follows from Proposition 5.1 that for each prime p which divides a or b, the group $\langle \mathrm{diag}(p, p^{-1}) \rangle$

has fixed points in X. Consequently, the group H_0 is generated by its elements of the form $\mathrm{diag}(p, p^{-1})$ with p prime, and the short exact sequence $1 \to H_0 \to H \to H/H_0 \to 1$ splits. Since H/H_0 is of finite rank, the result follows. □

5.1.2 The rigidity theorem for $SL_2(\mathbb{Q})$

Examples of actions of $SL_2(\mathbb{Q})$ on CAT(0) polyhedral complexes can be obtained by taking diagonal actions of $SL_2(\mathbb{Q})$ on products of trees; the $SL_2(\mathbb{Q})$-action on each factor of such a product is governed by Theorem 2.9. It seems reasonable to expect that these are essentially the only examples of $SL_2(\mathbb{Q})$-action on CAT(0) polyhedral complexes. The following theorem is a rigidity result of this kind, but probably not the best result one could hope in this direction.

Theorem 5.3. *Let $G := SL_2(\mathbb{Q})$ act by isometries on a CAT(0) polyhedral complex X. Then one of the following holds:*

(i) Every finitely generated subgroup of Γ has a fixed point in X.

(ii) There exists finitely many primes $p_1, \dots, p_n$ such that for each $i = 1, \dots, n$, there exists a G_{p_i}-equivariant embedding of the vertices of the Bruhat-Tits p_i-adic tree T_i in X, where $G_{p_i} = SL_2(\mathbb{Z}[\frac{1}{p_i}])$. Moreover, for each integer m prime to all p_i's, the group $SL_2(\mathbb{Z}[\frac{1}{m}])$ has fixed points in X.

If (i) holds in Theorem 5.3, then there exists a decreasing family $(X_i)_{i\in\mathbb{N}}$ of closed convex subsets of X such that every element of Γ acts trivially on at least one of the X_i's. If X is a tree, then either $\bigcap_{i\in\mathbb{N}} X_i$ is nonempty or $\bigcap_{i\in\mathbb{N}} X_i$ contains a unique end of X. Similarly, if X is locally compact and $\bigcap_{i\in\mathbb{N}} X_i$ is empty, then $\bigcap_{i\in\mathbb{N}} X_i$ contains (possibly several) ends of X. Thus, in those cases, (i) is equivalent to the fact that Γ fixes a point of X or a point of the visual boundary ∂X.

The proof of Theorem 5.3 requires the following two lemmas.

Lemma 5.4. *Let m be a positive integer. The group $SL_2(\mathbb{Z}[\frac{1}{m}])$ is boundedly generated by its subgroups*

$$U_+(m) := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{Z}[\frac{1}{m}] \right\}$$

and

$$U_-(m) := \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in \mathbb{Z}[\frac{1}{m}] \right\}.$$

Proof. This follows from a theorem of D. Carter, G. Keller and E. Paige which is stated and proved in [Mor05]. □

Lemma 5.5. *Let p be a prime. Then $SL_2(\mathbb{Z})$ is a maximal proper subgroup of $SL_2(\mathbb{Z}[\frac{1}{p}])$.*

Proof. This is an exercise of matrix computations, using the fact that $SL_2(\mathbb{Z}[\frac{1}{p}])$ is generated by $SL_2(\mathbb{Z})$ together with $\mathrm{diag}(p, p^{-1})$. □

Proof of Theorem 5.3. For each positive integer m, define $U_+(m)$ and $U_-(m)$ as in Lemma 5.4.

Let Π be the set consisting of the primes p such that $\mathrm{diag}(p, p^{-1})$ has no fixed point in X. By Corollary 5.2, Π is finite. By Proposition 5.1, if m is prime to every element of

Π, then $U_+(m)$ has a global fixed point. Since $U_+(m)$ and $U_-(m)$ are conjugate, $U_-(m)$ has also fixed points in X. It then follows from Lemma 5.4 together with Corollary 2.5 that $SL_2(\mathbb{Z}[\frac{1}{m}])$ has a global fixed point in X.

If Π is empty, (i) follows because any finitely generated subgroup of $SL_2(\mathbb{Q})$ is contained in $SL_2(\mathbb{Z}[\frac{1}{m}])$ for some positive integer m.

If Π is nonempty, let $p_1, \ldots, p_n$ be its elements and let q be a prime which does not belong to Π. Let $\Gamma = SL_2(\mathbb{Z})$ and for each $i = 1, \ldots, n$, let $G_i := SL_2(\mathbb{Z}[\frac{1}{p_i}])$ and

$$\Gamma_i = \left\{ \begin{pmatrix} a & bp_i \\ cp_i^{-1} & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\} \cap G_i.$$

The group G_i is generated by Γ and Γ_i. Moreover, conjugation by the matrix $\operatorname{diag}(p_i, 1)$ is a diagonal automorphism of G_i which maps Γ to Γ_i. It follows from Lemma 5.5 that Γ and Γ_i are maximal proper subgroups of G_i.

Since $q \notin \Pi$, it follows from what we have seen above that $SL_2(\mathbb{Z}[\frac{1}{q}])$ has a global fixed point in X. By similar arguments, one can show that the group $\operatorname{diag}(p_i, 1) SL_2(\mathbb{Z}[\frac{1}{q}]) \operatorname{diag}(p_i^{-1}, 1)$ has global fixed points in X. In particular, the groups Γ and Γ_i have fixed points in X. Now, Assertion (ii) follows from the fact that Γ and Γ_i are the respective stabilizers in G_i of two adjacent vertices of the p_i-adic Bruhat-Tits. □

5.2 Homomorphisms of Chevalley groups over $\mathbb{Q}$ to Kac-Moody groups

5.2.1 Unipotent subgroups

Lemma 5.6. *Let $\mathcal{G}$ be a Tits functor, $\mathbb{K}$ be a field of characteristic 0, $u \in G := \mathcal{G}(\mathbb{K})$ be a bounded unipotent element and $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be the twin building associated with G. For any $n \in \mathbb{Z}_{>0}$, the cyclic groups $\langle u \rangle$ and $\langle u^n \rangle$ have the same fixed point sets in $\mathcal{B}_+$ (resp. $\mathcal{B}_-$).*

Proof. Let $c_+ \in \mathcal{B}_+$ and $c_- \in \mathcal{B}_-$ be chambers fixed by $\langle u \rangle$. Let $n \in \mathbb{Z}_{>0}$. We have to prove that any spherical residue stabilized by $\langle u^n \rangle$ is also stabilized by $\langle u \rangle$. Given a residue ρ stabilized by $\langle u^n \rangle$, then $\langle u^n \rangle$ fixes the chambers $\operatorname{proj}_\rho(c_+)$ and $\operatorname{proj}_\rho(c_-)$. Thus it suffices to prove that any *chamber* fixed by $\langle u^n \rangle$ is also fixed by $\langle u \rangle$. Suppose by contradiction that this fails and let $\epsilon \in \{+, -\}$ and $x \in \mathcal{B}_\epsilon$ be a chamber fixed by $\langle u^n \rangle$ but not by $\langle u \rangle$, and at minimal numerical distance from c_ϵ among all chambers satisfying these properties. Let y be a chamber adjacent to x and closer to c_ϵ than x. Let π be the panel containing x and y. By the definition of x, $\langle u \rangle$ fixes y and, hence, stabilizes π. By Lemma 3.3, u is contained in the unipotent radical of the Borel group $B(y) := \operatorname{Stab}_G(y)$. Let $L(\pi) \ltimes U(\pi)$ be a Levi decomposition of $\operatorname{Stab}_G(\pi)$ (see Proposition 3.1). The action of u on π coincides with the action of the projection of u to $L(\pi) \simeq \operatorname{Stab}_G(\pi)/U(\pi)$. Since u is contained in the unipotent radical of $B(y)$, this projection is contained in the unipotent radical of $B(y) \cap L(\pi)$, which is isomorphic to the additive group of $\mathbb{K}$ and acts freely on $\pi \backslash \{y\}$. Since $\operatorname{char}(\mathbb{K}) = 0$ by hypothesis, it follows that u and all of its positive powers either fix all chambers of π or fix no chamber of π besides y. This is a contradiction and the lemma is proven. □

Proposition 5.7. *Let $\mathcal{G}$ be a Tits functor, $\mathbb{K}$ be a field and $\varphi : SL_2(\mathbb{Q}) \to \mathcal{G}(\mathbb{K})$ be a nontrivial homomorphism. Then $\mathbb{K}$ is of characteristic 0 and φ maps the group $U := \langle \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Q} \rangle$ to a bounded unipotent subgroup of $\mathcal{G}(\mathbb{K})$.*

Proof. Let Z be the center of $\mathcal{G}(\mathbb{K})$. If we prove the result for the adjoint group $\mathcal{G}(\mathbb{K})/Z$, then the result for $\mathcal{G}(\mathbb{K})$ itself will follow, by an easy argument using Jordan decomposition together with the fact that the group U is divisible. Hence we may assume without loss of generality that $\mathcal{G}(\mathbb{K})$ is center-free.

The group $\mathcal{G}(\mathbb{K})$ acts on a twin building $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$. By Corollary 5.2, there exists a prime p such that $\varphi(D)$ has fixed points both in $\mathcal{B}_+$ and in $\mathcal{B}_-$, where $D := \langle \mathrm{diag}(p, p^{-1}) \rangle$. By Proposition 5.1 this implies that $\varphi(B_p)$ has fixed points both in $\mathcal{B}_+$ and in $\mathcal{B}_-$, where $B_p := D \ltimes U_p$ and $U_p := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{Z}[\frac{1}{p}] \right\}$. Equivalently, there exist finite type parabolic subgroups P_+, P_- of $\mathcal{G}(\mathbb{K})$ such that $\varphi(B_p)$ is contained in $A := P_+ \cap P_-$.

Applying Lemma 3.9(i) to the group B_p (this can be done in view of Proposition 3.6), we deduce that φ maps an infinite p-divisible subgroup of U_p, say U_p^0, to an $\mathbf{Ad}_{\mathbb{K}}$-locally unipotent subgroup of $\mathcal{G}(\mathbb{K})$. In particular, this shows that $\mathrm{char}(\mathbb{K}) = 0$. Thus the field $\mathbb{K}$ is perfect and, since $\mathcal{G}(\mathbb{K})$ is center-free, we deduce from Theorem 3.7(iii) that the group $\varphi(U_p^0)$ is bounded unipotent.

Let $u := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Note that u is $SL_2(\mathbb{Q})$-conjugate to every element of the form $u := \begin{pmatrix} 1 & q^2 \\ 0 & 1 \end{pmatrix}$, with $q \in \mathbb{Q}$. Clearly U_p^0 contains an element of this form, from which it follows that $\varphi(u)$ is bounded unipotent. Now every element g of U is $SL_2(\mathbb{Q})$-conjugate to an element of $\langle u \rangle$ and, hence, $\varphi(g)$ is bounded unipotent for all $g \in U$. By Lemma 5.6 and since U is isomorphic to the additive group of $\mathbb{Q}$, we deduce that all elements of $\varphi(U)$ have the same fixed points in both halves of the twin building of $\mathcal{G}(\mathbb{K})$. This finally implies that $\varphi(U)$ is a bounded unipotent subgroup of $\mathcal{G}(\mathbb{K})$. □

Corollary 5.8. *Let $\mathcal{G}_0$ be a Chevalley group scheme, $\mathcal{G}$ be a Tits functor, $\mathbb{K}$ be a field and $\varphi : \mathcal{G}_0(\mathbb{Q}) \to \mathcal{G}(\mathbb{K})$ be a nontrivial homomorphism. Then $\varphi(\mathcal{G}_0(\mathbb{Q}))$ is a bounded subgroup of $\mathcal{G}(\mathbb{K})$.*

Proof. It follows from the Bruhat decomposition that a Chevalley group is boundedly generated by finitely many maximal unipotent subgroups. Now each maximal unipotent subgroup decomposes as a product of finitely many one-dimensional unipotent subgroups, which are all bounded by Proposition 5.7. Thus the result is a consequence of Corollary 2.5. □

5.2.2 Homomorphisms of $SL_2(\mathbb{Q})$ to algebraic groups

The following lemma elaborates on the fact that every abstract representation of $SL_2(\mathbb{Q})$ is rational. This result appears in [Ste85, p. 343] with an elegant proof of elementary nature. In view of the importance of this lemma in the sequel, we provide it with a complete proof, inspired by loc. cit.

Lemma 5.9. *Let $\mathbb{F}$ be a field, n be a positive integer and $\varphi : SL_2(\mathbb{Q}) \to GL_n(\mathbb{F})$ be a nontrivial homomorphism. Then $\mathbb{F}$ is of characteristic 0 and there exist nilpotent elements $X, Y \in \mathfrak{gl}_n(\mathbb{F})$ and a diagonalizable $Z \in \mathfrak{gl}_n(\mathbb{F})$ such that*

(i) $[X,Y] = Z$, $[Z,X] = 2X$ *and* $[Z,Y] = -2Y$.

(ii) $\varphi(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}) = \exp(tX)$ *and* $\varphi(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}) = \exp(tY)$ *for all* $t \in \mathbb{Q}$.

(iii) The Zariski closure of the images of the diagonal matrices is a one-dimensional $\mathbb{F}$-split torus with Lie algebra $\bar{\mathbb{F}}Z$.

Proof. For $t \in \mathbb{Q}$, let $u_+(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $u_-(t) := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. Let $U_+ := \{u_+(t) | \ t \in \mathbb{Q}\}$ and $U_- := \{u_-(t) | \ t \in \mathbb{Q}\}$. By Lemma 3.9, the Zariski closure of $\varphi(U_+)$ and $\varphi(U_-)$ is unipotent and $\mathbb{F}$ is of characteristic 0.

Let exp denote the exponential map. Since $\varphi(u_+(1))$ is unipotent, there exists a nilpotent $X \in \mathfrak{gl}_n(\mathbb{F})$ such that $\varphi(u_+(1)) = \exp(X)$. Since exp is injective on nilpotent elements (see for example [Hoc81, Theorem VIII.1.1]) and $\varphi(U_+) \simeq U_+ \simeq \mathbb{Q}, +$, it follows that $\varphi(u_+(t)) = \exp(tX)$ for all $t \in \mathbb{Q}$. Similarly, there exists a nilpotent $Y \in \mathfrak{g}(\mathbb{F})$ such that $\varphi(u_-(t)) = \exp(tY)$ for all $t \in \mathbb{Q}$.

For all $t \in \mathbb{Q}^\times$ one has the identity

$$u_+(t)u_-(-t^{-1})u_+(-t) = u_-(t^{-1})u_+(t)u_-(-t^{-1}).$$

One obtains successively

$$\exp(\mathrm{Ad}(\exp(tX)).(-t^{-1}Y)) = \exp(\mathrm{Ad}(\exp(t^{-1}Y)).(tX))$$

and

$$\exp(\exp(t\,\mathrm{ad}(X))(-t^{-1}Y)) = \exp(\exp(t^{-1}\,\mathrm{ad}(Y))(tX)).$$

Using again the fact that exp is injective on nilpotent elements, it follows that

$$(\exp t\,\mathrm{ad}(X))(-t^{-1}Y) = (\exp t^{-1}\,\mathrm{ad}(Y))(tX).$$

Expanding the exponentials, we obtain

$$-t^{-1}Y - \frac{t}{2}[X,[X,Y]] - \cdots = tX + \frac{t^{-1}}{2}[Y,[Y,X]] + \ldots$$

with a finite number of terms on both sides, because X and Y are nilpotent. Since the latter holds for all $t \in \mathbb{Q}^\times$, we finally deduce $[Z,X] = 2X$ and $[Z,Y] := -2Y$, where $Z := [X,Y]$. Thus we have (i) and (ii). Note that φ extends to a rational homomorphism from $SL_2(\mathbb{F})$ to $GL_n(\mathbb{F})$. Thus (iii) follows as well. □

Corollary 5.10. *Let $\mathbb{F}$ be a field of characteristic 0 and $G = \mathrm{R}_u(G) \rtimes P$ be an affine algebraic $\mathbb{F}$-split $\mathbb{F}$-group, where $\mathrm{R}_u(G)$ is the unipotent radical of G and P is reductive. Let $\varphi : SL_2(\mathbb{Q}) \to G(\mathbb{F})$ be a nontrivial homomorphism. Then there exists a rational $\mathbb{F}$-morphism $\psi : \mathrm{SL}_2 \to G$ such that $\varphi = \psi \circ SL_2(\iota)$, where $\iota : \mathbb{Q} \to \mathbb{F}$ denotes the canonical inclusion. In particular, there exists $t \in \mathrm{R}_u(G)(\mathbb{F})$ such that $t\varphi(SL_2(\mathbb{Q}))t^{-1} \leq P(\mathbb{F})$.*

Proof. The first assertion follows from Lemma 5.9. For the second one, see for example [Hoc81, Proposition VIII.4.2]. □

5.2.3 Homomorphisms of Chevalley groups over $\mathbb{Q}$ to Kac-Moody groups

Combining the results of the preceding paragraphs, one obtains the following.

Theorem 5.11. *Let $\mathcal{G}_0$ be a simply connected Chevalley group scheme, $\mathcal{G}$ be a Tits functor and $\mathbb{K}$ be a field. Let $\varphi : \mathcal{G}_0(\mathbb{Q}) \to G := \mathcal{G}(\mathbb{K})$ be a nontrivial homomorphism. Then $\mathbb{K}$ is of characteristic 0, the image $\varphi(\mathcal{G}_0(\mathbb{Q}))$ is contained in a Levi subgroup of finite type of G and diagonalizable subgroups of $\mathcal{G}_0(\mathbb{Q})$ are mapped by φ to diagonalizable subgroups of G.*

Proof. We know from Proposition 5.7 that $\mathrm{char}(\mathbb{K}) = 0$ and from Corollary 5.8 that $\varphi(\mathcal{G}_0(\mathbb{Q}))$ is a bounded subgroup of G. In the case where $\mathcal{G}_0 = \mathrm{SL}_2$, the other assertions follow from Corollary 5.10 together with Proposition 3.6. For other types of Chevalley groups, the proof is similar, the crucial point being that Lemma 5.9 holds for all types of Chevalley groups (with essentially the same proof, see the introduction of [Ste85]). Since we will only need to apply the theorem for $\mathcal{G}_0 = SL_2$ in the sequel, we omit details here. □

5.2.4 Homomorphic images of diagonalizable subgroups

Theorem 5.11 has the following consequence, which implies in particular the validity of Proposition 4.18 for fields of characteristic zero.

Corollary 5.12. *Let $(\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$ be the basis of a Tits functor $\mathcal{G}$ and let $\mathbb{K}$ be a field of characteristic 0. Given a Tits functor $\mathcal{G}'$, a field $\mathbb{K}'$ and a homomorphism $\varphi : \mathcal{G}(\mathbb{K}) \to \mathcal{G}'(\mathbb{K}')$, then there exists a finite index subgroup $X \subset \mathbb{Q}^\times$ such that φ maps*

$$H := \langle \varphi_i(\mathrm{diag}(x, x^{-1})|\ i \in I, x \in X\rangle$$

to a diagonalizable subgroup of $\mathcal{G}'(\mathbb{K}')$.

Proof. For each $i \in I$, the composite $\varphi \circ \varphi_i$ is a homomorphism $SL_2(\mathbb{K}) \to \mathcal{G}'(\mathbb{K}')$. Applying Theorem 5.11 to its restriction to $SL_2(\mathbb{Q})$, one deduces that φ maps the group $H_i := \langle \varphi_i(\mathrm{diag}(x, x^{-1})|\ x \in \mathbb{Q}^\times\rangle$ to a diagonalizable subgroup of $\mathcal{G}'(\mathbb{K}')$. The desired result follows from Proposition 4.4. □

5.3 Regularity of diagonalizable subgroups

Proposition 5.13. *Let $\mathcal{G}$ be a Tits functor, let $\mathbb{K}$ be a field of characteristic 0 and let H be a diagonalizable subgroup of $G := \mathcal{G}(\mathbb{K})$. Then H is regular if and only if there exists no nontrivial homomorphism $SL_2(\mathbb{Q}) \to G^H$, where $G^H = C_G(H)$.*

Proof. Suppose H is not regular. By Proposition 4.6, the group G^H is endowed with a non-degenerate twin root datum which is locally $\mathbb{K}$-split. In particular, if U_α and $U_{-\alpha}$ are opposite root groups of G^H then $\langle U_\alpha \cup U_{-\alpha}\rangle$ is isomorphic to a nontrivial quotient of $SL_2(\mathbb{K})$, which contains a nontrivial quotient of $SL_2(\mathbb{Q})$.

Assume now that H is regular and suppose there exists a nontrivial homomorphism $\varphi : SL_2(\mathbb{Q}) \to G^H$. Let $\mathcal{A}$ be the unique twin apartment of the building $\mathcal{B}$ associated with G, all of whose chambers are fixed by H. It follows that $\varphi(SL_2(\mathbb{Q})) \le \mathrm{Stab}_G(\mathcal{A})$ because G^H stabilizes $\mathcal{A}$. On the other hand, by Corollary 5.8, $\varphi(SL_2(\mathbb{Q}))$ is a bounded subgroup of G, which implies by Lemma 2.2 that $\varphi(SL_2(\mathbb{Q}))$ has fixed points in $|\mathcal{A}|$. This is impossible, since the subgroups of $\mathrm{Stab}_G(\mathcal{A})$ which have fixed points in $|\mathcal{A}|$ are virtually abelian. $\square$

Proposition 5.14. *Let $\mathcal{G}$ be a Tits functor, let $\mathbb{K}$ be a field of characteristic 0 and let H be a non-regular diagonalizable subgroup of $G := \mathcal{G}(\mathbb{K})$. Then H is coregular if and only if there exists no homomorphism with finite kernel $\mathcal{G}_0(\mathbb{Q}) \to G^H$ with $\mathcal{G}_0$ a Chevalley group scheme of rank 2 and $G^H = C_G(H)$.*

Proof. Suppose H is not coregular. By Proposition 4.6, the group G^H is endowed with a non-degenerate twin root datum $\mathcal{Z}^H = (G^H, (U_\alpha)_{\alpha \in \Phi^H})$ which is locally $\mathbb{K}$-split. Since H is not coregular, there exist $\alpha, \beta \in \Phi^H$ such that $\alpha \neq \pm\beta$ and $s^\alpha s^\beta$ is of finite order. It follows that the group $X_{\alpha,\beta}$ generated by $U_\alpha \cup U_{-\alpha} \cup U_\beta \cup U_{-\beta}$ is isomorphic to $\mathcal{G}_0(\mathbb{K})$, where $\mathcal{G}_0$ is a Chevalley group scheme of rank 2. Now $X_{\alpha,\beta}$ has a subgroup isomorphic to $\mathcal{G}_0(\mathbb{Q})$ which is contained in G^H since $X_{\alpha,\beta}$ is.

Assume now that H is coregular and suppose there exists a homomorphism with finite kernel $\varphi : X(\mathbb{Q}) \to G^H$, where $X(\mathbb{Q})$ is a Chevalley group of rank 2 over $\mathbb{Q}$. By Corollary 5.8, $\varphi(X(\mathbb{Q}))$ is a bounded subgroup of G, which implies by Proposition 4.6(iv) that $\varphi(X(\mathbb{Q}))$ is a bounded subgroup of G^H (with respect to the twin root datum $\mathcal{Z}^H$). Let P be a finite type parabolic subgroup of G^H which contains $\varphi(X(\mathbb{Q}))$ and let $P = L \ltimes U$ be a Levi decomposition of P (see Proposition 3.1). Replacing $X(\mathbb{Q})$ by its derived subgroup if necessary, we may assume without loss of generality that the canonical image of $\varphi(X(\mathbb{Q}))$ in $L = P/U$ is contained in the derived subgroup L' of L. Since the twin root datum $\mathcal{Z}^H$ is locally $\mathbb{K}$-split, the group L' is a nontrivial quotient of $SL_2(\mathbb{K})$. Now it follows from Lemma 3.11 that L' contains no subgroup isomorphic to $\varphi(X(\mathbb{Q}))$, which is a contradiction. $\square$

5.4 Proof of the isomorphism theorem

Let us place ourself in the setting of the isomorphism theorem (see §4.1.1) and let $\varphi : G \to G'$ be an isomorphism. We also assume that $\mathrm{char}(\mathbb{K}) = 0$.

We will use the following notation:

- $\Pi = \{\alpha_i |\ i \in I\}$ (resp. $\Pi' = \{\alpha'_i |\ i \in I'\}$) is the basis of Φ (resp. Φ') which is standard with respect to $\mathcal{F}$ (resp. $\mathcal{F}'$).
- $T := \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$ and $T' := \bigcap_{\alpha \in \Phi'} N_{G'}(U'_\alpha)$.
- Given $\alpha \in \Phi$ (resp. $\alpha \in \Phi'$), let $X_\alpha := \langle U_\alpha \cup U_{-\alpha} \rangle$.

For the sake of clarity, the proof is divided into several short steps.

Step 1. $\mathrm{char}(\mathbb{K}') = 0$.

Follows from Propositions 4.15 and 4.16.

Step 2. *The multiplicative group $\mathbb{Q}^\times$ contains a finite index subgroup X such that*

$$H := \langle \varphi_i(x, x^{-1}) |\ i \in I, x \in X \rangle$$

and $\varphi(H)$ are both diagonalizable and regular.

The existence of $X \subset \mathbb{Q}^\times$ such that H and $\varphi(H)$ are both diagonalizable follows from Proposition 5.12. Lemma 4.9(ii) shows that H is regular, and so is $\varphi(H)$ by Proposition 5.13.

Step 3. *Given $\alpha \in \Pi$, let $H_\alpha^\vee := \{t \in H | \; [t, X_\alpha] = 1\}$. Then $H_\alpha^\vee$ and $\varphi(H_\alpha^\vee)$ are both coregular.*

The fact that $H_\alpha^\vee$ is coregular follows from the definition of H and from Lemma 4.9(iii). By Proposition 5.14, $\varphi(H_\alpha^\vee)$ is coregular in G'.

Step 4. *Conclusion.*

The preceding two steps show that the system $(H, (H_\alpha^\vee)_{\alpha\in\Pi})$ satisfies the hypotheses of Proposition 4.19. Theorem 4.1 follows (see Remark 4.20). $\square$

Chapter 6

Isomorphisms of Kac-Moody groups in positive characteristic

As explained in the introduction, the proof of the isomorphism theorem in positive characteristic goes roughly along the same lines as in characteristic zero. The main new difficulty one has to face here is to prove that a Kac-Moody group isomorphism maps a regular diagonalizable subgroup to a regular diagonalizable subgroup. In characteristic zero, this was achieved by combining the rigidity theorem for actions of $SL_2(\mathbb{Q})$ on arbitrary buildings together with the rationality of abstract linear representations of $SL_2(\mathbb{Q})$. In positive characteristic, we proceed differently. By elementary cardinality considerations using Jordan decomposition, we are able to prove that a Kac-Moody group isomorphism maps a certain regular diagonalizable subgroup to a regular *completely reducible* subgroup. This allows to mimic some of the arguments used in characteristic zero, but is also responsible for some technical difficulties due to the fact that the centralizer of a completely reducible subgroup is not necessarily locally split, but might contain an anisotropic kernel. The discussion which allows to overcome these difficulties occupies the major part of the present chapter. An essential step is the proof of Theorem 6.6 below, which is the main result of this chapter and rests heavily on the results presented in §4.3.

Another important divergence between zero and positive characteristic is the existence of "exceptional" diagram automorphisms in the second case. We call a diagram automorphism of a Kac-Moody group **exceptional** if this diagram automorphism cannot be lifted to an automorphism of the Kac-Moody Lie algebra of the corresponding type. Otherwise, a diagram automorphism is called **standard**. The fact is that standard diagram automorphisms are functorial, i.e. may be viewed as automorphisms of Tits functors. In particular, they exist over arbitrary fields. On the contrary, exceptional diagram automorphisms exist only over fields whose characteristic is positive and depends on the generalized Cartan matrix which defines the type of the group under consideration.

The problem of proving the existence of exceptional diagram automorphisms is nontrivial and will not be considered here. Certain classes of exceptional diagram automorphisms were constructed by J.-Y. Hée [Hée90] and by A. Chosson [Cho00] (on this topic, see also [CM05a, §8.3]).

6.1 On bounded subgroups of Kac-Moody groups

6.1.1 Characterization of maximal bounded subgroups

One of the main results of [CM05b] is the complete description of all maximal bounded subgroups of a group endowed with a twin root datum. (By a maximal bounded subgroup, we mean a bounded subgroup which is not properly contained in any other bounded subgroup.) It follows from this description that the maximal bounded subgroups having a trivial unipotent radical are precisely the maximal Levi subgroups of finite type. This yields the following.

Proposition 6.1. *Let $\mathcal{G}$ be a Tits functor and $\mathbb{K}$ be a field of characteristic $p > 0$. Let H be a subgroup of $G := \mathcal{G}(\mathbb{K})$. Then H is a maximal Levi factor of finite type if and only if H is a maximal bounded subgroup of G having no nontrivial normal p-subgroup.*

Proof. This follows from the results of [CM05b, §4]. □

6.1.2 The case of finite fields

In the case of Kac-Moody groups over finite fields, one has the following simple characterizations of bounded subgroups.

Lemma 6.2. *Let $\mathcal{G}$ be a Tits functor and $\mathbb{K}$ be a finite field. A subgroup H of $\mathcal{G}(\mathbb{K})$ is bounded if and only if it is finite.*

Proof. It follows from Theorem 2.3 that a finite subgroup of $G := \mathcal{G}(\mathbb{K})$ is bounded. On the other hand, it is a consequence of Proposition 3.2 that a bounded subgroup is contained in a semidirect product of the form $L \ltimes U$ with L a Levi factor of finite type and U a bounded unipotent subgroup. Since the field $\mathbb{K}$ is finite, the groups L and U are both finite and the conclusion follows. □

6.2 Homomorphisms of certain algebraic groups to Kac-Moody groups

6.2.1 Recognition of locally finite fields

A field $\mathbb{K}$ is called **locally finite** if every finite subset of $\mathbb{K}$ generates a finite subfield. It is easily seen that a field is locally finite if and only if it is an algebraic extension of a finite field.

Lemma 6.3. *Let $\mathcal{G}$ be a Tits functor and $\mathbb{K}$ be a field of characteristic $p > 0$. Then $\mathbb{K}$ is locally finite if and only if there exists no nontrivial homomorphism $\varphi : SL_2(\mathbb{F}_p(t)) \to \mathcal{G}(\mathbb{K})$.*

Proof. If $\mathbb{K}$ is not locally finite then it contains an element t which is transcendental over $\mathbb{F}_p$. Now the derived group of a rank one Levi subgroup of $G := \mathcal{G}(\mathbb{K})$ is isomorphic to $SL_2(\mathbb{K})$ or $PSL_2(\mathbb{K})$. Thus there exists a nontrivial homomorphism $\varphi : SL_2(\mathbb{F}_p(t)) \to G$.

Let now $\mathbb{K}$ be locally finite and suppose by contradiction that there exists a nontrivial homomorphism $\varphi : SL_2(\mathbb{F}_p(t)) \to G$. The subgroup of $SL_2(\mathbb{F}_p(t))$ consisting of diagonal matrices contains a free abelian group of infinite rank. Therefore, it follows from Proposition 2.8 that φ maps some diagonal matrix of $SL_2(\mathbb{F}_p(t))$ to a bounded element of infinite order in G. On the other hand, the fact that $\mathbb{K}$ is locally finite implies that every bounded subgroup of G is locally finite (this uses Propositions 3.2 and 3.6). In particular, every bounded element is of finite order, a contradiction. $\square$

6.2.2 Regularity of completely reducible subgroups

Proposition 6.4. *Let $\mathcal{G}$ be a Tits functor, $\mathbb{K}$ be an infinite field of characteristic $p > 0$ and H be a completely reducible subgroup of $G := \mathcal{G}(\mathbb{K})$. We have the following.*

(i) If H is non-regular, then there exists no nontrivial homomorphism $\mathbf{G}(\mathbb{F}) \to G^H := C_G(H)$, where $\mathbb{F}$ is an infinite field of characteristic p and $\mathbf{G}$ is a $\mathbb{F}$-split simple algebraic $\mathbb{F}$-group of $\mathbb{F}$-rank 1.

(ii) If H is regular, then the image of every homomorphism $\mathbf{G}(\mathbb{F}) \to G^H$ as above is contained in the anisotropic kernel of G^H.

Proof. Suppose H is not regular. By Lemma 4.14 and Theorem 4.12, the group $C_G(H)$ is endowed with a twin root datum. By Lemma 4.13, any pair of opposite root groups of this twin root datum generates a subgroup of $C_G(H)$ which is isomorphic to the $\mathbb{K}$-points of a semisimple algebraic $\mathbb{K}$-group of relative rank 1. Thus (i) holds.

Assume now that H is regular and let $\varphi : \mathbf{G}(\mathbb{F}) \to G^H := C_G(H)$ be a nontrivial homomorphism, where $\mathbb{F}$ is some infinite field of characteristic p. Let $\mathcal{A}$ be the combinatorial convex closure of two opposite H-chambers in the twin building $\mathcal{B}$ associated with G. Since $\varphi(\mathbf{G}(\mathbb{F}))$ centralizes H, it stabilizes $\mathcal{A}$.

Since H is regular it follows that every spherical residue of $\mathcal{B}$ contains finitely many H-chambers. Furthermore an element of G^H which fixes a H-chamber must fix all of them (see Lemma 4.14). Therefore, there exists a positive integer n such that every bounded subgroup of G^H has a subgroup of index at most n which acts trivially on $\mathcal{A}$.

Let now $(\mathbf{G}(\mathbb{F}), (U_\epsilon^{\mathbb{F}})_{\epsilon \in \{+,-\}})$ be the rank one twin root datum whose existence is assumed by hypothesis. The root groups $U_+^{\mathbb{F}}$ and $U_-^{\mathbb{F}}$ are infinite nilpotent p-groups. They contain finite p-subgroups of arbitrarily large order. It follows that the kernel of the action of $\varphi(U_\epsilon^{\mathbb{F}})$ on $\mathcal{A}$ is of finite index; in particular Theorem 2.3 implies that $\varphi(U_\epsilon^{\mathbb{F}})$ is bounded, where $\epsilon \in \{+, -\}$. Using the Bruhat decomposition of $\mathbf{G}(\mathbb{F})$, one verifies that $\mathbf{G}(\mathbb{F})$ is boundedly generated by $U_+^{\mathbb{F}}$ and $U_-^{\mathbb{F}}$. Hence, we deduce from Corollary 2.5 that $\varphi(\mathbf{G}(\mathbb{F}))$ is bounded.

As $\mathbf{G}(\mathbb{F})$ is quasisimple, it has no finite index subgroup. Hence it follows from what we have seen above that $\varphi(\mathbf{G}(\mathbb{F}))$ acts trivially on $\mathcal{A}$. Now the result follows because $\mathbf{G}(\mathbb{F})$ is perfect. $\square$

Lemma 6.5. *Let $\mathcal{G}$ be a Tits functor, $\mathbb{K}$ be an infinite field of characteristic $p > 0$ and H be a non-regular completely reducible subgroup of $G := \mathcal{G}(\mathbb{K})$. If H is not coregular, then there exists a homomorphism with finite kernel $\mathbf{G}(\mathbb{F}) \to C_G(H)$, where $\mathbb{F}$ is an infinite field of characteristic p and $\mathbf{G}$ is a semisimple algebraic $\mathbb{F}$-group of relative rank 2.*

Proof. Suppose H is not coregular. Since H is not regular, Lemma 4.14 and Theorem 4.12 imply that the group $C_G(H)$ is endowed with a twin root datum. Since H is not coregular, this twin root datum possesses two pairs of opposite root groups which generate a group equipped with a rank 2 twin root datum of spherical type. By Lemma 4.13, this group is isomorphic to the $\mathbb{K}$-points of a semisimple algebraic $\mathbb{K}$-group of relative rank 2. $\square$

6.2.3 Homomorphisms of rank one groups to Kac-Moody groups

The main result of this chapter is the following theorem. It may be viewed as a non-split version of Proposition 4.17. Actually, the proof of both results work along the same lines; they both rely heavily on Tits' rigidity theorem (see Theorem 2.9). It is the cornerstone of the proof of the isomorphism theorem in positive characteristic.

Theorem 6.6. *Let $\mathcal{G}$ be a Tits functor and $\mathbb{K}$ be an infinite field of characteristic $p > 0$. Let H be a completely reducible subgroup of $G := \mathcal{G}(\mathbb{K})$ and $\mathcal{Z}^H = (C_G(H), (U_\psi)_{\psi\in\Psi^H})$ be the associated twin root datum (see Theorem 4.12). Let $\varphi : \mathbf{G}(\mathbb{F}) \to C_G(H)$ be a nontrivial homomorphism, where $\mathbf{G}$ is a semisimple algebraic $\mathbb{F}$-group of positive $\mathbb{F}$-rank and $\mathbb{F}$ is an infinite field of characteristic p. Suppose that*

- *H is coregular.*
- *$\mathbf{G}$ possesses a maximal $\mathbb{F}$-split torus $\mathbf{S}$ such that φ maps a subgroup of $\mathbf{S}(\mathbb{F})$ to a regular diagonalizable subgroup of $C_G(H)$ (with respect to $\mathcal{Z}^H$).*
- *The supremum of the set of orders of finite p-subgroups of the anisotropic kernel of $C_G(H)$ is finite.*

Then $\mathbf{G}$ is of relative rank one and the image of φ is contained in a finite type Levi factor of G which contains H; in particular $\varphi(\mathbf{G}(\mathbb{F}))$ is bounded.

Proof. Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be the twin building associated with $\mathcal{Z}^H$. Since H is coregular, $\mathcal{B}$ is of universal type and the geometric realizations of $\mathcal{B}_+$ and $\mathcal{B}_-$ are trees, which are acted upon by $\varphi(\mathbf{G}(\mathbb{F}))$.

Since φ is nontrivial, the group $X := \varphi(\mathbf{G}(\mathbb{F}))$ is endowed with a natural twin root datum $(X, (U_\alpha)_{\alpha\in\Phi(X)})$. Let $S := \varphi(\mathbf{S}(\mathbb{F}))$. We may assume without loss of generality that S is contained in $\bigcap_{\alpha\in\Phi(X)} N_X(U_\alpha)$. By the second hypothesis, S possesses a subgroup S^0 which is regular diagonalizable in G^H.

Step 1. *If X fixes a chamber of $\mathcal{B}_\epsilon$, then it fixes pointwise every panel containing that chamber ($\epsilon \in \{+, -\}$).*

Let C be a chamber of $\mathcal{B}_\epsilon$ fixed by X and let σ be a panel containing C. Let $\mathrm{Stab}_{G^H}(\sigma) = L \ltimes U(\sigma)$ be a Levi decomposition of the stabilizer of σ in G^H. Suppose X acts non-trivially on σ. Then the image of X under the canonical projection $L \ltimes U(\sigma) \to L$ is a central quotient of X which is contained in $\mathrm{Stab}_L(C)$.

Let now $\mathrm{Stab}_L(C) = T(C) \ltimes U_L(C)$ be a Levi decomposition of the stabilizer of C in L. Since $U_L(C)$ is nilpotent, we deduce that the projection of X to $T(C)$ is a central quotient of X which is contained in the derived group of $T(C)$. This contradicts the hypotheses, because $T(C)$ is the stabilizer in G^H of a pair of opposite H-chambers and its derived group is thus isomorphic to the anisotropic kernel of G^H.

Step 2. *X fixes no chamber of $\mathcal{B}_\epsilon$.*

It follows from the preceding step that, if X fixes a chamber of $\mathcal{B}_\epsilon$, then it acts trivially on $\mathcal{B}_\epsilon$. This implies that it acts trivially on $\mathcal{B}$. By Lemmas 1.7 and 4.13, the kernel of the action of G^H on $\mathcal{B}$ is central, whence abelian. In particular, it cannot contain X, which is quasisimple.

Step 3. *If X fixes an end of the tree $|\mathcal{B}_\epsilon|$, then it fixes no vertex and no end of $|\mathcal{B}_{-\epsilon}|$.*

Assume that X fixes an end ξ of $|\mathcal{B}_\epsilon|$.

Let $U_1, U_2, \ldots, U_n$ be a finite collection of finite subgroups of X. They fix a common ray of $|\mathcal{B}_\epsilon|$, and hence, a common chamber C of $\mathcal{B}_\epsilon$. Suppose now that X fixes an end ξ' of $|\mathcal{B}_{-\epsilon}|$. Then, by the same argument, we deduce that the U_j's fix a common chamber C' of $\mathcal{B}_{-\epsilon}$. Similarly, if X fixes a vertex v of $|\mathcal{B}_{-\epsilon}|$, and if σ is the residue of $\mathcal{B}_{-\epsilon}$ such that v belongs to $|\sigma|$ then the U_j's all fix the chamber $C' := \mathrm{proj}_\sigma(C)$ of $\mathcal{B}_{-\epsilon}$.

Assume now by contradiction that X fixes a vertex or an end of $|\mathcal{B}_{-\epsilon}|$. We have just shown that, in this case, every finite collection of finite subgroups of X stabilizes a common pair of chambers of opposite signs of $\mathcal{B}$. We now show that this latter property yields a contradiction. There are two cases.

Suppose first that $\mathbb{F}$ is locally finite. By the property above, every finite subgroup of X fixes a pair of chambers C, C' of opposite signs in $\mathcal{B}$. Moreover, it follows from Proposition 3.2 that the double stabilizer $\mathrm{Stab}_{G^H}(C, C')$ decomposes as a semi-direct product $L \ltimes U$, where U is a nilpotent p-group and L is the stabilizer in G^H of a pair of opposite H-chambers. Hence every finite simple subgroup of X embeds in the anisotropic kernel of G^H. Since $\mathbb{F}$ is an infinite locally finite field, X is a countable union of finite simple subgroups. This yields a contradiction with the third hypothesis, because X contains infinite p-subgroups.

Suppose now that $\mathbb{K}$ is not locally finite. Then it is possible to choose a finite subgroup U^0_ψ of U_ψ for each $\psi \in \Phi(X)$ in such a way that $\varphi^{-1}(X^0)$ is Zariski dense in $\mathbf{G}$, where X^0 denotes the group generated by all U^0_ψ. (The existence of the family $(U^0_\psi)_{\psi\in\Phi(X)}$ is essentially a consequence of [Bor91, Proposition 9.3], together with [TW02, §33.9].) In particular X^0 has no nontrivial normal p-subgroup (see [Bor91, Proposition 2.4]). The same conclusion holds for the group X^1 generated by a family of subgroups of the form $(U^1_\psi)_{\psi\in\Phi(X)}$, where $U^0_\psi \leq U^1_\psi \leq U_\psi$. In particular, the groups U^1_ψ can be chosen to be a finite p-subgroups of arbitrarily large order. On the other hand, by the property above, the group X^1 fixes a pair of H-chambers of opposite signs. Using a Levi decomposition of the corresponding double stabilizer, we obtain an embedding of X^1 in the stabilizer of a pair of opposite H-chambers, which contradicts again the hypotheses.

Step 4. *X fixes no end of $|\mathcal{B}_\epsilon|$.*

Assume by contradiction that X fixes an end ξ of $|\mathcal{B}_\epsilon|$. In view of the preceding step, Theorem 2.9 implies that $\mathbf{G}$ is of relative rank one and that the twin root datum of X admits a discrete valuation η such that the Bruhat-Tits tree T_η associated with (X, η) has a X-equivariant embedding in $|\mathcal{B}_{-\epsilon}|$.

Let λ be the unique line of $|\mathcal{B}_{-\epsilon}|$ stabilized by $K := \bigcap_{\psi\in\Phi(X)} N_X(U_\psi)$. Since S is central in K, it follows that K normalizes S^0. Moreover, the hypotheses imply that the set of all chambers of $\mathcal{B}$ fixed by S^0 constitute a twin apartment $\mathcal{A}$ of $\mathcal{B}$. We deduce that both ξ and λ must be contained in $|\mathcal{A}|$. Now the normalizer of K in X contains elements which act as reflections on λ and act on S by $s \mapsto s^{-1}$. Therefore, these elements stabilize $\mathcal{A}$ and, since $\mathcal{A}$ is of universal type, they act as reflections of $\mathcal{A}$. In particular they fix no end of $|\mathcal{A}|$. This shows that X does not fix ξ, a contradiction.

Step 5. *If X fixes a vertex of the tree $|\mathcal{B}_\epsilon|$, then it fixes a vertex of $|\mathcal{B}_{-\epsilon}|$.*

Assume by contradiction that X fixes a vertex v of $|\mathcal{B}_\epsilon|$ but no vertex of $|\mathcal{B}_{-\epsilon}|$. By the preceding step together with Theorem 2.9, this implies that $\mathbf{G}$ is of relative rank one and that the twin root datum of X admits a discrete valuation η such that the Bruhat-Tits tree T_η associated with (X, η) has a X-equivariant embedding in $|\mathcal{B}_{-\epsilon}|$.

Let λ be the unique line of $|\mathcal{B}_{-\epsilon}|$ stabilized by K. As in the preceding step, we deduce that both v and λ must be contained in the twin apartment $|\mathcal{A}|$. Now all elements of K stabilize $\mathcal{A}$ and fix v; hence they fix the unique vertex of $\mathcal{A}$ opposite v. In particular no element of K acts as a non-trivial translation along λ, which is absurd.

Step 6. *X fixes a vertex of the tree $|\mathcal{B}_\epsilon|$.*

Assume by contradiction that X fixes no vertex of the tree $|\mathcal{B}_\epsilon|$. By the preceding steps together with Theorem 2.9, $\mathbf{G}$ is of relative rank one and the twin root datum of X admits a discrete valuation η_+ (resp. η_-) such that the Bruhat-Tits tree T_+ (resp. T_-) associated with (X, η_+) (resp. (X, η_-)) has an X-equivariant embedding in $|\mathcal{B}_\epsilon|$ (resp. $|\mathcal{B}_{-\epsilon}|$).

Given a vertex x of T_ϵ, then the embedding $T_\epsilon \hookrightarrow |\mathcal{B}_\epsilon|$ maps x in the interior of $|\sigma_\epsilon(x)|$ for a (uniquely defined) panel $\sigma_\epsilon(x)$ of $\mathcal{B}_\epsilon$. Indeed, if x corresponded to a chamber, then $\mathrm{Stab}_{G^H}(x)$ would act trivially on the neighbors of x in $\mathcal{B}_\epsilon$.

The group K stabilizes a unique line λ_+ of T_+ (resp. λ_- of T_-), and an argument as above shows that λ_+ (resp. λ_-) is contained in $|\mathcal{A}|$.

Since $\mathbf{G}$ is of relative rank one, the root system $\Phi(X)$ is reduced to a pair of opposite roots, say $\{\psi, -\psi\}$. Let M be the subset of X consisting of all elements m such that conjugation by m swaps U_ψ and $U_{-\psi}$. Each element m of M acts as a reflection on the lines λ_+ and λ_-; the unique point of λ_ϵ fixed by m is noted x^m_ϵ. Since $\mathcal{B}$ is of universal type, it follows that the panels $\sigma_+(x^m_+)$ and $\sigma_-(x^m_-)$ are opposite. Therefore, via the maps $x \mapsto \sigma_+(x)$ and $x \mapsto \sigma_-(x)$, the opposition relation induces a K-equivariant isometry between λ_+ and λ_-. Since the valuations η_+ and η_- are completely determined by the action of K on λ_+ and λ_- respectively, we deduce that η_+ and η_- coincide. In particular it follows that, given a pair x_+, x_- of vertices of λ_+, λ_- respectively such that $\sigma_+(x_+)$ and $\sigma_-(x_-)$ are opposite, we have $\mathrm{Stab}_X(x_+) = \mathrm{Stab}_X(x_-)$.

Let now $e = \{x, y\}$ be an edge of λ_+ and let $e' = \{x', y'\}$ be the edge of λ_- such that $\sigma_+(x)$ and $\sigma_-(x')$ (resp. $\sigma_+(y)$ and $\sigma_-(y')$) are opposite. Let c_e (resp. $c_{e'}$) be the unique chamber of σ_x (resp. σ'_x) such that $|c_e| \cap e$ (resp. $|c_{e'}| \cap e'$) is not reduced to x (resp. x') in the geometric realization $|\mathcal{B}_\epsilon|$ (resp. $|\mathcal{B}_{-\epsilon}|$). Since $\sigma_+(x)$ and $\sigma_-(x')$ (resp. $\sigma_+(y)$ and $\sigma_-(y')$) are opposite, it follows that c_e and $c_{e'}$ are opposite. In particular $\mathrm{proj}_{\sigma_+(x)}(c_{e'}) \neq c_e$. Since $\mathrm{Stab}_X(e) = \mathrm{Stab}_X(e')$ fixes $\mathrm{proj}_{\sigma_+(x)}(c_{e'})$, we deduce that $\mathrm{Stab}_X(e)$ fixes both of the edges of λ_+ containing x. This contradicts the fact that e is the unique fixed edge of $\mathrm{Stab}_X(e)$ in the Bruhat-Tits tree T_+.

Step 7. *X stabilizes a unique pair σ_+, σ_- of opposite panels of $\mathcal{B}$.*

The preceding steps imply that X stabilizes a vertex in both $|\mathcal{B}_\epsilon|$ and $|\mathcal{B}_{-\epsilon}|$. Moveover we know that X stabilizes no chamber of $|\mathcal{B}_\epsilon|$ or $|\mathcal{B}_{-\epsilon}|$. It follows that X stabilizes a panel σ_+ of $|\mathcal{B}_+|$ and a panel σ_- of $|\mathcal{B}_-|$, such that $\mathrm{proj}_{\sigma_+}(\sigma_-) = \sigma_+$ and $\mathrm{proj}_{\sigma_-}(\sigma_+) = \sigma_-$. Since $\mathcal{B}$ is of universal type, this implies that σ_+ and σ_- are opposite, and are the only panels of $\mathcal{B}$ stabilized by X.

Step 8. *$\mathcal{G}$ is of relative rank one, and X is contained in a finite type Levi subgroup of $G = \mathcal{G}(\mathbb{K})$.*

The fact that X is contained in a finite type Levi subgroup of G is a consequence of the preceding step, together with Lemma 4.13. This lemma also allows to apply the main result of [BT73], which implies that $\mathcal{G}$ is of relative rank one. Note however that we won't need this fact in full generality in the sequel; the only special case relevant to our needs is the split case, for which we can appeal to Lemma 3.11.

This finishes the proof of the theorem. □

Remark 6.7. The hypotheses on the anisotropic kernel of G^H in the statement of Theorem 6.6 are satisfied for 'most fields'. Indeed, recall from Lemma 4.13 that this anisotropic kernel is isomorphic to the $\mathbb{K}$-points of a $\mathbb{K}$-anisotropic semisimple group. If $\mathbb{K}$ is perfect, it follows from the main result of [BT71] that such an anisotropic group contains no non-trivial p-subgroup. If $p > 5$ then [Tit87a, Corollary 2.6] implies that a semisimple simply connected group $\mathbf{G}$ which is $\mathbb{K}$-anisotropic has no $\mathbb{K}$-unipotent elements; in particular, even if $\mathbf{G}$ is not necessarily simply connected, the group $\mathbf{G}(\mathbb{K})$ has no p-subgroups of arbitrarily large order. Finally, if $p \in \{2,3,5\}$ and moreover, one has $[\mathbb{K} : \mathbb{K}^p] \leq p$, then the same conclusions hold in view of [Gil02, Theorem 2].

6.3 Images of certain small subgroups under Kac-Moody group isomorphisms

6.3.1 Images of certain diagonalizable subgroups

We place ourself in the setting of the isomorphism theorem (see §4.1.1).

Lemma 6.8. *Let $\varphi : G \to G'$ be an isomorphism. Suppose that the field $\mathbb{K}$ is infinite of characteristic $p > 0$. Then there exists a subgroup $X \subset \mathbb{K}^\times$ of cardinality ≥ 10 such that φ maps the group $H := \langle \varphi_i(\mathrm{diag}(x, x^{-1})) | i \in I, x \in X\rangle$ to a completely reducible subgroup of G'.*

Proof. Note that $p = \mathrm{char}(\mathbb{K}')$ by Proposition 4.16.

Assume first that $\mathbb{K}$ is locally finite. Let P be the set of primes $\pi \neq p$ such that $\mathbb{K}$ possesses a primitive π^{th} root of unity. We consider two cases.

Suppose first that P is finite. Since $\mathbb{K}$ is infinite, there exists a prime π in P such that $\mathbb{K}$ possesses primitive π^n roots of unity for n arbitrarily large, and we let $q \in P$ be such a prime π. Let $m \in \mathbb{Z}_{>5}$ be sufficiently large so that the Weyl groups of G and G' have no element of order q^{m-5} and let $Y := \{y \in \mathbb{K}^\times | \ y^{q^m} = 1\}$. Since $\mathrm{char}(\mathbb{K}) = \mathrm{char}(\mathbb{K}')$ it follows from Jordan decomposition that φ maps $\langle \varphi_i(\mathrm{diag}(y, y^{-1})) | y \in Y\rangle$ to a diagonalizable subgroup of $\mathcal{G}'(\bar{\mathbb{K}}')$ for each $i \in I$, where $\bar{\mathbb{K}}'$ denotes an algebraic closure of $\mathbb{K}'$. Applying Proposition 4.4 to these diagonalizable subgroups of $\mathcal{G}'(\bar{\mathbb{K}}')$, we deduce that the image under φ of $\langle \varphi_i(\mathrm{diag}(y, y^{-1})) | y \in Y\rangle$ normalizes some maximal diagonalizable subgroup T'' of $\mathcal{G}'(\bar{\mathbb{K}}')$. By construction, the Weyl group of $\mathcal{G}'(\bar{\mathbb{K}}')$, which coincides with the Weyl group of G', has no element of order q^{m-5}. It follows that $\varphi \circ \varphi_i(\mathrm{diag}(y, y^{-1}))^{q^{m-4}} \in T''$ for all $i \in I$ and $y \in Y$. Let $X := \{x \in \mathbb{K}^\times | \ x^{q^4} = 1\}$. In view of Lemma 4.10, we have established the proposition in this case.

Suppose now that P is infinite. Let us choose $q \in P$ such that $q > 10$ and that q is prime to every finite subgroup of the Weyl groups of both G and G'. Note that these Weyl groups are finitely generated Coxeter groups, from which it follows that they have finitely many conjugacy classes of finite subgroups. Thus q is well defined in this case. Let X be the q-torsion subgroup of $\mathbb{K}^\times$. The fact that φ maps $H := \langle \varphi_i(\mathrm{diag}(x, x^{-1})) | i \in I, x \in X\rangle$ to a completely reducible subgroup of G' follows from Jordan decomposition together with Proposition 4.4 and Lemma 4.10, which establishes the lemma in this case.

Now we assume that $\mathbb{K}$ is not locally finite. The multiplicative group $\mathbb{K}^\times$ then contains a free abelian subgroup of infinite rank. Given $i \in I$, let $Y_i \subset \mathbb{K}^\times$ be the subset consisting of those elements y such that $\varphi \circ \varphi_i(\mathrm{diag}(y, y^{-1}))$ is bounded in G'. In view of Proposition 2.8, Y_i is a subgroup of $\mathbb{K}^\times$ and the intersection $Y := \bigcap_{i \in I} Y_i$ contains a free abelian subgroup of infinite rank. Let then $y \in Y$ be transcendental over the prime subfield of $\mathbb{K}$. Using Jordan decomposition together with Lemma 4.10, we obtain a positive integer n such that $\varphi \circ \varphi_i(\mathrm{diag}(y^n, y^{-n}))$ is semisimple for each $i \in I$. By Proposition 4.4, there is some multiple m of n such that φ maps the group $\langle \varphi_i(\mathrm{diag}(t^m, t^{-m}) | i \in I\rangle$ to a completely reducible subgroup of G'. Thus, the proposition holds with $X := \langle t^m \rangle$ in this case. □

6.3.2 Images of rank one subgroups

Let us consider again in the setting of the isomorphism theorem (see §4.1.1). We let $\varphi : G \to G'$ be an isomorphism.

Lemma 6.9. *Suppose that the field $\mathbb{K}$ is infinite of characteristic $p > 0$. Given $i \in I$, let $X_i := \varphi_i(SL_2(\mathbb{K}))$. Then φ maps X_i to a bounded subgroup of G'.*

Proof. We consider the following:

- $X \subset \mathbb{K}^\times$ is as in Lemma 6.8.
- $H := \langle \varphi_i(\mathrm{diag}(x, x^{-1})) | i \in I, x \in X \rangle$.
- $H_i^\vee := C_H(X_i)$ for each $i \in I$.

It follows from Lemma 6.8 that H is a completely reducible subgroup of G'.

Assume first that $\varphi(H_i^\vee)$ is regular. Then $\varphi(X_i)$ is bounded by Proposition 6.4.

Assume now that $\varphi(H_i^\vee)$ is not regular. Then the twin root datum

$$(\mathcal{Z}')^i = ((G')^i, (U'_\psi)_{\psi \in \Psi^i})$$

associated with the centralizer of $\varphi(H_i^\vee)$ in G' by Theorem 4.12 is 'non-degenerate' in the sense that the corresponding root system Ψ^i is nonempty (see Lemma 4.14).

Let $\{\psi, -\psi\}$ be a pair of opposite roots of Ψ^i and denote by X_ψ the group generated by $U'_\psi \cup U'_{-\psi}$. By Lemma 4.13, the group X_ψ is isomorphic to the $\mathbb{K}'$-points of a semisimple algebraic $\mathbb{K}'$-group of relative rank one. The restriction of φ^{-1} to X_ψ is a homomorphism of X_ψ to $C_G(H_i^\vee)$. The goal is now to prove that this restricted homomorphism satisfies the hypotheses of Theorem 6.6.

Let $\bar{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$ and identify G to a subgroup of $\bar{G} := \mathcal{G}(\bar{\mathbb{K}})$ by means of (KMG4). By Lemma 4.9, the group $H_i^\vee$ is diagonalizable and coregular as a subgroup of $\bar{G}$. Let $\bar{\mathcal{Z}}^i = (C_{\bar{G}}(H_i^\vee), (U_\alpha)_{\alpha \in \Psi^i})$ be the twin root datum associated with $H_i^\vee$ as in Proposition 4.6. Since $\bar{\mathcal{Z}}^i$ is locally split, the anisotropic kernel of $C_{\bar{G}}(H_i^\vee)$ is trivial.

Let T_ψ be a subgroup of $N_{X_\psi}(U'_\psi) \cap N_{X_\psi}(U'_{-\psi})$ which is isomorphic to the $\mathbb{K}'$-points of a one-dimensional $\mathbb{K}'$-split torus. Using Proposition 2.8 and Jordan decomposition as in the proof of Lemma 6.8, we obtain an infinite subgroup T^0_ψ of T_ψ which is diagonalizable in $\bar{G}$. By Lemma 4.13 together with [Bor91, Proposition 2.4], the group T^0_ψ and T_ψ have the same centralizers in X_ψ; by [Bor91, Proposition 20.4] this centralizer intersects U'_ψ trivially. In particular, this shows that the intersection of T^0_ψ with the center of $C_{\bar{G}}(H_i^\vee)$ is finite. In view of Proposition 4.6 and Lemma 4.9, we deduce that T^0_ψ is regular with respect to $\bar{\mathcal{Z}}^i$.

In view of the conclusions of the preceding two paragraphs, the restriction of φ^{-1} to X_ψ satisfies the hypotheses of Theorem 6.6. We deduce that $\varphi^{-1}(X_\psi)$ is contained in a finite type Levi subgroup of $C_{\bar{G}}(H_i^\vee)$. The derived group of the latter is a central quotient of $SL_2(\bar{\mathbb{K}})$. Therefore, it follows from Lemma 3.9 that the anisotropic kernel of X_ψ is trivial. It also follows from Theorem 6.6 that every finite type root subsystem of Ψ^i is of rank one. In other words, the completely reducible subgroup $\varphi(H_i^\vee)$ is coregular.

It follows easily from Proposition 6.4 that $\varphi(H)$ is regular in G', which implies that $\varphi(H_i)$ is regular in $C_{G'}(\varphi(H_i^\vee))$, where $H_i := \langle \varphi_i(\mathrm{diag}(x, x^{-1}) \mid x \in X \rangle$. This, together with the conclusions of the preceding paragraph, means that the restriction of φ to X_i satisfies that hypotheses of Theorem 6.6, which yields the desired result. □

6.3.3 Images of bounded subgroups

The following result is a consequence of Lemma 6.9.

Lemma 6.10. *Let $\mathcal{G}$, $\mathcal{G}'$ be Tits functors and $\mathbb{K}$, $\mathbb{K}'$ be fields. Suppose that* $\mathrm{char}(\mathbb{K}) = p > 0$. *Let $\varphi : \mathcal{G}(\mathbb{K}) \to \mathcal{G}'(\mathbb{K}')$ be an isomorphism. Then φ maps every bounded subgroup of $\mathcal{G}(\mathbb{K})$ to a bounded subgroup of $\mathcal{G}'(\mathbb{K}')$.*

Proof. If $\mathbb{K}$ is finite, this is a consequence of Lemma 6.2. We assume now that $\mathbb{K}$ is infinite, and so is $\mathbb{K}'$ by Proposition 4.15.

Let $G := \mathcal{G}(\mathbb{K})$ and $G' := \mathcal{G}'(\mathbb{K}')$. Let $G^\dagger$ be the subgroup of G generated by the root groups. Then G decomposes as $Z(G).G^\dagger$, and every maximal diagonalizable subgroup T of G decomposes as $T = Z(G).T^\dagger$, where $T^\dagger := T \cap G^\dagger$ is a maximal diagonalizable subgroup of $G^\dagger$.

It follows from Lemma 6.9 that φ maps every rank one subgroup of G to a bounded subgroup of G'. In particular, this implies that the isomorphism φ maps $T^\dagger$, as well as every root subgroup of G, to a bounded subgroup of G'.

Since the center of G (resp. G') is bounded (see Lemma 1.7), it follows that φ maps maximal diagonalizable subgroups of G to bounded subgroups of G' (see Corollary 2.5).

Let now P_+, P_- be finite type parabolic subgroups of opposite signs of G. Let $P_+ \cap P_- = L \ltimes U$ be a Levi decomposition of $P_+ \cap P_-$, let $L' := [L, L]$ and let H be a maximal diagonalizable subgroup of L such that $L = H.L'$. The groups U and L' are boundedly generated by finitely root subgroups: this is obvious for U and follows immediately from the Bruhat decomposition for L'. Therefore $P_+ \cap P_-$ is boundedly generated by H together with finitely many root subgroups. In view of Corollary 2.5 and what we have seen in the preceding two paragraphs, this implies that φ maps $P_+ \cap P_-$ to a bounded subgroup of G'. □

6.3.4 Images of maximal diagonalizable subgroups

At this stage, we have shown that any Kac-Moody group isomorphism maps bounded subgroups to bounded subgroups. Therefore, we could apply the main results of [BT73] in their general form, which would finish the proof of the isomorphism theorem right away: this was observed in [CM05b]. Instead, we prefer to take advantage of the fact that the groups under consideration are split, which allows to avoid appealing to [BT73] in its full strength.

Lemma 6.11. *Let $\mathcal{G}$, $\mathcal{G}'$ be Tits functors and $\mathbb{K}$, $\mathbb{K}'$ be fields. Suppose that* $\mathrm{char}(\mathbb{K}) = p > 0$. *Let $\varphi : \mathcal{G}(\mathbb{K}) \to \mathcal{G}'(\mathbb{K}')$ be an isomorphism. Then φ maps maximal diagonalizable subgroups of $\mathcal{G}(\mathbb{K})$ to maximal diagonalizable subgroups of $\mathcal{G}'(\mathbb{K}')$.*

Proof. Let $G := \mathcal{G}(\mathbb{K})$ and $G' := \mathcal{G}'(\mathbb{K}')$. Let H be a maximal diagonalizable subgroup of G and choose a maximal finite type Levi factor L containing H. It follows from Proposition 6.1 and Lemma 6.10 that $L' := \varphi(L)$ is a maximal finite type Levi factor of G'. Hence φ induces an isomorphism of L to L', where L and L' are both Chevalley groups over fields of characteristic p (see Proposition 4.16).

If $\mathbb{K}$ is finite, then so is $\mathbb{K}'$ by Proposition 4.15 and the result follows because φ maps Sylow p-subgroups of L to Sylow p-subgroups of L' (see for example [Ste68, Theorem 30]).

If $\mathbb{K}$ is infinite we may apply [BT73, Proposition 7.2] which yields the result. $\square$

Obviously, the preceding lemma implies the validity of Proposition 4.18.

6.4 Proof of the isomorphism theorem

Let us place ourself in the setting of the isomorphism theorem (see §4.1.1) and let $\varphi : G \to G'$ be an isomorphism. We also assume that G is infinite, that $p := \mathrm{char}(\mathbb{K})$ is positive and that $\mathbb{K}$ is of cardinality at least 4.

We will use the following notation:

- $\Pi = \{\alpha_i |\ i \in I\}$ (resp. $\Pi' = \{\alpha'_i |\ i \in I'\}$) is the basis of Φ (resp. Φ') which is standard with respect to $\mathcal{F}$ (resp. $\mathcal{F}'$).
- $T := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$ and $T' := \bigcap_{\alpha\in\Phi'} N_{G'}(U'_\alpha)$.
- Given $\alpha \in \Phi$ (resp. $\alpha \in \Phi'$), let $X_\alpha := \langle U_\alpha \cup U_{-\alpha}\rangle$ (resp. $X'_\alpha := \langle U'_\alpha \cup U'_{-\alpha}\rangle$).

6.4.1 Finite fields

Note that the proof of the isomorphism theorem for Kac-Moody groups over finite fields was obtained in [CM05b]. We review it briefly in the present section.

Assume that $\mathbb{K}$ is finite. Then so is $\mathbb{K}'$ by Proposition 4.15. Moreover we have $\mathrm{char}(\mathbb{K}') = p$ by Proposition 4.16.

By Lemma 6.11, there exists an inner automorphism ν of G' such that $\nu \circ \varphi(T) = T'$. It also follows from Proposition 6.1 and Lemma 6.10 that there exists a map $\pi : \Phi \to \Phi'$ such that $\nu \circ \varphi(U_\alpha) = U'_{\pi(\alpha)}$ for each $\alpha \in \Phi$ (see [Ste68, Theorem 30]). In particular the hypotheses of Theorem 1.5 are satisfied. It is not difficult to deduce that all conclusions of Theorem 4.1 hold. $\square$

6.4.2 Infinite fields

We now suppose that $\mathbb{K}$ is infinite. Thus $\mathbb{K}'$ is infinite and $\mathrm{char}(\mathbb{K}') = p$.

Given $\alpha \in \Pi$, we set $T_\alpha := T \cap X_\alpha$ and $T^\vee_\alpha := C_T(X_\alpha)$.

By Lemma 6.11, $\varphi(T)$ is a maximal diagonalizable subgroup of G'. It follows from Lemma 4.9 and 6.5, Proposition 6.4 and Theorem 6.6 that the system $(H, (H^\vee_\alpha)_{\alpha\in\Pi})$ satisfies the hypotheses of Proposition 4.19. Theorem 4.1 follows (see Remark 4.20). $\square$

Chapter 7

Homomorphisms of Kac-Moody groups to algebraic groups

7.1 The non-linearity theorem

Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be a Kac-Moody root datum and $\mathcal{G}$ be a Tits functor of type $\mathcal{D}$.

Recall from [Kac90, Chapter 4] that indecomposable generalized Cartan matrices can be divided into three families: matrices of finite type, of affine type and of indefinite type (matrices of finite type are precisely the classical Cartan matrices). Kac-Moody groups of finite type are nothing but Chevalley groups over fields, while Kac-Moody groups of affine type can be realized as central extensions of Chevalley groups over rings of Laurent polynomials. In particular, Kac-Moody groups of finite or affine type are linear groups. The following theorem shows that this property is not shared by Kac-Moody groups of indefinite type.

Theorem 7.1. *Let $\mathbb{K}$ be an infinite field. Suppose there exists a linear representation $\varphi : \mathcal{G}(\mathbb{K}) \to GL_n(\mathbb{F})$ with central kernel, where $\mathbb{F}$ is a field and $n \in \mathbb{Z}_{>0}$. Then each indecomposable submatrix of the generalized Cartan matrix A is of finite or affine type.*

The proof of this theorem, which will be given in §7.4.3 below, can be outlined as follows. By arguments similar to the ones used in the proof of the isomorphism theorem, we show that any faithful linear representation of a Kac-Moody group G maps a regular diagonalizable subgroup $T < G$ to a diagonalizable algebraic group (for simplicity, think of T as maximal diagonalizable in G). Using the rigidity theorem for diagonalizable algebraic groups [Bor91, §III.8.10, Corollary 2] (recalled in Proposition 7.11), we deduce that the kernel of the action of the normalizer $N_G(T)$ on T is of finite index in $N_G(T)$. (Note that this condition is empty if T is finite, which would be the case if G were a Kac-Moody group over a finite field.) The action of $N_G(T)$ on T can be appropriately described in terms of the action of the Weyl group on an abstractly defined root system. In this way, we prove that the existence of a faithful linear representation of G implies that its Weyl group has an action on an abstract root system whose kernel is of finite index. It remains to show the only Weyl groups having such an action are of finite or affine type. This is done separately, in a 'Coxeter group context' which is free of any reference to Kac-Moody theory.

7.2 A combinatorial characterization of affine Coxeter groups

7.2.1 Statement of the result

An irreducible Coxeter group is called **affine** if its Coxeter complex can be realized as a triangulation of a finite-dimensional Euclidean space. Here, we shall define more generally a Coxeter group to be **affine** if each of its irreducible components is either spherical or affine. In particular, finite Coxeter groups are affine according to this terminology. Although this is not standard in the literature, it is quite natural, especially with respect to our purposes. There are several ways of characterizing affine Coxeter groups. For example, it is known that a (finitely generated) Coxeter group is affine if and only if it is virtually solvable, or, that an irreducible Coxeter system is affine if the associated Tits' bilinear form is positive definite or semi-definite. However, there does not seem to exist in the literature a combinatorial feature of affine Coxeter groups which distinguishes them from other Coxeter groups. The following result proposes such a characterization.

Theorem 7.2. *Let (W, S) be a Coxeter system such that S is finite and Σ be its Coxeter complex. The following conditions are equivalent:*

(i) W is affine.

(ii) There exists no set of three pairwise disjoint half-spaces of Σ.

(iii) Given half-spaces α, β, γ of Σ, if $\alpha \subset \beta$ and $\alpha \subset \gamma$, then $\beta \subset \gamma$ or $\gamma \subset \beta$.

(iv) Any finite set of points of Σ is contained in the convex hull of two chambers.

Recall that the **convex hull** of two chambers of Σ coincides with the intersection of all half-spaces containing them.

In an unpublished work by P. Abramenko and H. Van Maldeghem, it is shown that Property (iv) above is equivalent to the following statement, where $\mathcal{B}$ is a thick building of type (W, S): *For any choice of an apartment system $\mathfrak{A}$, the convex closure of any two chambers of $\mathcal{B}$ coincides with the intersection of all apartments of $\mathfrak{A}$ containing them.*

We will need a reformulation of these characterizations in terms of root systems. In order to make an accurate statement, we briefly review the corresponding terminology.

Given a Coxeter system (W, S), a representation of W on a real vector space V is called **geometric** if there exists a linearly independent subset $B = \{e_s\}_{s \in S}$ of V such that the following conditions are satisfied:

(GR1) Each $s \in S$ fixes a hyperplane of V, preserves the subspace spanned by B and maps e_s to its opposite.

(GR2) For each $w \in W$ and each $s \in S$, the vector $w.e_s$ has all its coordinates either ≥ 0 or ≤ 0 when expressed in terms of the basis B.

The set $\Phi := \bigcup_{s \in S} W.e_s$ is called the **root system** of this geometric representation; its elements are called **roots** and the set $B \subset \Phi$ is called a **basis** of Φ. A root is called **positive** if it is contained in $\sum_{s \in S} \mathbb{R}_+ e_s$. A **root subsystem** is a subset of Φ which coincides with the root system of a reflection subgroup of W. The root subsystem generated by a subset of Φ is the intersection of all root subsystems containing it.

A well known theorem of J. Tits insures that every Coxeter group W possesses a faithful geometric representation (see [Bou81, §V.4]). The consequence of Theorem 7.2 we will need later is the following.

Corollary 7.3. *Let (W, S) be a Coxeter system such that S is finite and Φ be the root system associated to a geometric representation of W. Then W is non-affine if and only if there exist three positive roots $\alpha, \beta, \gamma \in \Phi$ such that $\{\alpha, \beta, \gamma\}$ is a basis of the root subsystem it generates and that each 2-subset of $\{\alpha, \beta, \gamma\}$ generates an infinite root subsystem.*

7.2.2 Two lemmas on Coxeter groups

Lemma 7.4. *Let (W, S) be a Coxeter system such that S is finite. There exists a constant $N = N(W, S)$ such that given any set of reflections $R \subset W$, if $|R| > N$ then R contains two reflections whose product has infinite order.*

Proof. This follows from an easy computation using a faithful geometric representation of W. The crucial point is that, since S is finite, there is only finitely many possibilities for the order of the product of two reflections of W. See for example [NR03, Lemma 3] for more details. □

Following a standard convention in the theory of Coxeter groups, we denote by $-\alpha$ the complement of a half-space α in a Coxeter complex Σ.

Lemma 7.5. *Let (W, S) be a Coxeter system and Σ be its Coxeter complex. Let α, β, γ be half-spaces of Σ such that $\alpha \subset \beta$ and $\alpha \subset \gamma$ and $\beta \neq \alpha \neq \gamma$. If $\beta \not\subset \gamma$ and $\gamma \not\subset \beta$, then Σ possesses three pairwise disjoint half-spaces.*

Proof. The easiest way to prove this lemma is to notice that the subgroup of W generated by the reflections r_α, r_β and r_γ is a rank 3 hyperbolic Coxeter group. It is a general fact that a subgroup of W generated by k reflections is isomorphic to a Coxeter group of rank at most k (see [Deo89]). Moreover, Coxeter groups of rank at most 3 are either finite, or affine, or hyperbolic [Bou81]. Now the conditions imposed on the half-spaces α, β and γ imply that $r_\alpha r_\beta$ is of infinite order, and that $\langle r_\alpha, r_\beta, r_\gamma\rangle$ is not of affine type (otherwise $\alpha \subset \beta$ and $\alpha \subset \gamma$ would imply $\beta \subset \gamma$ or $\gamma \subset \beta$). Thus $\langle r_\alpha, r_\beta, r_\gamma\rangle$ is indeed a hyperbolic Coxeter group of rank 3, namely a hyperbolic triangle group. Now the desired result is obtained by considering a realization of this Coxeter group as a reflection group of the hyperbolic plane $\mathbb{H}^2$.

(A more elementary proof can be obtained by easy computations in the root system of a faithful geometric representations of W, using a case-by-case discussion on the relative positions of β and γ.) □

7.2.3 Proof of Theorem 7.2

(i) ⇒ (ii). Clear.

(ii) ⇒ (iii). Follows from Lemma 7.5.

(iii) ⇒ (i). Let Φ be the set of all half-spaces of Σ. The hypothesis implies that the relation on Φ defined by

$$\alpha \sim \beta \quad \Leftrightarrow \quad \alpha \subset \text{ or } \beta \subset \alpha$$

is an equivalence relation. We denote by Φ^∞ the corresponding quotient and by $p : \Phi \to \Phi^\infty$ the canonical projection. It follows from Lemma 7.4 that any sufficiently large subset of Φ contains two half-spaces such that one is contained in the other. In particular, the set Φ^∞ is finite. Clearly the group W acts on Φ and on Φ^∞. Given any $\alpha \in \Phi^\infty$, then the group $\mathrm{Fix}_W(\alpha)$ acts on the set $p^{-1}(\alpha)$ as an infinite cyclic group.

Let now T be the kernel of the action of W on Φ^∞. Given $\alpha \in \Phi^\infty$, the actions of any two elements of T on $p^{-1}(\alpha)$ commute. It follows that the actions of any two elements of T on Φ commute. Since W acts faithfully on Φ, we deduce that T is abelian. These arguments also show that T is torsion free. Moreover, since Φ^∞ is finite, T is of finite index in W. Thus W has a finite index subgroup which is free abelian. It is well known that this implies that W is affine (see for example [Kra94, §6.8])

(i) $\Rightarrow$ (iv). Without loss of generality, we may assume that (W, S) is irreducible. If W is finite the result is clear. Otherwise, W is affine and its Coxeter complex can be realized as a tessellation of some finite-dimensional Euclidean space $\mathbb{E}$. The walls of Σ are realized as hyperplanes of $\mathbb{E}$, and the fact that S is finite implies that there are finitely many directions of hyperplanes in Σ. In particular, there exists a line ℓ of $\mathbb{E}$ which meets each wall of Σ in exactly one point. It follows from this construction that if x and y are two point of ℓ, then the diameter of the convex hull of x and y in Σ tends to infinity when the distance between x and y increases. Thus (iv) holds.

(iv) $\Rightarrow$ (ii). Suppose by contradiction that (ii) does not hold and let α, β, γ be three pairwise disjoint half-spaces. Let A, B, C be three chambers of Σ which are respectively contained in α, β and γ. Let X and Y be chambers such that the convex hull $\mathcal{C}$ of $\{X, Y\}$ contains A, B and C. Up to a permutation of the set $\{\alpha, \beta, \gamma\}$ we may assume that X and Y are contained in $-\gamma$. This implies that $\mathcal{C}$ is contained in $-\gamma$, which contradicts the fact that $\mathcal{C}$ contains C.

This finishes the proof of Theorem 7.2. □

The equivalence between Theorem 7.2 and Corollary 7.3 follows from a dictionary correspondence between the root system of a geometric representation and the geometry of half-spaces of a Coxeter complex (a standard way of establishing this correspondence is to use the Tits' cone of a geometric representation). We do not go into details here.

7.3 On infinite root systems

The purpose of this section is to obtain some results which allow to describe the action of the Weyl group of a Kac-Moody group on the corresponding maximal diagonalizable subgroup. For Kac-Moody groups over infinite fields, this action is appropriately described by the action of the Weyl group on the root and coroot lattice of this maximal diagonalizable subgroup, which in turn can be defined abstractly, starting from a Kac-Moody root datum and with no reference to Kac-Moody groups (see Lemma 7.10 below). A similar abstract point of view is adopted in [MP95, Chapter 5], to study the root system of a Kac-Moody Lie algebra. Nevertheless, the axioms of an abstract root system considered in loc. cit. are too strong to apply directly to Kac-Moody root data as defined here. However, our purposes require only some partial results from this general theory of abstract root systems and we obtain them by straightforward adaptations of arguments given in loc. cit.

7.3.1 The Weyl group of a Kac-Moody root datum

Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum. We make the following definitions (see B. Rémy [Rém02b, §7.1.3 and 7.1.4]).

- The group $Q(\mathcal{D}) := \sum_{i\in I} \mathbb{Z}c_i \subset \Lambda$ is called the **root lattice** of the Kac-Moody root datum $\mathcal{D}$.
- The group $Q(\mathcal{D})^\vee := \sum_{i\in I} \mathbb{Z}h_i \subset \Lambda^\vee$ is called the **coroot lattice** of $\mathcal{D}$.
- The **Weyl group** of $\mathcal{D}$ is the Coxeter group W with standard generating set $S := \{s_i\}_{i\in I}$, whose type is the Coxeter matrix $M = (m_{ij})_{i,j\in I}$ defined by $m_{ij} := 2, 3, 4, 6$ or ∞ if $i \neq j$ and $A_{ij}A_{ji} = 0, 1, 2, 3$ and ≥ 4 respectively.
- A **standard realization** of A is a triple $(V, \Pi, \Pi^\vee)$ where V is a $\mathbb{Q}$-vector space of dimension $2.|I| - \mathrm{rank}(A)$, $\Pi = \{a_i\}_{i\in I}$ is a linearly independent subset of V and $\Pi^\vee = \{a_i^\vee\}_{i\in I}$ is a linearly independent subset of the dual V^*, such that $\langle a_i, a_j^\vee\rangle = A_{ji}$ for all $i, j \in I$.
- The group $Q(A) := \sum_{i\in I} \mathbb{Z}a_i \subset V$ is called the **root lattice** of the generalized Cartan matrix A.
- The group $Q(A)^\vee := \sum_{i\in I} \mathbb{Z}a_i^\vee \subset V^*$ is called the **coroot lattice** of A.
- Let $c : Q(A) \to Q(\mathcal{D})$ be the surjective homomorphism induced by the assignments $c : a_i \mapsto c_i$ $(i \in I)$.
- Let $h : Q(A)^\vee \to Q(\mathcal{D})^\vee$ be the surjective homomorphism induced by the assignments $c : a_i^\vee \mapsto h_i$ $(i \in I)$.
- Given $i \in I$, let

$$\begin{aligned}
\tau(s_i) &: \ \Lambda \to \Lambda \ : \lambda \mapsto \lambda - \langle \lambda, h_i\rangle c_i,\\
\tau^\vee(s_i) &: \Lambda^\vee \to \Lambda^\vee : \lambda \mapsto \lambda - \langle c_i, \lambda\rangle h_i,\\
\hat\tau(s_i) &: \ V \to V \ : v \mapsto v - \langle v, a_i^\vee\rangle a_i,\\
\hat\tau^\vee(s_i) &: V^* \to V^* : v \mapsto v - \langle a_i, v\rangle a_i^\vee.
\end{aligned}$$

The following lemma collects some standard facts.

Lemma 7.6. *(i) The assignments τ (resp. $\tau^\vee$, $\hat\tau$, $\hat\tau^\vee$) extend to a well defined action of the Weyl group W on Λ (resp. $\Lambda^\vee$, V, V^*) which preserves $Q(\mathcal{D})$ (resp. $Q(\mathcal{D})^\vee$, $Q(A)$, $Q(A)^\vee$) and such that the diagrams*

$$\begin{array}{ccc}
Q(A) & \xrightarrow{c} & Q(\mathcal{D})\\
{\scriptstyle\hat\tau(w)}\downarrow & & \downarrow{\scriptstyle\tau(w)}\\
Q(A) & \xrightarrow{c} & Q(\mathcal{D})
\end{array}
\quad \textit{and} \quad
\begin{array}{ccc}
Q(A)^\vee & \xrightarrow{h} & Q(\mathcal{D})^\vee\\
{\scriptstyle\hat\tau^\vee(w)}\downarrow & & \downarrow{\scriptstyle\tau^\vee(w)}\\
Q(A)^\vee & \xrightarrow{h} & Q(\mathcal{D})^\vee
\end{array}$$

commute for all $w \in W$.

(ii) The actions $\hat\tau$ and $\hat\tau^\vee$ are both faithful.

(iii) *For all $w \in W$ and all $\lambda \in \Lambda$, $h \in \Lambda^\vee$, we have $\langle \tau(w)\lambda | \tau^\vee(w)h\rangle = \langle \lambda | h\rangle$.*

(iv) *Let $\Delta := \bigcup_{i\in I} \hat{\tau}(W).a_i$, $\Delta^\vee := \bigcup_{i\in I} \hat{\tau}^\vee(W).a_i^\vee$ and for $\epsilon \in \{+,-\}$, let $Q(A)_\epsilon := \sum_{i\in I} \mathbb{Z}_{\geq 0}\epsilon a_i$, $Q(A)_\epsilon^\vee := \sum_{i\in I} \mathbb{Z}_{\geq 0}\epsilon a_i^\vee$, $\Delta_\epsilon := \Delta \cap Q(A)_\epsilon$ and $\Delta_\epsilon^\vee := \Delta^\vee \cap Q(A)_\epsilon^\vee$. Then $\Delta = \Delta_+ \sqcup \Delta_-$ and $\Delta^\vee = \Delta_+^\vee \sqcup \Delta_-^\vee$. Moreover, there exists a unique bijection $^\vee : \Delta \to \Delta^\vee : a \mapsto a^\vee$ such that the diagram*

$$\begin{tikzcd} \Delta \arrow[r, "\vee"] \arrow[d, "\hat{\tau}(w)"'] & \Delta^\vee \arrow[d, "\hat{\tau}^\vee(w)"] \\ \Delta \arrow[r, "\vee"] & \Delta^\vee \end{tikzcd}$$

commutes for every $w \in W$. For $\epsilon \in \{+,-\}$ one has $\Delta_\epsilon^\vee := \{\alpha^\vee | \alpha \in \Delta_\epsilon\}$.

(v) *Given $a, b \in \Delta$, one has $\langle a | b^\vee\rangle = \langle c(a) | h(b^\vee)\rangle$.*

Proof. For (i), (iii) and (v) see [Rém02b, Lemma 7.1.5]. Assertion (ii) is a consequence of [MP95, §5.3, Theorem 1] and Assertion (iv) follows from [MP95, Propositions 5.1.4 and 5.2.6]. □

We insist on the fact that the actions τ and $\tau^\vee$ are not faithful in general. However, the following lemma shows that the conditions under which the action of the product of two reflections of the Weyl group degenerates, are rather restrictive.

Lemma 7.7. *Given $\alpha, \beta \in \Delta$, let $N_{\alpha\beta} := \langle \alpha, \beta^\vee\rangle\langle \beta, \alpha^\vee\rangle$ and let $r_\alpha : V \to V : v \mapsto v - \langle v, \alpha^\vee\rangle\alpha$. We have the following.*

(i) *$r_\alpha \in \hat{\tau}(W)$.*

(ii) *Let $\alpha, \beta \in \Delta$, $\alpha \neq \pm\beta$ and suppose that $N_{\alpha,\beta} = 0$ (resp. $N_{\alpha,\beta} = 1$, $N_{\alpha,\beta} = 2$, $N_{\alpha,\beta} = 3$, $N_{\alpha,\beta} \geq 5$). Then the cyclic subgroup generated by the element $\hat{\tau}^{-1}(r_\alpha r_\beta)$ of the Weyl group acts faithfully on $Q(\mathcal{D})$ and on $Q(\mathcal{D})^\vee$ as an automorphism of order 2 (resp. $3, 4, 6, \infty$).*

Proof. Let $i \in I$ and $w \in W$ be such that $\hat{\tau}(w)(a_i) = \alpha$. It is easy to see, using Lemma 7.6, that $\hat{\tau}(ws_iw^{-1}) = r_\alpha$, whence (i).

Let us now turn to (ii). We claim that the hypotheses imply that $c(\alpha)$ and $c(\beta)$ are linearly independent, i.e. they generate a free abelian group of rank 2. Suppose the contrary. Then there exists $e \in Q(\mathcal{D})$ and integers $n_\alpha, n_\beta \in \mathbb{Z}$ such that $c(\alpha) = n_\alpha.e$ and $c(\beta) = n_\beta.e$. Applying Lemma 7.6(i) to the action of r_α and r_β on $\{\alpha, \beta\}$, we obtain the equations $\frac{n_\alpha}{n_\beta}\langle \beta, \alpha^\vee\rangle = 2$ and $\frac{n_\beta}{n_\alpha}\langle \alpha, \beta^\vee\rangle = 2$. It follows that $N_{\alpha\beta} = 4$, which contradicts the hypotheses, whence the claim.

Now, we follow the arguments of the proof of Proposition 5.1.11 in [MP95]. Let $x \in Q(\mathcal{D})$ and set $U(x) := \mathbb{Q}x + \mathbb{Q}c(\alpha) + \mathbb{Q}c(\beta) \subset Q(\mathcal{D}) \otimes_{\mathbb{Z}} \mathbb{Q}$. If $U(x) \not\subset U(\alpha)$, then the set $\{x, c(\alpha), c(\beta)\}$ is linearly independent and the matrix of the restriction of $\tau \circ \hat{\tau}^{-1}(r_\alpha r_\beta)$ to $U(x)$ relative to the ordered basis $(x, c(\alpha), c(\beta))$ is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ * & -1 + N_{\alpha\beta} & \langle \beta, \alpha^\vee\rangle \\ * & -\langle \alpha, \beta^\vee\rangle & -1 \end{pmatrix}.$$

Its characteristic polynomial is $(X-1)(X^2 + (2 - N_{\alpha\beta})X + 1)$. If $U(x) \subset U(\alpha)$ then we obtain the characteristic polynomial $X^2 + (2 - N_{\alpha\beta})X + 1$. For $N_{\alpha\beta} = 0, 1, 2$ and 3, the roots of the polynomial $X^2 + (2 - N_{\alpha\beta})X + 1$ are square, third, fourth and sixth roots of unity, and at least one of them is primitive. For $N_{\alpha\beta} \geq 5$, the roots of $X^2+(2-N_{\alpha\beta})X+1$ are distinct and are not roots of unity. Assertion (ii) follows for the root lattice $Q(\mathcal{D})$. The arguments for the coroot lattice are similar. $\square$

The only technical result on root systems that we will refer to in the sequel, is the following.

Lemma 7.8. *Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum and W be its Weyl group. Suppose that A has at least one component of indefinite type. Then there exists an element $w \in W$ of infinite order which generates a cyclic group acting faithfully on the coroot lattice $Q(\mathcal{D})^\vee$.*

Proof. If the Coxeter group W is affine in the sense of §7.2.1 (this happens when the generalized Cartan matrix A has a rank 2 component of hyperbolic type), then it follows from the classification of generalized Cartan matrices (see [Kac90, Chapter 4]) that there exist $i, j \in I$, $i \neq j$, such that $A_{ij}A_{ji} \geq 5$. It then follows from Lemma 7.7 that s_is_j generates a cyclic group acting faithfully on the root and coroot lattices, whence the claim.

If the Coxeter group W is non-affine, then it follows from Corollary 7.3 that there exist $\alpha, \beta, \gamma \in \Delta_+$ (notation of Lemma 7.6(v)) such that each 2-subset of $\{\alpha, \beta, \gamma\}$ generates an infinite root subsystem of Δ (in the sense of [MP95, §5.7]) and is a basis of the root subsystem it generates. Up to a permutation of the set $\{\alpha, \beta, \gamma\}$, we may and shall assume that $\langle \alpha, \beta^\vee \rangle \leq \langle \phi, \psi^\vee \rangle$ for all $\phi, \psi \in \{\alpha, \beta, \gamma\}$. By the definition of α, β and γ, we have $\langle \gamma, \beta^\vee \rangle, \langle \gamma, \alpha^\vee \rangle \in \mathbb{Z}_{<0}$. Moreover, the assumption on $\langle \alpha, \beta^\vee \rangle$ implies that if $\langle \alpha, \beta^\vee \rangle = -2$ then $\langle \gamma, \beta^\vee \rangle = \langle \gamma, \alpha^\vee \rangle = -2$. It follows that $\langle r_\alpha(\gamma), \beta^\vee \rangle = \langle \gamma, \beta^\vee \rangle - \langle \gamma, \alpha^\vee \rangle \langle \alpha, \beta^\vee \rangle \leq -5$. Since $\langle \beta, r_\alpha(\gamma)^\vee \rangle \in \mathbb{Z}_{<0}$, we deduce $N_{\beta, r_\alpha(\gamma)} \geq 5$. The conclusion follows from Lemma 7.7(ii). $\square$

7.3.2 The Weyl group action on diagonalizable subgroups of Kac-Moody groups

Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum and $\mathcal{F} = (\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$ be the basis of a Tits functor $\mathcal{G}$ of type $\mathcal{D}$.

Lemma 7.9. *Let $\mathbb{K}$ be a field, let $i, j \in I$ and $w \in W$, where W is the Weyl group of $\mathcal{D}$. For all $t \in \mathbb{K}^\times$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{K})$, we have the following:*

$$\begin{aligned} & w\varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} w^{-1}.\varphi_j \begin{pmatrix} a & b \\ c & d \end{pmatrix} .w\varphi_i \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} w^{-1} \\ = \;& \varphi_j \begin{pmatrix} a & t^n.b \\ t^{-n}.c & d \end{pmatrix}, \end{aligned}$$

where $n = \langle a_j, w(a_i^\vee) \rangle$.

Proof. This follows from the defining relations of $\mathcal{G}(\mathbb{K})$ (see [Tit87b, §3.6]). $\square$

Lemma 7.10. *Let $\mathbb{K}$ be a field and X be an infinite subset of $\mathbb{K}^\times$. Let $G := \mathcal{G}(\mathbb{K})$, let $T := \eta(\mathcal{T}_\Lambda(\mathbb{K}))$ be the standard maximal diagonalizable subgroup of G and $H := \langle \varphi_i(\mathrm{diag}(x, x^{-1}))|\ i \in I, x \in X \rangle$. Then we have the following:*

(i) $N_G(H) = N_G(T)$.

(ii) $C_G(H) = C_G(T)$.

(iii) $C_G(H)/T$ *coincides with the kernel of the action of the Weyl group of* $\mathcal{D}$ *on the coroot lattice* $Q(\mathcal{D})^\vee$.

Proof. Let $\mathcal{B}$ be the twin building associated with G and let $\mathcal{A}$ be the standard twin apartment.

It follows from Lemma 4.9(i) that H is regular. Thus the set of fixed chambers of H in $\mathcal{B}$ is $\mathcal{A}$. Therefore $N_G(H) \leq \mathrm{Stab}_G(\mathcal{A}) = N_G(T)$, where the equality follows from Lemma 4.7(ii). On the other hand Lemma 4.9(ii) implies that $N_G(T)$ normalizes H. Thus $N_G(T) \leq N_G(H)$ and Assertion (i) follows.

Note that (ii) is a consequence of (iii). Hence, it remains to prove Assertion (iii).

Since H is a subgroup of T and T is abelian, we have $T \leq C_G(H)$ and it makes sense to consider the quotient $C_G(H)/T$. Moreover, by (i) we have $C_G(H) \leq N_G(T)$ from which it follows that $C_G(H)/T$ identifies to a subgroup of the Weyl group $W \simeq N_G(T)/T$.

Let us identify T with $\mathcal{T}_\Lambda(\mathbb{K}) = \mathrm{Hom}(\Lambda, \mathbb{K}^\times)$ (see Axiom KMG2). As in §1.1.3, we denote by x^{h_i} the element of T defined by $\lambda \mapsto x^{\langle \lambda, h_i \rangle}$ ($x \in \mathbb{K}^\times$). Note that $x^{h_i} = \varphi_i(\mathrm{diag}(x, x^{-1}))$ by (KMG3). Lemma 7.9 implies that:

$$w x^{h_i} w^{-1} = x^{w(h_i)}$$

for all $i \in I$, $x \in \mathbb{K}^\times$ and $w \in W$, where the action of W on $\Lambda^\vee$ is the one described in §7.3.1. It follows that a given $w \in W$ centralizes H if and only if $x^{h_i} = x^{w(h_i)}$ for all $x \in X$ and $i \in I$. Equivalently $x^{\langle \lambda, h_i - w(h_i) \rangle} = 1$ for all $x \in X$, $i \in I$ and $\lambda \in \Lambda$. Since X is infinite, this implies $w(h_i) = h_i$ for all $i \in I$, and Assertion (iii) follows. □

7.4 Proof of the non-linearity theorem

7.4.1 Rigidity of diagonalizable algebraic groups

Let $\mathbb{K}$ be an algebraically closed field. A linear algebraic $\mathbb{K}$-group $\mathbf{G}$ is called **diagonalizable** if is it isomorphic to a closed subgroup of the subgroup of diagonal matrices of $\mathrm{GL}_n(\mathbb{K})$ for some n.

The following classical result is a standard consequence of the rigidity theorem for diagonalizable algebraic groups.

Proposition 7.11. *Let* $\mathbf{G}$ *be a linear algebraic group and* $\mathbf{H}$ *be a diagonalizable subgroup. Then* $N_{\mathbf{G}}(\mathbf{H})/C_{\mathbf{G}}(\mathbf{H})$ *is finite.*

Proof. See [Bor91, §III.8.10, Corollary 2]. □

7.4.2 Linear images of diagonalizable subgroups of Kac-Moody groups

Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be a Kac-Moody root datum and $\mathcal{F} = (\mathcal{G}, (\varphi_i)_{i \in I}, \eta)$ be the basis of a Tits functor $\mathcal{G}$ of type $\mathcal{D}$.

Lemma 7.12. *Let $\mathbb{K}$ be an infinite field, let $G := \mathcal{G}(\mathbb{K})$ and $T := \eta(\mathcal{T}_\Lambda(\mathbb{K}))$. Let $\varphi : G \to GL_n(\mathbb{F})$ be a nontrivial homomorphism, where n is a positive integer and $\mathbb{F}$ is an algebraically closed field. Suppose that the image of φ is non-abelian. Then* $\mathrm{char}(\mathbb{F}) = \mathrm{char}(\mathbb{K})$ *and $\mathcal{G}(\mathbb{K})$ possess a subgroup H contained in T such that the following conditions hold:*

(i) $N_G(H) = N_G(T)$, $C_G(H) = C_G(T)$ and $C_G(H)/T$ coincides with the kernel of the action of the Weyl group of $\mathcal{D}$ on the coroot lattice $Q(\mathcal{D})^\vee$.

(ii) The Zariski closure of $\varphi(H)$ is diagonalizable in $GL_n(\mathbb{F})$.

Proof. Since the image of φ is non-abelian, it follows that G is generated by the images of the φ_i's. Since φ is nontrivial, at least one of the composites $\varphi \circ \varphi_i$ is nontrivial, which implies, using Lemma 3.9, that $\mathrm{char}(\mathbb{F}) = \mathrm{char}(\mathbb{K})$.

If $\mathrm{char}(\mathbb{K}) = 0$, set $H_i := \langle \varphi_i(\mathrm{diag}(x, x^{-1}))|\ x \in \mathbb{Q}^\times \rangle$ and $H := \langle H_i|\ i \in I\rangle$. By Lemma 5.9 applied to the restriction of $\varphi \circ \varphi_i$ to $SL_2(\mathbb{Q})$, the Zariski closure of H_i in $GL_n(\mathbb{F})$ is a torus. Since the H_i's centralize each other, so do the Zariski closures of their images under φ (see [Bor91, Proposition I.2.4)], from which it follows that the Zariski closure of $\varphi(H)$ is a torus. Thus (ii) holds in this case.

If $\mathrm{char}(\mathbb{K}) = p > 0$ and $\mathbb{K}$ is locally finite, set $H := T$. Each element of T is of finite order prime to p; the image of such an element under φ is thus diagonalizable because $\mathbb{F}$ is algebraically closed. Since T is abelian, we that $\varphi(T)$ is conjugate to the subgroup of diagonal matrices of $GL_n(\mathbb{F})$, whence (ii).

If $\mathrm{char}(\mathbb{K}) = p > 0$ and $\mathbb{K}$ is not locally finite, choose an element $t \in \mathbb{K}$ which is transcendental over $\mathbb{F}_p$. Using Jordan decomposition in $GL_n(\mathbb{F})$ and the fact that $\mathrm{char}(\mathbb{F}) = p$, we see that for each $i \in I$ there exists $n_i \in \mathbb{Z}_{>0}$ such that $\varphi \circ \varphi_i(\mathrm{diag}(t^{n_i}, t^{-n_i}))$ is semisimple, hence diagonalizable. Let $n := \prod_{i \in I} n_i$ and $H := \langle \varphi_i(\mathrm{diag}(t^n, t^{-n}))|\ i \in I\rangle$. As before, since H is abelian we deduce that the Zariski closure of $\varphi(H)$ is diagonalizable.

In each of the three cases above, Assertion (i) follows from the definition of H and from Lemma 7.10. □

7.4.3 Proof of the non-linearity theorem

We now proceed to the proof of Theorem 7.1. Let thus $\mathcal{D}, \mathcal{F}, \mathcal{G}$ and $\mathbb{K}$ be as in §7.1 and let $\varphi : \mathcal{G}(\mathbb{K}) \to GL_n(\mathbb{F})$ be a central homomorphism. Without loss of generality, we may and shall assume that $\mathbb{F}$ is algebraically closed. Moreover, since any central quotient of a Kac-Moody group is again a Kac-Moody group whose type is given by the same generalized Cartan matrix, we may and shall assume without loss of generality that the representation φ is faithful.

Let $G := \mathcal{G}(\mathbb{K})$, $T := \eta(\mathcal{T}_\Lambda(\mathbb{K}))$ and $N := N_G(T)$. Let $H \leq T$ be the subgroup of G provided by Lemma 7.10. Then $\varphi(N_G(H))$ normalizes $\varphi(H)$ and its Zariski closure. Therefore, by Proposition 7.11, it has a finite index subgroup which centralizes $\varphi(H)$. Since φ is injective, we deduce from Lemma 7.10 that the Weyl group W of $\mathcal{D}$ has a finite index subgroup which acts trivially on the coroot lattice $Q(\mathcal{D})^\vee$. By Lemma 7.8, this implies that each component of the generalized Cartan matrix A is of finite or affine type. □

Chapter 8

Unitary forms of Kac-Moody groups

8.1 Introduction

So far, the only Kac-Moody groups we have considered were split. The purpose of this chapter is to illustrate the possibility of generalizing the ideas developed in this work in order to obtain a theory of "abstract" homomorphisms which applies to all "forms" of Kac-Moody groups (in the sense of [Rém02b]). Here, we restrict ourself to the unitary forms of complex Kac-Moody groups. These unitary forms generalize the class of compact semisimple Lie groups (in particular, they are "anisotropic"). By adapting the ideas and tools developed in the preceding chapters, we obtain an isomorphism theorem and a non-linearity theorem for these unitary forms (see Theorems 8.2 and 8.8 below). Most of the necessary adaptations are straightforward.

8.2 Definitions

8.2.1 The compact involution

Let A be a generalized Cartan matrix, let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum of type A and $\mathcal{G}$ be a Tits functor of type $\mathcal{D}$ with basis $\mathcal{F} = (\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$.

Let $\mathfrak{g}_A$ be the Kac-Moody algebra of type A and let e_i, f_i, h_i ($i \in I$) be its standard generators (see §1.1.2). It is immediate to check that the assignments $e_i \mapsto -f_i$, $f_i \mapsto -e_i$, $h_i \mapsto -h_i$ extend to a well defined involutory automorphism ω of $\mathfrak{g}_A$ which is called the **Chevalley involution**. Composing ω with the complex conjugation, one obtains a well defined involutory automorphism ω_c of $\mathfrak{g}_A$ which is called the **compact involution**.

The Chevalley involution (resp. the complex conjugation) lifts to a well defined sign automorphism (resp. field automorphism) of the Kac-Moody groups $\mathcal{G}(\mathbb{C})$. Hence the compact involution lifts to a well defined automorphism of $\mathcal{G}(\mathbb{C})$, which is also called the **compact involution** and noted ω_c.

The **unitary form** (or **compact form**) of the complex Kac-Moody group $\mathcal{G}(\mathbb{C})$ is the subgroup, noted $\mathcal{K}(\mathcal{D})$, consisting of the fixed points of ω_c in $\mathcal{G}(\mathbb{C})$. If the root datum $\mathcal{D}$ is simply connected, the corresponding complex Kac-Moody group is noted $\mathcal{G}(A)$ and its unitary form $\mathcal{K}(A)$. If $\mathcal{D}$ is any other Kac-Moody root datum, then there exists a canonical map $\mathcal{K}(A) \to \mathcal{K}(\mathcal{D})$ whose image coincides with the derived subgroup of $\mathcal{K}(\mathcal{D})$.

8.2.2 The topology of $\mathcal{K}(A)$

We endow the group $\mathcal{G}(A)$ with the finest topology such that for each $n \in Z_{>0}$ and each n-tuple $(i_1, \dots, i_n) \in I^n$ of elements of I, the map

$$SL_2(\mathbb{C}) \times \cdots \times SL_2(\mathbb{C}) \to \mathcal{G}(A) : (x_1, \dots, x_n) \mapsto \varphi_{i_1}(x_1) \dots \varphi_{i_n}(x_n)$$

is continuous. In this way, $\mathcal{G}(A)$ is endowed with a structure of a connected simply-connected Hausdorff topological group (see [KP83, §4G]).

Consider the following subgroup of $\mathcal{G}(A)$:

$$H_+ := \langle \varphi_i(\mathrm{diag}(t, t^{-1})) |\ i \in I, t \in \mathbb{R}_{>0} \rangle.$$

Let also U_+ be the unipotent radical of the standard Borel subgroup of $\mathcal{G}(A)$ of positive sign. The subgroups H_+, U_+ and $\mathcal{K}(A)$ are closed in $\mathcal{G}(A)$. Moreover, H_+ and U_+ are contractible and the multiplication map

$$\mathcal{K}(A) \times H_+ \times U_+ \to \mathcal{G}(A)$$

is a homeomorphism (see loc. cit.). In particular, $\mathcal{K}(A)$ is a connected simply-connected Hausdorff topological group.

The group $\mathcal{K}(A)$ is compact if and only if A is of finite type. In that case $\mathcal{K}(A)$ is nothing but the connected simply connected compact semisimple Lie group of type A.

8.2.3 The subgroup structure of $\mathcal{K}(A)$

The group $\mathcal{K}(A)$ is **anisotropic** in the sense that it intersects the unipotent radical of every Borel subgroup of $\mathcal{G}(A)$ trivially.

Given $i \in I$, the map φ_i yields by restriction a continuous homomorphism $\varphi_i : SU_2 \to \mathcal{K}(A)$ whose image is noted K_i. The image of $\{\mathrm{diag}(t, t^{-1}) |\ t \in \mathbb{C},\ ||t|| = 1\}$ under φ_i is noted H_i, and one sets $T := \langle H_i |\ i \in I \rangle$.

Given $J \subset I$, let P_+^J (resp. P_-^J) be the standard parabolic subgroup of type J and sign $+$ (resp. $-$) of $\mathcal{G}(A)$. One has $P_+^J \cap \mathcal{K}(A) = P_-^J \cap \mathcal{K}(A)$ and the common value is denoted by $\mathcal{K}(A)_J$. For each $i \in I$, one has $\mathcal{K}(A)_{\{i\}} = TK_i$, and moreover $\mathcal{K}(A)_\emptyset = T$. The subgroup $\mathcal{K}(A)_J$ is compact if and only if J is of finite type (i.e. the matrix $A_J := (A_{ij})_{i,j \in J}$ is of finite type), in which case $\mathcal{K}(A)_J$ is a isomorphic to a connected simply-connected reductive compact Lie group of type A_J.

A compact connected abelian subgroup of $\mathcal{K}(A)$ is called a **torus**. The following result is due to Kac and Peterson.

Proposition 8.1. *(i) Every compact subgroup of $\mathcal{K}(A)$ (resp. $\mathcal{G}(A)$) is contained in a maximal compact subgroup of $\mathcal{K}(A)$ (resp. $\mathcal{G}(A)$).*

(ii) A compact subgroup of $\mathcal{K}(A)$ (resp. $\mathcal{G}(A)$) is maximal if and only if it is conjugate to a subgroup of the form $\mathcal{K}(A)_J$, where J is maximal among all finite-type subsets of I.

(iii) T is a maximal torus of $\mathcal{K}(A)$, and every torus of $\mathcal{K}(A)$ is conjugate to a subgroup of T.

Proof. See [KP87, Proposition 3.5] and the subsequent remarks. □

Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be the twin building associated with $\mathcal{G}(A)$. It is easy to see that $\mathcal{K}(A)$ is transitive on the chambers of $\mathcal{B}_+$ and the chambers of $\mathcal{B}_-$ (this follows from the fact that SU_2 is transitive on the projective line $\mathbb{C}P^1$, together with the fact that a building is connected as a chamber system). This observation, together with a lemma of Tits (see [Tit86, §14]), yields a simple proof of the fact that $\mathcal{K}(A)$ is the amalgam of its subgroups of the form $\mathcal{K}(A)_J$ for $J \subset I$, J of finite type and $|J| \leq 2$. Kac and Peterson proved an elegant refinement of this amalgam presentation of $\mathcal{K}(A)$ (see [KP85, Proposition 5.1(e)]) but we won't need their specific presentation here.

8.3 Isomorphisms of unitary forms

8.3.1 The isomorphism theorem for unitary forms

The setting is the following:

- $A = (A_{ij})_{i,j\in I}$ is a generalized Cartan matrix.
- $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ is a Kac-Moody root datum of type A.
- $\mathcal{F} = (\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$ is the basis of a Tits functor $\mathcal{G}$ of type $\mathcal{D}$.
- $K := \mathcal{K}(\mathcal{D})$ is the unitary form of $\mathcal{G}(\mathbb{C})$.

Let also $\mathcal{F}' = (\mathcal{G}', (\varphi'_i)_{i\in I'}, \eta')$ be the basis of a Tits functor $\mathcal{G}'$ of type $\mathcal{D}' = (I', A', \Lambda', (c'_i)_{i\in I'}, (h'_i)_{i\in I'})$, and let $K' := \mathcal{K}(\mathcal{D}')$ denote the unitary form of $\mathcal{G}'(\mathbb{C})$.

Theorem 8.2. *Let $\varphi : K \to K'$ be an isomorphism. Then there exists a bijection $\pi : I \to I'$, an inner automorphism ν of K' and for each $i \in I$, a diagonal-by-sign automorphism δ_i of SU_2, such that the diagram*

$$\begin{tikzcd} SU_2 \arrow[r, "\delta_i"] \arrow[d, "\varphi_i"'] & SU_2 \arrow[d, "\varphi'_{\pi(i)}"] \\ K \arrow[r, "\nu\circ\varphi"] & K' \end{tikzcd}$$

commutes for every $i \in I$. Furthermore one has $A_{ij} = A'_{\pi(i)\pi(j)}$ for all $i, j \in I$.

If moreover A is indecomposable then there exist a diagonal automorphism δ and a sign automorphism σ of K' such that the diagram

$$\begin{tikzcd} SU_2 \arrow[r, "\mathrm{id}"] \arrow[d, "\varphi_i"'] & SU_2 \arrow[d, "\varphi'_{\pi(i)}"] \\ K \arrow[r, "\delta\circ\sigma\circ\nu\circ\varphi"] & K' \end{tikzcd}$$

commutes for every $i \in I$.

It was observed by Kac and Peterson, as a consequence of Proposition 8.1, that a continuous isomorphism $\mathcal{K}(A) \to \mathcal{K}(A')$ between unitary forms of complex Kac-Moody groups must satisfy the conclusions of the theorem above (see [KP87, Remark (f) on p. 136]). Actually, it follows from Theorem 8.2 that any abstract isomorphism of unitary forms is a homeomorphism. This is well known in the finite-dimensional case.

8.3.2 Preparatory lemmas

Throughout this section, we place ourself in the setting of §8.3.1. We also keep the notation of §8.2.3. In particular, T is the standard maximal torus of $\mathcal{K}(\mathcal{D})$.

Lemma 8.3. *Let $p > 10$ be a prime such that the Weyl group of $\mathcal{D}$ has no element of order p. We have the following.*

(i) The p-torsion subgroup of T is a maximal elementary abelian p-subgroup of $\mathcal{G}(\mathbb{C})$. It is diagonalizable and regular.

(ii) Any two maximal elementary abelian p-subgroups of $\mathcal{K}(\mathcal{D})$ are conjugate in $\mathcal{K}(\mathcal{D})$.

Proof. Let H be the p-torsion subgroup of T. Since T is a direct product of circle groups, it is clear that H is an elementary abelian p-group. Moreover, it is clearly diagonalizable. The fact that it is regular is a consequence of Lemma 4.9(ii). The remaining assertions of the lemma will follow if we prove that any elementary abelian p-subgroup of $\mathcal{K}(\mathcal{D})$ is conjugate in $\mathcal{K}(\mathcal{D})$ to a subgroup of T.

Let H' be an elementary abelian p-subgroup of $\mathcal{K}(\mathcal{D})$. Since H' is finite, it is bounded in $\mathcal{G}(\mathbb{C})$. Therefore, it follows from Proposition 3.12 that H' is diagonalizable in $\mathcal{G}(\mathbb{C})$. Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be the twin building associated with $\mathcal{G}(\mathbb{C})$. It follows that H' is contained in the stabilizer of a chamber of $\mathcal{B}_+$ in $\mathcal{K}(\mathcal{D})$. But we have seen that $\mathcal{K}(\mathcal{D})$ is transitive on $\mathcal{B}_+$ and that T is the stabilizer of the standard chamber of $\mathcal{B}_+$ (see §8.2.3). It follows that H' is conjugate to a subgroup of T. □

Lemma 8.4. *Let $p > 10$ be a prime such that the Weyl group of $\mathcal{D}$ has no element of order p, let H be the p-torsion subgroup of T and let $i \in I$. Then the centralizer of $K_i := \varphi_i(SU_2)$ in H is a coregular diagonalizable subgroup of $\mathcal{G}(\mathbb{C})$.*

Proof. It follows from Lemma 7.9 that $C_H(K_i)$ also centralizes $\varphi_i(SL_2(\mathbb{C}))$. Thus the lemma is a consequence of Lemma 4.9(iii). □

Lemma 8.5. *Let T be a simplicial tree, G be a connected compact Lie group and suppose that G acts (not necessarily continuously) on T by isometries. Then G fixes a vertex of T or an end of T.*

Proof. This is a special case of the following result: Any locally compact connected Hausdorff topological group acting on T fixes a vertex or an end of T (see the main result of [Alp82]). However, in the case of a connected compact Lie group, the following simple argument yields a proof of the lemma. Tori are divisible, hence any element of a tori must be elliptic because T is simplicial. Since every element is contained in a torus, the conclusion is an easy consequence of [Tit77, Lemma 1.6]. □

Lemma 8.6. *Every automorphism of SU_2 decomposes as the product of an inner automorphism and a diagonal automorphism, i.e. an automorphism of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & t.b \\ t^{-1}.c & d \end{pmatrix}$$

for some $t \in \mathbb{C}$ with $||t|| = 1$.

Proof. This follows from a well known theorem of H. Freudenthal. □

The following lemma will serve as a substitute for Theorem 1.5 in the case of unitary forms of Kac-Moody groups.

Lemma 8.7. *Let $\varphi : \mathcal{K}(\mathcal{D}) \to \mathcal{K}(\mathcal{D}')$ be an isomorphism. Let T (resp. T') be the standard maximal torus of $\mathcal{K}(\mathcal{D})$ (resp. $\mathcal{K}(\mathcal{D}')$). Suppose that*

$$\begin{aligned} & \{\varphi(g\varphi_i(SU_2)g^{-1})|\ i \in I,\ g \in N_{\mathcal{K}(\mathcal{D})}(T)\} \\ = \ & \{g\varphi'_i(SU_2)g^{-1}|\ i \in I',\ g \in N_{\mathcal{K}(\mathcal{D}')}(T')\}. \end{aligned}$$

Then there exist an element $g \in N_{\mathcal{K}(\mathcal{D}')}(T')$, a bijection $\pi : I \to I'$ and for each $i \in I$, an automorphism α_i of SU_2 which normalizes the standard maximal torus, such that the diagram

$$\begin{tikzcd} SU_2 \arrow[r, "\alpha_i"] \arrow[d, "\varphi_i"'] & SU_2 \arrow[d, "\varphi'_{\pi(i)}"] \\ \mathcal{K}(\mathcal{D}) \arrow[r, "(\operatorname{Inn} g)\circ\varphi"] & \mathcal{K}(\mathcal{D}') \end{tikzcd}$$

commutes for all $i \in I$, where $\operatorname{Inn} g$ *denotes the conjugation by g. Furthermore, one has $A_{ij} = A'_{\pi(i)\pi(j)}$ for all $i, j \in I$.*

Proof. It suffices to prove the lemma for indecomposable generalized Cartan matrices A and A', and we assume from now on that A and A' are indecomposable.

We set $K := \mathcal{K}(\mathcal{D})$ and $K' := \mathcal{K}(\mathcal{D}')$. Let W (resp. W') be the Weyl group of $\mathcal{D}$ (resp. $\mathcal{D}'$) and $\Delta \subset Q(A)$ (resp. $\Delta' \subset Q(A')$) be as in Lemma 7.6(iv). Given $a \in \Delta$, let $w \in W$ and $i \in I$ be such that $a = w.a_i$ and set $K_a := w\varphi_i(SU_2)w^{-1}$; then K_a depends only on a and not on the choice of i and w. Define similarly K'_b for $b \in \Delta'$. Note that $K_a = K_{-a}$ for all $a \in \Delta$ (resp. $K'_a = K'_{-a}$ for all $a \in \Delta'$).

By hypothesis, given $a \in \Delta$, one has $\varphi(K_a) = K'_{a'}$ for some a' which is uniquely defined up to a sign. Hence the isomorphism φ induces a bijection π_0 between pairs of opposite roots of Δ and pairs of opposite roots of Δ'. Since moreover $\varphi(T) = T'$, it follows that π_0 induces a reflection-preserving isomorphism π_1 between the Weyl groups $W \simeq N_K(T)/T$ of $\mathcal{D}$ and $W' \simeq N_{K'}(T')/T'$ of $\mathcal{D}'$. We shall now construct a bijection $\Delta \to \Delta'$ which is π_1-equivariant.

Let $i \in I$ and choose $b_i \in \Delta'$ be such that $\pi_0(\{\pm a_i\}) = \{\pm b_i\})$. Let $w \in W'$ and $i' \in I'$ be such that $w.a'_{i'} = b_i$, and let δ_i be an automorphism of SU_2 such that the diagram

$$\begin{tikzcd} SU_2 \arrow[r, "\delta_i"] \arrow[d, "\varphi_i"'] & SU_2 \arrow[d, "w\varphi'_{i'}w^{-1}"] \\ K \arrow[r, "\varphi"] & K' \end{tikzcd}$$

commutes. Since $\varphi(T) = T'$, it follows from Lemma 8.6 that there exists a unique inner automorphism ι_i of SU_2 which is either trivial or conjugation by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and such that $\delta_i \circ \iota_i$ is a diagonal automorphism of SU_2. In other words, δ_i is diagonal-by-sign. We set $\pi(a_i) := b_i$ if ι_i is trivial and $\pi(a_i) := -b_i$ otherwise.

It is easy to check that the assignments $a_i \mapsto \pi(a_i)$ extend uniquely to a bijection $\pi : \Delta \to \Delta'$ such that for all $a \in \Delta$ and $w \in W$ one has $\pi(w.a) = \pi_1(w).\pi(a)$. Furthermore, it follows from the definition of π, together with Lemma 7.9, that for all $i, j \in I$ one has $\langle a_i, a_j^\vee\rangle = \langle \pi(a_i), \pi(a_j)^\vee\rangle$. In particular, one deduces that the set $\{\pi(a_i)|\ i \in I\}$

is a root basis of Δ', or in other words, that every element of Δ' can be written as a linear combination $\pm\sum_{i\in I}\lambda_i\pi(a_i)$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$ for all $i \in I$. Now it follows from Kac' conjugation theorem of root bases (see [Kac90, Proposition 5.9]) that there exists $w_0 \in W'$ and $\epsilon \in \{+,-\}$ such that $w_0.\pi(\epsilon a_i) \in \{a'_j |\ j \in I'\}$. Let $g \in N_{K'}(T')$ be such that $g.T' = w_0$. Then, for each $i \in I$, the isomorphism $(\operatorname{Inn} g) \circ \varphi$ maps K_{a_i} to $K'_{a'_j}$ for some $j \in I'$, and the assertions of the lemma follow easily. $\square$

8.3.3 Proof of Theorem 8.2

Let $p > 10$ be a prime such that no element of Weyl groups of $\mathcal{D}$ and $\mathcal{D}'$ is of order p. Let H be the p-torsion subgroup of T. By Lemma 8.3 there exists $\nu \in \operatorname{Inn}(K')$ such that $\nu \circ \varphi$ maps H to the p-torsion subgroup H' of T'. Let $\varphi' := \nu \circ \varphi$.

Let now $i \in I$ and $H_i^\vee := C_H(K_i)$, where $K_i := \varphi_i(SU_2)$. By Lemma 8.4, $H_i^\vee$ is coregular. We claim that $\varphi'(H_i^\vee)$ is coregular.

Let $\mathcal{B}' = (\mathcal{B}'_+, \mathcal{B}'_-, \delta^*)$ be the twin building associated with $\mathcal{G}'(\mathbb{C})$ and let $\mathcal{A}'$ be the standard twin apartment.

Suppose first that $\varphi'(H_i^\vee)$ is regular. Then $\varphi'(K_i)$ stabilizes $\mathcal{A}'$. Since every element of K_i is contained in a torus of K_i, it follows that for all $g \in K_i$ and $n \in \mathbb{Z}_{>0}$, there exists $h \in \varphi'(K_i)$ such that $h^n = g$. This implies that $\varphi'(K_i)$ acts trivially on $\mathcal{A}'$, which is impossible since $\operatorname{Fix}_{K'}(\mathcal{A}') = T'$ is abelian. Thus $\varphi'(H_i^\vee)$ is not regular.

Suppose now that $\varphi'(H_i^\vee)$ is not coregular. Then the centralizer of $\varphi'(H_i^\vee)$ in K' contains a subgroup, say X' which is isomorphic to a connected compact semisimple Lie group of rank 2. Transforming by $(\varphi')^{-1}$ and using the fact that $H_i^\vee$ is coregular, together with Lemma 8.5, we obtain a contradiction. This proves the claim.

Mimicking the arguments of Steps 3–6 from the proof of Proposition 4.17 (the present situation is simpler in view of Lemma 8.5), one deduces from the claim that φ' satisfies the hypotheses of Lemma 8.7. This lemma yields the desired conclusions. $\square$

8.4 Non-linearity

Let A be a generalized Cartan matrix, $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum of type A, $\mathcal{G}$ be a Tits functor of type $\mathcal{D}$ and $K := \mathcal{K}(\mathcal{D})$ be the unitary form of $\mathcal{G}(\mathbb{C})$.

Theorem 8.8. *Let n be a positive integer, $\mathbb{F}$ be a field and $\varphi : K \to GL_n(\mathbb{F})$ be a homomorphism with central kernel. Then every indecomposable component of the generalized Cartan matrix A is of finite or affine type.*

Proof. We may assume without loss of generality that $\mathbb{F}$ is algebraically closed.

Let $T := \langle \varphi_i(\operatorname{diag}(t, t^{-1}))|\ i \in I,\ t \in \mathbb{C}, ||t|| = 1\rangle$. Let H be the subgroup of T generated by the elements of p^n-torsion for all primes $p \neq \operatorname{char}(\mathbb{F})$ and all $n \in \mathbb{Z}_{>0}$. Then $\varphi(H)$ is an abelian group consisting of diagonalizable elements of $GL_n(\mathbb{F})$; in particular its Zariski closure is diagonalizable.

On the other hand, it follows from Lemma 7.10 that $N_G(H) = N_G(T)$, $C_G(H) = C_G(T)$ and $C_G(H)/T$ coincides with the kernel of the action of the Weyl group of $\mathcal{D}$ on the coroot lattice $Q(\mathcal{D})^\vee$, where $G := \mathcal{G}(\mathcal{D})$. Therefore, the same arguments as in the proof of Theorem 7.1 show that every indecomposable component of the generalized Cartan matrix A is of finite or affine type. $\square$

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