The thick frame of a weak twin building

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1st April 2004

ABSTRACT. We prove the following extension to twin buildings of a result for spherical buildings which appears in [11] : *to every weak twin building* Δ*, is canonically associated a thick twin building* $\bar{\Delta}$ *whose Weyl group* $W(\bar{\Delta})$ *can be considered as a reflection subgroup of the Weyl group* $W(\Delta)$ *of* Δ *; conversely, it is possible to recover* Δ *from the thick twin building* $\bar{\Delta}$ *and from the inclusion* $W(\bar{\Delta}) \hookrightarrow W(\Delta)$.

1 Introduction

(1.1) In the course of the 1950's Jacques Tits developed the theory of buildings in order to study groups of Lie type from a combinatorial point of view. A building was then defined as a simplicial complex with an apartment system, and a certain thickness axiom was used in order to prove that the apartments are actually Coxeter complexes. A more recent approach consists in seeing a building as a metric space (Δ, δ) where the metric δ takes its values in the Coxeter group W of some fixed Coxeter system (W, S). In this new setting, the thickness axiom is no longer needed for the basic developments of the theory, and the consideration of *weak buildings* – namely buildings for which the axiom of thickness is not required – becomes quite natural.

In 1987, R. Scharlau provided a structure theorem for weak buildings of spherical type (see [11]), which can be seen as a definitive version of earlier results by G. Birkhoff [1], F. Buekenhout and A. Sprague [2] and S. Rees [9]. Scharlau describes in his paper how an arbitrary weak spherical building Δ of type (W, S) yields a thick spherical building $\bar{\Delta}$ of type $(\bar{W}, \bar{S})$ together with a *geometric inclusion* of $(\bar{W}, \bar{S})$ into (W, S) (see (3.4) for the definition of the latter notion). Conversely, given a thick spherical building $\bar{\Delta}$ of type $(\bar{W}, \bar{S})$ and a geometric inclusion of $(\bar{W}, \bar{S})$ in some Coxeter system (W, S), then there exists a weak spherical building Δ of type (W, S) whose associated thick building is Δ.

It turns out that this result cannot be generalized to arbitrary buildings: in fact, it is quite easy to produce counterexamples for buildings of universal type. The main purpose of this paper is to prove that the result remains however true for *twin buildings*.

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AMS subject classification codes (2000) : 51E24.
Keywords: Twin building, weak building.

Twin buildings were introduced by M. Ronan and J. Tits to study groups of Kac-Moody type, which are natural infinite-dimensional generalizations of Chevalley groups. Twin buildings are attached to Kac-Moody groups just as spherical buildings are associated with Chevalley groups. Our main result has found an application in this context to the solution of the isomorphism problem for Kac-Moody groups over algebraically closed fields (see [3], [4]). There, weak twin buildings appear as fixed point sets of finite subgroups of tori. The action of the centralizer of such a subgroup on the corresponding fixed weak building and on the associated thick frame turns out to be a crucial ingredient in the solution of this isomorphism problem.

As we have mentioned, twin buildings generalize spherical buildings in a natural way, so the main result of this paper does not seem surprising at first. Nevertheless, the methods presented in [11] do not apply in the more general situation of twin buildings. Hence we also provide a new approach to the spherical case.

To make the statement of the main result more precise, we first introduce some terminology which is specific to weak buildings, and notably inspired by [11].

(1.2) Let (W, S) be a Coxeter system and (Δ, δ) be a weak building of type (W, S). Two chambers of Δ are called **thick-adjacent** if they are contained in some thick panel. A gallery $\gamma = (x_0, x_1, \ldots, x_n)$ is called **thin** if $\{x_{i-1}, x_i\}$ is a thin panel of Δ for each $i \in [1, n]$. The set of all ordered pairs of chambers (x, y) such that x can be joined to y by a thin gallery, is an equivalence relation, which is called the **thin-equivalence**. The corresponding equivalence classes are called **thin-classes**. If $c \in \Delta$ is any chamber, we denote by $\bar{c}$ the thin-class of c; if $\Gamma \subset \Delta$ is any set of chambers, we denote by $\bar{\Gamma}$ the corresponding set of thin-classes. It is easy to see that any apartment containing c necessarily contains the whole thin-class $\bar{c}$.

Our main result is the following (see (3.4) for the definition of a geometric inclusion between Coxeter systems).

(1.3) Theorem. *Let $\Delta = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ be a twinned pair of weak buildings of type (W, S), for some Coxeter system (W, S). Let (c_+, c_-) be a fixed pair of opposite chambers in Δ, and denote by $\Sigma = (\Sigma_+, \Sigma_-)$ the corresponding twin apartment. Let $\bar{S}$ be the set of reflections of Σ_+ corresponding to thick panels which intersect $\bar{c}_+$ non trivially, and denote by $\bar{W}$ the group generated by $\bar{S}$. Then $(\bar{W}, \bar{S})$ is a Coxeter system, and $(\bar{\Delta}_+, \bar{\Delta}_-)$ is naturally endowed with a structure of thick twin building of type $(\bar{W}, \bar{S})$, which is called the **thick frame** of Δ and denoted by $\bar{\Delta}$. If (Σ'_+, Σ'_-) is a twin apartment of Δ then $(\bar{\Sigma}'_+, \bar{\Sigma}'_-)$ is a twin apartment of $\bar{\Delta}$; moreover, two thin-classes are opposite as chambers of $\bar{\Delta}$ if and only if they contain opposite chambers of Δ.*

Conversely, if $\bar{\Delta}$ is a thick building of type $(\bar{W}, \bar{S})$ and $(\bar{W}, \bar{S})$ is geometrically included in some Coxeter system (W, S), then $\bar{\Delta}$ is the thick frame of a weak twin building Δ of type (W, S).

(1.4) Remarks.

1. In the spherical case, [11] points out that the rank of the reflection subgroup $\bar{W}$ of the finite reflection group W is bounded by the rank of this latter group; namely, we have $|\bar{S}| \leq |S|$. This inequality is no longer true in the twin case, when the Coxeter group W is infinite. It is well known, for example, that Coxeter systems of universal type and arbitrary rank are geometrically included in some fixed Coxeter system (W, S)

of rank 3 and universal type. On the other hand, the existence of thick buildings of type (W, S) follows from the theory of Kac-Moody groups, and allows us to apply the converse part of Theorem (1.3).

2. In [11], Scharlau describes his construction for the purpose of classifying weak spherical buildings. Clearly, a result like Theorem (1.3) reduces such a classification to the one of thick buildings together with geometric inclusions of reflection groups $\bar{W} \hookrightarrow W$: this is exactly the approach adopted in loc. cit. In the general framework of twin buildings, however, a classification even in the thick case is impossible without important restrictions on the type (see [6], [7] and [15]). Thus, the classification of weak twin buildings cannot be realized in full generality.

3. In the special case when W is the infinite dihedral group, or in other words when Δ is a twin tree, the result was proved in [5].

4. As previously mentioned, we will avoid appealing to the geometric realization of the buildings we will meet. This approach will notably lead us to give a new, purely combinatorial, proof of the fact that two reflections in a Coxeter group stabilize a common spherical residue of rank 2 in the corresponding thin building, whenever their product has finite order (see (5.1)).

5. If Δ is thin, then its thick frame $\bar{\Delta}$ is degenerate: it is a twinned pair of trivial buildings possessing each exactly one chamber. The type of $\bar{\Delta}$ is the trivial Coxeter system $(\{1\}, \emptyset)$.

The author is very grateful to Bernhard Mühlherr for his kind support in the preparation of this work.

2 Preliminaries

In this section, we recall the basic notions and fix the notation to be used throughout. The main references are [10], [14] and [16].

(2.1) *Buildings.*
Let (W, S) be some fixed Coxeter system, and let ℓ denote the corresponding length function. Recall that a **building** of type (W, S) is a set Δ, whose elements are called **chambers**, endowed with a map $\delta : \Delta \times \Delta \to W$ called the **W-distance**, satisfying the following axioms, where $x, y \in \Delta$ and $w = \delta(x, y)$:

(Bu1) $w = 1 \Leftrightarrow x = y$;
(Bu2) if $z \in \Delta$ is such that $\delta(y, z) = s \in S,$ then $\delta(x, z) \in \{w, ws\}$;
if, furthermore, $\ell(ws) = \ell(w) + 1,$ then $\delta(x, z) = ws$;
(Bu3) if $s \in S,$ there exists $z \in \Delta$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

For chambers $x, y \in \Delta$, the **numerical distance** between x and y in Δ is the natural number $d(x, y) = \ell(\delta(x, y))$. Notice that (Δ, d) is a discrete metric space.

An **isomorphism** (or an **isometry**) between two buildings of the same type (W, S) is a bijection preserving the W-distance.

(2.2) For $c \in \Delta$ and $s \in S$, the set

$$\{x \in \Delta | \delta(x, c) \in \{1, s\}\}$$

is called an s**-panel** of Δ, or a **panel of type** s. A **panel** is an s-panel for some $s \in S$. Two chambers are called s**-adjacent** if they are contained in some s-panel, and simply **adjacent** if they are s-adjacent for some $s \in S$. More generally, for $c \in \Delta$ and $S' \subset S$, the set

$$\mathrm{Res}_{S'}(c) := \{x \in \Delta | \delta(x, c) \in \langle S' \rangle \leq W\}$$

is called the S'**-residue** of Δ which contains c. Its **rank** is the cardinality of the set S'; hence, residues of rank 1 are just panels. It is a fact that an S'-residue is itself a building of type $(\langle S' \rangle, S')$. Furthermore, for each residue R and each chamber c in Δ, there is a unique chamber z of R at minimal numerical distance from c. Hence, $d(c, z) = \min\{d(c, x) | x \in R\}$ and z is unique with respect to that property. The chamber z is called the **projection** of c onto R, and is denoted by $\mathrm{proj}_R(c)$. If $P \subset R$ are two residues, and c is any chamber, then we have $\mathrm{proj}_P(\mathrm{proj}_R(c)) = \mathrm{proj}_P(c)$.

A finite sequence of chambers $\gamma = (x_0, x_1, \ldots, x_n)$ such that two consecutive chambers of γ are adjacent, is called a **gallery** of length n. The gallery γ is called **minimal** if $n = d(x_0, x_n)$. A subset Γ of Δ is called **convex** if for all $x, y \in \Gamma$, any minimal gallery from x to y is completely contained in Γ. For example, residues are convex.

(2.3) A building is called **thin** (resp. **thick**) if its panels have cardinality 2 (resp. at least 3). It is called **weak** if no special assumption is made concerning thinness or thickness. The terms *thin* and *thick* are also used accordingly to qualify panels.

For example, if $\Delta = W$ then $\delta : \Delta \times \Delta \to W : (x, y) \mapsto x^{-1}y$ endows the set W with a canonical structure of thin building of type (W, S), which is denoted by $\Sigma(W, S)$. It is the only thin building of type (W, S), up to isomorphism. The group W naturally acts by automorphisms on $\Sigma(W, S)$ from the left. This action is regular. A non trivial element of W which stabilizes some panel of $\Sigma(W, S)$ is called a **reflection**. To each panel of $\Sigma(W, S)$, there corresponds a unique reflection of W. The set of all panels stabilized by a given reflection t is called the **wall** corresponding to that reflection, and is denoted by $P(t)$. A residue of Δ is stabilized by t if and only if it contains a panel of $P(t)$. We write $C(t) := \bigcup_{\pi \in P(t)} \pi$. If $\{x, y\} \in P(t)$, then the set $H(t, x) := \{c \in \Sigma(W, S) | d(c, x) < d(c, y)\}$ is called a **root** associated to t. We have $\Sigma(W, S) = H(t, x) \sqcup H(t, y)$ and $t(H(t, x)) = H(t, y)$. We also write $H(t, x) = -H(t, y)$ and $H(t, x) = H(t, z)$ for all $z \in H(t, x)$. Roots are convex sets of chambers. Moreover, the wall $P(t)$ is precisely the set of panels which intersect the root $H(t, x)$ in a single chamber.

(2.4) If (Δ, δ) is a building of type (W, S), then any isometric image of $\Sigma(W, S)$ in Δ is called an **apartment**. Apartments are convex sets of chambers, and any two chambers of Δ are contained in some apartment. By a *wall* or a *root in a building*, we mean a wall or a root in some apartment of that building. We will use the corresponding notations (such as $P(t)$, $C(t)$ or $H(t, x)$) accordingly.

Let c be a chamber in Δ and Σ be an apartment which contains c. Hence, there is an isometry $\iota : \Sigma(W, S) \to \Sigma$. Up to replacing ι by $\iota \circ w$ for some element $w \in W$, we may assume that $\iota(1) = c$. In that case, the mapping

$$\rho_{\Sigma,c} : \Delta \to \Sigma : x \mapsto \rho_{\Sigma,c}(x) := \iota(\delta(c, x))$$

is called the **retraction** of Δ onto Σ based at c. Its restriction on Σ is the identity, and it preserves the s-adjacency of chambers, for each $s \in S$.

(2.5) The building is called **spherical** or **of spherical type** when W is finite. In that case, W possesses a unique element of maximal length, denoted by w^0, which is an involution. Two chambers x, y such that $\delta(x, y) = w^0$ are called **opposite**.

(2.6) ***Chamber systems.***
Let S be a set. A **chamber system over** S is a pair $(C, (\sim_s)_{s\in S})$ which consists of a set C, whose elements are called **chambers**, and a collection of equivalence relations on C labelled by S. If x and y are chambers such that $x \sim_s y$, then x and y are called s**-adjacent**. A **morphism** between chamber systems over the same set S is just a mapping between the underlying sets of chambers, such that s-adjacent chambers are sent to s-adjacent chambers. **Automorphisms** and **isomorphisms** are defined as usual.

The following lemma shows that any building is a chamber system. It also insures that, provided a given chamber system is one coming from some building of type (W, S), then it is possible to recover the type (W, S) and the W-distance of this building from the chamber system itself.

(2.7) Lemma. *Let (Δ, δ) be a building of type (W, S). Then Δ is a chamber system of type S where the s-adjacency is defined by $x \sim_s y \Leftrightarrow \delta(x, y) \in \{1, s\}$. The type and the W-distance are uniquely determined by that chamber system alone.*

Proof. The first part follows directly from the definition of a building. For the second part, see Proposition (7.21) and the remark following that proposition in [17]. □

This lemma says in particular that if a chamber system $(C, (\sim_s)_{s\in S})$ is isomorphic *as a chamber system* to some building of type (W, S) (by considering that building as a chamber system according to the preceding lemma), then $(C, (\sim_s)_{i\in S})$ is itself a building of type (W, S).

(2.8) ***Twin buildings.***
As before, let (W, S) denote a fixed Coxeter system. A **twinned pair of buildings** or **twin building** of type (W, S) is a pair $((\Delta_+, \delta_+), (\Delta_-, \delta_-))$ of buildings of that type, endowed with a W**-codistance**

$$\delta^* : (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \to W$$

satisfying the following axioms, where $\epsilon \in \{+, -\}$, $x \in \Delta_\epsilon$, $y \in \Delta_{-\epsilon}$ and $w = \delta^*(x, y)$:

- **(Tw1)** $\delta^*(y, x) = w^{-1}$;
- **(Tw2)** if $z \in \Delta_{-\epsilon}$ is such that $\delta_{-\epsilon}(y, z) = s \in S$ and $\ell(ws) < \ell(w)$, then $\delta^*(x, z) = ws$;
- **(Tw3)** if $s \in S$, there exists $z \in \Delta_{-\epsilon}$ such that $\delta_{-\epsilon}(y, z) = s$ and $\delta^*(x, z) = ws$.

Two chambers $x \in \Delta_+$ and $y \in \Delta_-$ are called **opposite** if $\delta^*(x, y) = 1$. It can be proved that the W-codistance δ^* is completely determined by the opposition relation and the W-distances δ_+ and δ_- (see [14]). Two residues are called **opposite** if they are of the same type and contain opposite chambers. A pair of opposite residues of type S', endowed with the appropriate restriction of the W-codistance, is itself a twin building of type $(\langle S'\rangle, S')$.

(2.9) An ordered pair of apartments $\Sigma = (\Sigma_+, \Sigma_-)$ with $\Sigma_\epsilon \subset \Delta_\epsilon$ for $\epsilon = +, -$ is called a **twin apartment** if the restriction op_Σ of the opposition relation to Σ defines a bijection between Σ_+ and Σ_-. In that case, the appropriate restriction of the W-codistance δ^* endows Σ with a the structure of a twinned pair of thin buildings. If $(c_+, c_-) \in \Delta_+ \times \Delta_-$ is a pair of opposite chambers, then there exists a unique twin apartment (Σ_+, Σ_-) such that $(c_+, c_-) \in \Sigma_+ \times \Sigma_-$.

Abusing the notation, we often write $x \in \Sigma$ in place of $x \in \Sigma_+ \cup \Sigma_-$ for a chamber x. An apartment of Δ_ϵ is called **admissible** if it is involved in some twin apartment. Any two chambers of Δ_ϵ are contained in some admissible apartment.

(2.10) If $\Sigma = (\Sigma_+, \Sigma_-)$ is a twin apartment in a twin building $\Delta = (\Delta_+, \Delta_-, \delta^*)$ of type (W, S), then the group W acts naturally on both Σ_+ and Σ_-, and this action preserves the restriction of the W-codistance of Δ to Σ. Therefore, if $t \in W$ is a reflection of Σ_+, it is also a reflection of Σ_- and vice versa, and we say that t is a reflection of Σ. The symbol $P(t)$ is used to denote the union of the walls associated to t both in Σ_+ and Σ_-, and the definition of $C(t)$ is adapted accordingly. The symbol $H(t, c)$ still makes sense for a chamber $c \in C(t)$, and denotes a root of Σ_+ or Σ_- depending on whether c is in Δ_+ or Δ_-. A pair of roots $(\alpha, \tilde{\alpha})$ in the twin apartment Σ is called a **twin root** if we have $\tilde{\alpha} = -op_\Sigma \alpha$, where op_Σ denotes the restriction of the opposition relation to Σ. If $x \in \Delta_+$ and $y \in \Delta_-$ are two chambers such that $\delta^*(x, y) \in S$, then there exists a unique twin root $(\alpha, \tilde{\alpha})$ with $x \in \alpha$ and $y \in \tilde{\alpha}$. Moreover, if $(\alpha, \tilde{\alpha})$ is a twin root and if π is a panel in a wall associated to the root α (resp. $\tilde{\alpha}$), then for each chamber $x \in \pi \backslash \alpha$ (resp. $x \in \pi \backslash \tilde{\alpha}$) there exists a unique twin apartment which contains $(\alpha, \tilde{\alpha})$ and x (for proofs of the latter two facts, see [3], Lemmas 81 and 82).

(2.11) If (Δ, δ) is a building of spherical type (W, S) and if $w^0 \in W$ is the element of maximal length, then $(\Delta_+, \Delta_-) := ((\Delta, \delta), (\Delta, w^0 \delta w^0))$ is endowed with a structure of twin building of type (W, S) by defining $\delta^* : (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \to W$ by $\delta^* = w^0 \delta$ on $\Delta_+ \times \Delta_-$ and $\delta^* = \delta w^0$ on $\Delta_- \times \Delta_+$. In this case, both notions of opposite chambers which have been defined actually coincide. Two residues are called *opposite* in the spherical building Δ if they are opposite in the corresponding twin building. If Σ is an apartment of Δ then (Σ, Σ) is a twin apartment of $(\Delta_+, \Delta_-, \delta^*)$, and so any apartment of Δ_ϵ is admissible for $\epsilon = +, -$.

We end this section by recalling a classical result for spherical buildings.

(2.12) Lemma. *Let (Δ, δ) be a building of spherical type (W, S). Let A and B be two opposite residues in Δ. Then the restriction of proj_A to the residue B establishes a one-to-one correspondence between the chambers of these two residues, the inverse of which is the restriction of proj_B to A.*

Proof. This is Theorem 3.28 in [13]. See Proposition (9.11) in [17] for a proof in the context of chamber systems. □

(2.13) In particular, it follows from the lemma that opposite panels are in one-to-one correspondence in any twin building. This fact is easy to deduce directly from Definition (2.8): if (π_+, π_-) is a pair of opposite panels in some twin building $(\Delta_+, \Delta_-, \delta^*)$, and if $x \in \pi_+$ then it follows from (Tw2) and (Tw3) that all but one chamber of π_- are opposite x; the unique chamber $\tilde{x}$ of π_- which is not opposite x satisfies $\delta^*(x, \tilde{x}) = s$, where $s \in S$ is the type of the panels π_+ and π_-. A consequence of this fact is that any twin apartment containing x and intersecting π_- also contains $\tilde{x}$ (and vice versa).

3 A lemma of Tits

This section recalls a lemma of Tits, which gives sufficient conditions for a group acting on a set to be a Coxeter group.

(3.1) Let Σ be a set, and let W be a group acting on Σ from the left. A subset $D \neq \emptyset$ of Σ is called **prefundamental** (or a **prefundamental domain**) if, for $w \in W$, we have $w = 1$ whenever $wD \cap D \neq \emptyset$. We call D **fundamental** (or a **fundamental domain**) if, moreover, we have $\bigcup_{w \in W} wD = \Sigma$.

(3.2) Let now Ψ be a set of roots in the thin building Σ of type (W, S). We set $R(\Psi) = \{r_\psi | \psi \in \Psi\}$ (where r_ψ is the reflection associated to ψ) and call Ψ **2-geometric** if for all $\psi, \psi' \in \Psi$ the set $\psi \cap \psi'$ is a fundamental domain for the action of $\langle r_\psi, r_{\psi'} \rangle$ on $\Sigma(W, S)$. It is called **geometric** if it is 2-geometric and if, additionally, $\bigcap_{\psi \in \Psi} \psi$ is not empty.

(3.3) Lemma. *Let Ψ be a geometric set of roots in a thin building Σ of type (W_0, S_0). Then $D := \bigcap \Psi$ is a fundamental domain for the action of $W := \langle R(\Psi) \rangle$ on Σ, and $(W, R(\Psi))$ is a Coxeter system. Moreover, if we set $C := \{wD | w \in W\}$ and $\delta : C \times C \to W : (vD, wD) \mapsto \delta(vD, wD) := v^{-1}w$ then (C, δ) is isomorphic to the thin building $\Sigma(W, R(\Psi))$.*

Proof. This is essentially a consequence of Lemma 1 in [12]. See also Lemma 3.2 and Proposition 3.3 in [8]. □

(3.4) In the situation of Lemma (3.3), we say that the Coxeter system $(W, R(\Psi))$ is **geometrically included** in the Coxeter system (W_0, S_0). We write $(W, R(\Psi)) \hookrightarrow (W_0, S_0)$ and speak about a **geometric inclusion** between Coxeter systems.

4 Thickness of walls. Reflections and retractions

First we prove that all of the panels belonging to a given wall are in one-to-one correspondence in a canonical way. In particular, in a weak building, these panels are either all thin or all thick, so we can speak about *thick walls*. Next we see that if the weak building is actually a weak *twin* building, then reflections corresponding to thick walls preserve the thickness of the panels. Finally, we record that retractions also preserve the thickness of panels.

(4.1) Lemma. *Let (Σ, δ) be the thin building of type (W, S) and t be a reflection of Σ. Let π and π' be panels belonging to the wall $P(t)$. Then there exists panels $\pi = \pi_0, \pi_1, \ldots, \pi_n = \pi'$ in $P(t)$ such that, for $i = 1, \ldots, n$, π_{i-1} and π_i are contained in a common spherical residue of rank 2, in which they are opposite.*

Proof. Let $x, x', y, y' \in C(t)$ be chambers such that $\pi = \{x, y\}$, $\pi' = \{x', y'\}$ where x and x' belong to the same root associated to t. Set $\alpha := H(t, x) = H(t, x')$. Let $v = \delta(x, y)$ and choose $s \in S$ such that $\ell(s\delta(x, x')) < \ell(\delta(x, x')) = d(x, x')$. Let $R = \mathrm{Res}_{\{s,v\}}(x)$. Then, by definition of s, we have $\mathrm{proj}_R(x') \neq x$. Let $x_1 = \mathrm{proj}_R(x')$ and $y_1 = \mathrm{proj}_R(y')$. Hence, we have $x_1 \in \alpha$ and $y_1 \in -\alpha$. Furthermore, by definition of a root, we have $d(x', x) < d(x', y)$ and thus $\mathrm{proj}_\pi(x') = x$; similarly we have also $\mathrm{proj}_\pi(y') = y$. The composition law for projections now implies that $\mathrm{proj}_\pi(x_1) = x$ and $\mathrm{proj}_\pi(y_1) = y$. In particular $x_1 \neq y_1$, whence $d(x_1, y_1) = 1$. Therefore $\pi_1 := \{x_1, y_1\}$ is a panel, which also belongs to $P(t)$. Thus π and π' are two panels of the rank 2 residue R which belong to the wall $P(t)$. It is easy to see that the only thin buildings of rank 2 possessing such walls are the spherical ones, and that two panels belonging to the same wall in a such a rank 2 residue are opposite in that residue. The conclusion follows from an easy induction. □

(4.2) Corollary. *Let (Δ, δ) be a weak building, and let t be a reflection. The panels belonging to $P(t)$ are either all thick or all thin. We speak respectively about* ***thick walls*** *and* ***thin walls****.*

Proof. This follows from Lemma (2.12) which states the existence of canonical bijections between panels lying on a given wall. $\square$

We now prove that in a given twin apartment of a weak twin building, reflections through thick walls preserve the thickness of each panel in that apartment.

(4.3) Lemma. *Let $(\Delta_+, \Delta_-, \delta^*)$ be weak twin building, let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment and let t be a reflection of Σ such that the wall $P(t)$ is thick. Then t maps thick panels onto thick panels and thin panels onto thin panels.*

Proof. Let $(\alpha, \tilde{\alpha})$ be a twin root of Σ associated to t, let $\pi \in P(t)$ and choose a chamber $z \in \pi \backslash \Sigma$, which is possible since the wall $P(t)$ is thick by assumption. Let $\Sigma' = (\Sigma'_+, \Sigma'_-)$ be the unique twin apartment containing $(\alpha, \tilde{\alpha})$ and z (see (2.10)). Let $\tilde{\pi} = op_{\Sigma'}(\pi)$, and let $\tilde{z} \in \Sigma'$ be the unique chamber of $\tilde{\pi}$ which is not opposite z. Let β (resp. $\tilde{\beta}$) be the root complementary to α in Σ'_+ (resp. complementary to $\tilde{\alpha}$ in Σ'_-). Hence $(\beta, \tilde{\beta})$ is a twin root of Σ' which is opposite to $(\alpha, \tilde{\alpha})$ in that twin apartment. Moreover, it is the unique twin root which contains z and $\tilde{z}$. Define finally $\Sigma'' = (\Sigma''_+, \Sigma''_-)$ to be the unique twin apartment containing z and $(-\alpha, -\tilde{\alpha}) \subset \Sigma$. Hence, $\tilde{z} \in \Sigma''$ (see (2.13)) and therefore, the twin root $(\beta, \tilde{\beta})$ is contained in Σ''.

Let now x be a chamber in Σ_+. If x belongs to α then it is easy to deduce from the definitions that

$$t(x) = \mathrm{op}_{\Sigma''} \circ \mathrm{op}_{\Sigma'}(x).$$

Similarly, if x belongs to $-\alpha$ then

$$t(x) = \mathrm{op}_{\Sigma'} \circ \mathrm{op}_{\Sigma''}(x).$$

Since there are canonical one-to-one correspondences between opposite panels in any twin building (see (2.13)), we deduce that t preserves the thickness of the panels whose trace on Σ_+ is either in α or in $-\alpha$. Obviously, all of the other panels of Σ_+ are in $P(t)$, and so t leaves them invariant. The case of the action of t on Σ_- is similar. $\square$

We now study how retractions behave with respect to the thickness of panels. Notice that by definition, the image of a panel under a retraction always consists of two distinct chambers in some (other) panel.

(4.4) Lemma. *Let $(\Delta_+, \Delta_-, \delta^*)$ be weak twin building, let $\Sigma = (\Sigma_+, \Sigma_-)$ and $\Sigma' = (\Sigma'_+, \Sigma'_-)$ be twin apartments such that there is a sign $\epsilon \in \{+, -\}$ and chamber $c \in \Sigma_\epsilon \cap \Sigma'_\epsilon$. Then the restriction of the retraction $\rho_{\Sigma_\epsilon, c}$ to Σ'_ϵ is an isomorphism preserving the thickness of panels. In particular, retractions onto admissible apartments map thick panels to thick panels and thin ones to thin ones.*

Proof. Without loss of generality, we may assume that $\epsilon = +$. Suppose first that Σ and Σ' share a twin root $(\alpha, \tilde{\alpha})$. Let $(-\alpha, -\tilde{\alpha})$ be the twin root opposite $(\alpha, \tilde{\alpha})$ in Σ and $(\beta, \tilde{\beta})$ be the twin root opposite $(\alpha, \tilde{\alpha})$ in Σ'. As in the proof of the preceding lemma, there is a unique twin apartment $\Sigma'' = (\Sigma''_+, \Sigma''_-)$ which is the union of $(-\alpha, -\tilde{\alpha})$ and $(\beta, \tilde{\beta})$.

Now, let $z \in \Sigma'_+$. If $z \in \alpha$ then $\rho_{\Sigma_\epsilon, c}(z) = z$, but if $z \in \beta$ then $\rho_{\Sigma_\epsilon, c}(z) = t(z)$, where t denotes the reflection of Σ' corresponding to α. The conclusion follows from Lemma (4.3) in this case.

In the general situation, choose a sequence of twin apartments $\Sigma = \Sigma_0, \Sigma_1, \ldots, \Sigma_n = \Sigma'$ such that Σ_{i-1} and Σ_i share a twin root $(\alpha_i, \tilde{\alpha}_i)$ with $c \in \alpha_i$. Such a sequence exists in view of Proposition 5 of [14] (this can also be seen directly by an easy induction on the length of a minimal gallery from $c' = \mathrm{op}_{\Sigma'}(c)$ to a chamber of Σ_-). Repeated applications of the above-proven special case now produce the desired conclusion. $\square$

5 Pairs of reflections in a Coxeter system

In this section, we recall a well known result about pairs of reflections in Coxeter systems. This result can be seen as a consequence of the fact that finite subgroups of Coxeter groups are always contained in spherical parabolic subgroups, or, alternatively, of the geometric representation of finite Coxeter groups. Nevertheless, we give here a more elementary and purely combinatorial proof of the same result.

(5.1) Lemma. *Let (Σ, δ) be the thin building of type (W, S), and let t_1 and t_2 be two distinct reflections. Also, let α_2 be a root associated to t_2. Then the following statements are equivalent:*
(i) the order of the product $t_1 t_2$ is finite;
(ii) there exist panels π and π' in $P(t_1)$ such that $\pi \subset \alpha_2$ and $\pi' \subset (-\alpha_2)$.
If these statements are satisfied, then t_1 and t_2 stabilize a common rank 2 residue of spherical type.

Proof. Assume first that (ii) holds. By Lemma (4.1), there exist panels $\pi = \pi_0, \pi_1, \ldots, \pi_n = \pi'$ in $P(t_1)$ such that, for $i \in [1, n]$, π_{i-1} and π_i are contained in a common spherical residue of rank 2 that we denote by R_i. Choose a chamber $x \in \pi$, and let $y \in \pi' \cap H(t_1, x)$. Then there exists a gallery $(x = c_0, c_1, \ldots, c_m = y)$ joining x to y and which is entirely contained in $H(t_1, x) \cap (\bigcup_{i=1}^n R_i)$. By hypothesis we also have $x \in \alpha_2$ and $y \in -\alpha_2$, so this gallery must cross the wall $P(t_2)$. In other words, there is an $i \in [1, m]$ and a $j \in [1, n]$ such that the panel $\{c_{i-1}, c_i\}$ belongs to $P(t_2)$ and is contained in R_i. By construction the residue R_i also contains panels which belong to $P(t_1)$, so R_i is a spherical residue of rank 2 which is stabilized by both t_1 and t_2. The group generated by t_1 and t_2 has a finite orbit, so it is finite since W acts sharply transitively on Σ. Hence (i) holds.

If (ii) does not hold, then all panels of $P(t_1)$ are contained in α_2 (switching the notations for α_2 and $-\alpha_2$ if necessary). It follows from what we have already proved that all panels of $P(t_2)$ are contained in α_1, where α_1 denotes one of the roots associated to t_1.

To prove that $-\alpha_1 \subset \alpha_2$ and $-\alpha_2 \subset \alpha_1$, we now assume that there is a chamber $x \in (-\alpha_1) \cap (-\alpha_2)$. Let $y \in (-\alpha_1)$ be a chamber contained in a panel which belongs to $P(t_1)$; hence $y \in \alpha_2$ by assumption. A minimal gallery from x to y must cross the wall $P(t_2)$ but it is also entirely contained in $-\alpha_1$ by the convexity of a root. This says there is a panel of $P(t_2)$ contained in $-\alpha_1$, which is a contradiction.

Finally, set $A = \alpha_1 \cap \alpha_2$. An easy induction on m shows both that any product of m factors $t_1 t_2 t_1 \ldots$ maps A into $-\alpha_1$ and that any product of m factors $t_2 t_1 t_2 \ldots$ maps A into $-\alpha_2$. In particular, for each nonzero m we have $(t_1 t_2)^m(A) \cap A = \emptyset$ whence $t_1 t_2$ has infinite order. $\square$

(5.2) It follows from this lemma and its proof that if $t_1 t_2$ has infinite order, then $C(t_1)$ (resp. $C(t_2)$) is completely contained in one of the roots associated to t_2 (resp. t_1), say α_2 (resp. α_1), and that $\alpha_1 \cap \alpha_2$ is a fundamental domain for the group $\langle t_1, t_2 \rangle$.

6 More on thin-classes

We develop in this section more specific material necessary for the forthcoming proof of Theorem (1.3).

(6.1) Since any minimal gallery joining two thin-equivalent chambers must be a thin gallery, thin-classes are convex sets of chambers. Now consider an apartment Σ in a weak building Δ. If t is a reflection of Σ such that $P(t)$ is thick, then the thin-class of a chamber $c \in \Sigma$ is completely contained in one of the roots corresponding to t. This root is denoted by $H(t, \bar{c})$. If the set $C(t) \cap \bar{c}$ is not empty, then the reflection t is said to **border** the thin-class $\bar{c}$. Hence, a reflection borders a thin-class only if the wall corresponding to that reflection is thick.

From now on, we denote by $\Delta = (\Delta_+, \Delta_-, \delta^*)$ a weak twin building of given type (W, S).

(6.2) We also say that two distinct thin-classes are **adjacent** if they contain thick-adjacent chambers. If π is a thick panel, then $\bar{\pi}$ is called a **class-panel** and π is said to be a **support** for $\bar{\pi}$. Note that each thin-class in $\bar{\pi}$ contains exactly one chamber of π. The next lemma shows that any two distinct thin-classes in $\bar{\pi}$ determine the set of all possible supports for $\bar{\pi}$. In particular, given two adjacent thin-classes, it makes sense to speak about "the" class-panel containing them.

(6.3) Lemma. *Let π and π' be thick panels of Δ. If x and y are distinct chambers in π such that $\bar{x}$ and $\bar{y}$ both belong to $\bar{\pi}'$, then $\bar{\pi} = \bar{\pi}'$.*

Proof. We have to show both that if $z \in \pi$ then $\bar{z} \in \bar{\pi}'$, and conversely that if $z \in \pi'$ then $\bar{z} \in \bar{\pi}$.

Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment containing x and y. We may assume that $x, y \in \Sigma_+$. Let t be the reflection of Σ which stabilizes π. Let $(\alpha, \tilde{\alpha})$ be the twin root of Σ corresponding to t and such that $x \in \alpha$. Thus we have $\bar{x} \subset \alpha$ and $\bar{y} \subset -\alpha$. By Lemma (4.3), the reflection t maps $\bar{x}$ onto $\bar{y}$. Since both $\bar{x} \cap \pi'$ and $\bar{y} \cap \pi'$ are non-empty by assumption, π' also belongs to $P(t)$.

Obviously, we may assume that z is distinct from x. There is then a unique twin apartment $\Sigma' = (\Sigma'_+, \Sigma'_-)$ containing $(\alpha, \tilde{\alpha})$ and z. Denote by t' the reflection of Σ' to which $(\alpha, \tilde{\alpha})$ is associated. By assumption and Lemma (4.3), t' maps $\bar{x}$ onto $\bar{z}$. Because π' is in $P(t) = P(t')$ and $\pi' \cap \bar{x} \neq \emptyset$, we deduce that $\pi' \cap \bar{z} \neq \emptyset$. Hence $\bar{z} \in \bar{\pi}'$.

The converse statement follows by symmetry. $\square$

7 Proof of Theorem (1.3).

The idea of the proof is as follows. We first focus on the twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$, and prove that the sets of the corresponding thin-classes has a natural structure of thin twin building of type $(\bar{W}, \bar{S})$ (see (7.1)). Next, we use retractions to define a structure of thick chamber system on $\bar{\Delta}_+$ and $\bar{\Delta}_-$. This structure is then used to define a $\bar{W}$-distance on $\bar{\Delta}_+$ and on $\bar{\Delta}_-$. The key idea for that purpose, is to use twin apartments. The set of thin-classes of the chambers in a given twin apartment, is endowed with an induced structure of chamber system over $\bar{S}$. We observe that those chamber systems are actually all isomorphic to the thin building of type $(\bar{W}, \bar{S})$, just as $(\bar{\Sigma}_+, \bar{\Sigma}_-)$ is.

The proof is presented as a succession of lemmas and corollaries. The hypotheses and notations of Theorem (1.3) are kept throughout.

(7.1) Lemma. *The ordered pair $(\bar{W}, \bar{S})$ is a Coxeter system, and $\bar{\Sigma} := (\bar{\Sigma}_+, \bar{\Sigma}_-)$ has a canonical structure of thin twin building of type $(\bar{W}, \bar{S})$.*

Proof. We apply Tits' lemma. To this end, we first consider the set Ψ of all roots of Σ_+ of the form $H(t, \bar{c}_+)$ for a reflection t which borders $\bar{c}_+$. This gives $R(\Psi) = \bar{S}$. We now prove that Ψ is a 2-geometric set of roots.

Let $\psi, \psi' \in \Psi$ and denote by t and t' the corresponding reflections of Σ. Assume first that tt' has finite order, and let R be a spherical residue of rank 2 of Σ which is stabilized by t and t' (see (5.1)). To prove that $R \cap \psi \cap \psi'$ is a fundamental domain for the action of $\langle t, t' \rangle$ on R, it suffices to prove that there exists no reflection $r \in \langle t, t' \rangle$ such that $R \cap \psi \cap \psi'$ intersects $C(r)$. Assume on the contrary that such a reflection r exists. This means that $R \cap \psi \cap \psi'$ contains a panel of $P(r)$ as a subset. Now choose a chamber $y \in C(t) \cap \bar{c}_+$ (resp. $y' \in C(t') \cap \bar{c}_+$), and let $z = \mathrm{proj}_R(y)$ (resp. $z' = \mathrm{proj}_R(y')$). We have $z \in C(t) \cap \psi$ (resp. $z' \in C(t') \cap \psi'$). If α denotes the root associated with r and containing z, then $z' \in -\alpha$; in other words, z and z' are separated by the wall $P(r)$. This implies that $y \in \alpha$ and $y' \in -\alpha$, so any gallery from y to y' must cross the wall $P(r)$. But since r is conjugate to t or t', the latter wall is thick by (4.3), which contradicts the fact that y and y' are thin-equivalent (since they both belong to $\bar{c}_+$). This proves that $R \cap \psi \cap \psi'$ is a fundamental domain for the action of $\langle t, t' \rangle$ on R. Now, since $\psi \cap \psi' = \{x \in \Sigma_+ | \mathrm{proj}_R(x) \in \psi \cap \psi' \cap R\}$ and $\langle t, t' \rangle$ acts faithfully on R that $\psi \cap \psi'$ is fundamental for the action of $\langle t, t' \rangle$ on Σ_+.

Now suppose the order of tt' is infinite. By definition of Ψ, there exists a chamber $y \in \psi \cap C(t) \cap \bar{c}_+$ (resp. $y' \in \psi' \cap C(t') \cap \bar{c}_+$). Hence y and y' are thin-equivalent, and so $y, y' \in \psi \cap \psi'$ since $P(t)$ and $P(t')$ are thick. Lemma (5.1) gives $C(t) \subset \psi'$ and $C(t') \subset \psi$. By (5.2), this implies that the pair $\{\psi, \psi'\}$ is geometric. Hence Ψ is a 2-geometric set of roots.

Now we prove that $\bigcap \Psi = \bar{c}_+$. Observe that the inclusion $\bar{c}_+ \subset \bigcap \Psi$ is obvious. Assume there exists $x \in (\bigcap \Psi) \backslash \bar{c}_+$. Let $\gamma = (x = x_0, x_1, \dots, x_n = c_+)$ be a minimal gallery. Since $c \notin \bar{c}_+$ the gallery γ cannot be thin, and there is an $i \in [1, n]$ such that x_{i-1} and x_i are thick-adjacent. Let j be the maximal such i. If t denotes the reflection of Σ which switches x_{j-1} and x_j, then we have $H(t, x_j) = H(t, \bar{c}_+) \in \Psi$ while $x \in H(t, x_{j-1}) = -H(t, x_j)$. Thus, $x \notin H(t, \bar{c}_+)$ which contradicts the fact that $x \in \bigcap \Psi$, so we have $\bigcap \Psi = \bar{c}_+$.

By Lemma (3.3), $(\bar{W}, \bar{S})$ is a Coxeter system and $\bar{\Sigma}_+$ has a natural $\bar{W}$-codistance $\bar{\delta}_+$ which endows it with a structure of a thin building of type $(\bar{W}, \bar{S})$. Then define two thin-classes in $\bar{\Sigma}_+ \cup \bar{\Sigma}_-$ to be opposite if they contain opposite chambers, the opposition of classes establishes a one-to-one correspondence between $\bar{\Sigma}_+$ and $\bar{\Sigma}_-$. Henceforth, there are canonical $\bar{W}$-valued functions $\bar{\delta}_-$ and $\bar{\delta}^*$ such that $(\bar{\Sigma}_-, \bar{\delta}_-)$ is a thin twin building of type $(\bar{W}, \bar{S})$ and that $(\bar{\Sigma}_+, \bar{\Sigma}_-, \bar{\delta}^*)$ is a thin twin building of the same type, whose opposition relation is precisely the one we have just defined. □

(7.2) We are now able to define a structure of chamber system on $\bar{\Delta}_+$ and $\bar{\Delta}_-$. Two adjacent thin-classes $\bar{x}$ and $\bar{y}$ are called **$\bar{s}$-adjacent**, where $\bar{s} \in \bar{S}$, if the class-panel containing them is sent into an $\bar{s}$-class-panel of $\bar{\Sigma}$ under the retraction ρ_{Σ_+, c_+} or ρ_{Σ_-, c_-}. Notice that, by Lemma (4.4), the image of that class-panel under ρ_{Σ_+, c_+} or ρ_{Σ_-, c_-} determines a unique class-panel of $\bar{\Sigma}$, which has a well defined label in $\bar{S}$ thanks to Lemma (7.1). It is straightforward to check that the $\bar{s}$-adjacency of thin-classes endows $\bar{\Delta}_+$ and $\bar{\Delta}_-$ with a structure of a chamber system over $\bar{S}$. Therefore, any subset of $\bar{\Delta}_\epsilon$ also inherits of a structure of chamber system over $\bar{S}$, for $\epsilon = +$ or $-$.

(7.3) Lemma. *Let $\Sigma^0 = (\Sigma^0_+, \Sigma^0_-)$ be a twin apartment of Δ. Assume that each reflection t of Σ^0 such that $P(t)$ is a thick wall induces an isomorphism of the chamber system $\bar{\Sigma}^0_+$ or $\bar{\Sigma}^0_-$. Assume also that there is a chamber $c \in \Sigma^0_\epsilon$ for some sign ϵ such that the retraction $\rho_{\Sigma^0_\epsilon,c}$ induces a morphism of chamber systems $\bar{\Delta}_\epsilon \to \bar{\Sigma}^0_\epsilon$. Then we have the following:*

(i) for each chamber $d \in \Sigma^0_\epsilon$, the retraction $\rho_{\Sigma^0_\epsilon,d}$ also induces a morphism of chamber systems $\bar{\Delta}_\epsilon \to \bar{\Sigma}^0_\epsilon$;

(ii) if $\Sigma^1 = (\Sigma^1_+, \Sigma^1_-)$ is another twin apartment such that $c \in \Sigma^1_\epsilon$, then the retraction $\rho_{\Sigma^1_\epsilon,c}$ also induces a morphism of chamber systems $\bar{\Delta}_\epsilon \to \bar{\Sigma}^1_\epsilon$. Moreover, the restriction of the retraction $\rho_{\Sigma^0_\epsilon,c}$ to Σ^1_ϵ induces an isomorphism of chamber systems $\bar{\Sigma}^1_\epsilon \to \bar{\Sigma}^0_\epsilon$. In particular, each reflection t' of Σ^1 such that $P(t')$ is a thick wall induces an isomorphism of the chamber system $\bar{\Sigma}^1_\epsilon$.

Proof. The structure of a chamber system on $\bar{\Delta}_\epsilon$ is simply a labelling of its class-panels. Now, by Lemma (6.3), a class-panel is completely determined by any of its supports, so the image of such a support under a retraction is a support of the image of the class-panel itself. Thus we may restrict our consideration to images of thick panels under retractions. Let π be any thick panel of $\bar{\Delta}_\epsilon$, and let $x = \mathrm{proj}_\pi(c)$.

For (i), notice first that it suffices to prove the result in case d is adjacent to c because apartments are convex. Let $s = \delta_\epsilon(c, d)$. There are two cases.

First assume that $\delta_\epsilon(c, x) = s\delta_\epsilon(d, x)$. Then we have

$$\rho_{\Sigma^0_\epsilon,c}(x) = \rho_{\Sigma^0_\epsilon,d}(x).$$

Define a chamber $y \in \pi$ as follows: if $\mathrm{proj}_\pi(d) \neq x$, then $y = \mathrm{proj}_\pi(d)$ and if $\mathrm{proj}_\pi(d) = x$, then y is any chamber of π distinct from x. It is easy to check that

$$\rho_{\Sigma^0_\epsilon,c}(y) = \rho_{\Sigma^0_\epsilon,d}(y),$$

which gives

$$\rho_{\Sigma^0_\epsilon,c}(\pi) = \rho_{\Sigma^0_\epsilon,d}(\pi),$$

so $\rho_{\Sigma^0_\epsilon,d}$ preserves the type of the class-panel $\bar{\pi}$.

Assume now that $\delta_\epsilon(c, x) = \delta_\epsilon(d, x)$. In this case, we have $\mathrm{proj}_\pi(d) = x$. Let π' be the s-panel containing c and d. Since $c \neq d$, we know that $\mathrm{proj}_{\pi'}(x)$ is neither c nor d, and, particularly, that π' is thick. Let t be the reflection of Σ^0 switching c and d. We have

$$\rho_{\Sigma^0_\epsilon,d}(x) = t(\rho_{\Sigma^0_\epsilon,c}(x)),$$

whence

$$\rho_{\Sigma^0_\epsilon,d}(\pi) = t(\rho_{\Sigma^0_\epsilon,c}(\pi)),$$

since $\mathrm{proj}_\pi(d) = x$. Therefore $\rho_{\Sigma^0_\epsilon,d}$ preserves the type of the class-panel $\bar{\pi}$, because t preserves the types of the class-panels in $\bar{\Sigma}^0_\epsilon$ by assumption.

For (ii). the assumption clearly implies in view of Lemma (4.4) that the restriction of the retraction $\rho_{\Sigma^0_\epsilon,c}$ to Σ^1_ϵ induces an isomorphism of chamber systems $\bar{\Sigma}^1_\epsilon \to \bar{\Sigma}^0_\epsilon$. It is also apparent that the restriction of the retraction $\rho_{\Sigma^1_\epsilon,c}$ to Σ^0_ϵ is the inverse mapping of the restriction of $\rho_{\Sigma^0_\epsilon,c}$ to Σ^1_ϵ. Since the inverse mapping of an invertible morphism of chamber systems is itself a morphism of chamber systems, and because we have

$$\rho_{\Sigma^1_\epsilon,c} = \rho_{\Sigma^1_\epsilon,c} \circ \rho_{\Sigma^0_\epsilon,c},$$

we deduce that $\rho_{\Sigma^1_\epsilon,c}$ induces a morphism of chamber systems $\bar{\Delta}_\epsilon \to \bar{\Sigma}^1_\epsilon$. □

Observe that the twin apartment Σ, together with its chambers c_+ or c_-, satisfies the hypotheses of the preceding lemma (see (7.1) and (7.2)).

(7.4) Corollary. *Let $\Sigma' = (\Sigma'_+, \Sigma'_-)$ be any twin apartment of Δ. Then $\bar{\Sigma}' = (\bar{\Sigma}'_+, \bar{\Sigma}'_-)$, endowed with its induced structure of chamber system, is a thin twin building of type $(\bar{W}, \bar{S})$.*

Proof. The proof rests on (2.7). Thanks to Lemma (7.1), we already know the result for the twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$. Now let $d_+ \in \Sigma'_+$ and choose a twin apartment $\Sigma'' = (\Sigma''_+, \Sigma''_-)$ such that c_+ and d_+ both belong to Σ''_+. By Lemma (7.3)(ii), we see that $\bar{\Sigma}''_+$ (endowed with its induced structure of chamber system) is isomorphic to $\bar{\Sigma}_+$, namely that it is a thin building of type $(\bar{W}, \bar{S})$. We may now apply the part (i) of that lemma, with $\bar{\Sigma}''_+$ (resp. c_+) playing the role of Σ_+ (resp. c). As a result, we can apply the lemma a third time, now with d_+ in the role of c (and still $\bar{\Sigma}''_+$ in the role of Σ_+). This finally shows that $\bar{\Sigma}'_+$ is isomorphic to $\bar{\Sigma}_+$. Thus $\bar{\Sigma}'_+$, endowed with its induced structure of chamber system, is a thin twin building of type $(\bar{W}, \bar{S})$. The corresponding $\bar{W}$-distance is also denoted by $\bar{\delta}_+$. Similarly, $\bar{\Sigma}'_-$ is a thin building of type $(\bar{W}, \bar{S})$, with corresponding $\bar{W}$-distance denoted by $\bar{\delta}_-$. Finally, define two thin-classes of $\bar{\Sigma}'$ to be opposite if they contain opposite chambers. This induces a well defined $\bar{W}$-codistance on $\bar{\Sigma}'$, that we also denote by $\bar{\delta}^*$. The result clearly follows. □

End of the proof.

For $\epsilon \in \{+, -\}$, we define $\bar{\delta}_\epsilon : \bar{\Delta}_\epsilon \times \bar{\Delta}_\epsilon \to \bar{W}$ as follows: For x and y in Δ_ϵ, choose an admissible apartment Σ'_ϵ containing them both. Hence $\bar{x}$ and $\bar{y}$ both belong to $\bar{\Sigma}'_\epsilon$. By Corollary (7.4) applied to $\bar{\Sigma}'_\epsilon$, the expression $\bar{\delta}_\epsilon(\bar{x}, \bar{y})$ makes sense in $\bar{\Sigma}'_\epsilon$. Now, by Lemma (7.3)(ii), this element of $\bar{W}$ is actually independent of the choice of the admissible apartment Σ'_ϵ. Thus $\bar{\delta}_\epsilon$ is a well defined mapping $\bar{\Delta}_\epsilon \times \bar{\Delta}_\epsilon \to \bar{W}$.

Let now $x \in \Delta_+$ and $y \in \Delta_-$, and choose $\Sigma' = (\Sigma'_+, \Sigma'_-)$ to be a twin apartment containing x and y. By Corollary (7.4) applied to $\bar{\Sigma}'_+$, the expression $\bar{\delta}^*(\bar{x}, \bar{y})$ makes sense in $\bar{\Sigma}'$ and is equal to $\bar{\delta}_-(\bar{y}, \bar{x}') \in \bar{W}$, where $x' = \mathrm{op}_{\Sigma'}(x)$. But, if $\Sigma'' = (\Sigma''_+, \Sigma''_-)$ is another twin apartment containing x and y, and if $x'' = \mathrm{op}_{\Sigma''}(x)$, then $\bar{\delta}_-(\bar{y}, \bar{x}') = \bar{\delta}_-(\bar{y}, \bar{x}'')$ thanks to Lemma (7.3)(ii). Thus $\bar{\delta}^*(\bar{x}, \bar{y})$ is actually independent of the choice of Σ', and $\bar{\delta}^*$ is a well defined mapping $(\bar{\Delta}_+ \times \bar{\Delta}_-) \cup (\bar{\Delta}_- \times \bar{\Delta}_+) \to \bar{W}$.

It is now clear that axioms (Bu1), (Bu3), (Tw1) and (Tw3) are satisfied. Let us now prove (Bu2) for $\bar{\delta}_\epsilon$. Let thus x, y and z be chambers of Δ_ϵ such that $\bar{\delta}_\epsilon(\bar{x}, \bar{y}) = \bar{w} \in \bar{W}$ and that $\bar{\delta}_\epsilon(\bar{y}, \bar{z}) = \bar{s} \in \bar{S}$. Thus $\bar{y}$ and $\bar{z}$ belong to the same $\bar{s}$-class-panel $\bar{\pi}$. Without loss of generality, we may assume that y and z belong to π (which is a support for $\bar{\pi}$). Let $d = \mathrm{proj}_\pi(x)$. If $d \in \{y, z\}$ then $\bar{x}, \bar{y}$ and $\bar{z}$ are all subsets of a common admissible apartment, and (Bu2) follows from Corollary (7.4) in that case. Otherwise, choose admissible apartments Σ_ϵ and Σ'_ϵ containing x, y, and x, z respectively. Thus Σ_ϵ and Σ'_ϵ both contain d. We have $\rho_{\Sigma_\epsilon, x}(z) = y$, which implies, by Lemma (7.3)(ii), that $\bar{\delta}_\epsilon(\bar{x}, \bar{y}) = \bar{\delta}_\epsilon(\bar{x}, \bar{z})$ in that case. Thus (Bu2) holds. The proof of (Tw2) is similar. The fact that $\bar{\Delta}_+$ and $\bar{\Delta}_-$ are thick is obvious by construction.

This concludes the proof of the first part of Theorem (1.3).

It remains to prove the converse statement. Hence, let $\bar{\Delta} = ((\bar{\Delta}_+, \bar{\delta}_+), (\bar{\Delta}_-, \bar{\delta}_-), \bar{\delta}^*)$ be a thick building of type $(\bar{W}, \bar{S})$ and assume given a geometric inclusion of Coxeter systems $(\bar{W}, \bar{S}) \hookrightarrow (W, S)$. Thus we may identify $\bar{S}$ with the set of reflections $R(\Psi)$ of W, where Ψ is a geometric set of roots in $\Sigma(W, S)$; hence, $\bar{W}$ itself is identified with a subgroup of W. We set $D := \bigcap \Psi$. We have to construct a twin building $\Delta = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ of type (W, S) whose thick frame is isomorphic to $\bar{\Delta}$. For that purpose, we define

$$\Delta_\epsilon := \bar{\Delta}_\epsilon \times D$$

and

$$\delta_\epsilon : \Delta_\epsilon \times \Delta_\epsilon \to W : ((x,c),(y,d)) \mapsto c^{-1} \cdot \bar{\delta}_\epsilon(x,y) \cdot d$$

for $\epsilon \in \{+,-\}$. The latter expression makes sense because D consists of chambers of $\Sigma(W,S)$, which are actually elements of W. Finally, we put

$$\delta^* : (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \to W : ((x,c),(y,d)) \mapsto c^{-1} \cdot \bar{\delta}^*(x,y) \cdot d.$$

We have to prove that (Δ_+, δ_+) and (Δ_-, δ_-) satisfy the axioms (Bu1)–(Bu3) and that $\Delta = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ satisfies the axioms (Tw1)–(Tw3).

Let $\epsilon \in \{+,-\}$.

Let $(x,c),(y,d) \in \Delta_\epsilon$ and assume that $\delta_\epsilon((x,c),(y,d)) = 1$. This implies that $\bar{\delta}_\epsilon(x,y)d = c$ and, hence, that $c \in D \cap (\bar{\delta}_\epsilon(x,y)D)$. Therefore, we obtain $\bar{\delta}_\epsilon(x,y) = 1$ because D is a fundamental domain for $\bar{W}$. This proves that $(\Delta_\epsilon, \delta_\epsilon)$ satisfies (Bu1).

The fact that Δ satisfies (Tw1) follows immediately from the definition of δ^* and the fact that $\bar{\Delta}$ satisfies (Tw1).

Now, we prove that (Bu2) and (Tw2) hold.

Let $(y,d),(y',d') \in \Delta_\epsilon$ and assume that $s := \delta_\epsilon((y,d),(y',d')) \in S$. We distinguish two cases.

Case 1: $ds \in D$.
By the definition of s, this implies that $ds \in D \cap (\bar{\delta}_\epsilon(y,y')D)$ and thus, that $\bar{\delta}_\epsilon(y,y') = 1$ because D is a fundamental domain for $\bar{W}$. Therefore, we have $y = y'$ and $ds = d'$.

Given $(x,c) \in \Delta_\epsilon$, then, setting $w := \delta_\epsilon((x,c),(y,d))$, we obtain

$$\begin{aligned} \delta_\epsilon((x,c),(y',d')) &= \delta_\epsilon((x,c),(y,ds)) \\ &= c^{-1}\bar{\delta}_\epsilon(x,y)ds \\ &= ws, \end{aligned}$$

which shows that $(\Delta_\epsilon, \delta_\epsilon)$ satisfies (Bu2) in this case.

Similarly, given $(x,c) \in \Delta_{-\epsilon}$, then, setting $w := \delta^*((x,c),(y,d))$, we obtain $\delta^*((x,c),(y',d')) = ws$, thereby showing that $\bar{\Delta}$ satisfies (Tw2) in this case.

Case 2: $ds \notin D$.
In $\Sigma(W,S)$, the chamber $d \in D$ is s-adjacent to $ds \notin D$. Since D is a fundamental domain for $\bar{W}$, we deduce that the only element $\bar{s}$ of $\bar{W}$ such that $ds \in \bar{s}D$ is the reflection $\bar{s} = dsd^{-1}$ that stabilizes the s-panel containing d. Hence, $ds \in (\bar{s}D) \cap (\bar{\delta}_\epsilon(y,y')D)$ from which it follows that $\bar{\delta}_\epsilon(y,y') = \bar{s}$ and that $d = d'$. We also remark that $\bar{s}$ belongs to $\bar{S}$ in view of Lemma (3.3).

Let us check (Bu2). Given $(x,c) \in \Delta_\epsilon$, then $\delta_\epsilon((x,c),(y',d')) = c^{-1}\bar{\delta}_\epsilon(x,y')d$. Since $\bar{s} \in \bar{S}$ and since $(\bar{\Delta}_\epsilon, \bar{\delta}_\epsilon)$ satisfies (Bu2), it follows that $\bar{\delta}_\epsilon(x,y') \in \bar{\delta}_\epsilon(x,y)\{1,\bar{s}\}$. Setting $w := \delta_\epsilon((x,c),(y,s))$, we obtain

$$\delta_\epsilon((x,c),(y',d')) \in c^{-1}\bar{\delta}_\epsilon(x,y)\{1,\bar{s}\}d' = \{w, ws\}$$

because $\bar{s} = dsd^{-1}$ and $d = d'$. Thus the first part of (Bu2) holds in this case.

Let us now consider the additional assumption that $\ell(ws) = \ell(w) + 1$. Let d denote the numerical distance of $\Sigma(W,S)$. Then we have $\ell(w) = d(c, \bar{\delta}_\epsilon(x,y)d) < d(c, \bar{\delta}_\epsilon(x,y)ds) = \ell(ws)$. This means that in $\Sigma(W,S)$, the chambers c and $\bar{\delta}_\epsilon(x,y)ds = \bar{\delta}_\epsilon(x,y)\bar{s}d$ are separated by the wall $P(\bar{t})$, where $\bar{t} = \bar{\delta}_\epsilon(x,y)\bar{s}(\bar{\delta}_\epsilon(x,y))^{-1}$. Since $\bar{t} \in \bar{W}$, we deduce that the wall $P(\bar{t})$ separates also D from $\bar{\delta}_\epsilon(x,y)\bar{s}D$. By the last statement of Lemma (3.3), this implies that $\bar{\ell}(\bar{\delta}(x,y)\bar{s}) = \bar{\ell}(\bar{\delta}(x,y)) + 1$, where $\bar{\ell}$ denotes the length function of the Coxeter system

$(\bar{W},\bar{S})$. Thus we have $\bar{\delta}_\epsilon(x,y') = \bar{\delta}_\epsilon(x,y)\bar{s}$ because $(\bar{\Delta}_\epsilon,\bar{\delta}_\epsilon)$ satisfies (Bu2). Again setting $w := \delta_\epsilon((x,c),(y,s))$, we now obtain

$$\delta_\epsilon((x,c),(y',d')) \;=\; c^{-1}\bar{\delta}_\epsilon(x,y)\bar{s}d \;=\; ws.$$

This proves that $(\Delta_\epsilon,\delta_\epsilon)$ satisfies (Bu2).

Let us now check (Tw2). Given $(x,c) \in \Delta_{-\epsilon}$, then we put $w := \delta^*((x,c),(y,d))$ and we assume that $\ell(ws) < \ell(w)$. By a discussion similar to the one of the preceding paragraph, this implies that $\bar{\ell}(\bar{\delta^*}(x,y)\bar{s}) < \bar{\ell}(\bar{\delta^*}(x,y))$, where $\bar{\ell}$ denotes again the length function of the Coxeter system $(\bar{W},\bar{S})$. Since $\bar{s} \in \bar{S}$ and since $\bar{\Delta}$ satisfies (Tw2), it follows that $\bar{\delta^*}(x,y') = \bar{\delta^*}(x,y)\bar{s}$. Thus we obtain

$$\delta^*((x,c),(y',d')) \;=\; c^{-1}\bar{\delta^*}(x,y)\bar{s}d \;=\; ws.$$

This shows that Δ satisfies (Tw2) in this case.

We now check that (Bu3) and (Tw3) hold.

Let $(y,d) \in \Delta_\epsilon$ and let $s \in S$. As before, we distinguish two cases.

Case $1'$: $ds \in D$.

We have $\delta_\epsilon((y,d),(y,ds)) = s$. Moreover, by the discussion of Case 1 above, we know that for all $(x,c) \in \Delta_\epsilon$ we have $\delta_\epsilon((x,c),(y,ds)) = \delta_\epsilon((x,c),(y,d))s$. Hence, $(\Delta_\epsilon,\delta_\epsilon)$ satisfies (Bu3) in this case.

Similarly, for all $(x,c) \in \Delta_{-\epsilon}$ we have $\delta^*((x,c),(y,ds)) = \delta^*((x,c),(y,d))s$. Hence, Δ satisfies (Tw3) in this case.

Case $2'$: $ds \notin D$.

As in Case 2 above, this implies that $\bar{s} := dsd^{-1} \in \bar{S}$.

Given $(x,c) \in \Delta_{-\epsilon}$, then there exists $y' \in \bar{\Delta}_\epsilon$ such that $\bar{\delta}_\epsilon(y,y') = \bar{s}$ and that $\bar{\delta}_\epsilon(x,y') = \bar{\delta}_\epsilon(x,y)\bar{s}$, because $(\bar{\Delta}_\epsilon,\bar{\delta}_\epsilon)$ satisfies (Bu3). Thus $(y',d) \in \Delta_\epsilon$ and we have $\delta_\epsilon((y,d),(y',d')) = s$ and $\delta_\epsilon((x,c),(y',d)) = \delta_\epsilon((x,c),(y,d))s$. This shows that $(\Delta_\epsilon,\delta_\epsilon)$ satisfies (Bu3).

Finally, given $(x,c) \in \Delta_{-\epsilon}$, then there exists $y' \in \bar{\Delta}_\epsilon$ such that $\bar{\delta}_\epsilon(y,y') = \bar{s}$ and that $\bar{\delta^*}_\epsilon(x,y') = \bar{\delta^*}_\epsilon(x,y)\bar{s}$, because $\bar{\Delta}$ satisfies (Tw3). Thus $(y',d) \in \Delta_\epsilon$ and we have $\delta_\epsilon((y,d),(y',d')) = s$ and $\delta^*((x,c),(y',d)) = \delta^*((x,c),(y,d))s$. This shows that Δ satisfies (Tw3).

We have shown that Δ is a twin building of type (W,S). The fact that the thick frame of Δ is isomorphic to $\bar{\Delta}$ is immediate by construction of Δ and by definition of the thick frame. This concludes the proof of Theorem (1.3). □

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