ADDITIVE AND NON-ADDITIVE SET FUNCTIONS

by

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A DISSERTATION

Presented to the
Faculty of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance by the
Department of Mathematics

March 1953

Abstract.

The properties of real-valued functions on a Boolean ring are investigated with the aid of three kinds of limiting processes, based on (1) directed sets of partitions, ordered by refinement; (2) finite trees of successive partitions, ordered by extension; (3) directed sets of finite subrings, ordered by inclusion. Through (1) a theory of integration is developed. The integral takes the form of a linear operator, projecting general set functions onto their additive parts. Integration of real functions with respect to non-additive "measures" then becomes possible. Through (2) a notion of deviation from additivity for non-additive functions is obtained, leading to a canonical decomposition: $f = f^{+} + f^{-} + f^{0}$ of every function of finite deviation into its superadditive, subadditive, and additive parts. Through (3), certain new linear operators, called "imputations", are defined. These are also projections from general set functions to the additive functions; they have application to the problem of equitable distribution in economics and to the theory of n-person games.

Acknowledgement.

I am deeply indebted to Professors William Feller and A. W. Tucker for their encouragement and willing assistance. Early in my work I had the benefit of stimulating discussions with Prof. S. Bochner and Dr. John McCarthy. My topic grew out of research undertaken in 1951 for the RAND Corporation of Santa Monica, California; in applying the final touches I have been aided by the financial support of the Office of Naval Research.

0. Preface.

The material of this paper is arranged in seven chapters. The first three give the necessary elementary facts about sets, partitions, and set functions; the two chief novelties are the concept of partition tree, in Chapter 2, and the idea of the carrier of a set function, in Chapter 3. The fourth chapter deals with the integration of non-additive set functions, and their use in integrating point functions. A concluding section relates this material to the work of Burkill on integration of non-additive functions of intervals. In Chapter 5 partition trees are used to measure the deviation from additivity of non-additive set functions, and a fundamental decomposition theorem (Theorem 21) is thereby obtained. The main result of the sixth chapter is an inversion formula for step-function — set functions which are completely determined by their values on a finite collection of sets (Theorem 27). The seventh chapter develops a theory for imputation operators. The central result of the chapter, Theorem 34, permits their extension from the space of step-functions to a wide, not completely delineated subspace of the space of all set functions. An introductory section describes the relation of these operators to the "problem of imputation" in economics, and a final section gives an application to the theory of n-person games.

I have used examples quite freely to illustrate special points, and to indicate the scope and limitations of the theory. They may be omitted without damage to the logical continuity, but it is hoped that their presence will make for an easier understanding of the subject matter.

References to the literature will be found in the footnotes. In addition to the works cited in the text, I have consulted profitably Birkhoff's Lattice Theory (American Mathematical Society Colloquium Publications, Vol. 25, New York, 1940) and Sierpiñsky's Leçons sur les nombres transfinis (Paris, 1928).

Contents

1. Boolean rings.

"Sets" in this paper will be viewed as elements of an abstract Boolean ring, $\mathcal{B}$. While one can always construct an underlying space of "points", to populate the sets,[1] we shall not do so except, occasionally, for purposes of illustration. In particular, our excursion into measure theory in Chapter 4 will be wholly independent of the topological structure of the point space.[2]

Probably the shortest characterization of an abstract Boolean ring is the one due to Stone:[3] a ring $\mathcal{B}$ (in the usual algebraic sense) in which each element is its own square. The binary operations of intersection, union, and difference must then be defined in terms of multiplication and addition. If S, T are elements of $\mathcal{B}$, we have

$$S \cap T = ST$$
$$S \cup T = ST + (S + T)$$
$$S - T = ST + S.$$

The operations on the left then combine according to the familiar

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1. See the remark at the end of this chapter.

2. Compare the treatment of measure theory in J. von Neumann, Functional Operators, Vol. I, Annals of Mathematics Study No. 21, Princeton, 1950, pp. 83 et seq.

3. M. H. Stone, Theory of representations for Boolean algebras, Trans. Amer. Math. Soc. 40 (1936), pp. 37-111.

rules of set theory.[1] Henceforth we shall discard the algebraic, in favor of the set-theoretic notation. The empty set, 0, is the zero of the ring. The union, I, of all sets in $\mathcal{B}$, may of may not belong to $\mathcal{B}$; if it does it is the unit of the ring.[2]

If $S \cap T = S$, we call S a subset of T, or T a superset of S, and write $S \subseteq T$ or $T \supseteq S$. The symbols $\cap$, $\cup$, $-$, $\subseteq$, $\supseteq$ will be used with <u>collections</u> of sets (denoted by script capitals), as well as with the sets themselves (denoted by latin capitals). We shall use "$\{R, S, \ldots, T\}$" to indicate the collection whose members are just the sets R, S, ..., T. The empty collection, denoted by $\mathcal{O}$, should not be confused with the collection $\{0\}$ or with the set 0.

A <u>subring</u> of the ring $\mathcal{B}$ is a subcollection of $\mathcal{B}$ that is closed under the operations $\cup$ and $-$. An <u>ideal</u> is a subring that is closed under intersection by arbitrary elements of $\mathcal{B}$. Examples of ideals are $\mathcal{B}$, $\{0\}$, and the collection $\mathcal{B}_N$ of all subsets of a set $N \in \mathcal{B}$. The empty set 0 is common to all subrings. However, if $\mathcal{A}$ is a subring of $\mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ may have the same or different units, or either $\mathcal{A}$ or $\mathcal{B}$, or both, may fail to have a unit. Given any subcollection $\mathcal{S}$ of $\mathcal{B}$, we denote by $\overline{\mathcal{S}}$ the smallest subring of $\mathcal{B}$ which includes $\mathcal{S}$; we shall say that $\mathcal{S}$ generates $\overline{\mathcal{S}}$.

An atom of a Boolean ring $\mathcal{A}$ is a non-zero element whose only subsets in $\mathcal{A}$ are itself and 0. The collection of all

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1. Addition in the ring corresponds to the "symmetric difference" of sets: $S + T = (S \cup T) - (S \cap T)$.
2. A Boolean ring with unit is sometimes called a Boolean algebra. Only in a Boolean algebra do sets have complements.

atoms of $\mathcal{A}$ is denoted by $\mathcal{A}^*$. A finite ring is always generated by its atoms; an infinite ring, on the other hand, may possess no atoms at all (see Example 6, below).

Representations for abstract Boolean rings

It was shown by Stone[1] that every abstract Boolean ring is isomorphic to a concrete "algebra of classes" — that is, to a system of subsets of a set of points. For a proof, one constructs a transfinite nested sequence of sets of $\mathcal{B}$, extended as far as possible without including the empty set. The collection, $\mathcal{X}$, of all supersets of sets in this sequence is an ultrafilter in the sense of Bourbaki[2]; its complement is a prime ideal in $\mathcal{B}$. The collection $\mathcal{X}$ is identified with a "point" x by the rule

$$x \in S \quad \text{if and only if} \quad S \in \mathcal{X}.$$

It is easily shown (1) that if two sets are distinct, then there is a "point" belonging to one and not the other, and (2) that if two "points" are distinct, then there is a set containing one and not the other. $\mathcal{B}$ can therefore be realized as a system of subsets of the set of such "points".

Examples

EXAMPLE 1. All Boolean rings having the same, finite number of elements are isomorphic. The number of elements is of the form 2^n, n being the number of atoms.

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1. M. H. Stone, op. cit.

2. N. Bourbaki, Élements de Mathématique, Pt. I, Bk. III, Ch. I, Hermann et Cie., Paris, 1940, pp. 25-26.

EXAMPLE 2. Let $\mathcal{B}$ consist of all subsets of a point set I. Then $\mathcal{B}$ has cardinal number 2^i, where i denotes the cardinal number of I.

EXAMPLE 3. Let $\mathcal{B}$ consist of all finite subsets of an infinite point set I. Then $\mathcal{B}$ and I have the same cardinality. $\mathcal{B}$ has no unit, and is generated by its atoms. If I is adjoined to $\mathcal{B}$, the cardinality remains the same, but the extended ring $\overline{\mathcal{B} \cup \{I\}}$ is no longer generated by its atoms.

EXAMPLE 4. Consider the collection of all square-free integers as a Boolean ring, under the operations

$$S \cup T = \text{lcm}(S, T)$$
$$S \cap T = \text{gcd}(S, T)$$
$$S - T = S/\text{gcd}(S, T).$$

The integer 1 is the empty set. The primes are the atoms, and they generate the ring.

EXAMPLE 5. The Lebesgue-measurable subsets of the closed unit interval $[0, 1]$ form a Boolean ring. The points are the only atoms. The sets of measure zero form an ideal.

EXAMPLE 6. Take $\mathcal{B}$ as the ring generated by the half-open intervals $(a, b]$ with rational endpoints, $0 \leq a < b \leq 1$. This ring has no atoms, and $\aleph_0$ elements. The underlying point set of Stone's representation, however, contains $c = 2^{\aleph_0}$ elements. Among these one finds not only the ordinary, geometrical points of the interval $(0, 1]$, but also such abstruse objects as the "point" common to the intervals:

$$(a, a + 1/n], \qquad n = 1, 2, \ldots .$$

This illustrates the inherent inefficiency of the representation for an atomless ring $\mathcal{B}$. In the present instance, the rational points of $(0, 1]$ — or any other countable, dense subset of $(0, 1]$ — would have been a sufficient point basis for $\mathcal{B}$, satisfying conditions (1) and (2) on page 3.

We shall return occasionally to these examples in the following chapters.

2. Partitions and partition trees.

DEFINITION 1. The <u>cover</u> of a finite collection of sets is their union. The cover of the empty collection $\mathcal{O}$ is the empty set 0.

DEFINITION 2. The <u>product</u> $\mathcal{S}\cdot\mathcal{T}$ of two collections of non-empty sets is the collection consisting of all non-empty sets of the form $S \cap T$, with $S \in \mathcal{S}$, $T \in \mathcal{T}$.

THEOREM 1. The formation of products of collections of sets is associative and commutative. For any collection $\mathcal{S}$ we have

$$\mathcal{S} \subseteq \mathcal{S}\cdot\mathcal{S} \qquad \text{and} \qquad \mathcal{O}\cdot\mathcal{S} = \mathcal{O}.$$

If the covers of the finite collections $\mathcal{S}$ and $\mathcal{T}$ are disjoint, then $\mathcal{S}\cdot\mathcal{T} = \mathcal{O}$. For any collections $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$, if $\mathcal{S} = \mathcal{S}\cdot\mathcal{T}$ and $\mathcal{T} = \mathcal{T}\cdot\mathcal{U}$, then $\mathcal{S} = \mathcal{S}\cdot\mathcal{U}$.

We omit the proof of this theorem.

DEFINITION 3. Given two finite collections $\mathcal{S}$ and $\mathcal{T}$, we shall say that $\mathcal{S}$ is <u>finer than</u> $\mathcal{T}$, or a <u>refinement of</u> $\mathcal{T}$, if and only if they have the same cover, and $\mathcal{S} = \mathcal{S}\cdot\mathcal{T}$.

It is clear, by Theorem 1, that the refinement relation is a partial ordering of the finite subcollections of $\mathcal{B}$.

DEFINITION 4. A <u>partition</u> of a non-empty set N is a finite collection of non-empty, pairwise disjoint sets whose cover is N. The empty sets 0 has the formal partition $\mathcal{O}$.

Given a partition $\mathcal{P}$ of a set N, we shall sometimes speak of the induced partition $\{S\}\cdot\mathcal{P}$ of a subset S of N. The following theorem is a direct consequence of the definitions.

THEOREM 2. If $\mathcal{P}$ partitions S and $\mathcal{Q}$ partitions T, then $\mathcal{P}\cdot\mathcal{Q}$ partitions $S \cap T$. Any two partitions of the same set have a common refinement — namely, their product — which is again a partition of that set.

The last-mentioned property of partitions, sometimes referred to as the "Moore-Smith" property, will enable us to make use of the techniques of directed sets:

DEFINITION 5. Let $\phi(\mathcal{P})$ be a real valued function defined on the partitions of a fixed set $N \in \mathcal{B}$. Then the <u>directed limit</u> of ϕ, denoted by $\lim_{\mathcal{P}} \phi(\mathcal{P})$, is

(a) equal to the number λ with the property: for any $\varepsilon > 0$ there is a partition $\mathcal{P}_\varepsilon$ of N such that

$$|\phi(\mathcal{P}) - \lambda| < \varepsilon,$$

for every $\mathcal{P}$ finer than $\mathcal{P}_\varepsilon$, if such a number exists;

(b) equal to $+\infty$ (resp. $-\infty$) if, for any number β, there is a partition $\mathcal{P}_\beta$ of N such that

$$\phi(\mathcal{P}) > \beta \qquad (\text{resp. } \phi(\mathcal{P}) < \beta),$$

for every $\mathcal{P}$ finer that $\mathcal{P}_\beta$; or

(c) nonexistent, if neither (a) nor (b) can be satisfied.

The Moore-Smith property of partitions guarantees that the directed limit is unique when it exists.

Partition trees.

Let π be a function — called a partition function — that associates with each $S \in \mathcal{B}$ a collection $\pi(S) \subseteq \mathcal{B}$ which partitions S. A collection $\mathcal{T}$ is said to be closed with respect to π if $S \in \mathcal{T}$ always implies $\pi(S) \subseteq \mathcal{T}$. The notation $\tau(\pi, N)$ will denote the smallest collection containing N and closed with respect to π.[1]

DEFINITION 6. A partition tree is a finite collection of the form $\tau(\pi, N)$, for some partition function π and set N.

Given a partition tree $\mathcal{T}$, the π and N giving rise to it are essentially determined. In fact, N must be the cover of $\mathcal{T}$. On a minimal set[2] P of $\mathcal{T}$, the function π must operate trivially: $\pi(P) = \{P\}$. For any non-minimal set S of $\mathcal{T}$, $\pi(S)$ must consist precisely of all the maximal proper subsets of S in $\mathcal{T}$. For sets S outside of $\mathcal{T}$, the value of $\pi(S)$ is of course not determined.

DEFINITION 7. The collection of minimal sets in a partition tree $\tau(\pi, N)$ is a partition of the maximal set N; we call it the ground partition of N and denote it by $\pi^*(N)$.

We wish next to express the relationship that exists between two trees when one can be obtained from the other by adding partitions of the elements of the latter's ground partition, or by a finite sequence of such extensions. For this purpose

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1. I. e., the intersection of all such collections.
2. With respect to the partial ordering of sets by inclusion.

we define a class of partial orderings of the partition functions.

DEFINITION 8. If π and π' are partition functions, then the expression

$$\pi' \geq_N \pi$$

is to mean that $\tau(\pi', N)$ and $\tau(\pi, N)$ are partition trees, and that

$$\pi'(S) = \pi(S)$$

holds for all sets S in $\tau(\pi, N) - \pi^*(N)$. It follows that

$$\tau(\pi', N) \supseteq \tau(\pi, N);$$

we call the former partition tree an <u>extension</u> of the latter.

Two partition trees with the same cover, N, do not necessarily have a common extension. The directed limit of a function of partition trees therefore cannot be defined. The more complicated limiting process which we are forced to go through in Chapter 5 is based on the next two theorems.

THEOREM 3. If $\tau(\pi, N)$ is a partition tree, and if $\mathcal{P}$ is any refinement of $\pi^*(N)$, then a partition function π' can be found such that $\pi' \geq_N \pi$ and $\pi'^*(N) = \mathcal{P}$.

Proof. Take the function π' as follows:

$$\pi'(S) = \{S\} \cdot \mathcal{P}, \qquad \text{if} \quad S \in \pi^*(N),$$

$$\pi'(S) = \{S\}, \qquad \text{if} \quad S \in \mathcal{P},$$

$$\pi'(S) = \pi(S), \qquad \text{if} \quad S \notin \pi^*(N) \cup \mathcal{P}.$$

(The first two cases may overlap.) One now verifies immediately that π' has the required properties.

THEOREM 4. If $\tau(\pi_1, N)$ and $\tau(\pi_2, N)$ are partition trees, then partition functions π_1' and π_2' can be found such that $\pi_1' \geq_N \pi_1$, $\pi_2' \geq_N \pi_2$, and $\pi_1'^*(N) = \pi_2'^*(N)$.

Proof. Take $\mathcal{P}$ to be the product $\pi_1^*(N)\cdot\pi_2^*(N)$, then apply Theorem 3 twice.

Example.

Taking $\mathcal{B}$ as in Example 6 (page 4, above), let $\mathcal{P}_n$ be the collection of intervals

$$((k-1)2^{-n}, k2^{-n}], \qquad k = 1, 2, \ldots, 2^n.$$

(Thus, $\mathcal{P}_n$ partitions I.) Define $\pi(S)$ to be the induced partition $\{S\}\cdot\mathcal{P}_n$, where n is the least integer such that $\mathcal{P}_n \cap [\{S\}\cdot\mathcal{P}_n] \neq \mathcal{O}$. Then $\pi(S) = \{S\}$ if and only if $S \in \mathcal{P}_n$, for some $n = 0, 1, 2, \ldots$. It follows that $\tau(\pi, N)$ is a partition tree if and only if N is a set of the subring generated by the collection of intervals of the form

$$(0, p2^{-q}] \qquad \text{(p and q integers).}$$

For example, if $N = (0, 7/16]$ then $\tau(\pi, N)$ is the tree consisting of the five sets

$$N,\ (0, 1/4],\ (1/4, 7/16],\ (1/4, 3/8],\ (3/8, 7/16].$$

If on the other hand $N = (0, 1/3]$, then $\tau(\pi, N)$ is the infinite collection:

$$\{N, (0, 1/4], (1/4, 1/3], (1/4, 5/16], (5/16, 1/3], (5/16, 21/64], \dots \}.$$

3. Set functions and carriers.

The real-valued functions f defined on $\mathcal{B}$, subject to the restriction

$$f(0) = 0,$$

form a linear space, which we shall denote by F.

DEFINITION 9. A set function $f \in F$ is <u>additive</u> if, for every pair of disjoint sets S, T in $\mathcal{B}$,

$$f(S \cup T) = f(S) + f(T).$$

The additive functions form a subspace of F, denoted by FA.

We emphasize that this is "finite" additivity. Questions involving "countable" additivity of set functions will not be considered in this paper.

DEFINITION 10. A set function $f \in F$ is said to be <u>superadditive</u> if, for every pair of disjoint sets S, T in $\mathcal{B}$,

$$f(S \cup T) \geq f(S) + f(T).$$

DEFINITION 11. A set function $f \in F$ is said to be <u>subadditive</u> if, for every pair of disjoint sets S, T in $\mathcal{B}$,

$$f(S \cup T) \leq f(S) + f(T).$$

The superadditive and subadditive functions form convex cones in F, whose intersection is FA.

DEFINITION 12. A set function $f \in F$ is <u>monotone</u> if, for every pair S, T in $\mathcal{B}$ with $S \subseteq T$,

$$f(S) \leq f(T).$$

We shall also speak of non-negative, and bounded set functions. A monotone function is necessarily non-negative; a non-negative, superadditive function is monotone.

DEFINITION 13. A non-negative, additive set function, not identically zero, is called a <u>measure</u>.

When we say that a function is <u>additive on the set</u> N, we shall mean that, if it were restricted to the subring $\mathcal{B}_N$, it would be additive. Analogous variants of the other definitions will also be used occasionally.

<u>Existence of measures</u>.

A measure is easily constructed on an arbitrary ring $\mathcal{B}$ by means of Stone's representation theorem (see page 3). Let x be an element of the underlying point space; and let $m(S) = 1$ if $x \in S$, $m(S) = 0$ if $x \notin S$. Then clearly m is a measure.

EXAMPLE 7. Let $\mathcal{B}$ be the collection of finite subsets of a point set I, as in Example 1 or Example 3 (pages 3-4). Let $j(S)$ be the number of points in S, for each S. Then j is a measure on $\mathcal{B}$, not necessarily bounded.

<u>Carriers</u>.

DEFINITION 14. A <u>carrier</u> of a function $f \in F$ is any set $N \in \mathcal{B}$ such that

$$f(N \cap S) = f(S)$$

holds for every $S \varepsilon \mathcal{B}$.

Thus, a function vanishes on all sets that are entirely outside any one of its carriers.

THEOREM 5. (a) If N carries f, then any superset of N carries f.

(b) If M and N each carry f, then $M \cap N$ carries f.

(c) If M carries f and N carries g, and if α and β are real numbers, then $M \cup N$ carries $\alpha f + \beta g$.

(d) Every set carries the function 0.

(e) If $I \varepsilon \mathcal{B}$, then I carries every function.

The proofs present no difficulty.

The intersection of all carriers of a function need not be a carrier of that function, even though it be a well defined set of $\mathcal{B}$ (see Example 8, below). The function j of Example 7 (above) has no carrier at all if I is infinite. Example 9 presents another instance of a function that has no carrier, even though it vanishes on almost all sets.

EXAMPLE 8. Let $\mathcal{B}$ be the collection of all subsets of an infinite set I, and define $f(S)$ to be 0 if S is finite, 1 if S is infinite. Then any set which is the complement of a finite set carries f. But the intersection of all these carriers is 0, which is not a carrier of f.

EXAMPLE 9. Let R be a non-empty set in an arbitrary ring $\mathcal{B}$. Define the function $d_R \in F$ to be 1 on R and 0 on all other sets. The only possible carrier of d_R is the unit set I: if $\mathcal{B}$ has no unit, then d_R has no carrier.

Proof. Suppose N carries d_R. If $N \neq I$ then a non-empty set P can be found which intersects neither N nor R. This produces the contradiction:

$$1 = d_R(R) = d_R(N \cap R) = d_R(N \cap (R \cup P)) = d_R(R \cup P) = 0.$$

EXAMPLE 10. Define e_R to be 1 on the supersets of R, and 0 on all other sets (see Chapter 6), and put $f_R = e_R - d_R$. First consider the ring $\mathcal{B}$ of Example 6 (page 4) generated by the half-open, rational subintervals of $(0, 1]$. If V denotes the interval $(0, 1/\sqrt{2}]$, then e_V and f_V are identical and are carried by any set of $\mathcal{B}$ containing V. The intersection of all carriers is V itself, which is not in $\mathcal{B}$. Hence neither function has a least carrier.

Now consider the extended ring $\mathcal{B}' = \overline{\mathcal{B} \cup \{V\}}$. The functions e_V and f_V are now different. The former has all supersets of V in $\mathcal{B}'$ as carriers, as before; the latter has the unique carrier I. In both cases there is now a least carrier.

4. Integration.

We introduce for convenience the abbreviation

$$J(f, \mathcal{S}) \qquad \text{for} \qquad \sum_{S \in \mathcal{S}} f(S).$$

Only sums over finite collections $\mathcal{S}$ are contemplated.

DEFINITION 15. The <u>integral</u> over the set N of the function f, is given by

$$\int_N f = \lim_{\mathcal{P}} J(f, \mathcal{P}),$$

the directed limit being taken over the partitions $\mathcal{P}$ of N, ordered by refinement. (See Definition 5, page 7.)

DEFINITION 16. A function $f \in F$ for which $\int_N f$ exists and is finite is said to be integrable on N. A function which is integrable on every set in $\mathcal{B}$ is said simply to be <u>integrable</u>. The class of integrable functions is denoted by FI.

THEOREM 6. If f and g are integrable on N, then $\alpha f + \beta g$ is integrable on N and

$$\int_N (\alpha f + \beta g) = \alpha \int_N f + \beta \int_N g.$$

for any real numbers α and β.

Thus, the integral is a linear operator, and FI is a linear subspace of F. The proof of the theorem is immediate.

THEOREM 7. (a) If f is integrable on N, then f is integrable on every subset of N.

(b) If f is integrable on M and on N, then f is integrable on $M \cup N$.

Proof. (a) Let S be a proper non-empty subset of N. Given any $\varepsilon > 0$ choose $\mathcal{P}_\varepsilon$ so that

$$|J(f, \mathcal{P}) - \int_N f| < \varepsilon$$

for every $\mathcal{P}$ finer than $\mathcal{P}_\varepsilon$. Write $\mathcal{Q}_\varepsilon$ and $\mathcal{R}_\varepsilon$ for the induced partitions $\{S\}\cdot\mathcal{P}_\varepsilon$ and $\{N-S\}\cdot\mathcal{P}_\varepsilon$, respectively. Since $\mathcal{Q}_\varepsilon \cup \mathcal{R}_\varepsilon$ is a partition of N finer than $\mathcal{P}_\varepsilon$, we have

$$|J(f, \mathcal{Q}_\varepsilon) + J(f, \mathcal{R}_\varepsilon) - \int_N f| < \varepsilon.$$

We also have

$$|J(f, \mathcal{Q}) + J(f, \mathcal{R}_\varepsilon) - \int_N f| < \varepsilon$$

for any partition $\mathcal{Q}$ of S finer than $\mathcal{Q}_\varepsilon$. Hence

$$|J(f, \mathcal{Q}) - J(f, \mathcal{Q}_\varepsilon)| < 2\varepsilon.$$

The existence and finiteness of the integral $\int_S f$ is now apparent.

(b) By part (a) we may suppose without loss of generality that M and N are disjoint. Given $\varepsilon > 0$ we can choose $\mathcal{Q}_\varepsilon$ and $\mathcal{R}_\varepsilon$ partitioning M and N respectively so that

$$|J(f, \mathcal{Q}) - \int_M f| < \varepsilon$$

$$|J(f, \mathcal{R}) - \int_N f| < \varepsilon$$

hold for all refinements $\mathcal{Q}$ of $\mathcal{Q}_\varepsilon$ and $\mathcal{R}$ of $\mathcal{R}_\varepsilon$. Since the collection $\mathcal{P}_\varepsilon = \mathcal{Q}_\varepsilon \cup \mathcal{R}_\varepsilon$ is a partition of $M \cup N$, and any refinement $\mathcal{P}$ of $\mathcal{P}_\varepsilon$ is a union of refinements of $\mathcal{Q}_\varepsilon$ and $\mathcal{R}_\varepsilon$, we have

$$|J(f, \mathcal{P}) - (\int_M f + \int_N f)| < 2\varepsilon.$$

This shows not only that f is integrable on $M \cup N$, but also that $\int_{M \cup N} f = \int_M f + \int_N f$.

DEFINITION 17. The <u>additive part</u> of $f \in FI$ is the function $f^0 \in F$ defined by

$$f^0(S) = \int_S f.$$

THEOREM 8. The collection of sets on which a given function $f \in F$ is integrable is an ideal, on which $\int_S f$ is additive in S. In particular, if $f \in FI$, then $f^0 \in FA$: the additive part of an integrable set function is additive.

The proof is an application of the preceding theorem, and of the concluding remark of its proof.

THEOREM 9. If $f \in FA$, then $f \in FI$ and $f^0 = f$.

The proof is trivial.

THEOREM 10. If f is superadditive, then $\int_S f$ exists and does not exceed $f(S)$. Likewise, if f is subadditive, then $\int_S f$ exists and is not less than $f(S)$.

We omit the proof. The difference $|f(S) - \int_S f|$ turns out in each case to be what is called in Chapter 5 the "deviation" of f; it may be finite or infinite. (See Example 11, below.)

THEOREM 11. If $f \in FI$ is non-negative, then f^0 is monotone.

The proof is trivial.

EXAMPLE 11. Let $f \in F$ have the value 1 on all non-empty sets. Then f is subadditive. But f is not integrable on any set which has more than a finite number of subsets.

Absolutely integrable functions.

DEFINITION 18. A function $f \in F$ is absolutely integrable (AI) if $|f| \in FI$. A function $f \in F$ is absolutely integrable on N if $|f|$ is integrable on N.

An absolutely integrable function is integrable, and we have

$$|f^0(S)| \leq \int_S |f|$$

for every $f \in AI$, $S \in \mathcal{B}$.

THEOREM 12. The sum of two absolutely integrable functions is absolutely integrable. If

$$\int_N |f| = 0,$$

then f^0 is identically 0 on $\mathcal{B}_N$; conversely, if f^0 is identically 0 on $\mathcal{B}_N$, then f is absolutely integrable on N, and $\int_N |f| = 0$.

Proof. The first statement is an easy consequence of the inequality

$$|f(S) + g(S)| \leq |f(S)| + |g(S)|.$$

To prove the first part of the second statement, we note that $\int_S |f|$ must vanish for all subsets of N, since the absolute integral is obviously a monotone set function. Given $\varepsilon > 0$, let $\mathcal{P}_\varepsilon$ be a partition of $S \subseteq N$ such that

$$J(|f|, \mathcal{P}) < \varepsilon$$

holds for all $\mathcal{P}$ finer than $\mathcal{P}_\varepsilon$. Then also $|J(f, \mathcal{P})| < \varepsilon$. Hence $f^o(S) = 0$, as was to be shown. To prove the second part of the second statement, choose $\mathcal{P}_\varepsilon$ partitioning N so that

$$|J(f, \mathcal{P})| < \varepsilon$$

holds for all $\mathcal{P}$ finer than $\mathcal{P}_\varepsilon$. Let $\mathcal{R}$ be a subcollection of $\mathcal{P}$. If $|J(f, \mathcal{R})|$ were greater than ε, then by refining the complementary subcollection $\mathcal{P} - \mathcal{R}$ we would obtain a contradiction between the two requirements $f^o(N) = 0$ and $f^o(S) = 0$, S being the cover of $\mathcal{P} - \mathcal{R}$. Therefore we have

$$|J(f, \mathcal{R})| \leq \varepsilon.$$

Taking $\mathcal{R}$ to consist, in turn, of those $S \in \mathcal{P}$ with $f(S) \geq 0$, and those with $f(S) < 0$, yields the inequality

$$J(|f|, \mathcal{P}) \leq 2\varepsilon.$$

The result now follows.

This theorem shows that AI is a subspace of FI. The absolute integral $\int_N |f|$ is a norm, not on AI, but on the smaller subspace of additive, absolutely integrable functions with carrier N.

THEOREM 13. If $f \in FI$, then $f \in AI$ if and only if $f^o \in AI$.

Proof. If $f \in FI$, then the integral of $f - f^o$ is identically zero. By the second part of Theorem 12, $f - f^o \in AI$. Hence, by the first part of Theorem 12, $f^o \in AI$ implies that $f = (f - f^o) + f^o \in AI$, and conversely.

THEOREM 14. A set function is absolutely integrable if and only if it is integrable and its additive part is the difference of two measures.

Proof. Express $f \in F$ as the difference of two non-negative functions: $f = f_1 - f_2$, where

$$f_1 = \tfrac{1}{2}(|f| + f), \qquad f_2 = \tfrac{1}{2}(|f| - f).$$

If $f \in AI$, then f_1 and f_2 are absolutely integrable, and hence integrable, and we have

$$f^o = f_1{}^o - f_2{}^o.$$

But $f_1{}^o$ and $f_2{}^o$ are measures, by Theorem 11. Conversely, suppose that f^o is the difference of two measures. By Theorem 9 they are integrable. Being non-negative, they are absolutely integrable. By Theorem 12 their difference f^o is therefore absolutely integrable. By Theorem 13 the original function f is also absolutely integrable.

Not all integrable functions are absolutely integrable.

EXAMPLE 12. Define $g(S)$ on the half-open subintervals of $(0, 1]$ by:

$$\begin{cases} g((0, a]) = 1/a, \\ g((a, b]) = 1/b - 1/a, \end{cases}$$

$0 < a < b \leq 1$. Extend g by additivity to the ring generated by these intervals — this can be done in a unique way. Then g is additive, but not the difference of measures. Hence g is integrable, but $|g|$ is not.

Integration of point functions.

The purpose of this section is to describe how Stieltjes integration of point functions can be defined in terms of non-additive set functions.

THEOREM 15. If $f \in AI$ and if h is monotone, then $hf \in AI$. Moreover, for any $S \in \mathcal{B}$, we have

$$\int_S hf = \int_S hf^o.$$

Proof. We shall show by separate arguments that hf^o and $hf - hf^o$ are absolutely integrable. The absolute integrability of hf will follow by Theorem 12. (a) For the first, it suffices to show that $hm \in AI$ for measures m. Let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of the same set N, with $\mathcal{P}'$ finer than $\mathcal{P}$. Then

$$\sum_{T \in \mathcal{P}'} h(T)m(T) = \sum_{S \in \mathcal{P}} \sum_{\substack{T \in \mathcal{P}' \\ T \subseteq S}} h(T)m(T) \leq \sum_{S \in \mathcal{P}} \sum_{\substack{T \in \mathcal{P}' \\ T \subseteq S}} h(S)m(T) = \sum_{S \in \mathcal{P}} h(S)m(S),$$

the inequality resulting from the monotonicity of h. Therefore

$$0 \leq J(hm, \mathcal{P}') \leq J(hm, \mathcal{P}).$$

We therefore have convergence to the directed limit $\int_N hm$. Hence the function $hm = |hm|$ is integrable. (b) Consider the function $hf - hf^o$. By the monotonicity of h we have

$$0 \leq J(|hf-hf^o|, \mathcal{P}) \leq h(N)J(|f - f^o|, \mathcal{P})$$

for any partition $\mathcal{P}$ of the set N. The right member of this inequality converges to zero, since $(f - f^o)^o$ is identically zero (Theorems 9 and 12). Therefore $\int_N |hf - hf^o| = 0$, and the function $hf - hf^o$ is absolutely integrable. (c) To prove the last statement of the theorem, we need only observe that

$$(hf - hf^o)^o = 0,$$

—by Theorem 12 and what we have just proved.

Now let $\phi(x)$ be a bounded, non-negative function on a given point set I. Let $\mathcal{B}$ be any ring of subsets of I. Define $\bar{h}_\phi \in F$ by:

$$\bar{h}_\phi(S) = \sup_{x \in S} \phi(x);$$

clearly $\bar{h}_\phi$ is monotone. If f is any absolutely integrable function on $\mathcal{B}$, then $\bar{h}_\phi f$ is integrable by Theorem 15. We define the <u>upper Stieltjes integral</u> to be

$$\bar{\int_N} \phi \, df = \int_N \bar{h}_\phi f.$$

The rest is straightforward: after removing the restriction to non-negative functions ϕ,[1] we can proceed to define the lower

1. Required by the particular way in which monotone set functions were defined in Chapter 3, but not essential in the present context.

Stieltjes integral to be.

$$\underline{\int}_N \phi \, df = -\overline{\int}_N -\phi \, df.$$

When upper and lower integrals are equal, they define the Stieltjes integral itself. The existence of the latter will depend in general on whether the ring $\mathcal{B}$ is fine enough to "resolve" the function $\phi(x)$. A necessary condition for existence is that, for every $y \in N$,

$$\underset{S \ni y}{\text{limit}} \left[\sup_{x \in S} \phi(x) - \inf_{x \in S} \phi(x) \right] = 0,$$

the limit being taken over the directed set of sets S containing the point y, ordered by inclusion. This condition is not sufficient, as the following example reveals.

EXAMPLE 13. Let $\mathcal{B}$ be the ring generated by the collection consisting of the subintervals of $I = [0, 1]$ and the individual points in I. Then the condition of the preceding paragraph is always met. But if $\phi(x)$ is the function which is 1 on the rational points and 0 on the irrational points, and if f is ordinary Lebesgue measure, then we have, for each N in $\mathcal{B}$,

$$\underline{\int}_N \phi \, df = 0, \quad \overline{\int}_N \phi \, df = f(N).$$

On the other hand, if $\mathcal{B}$ is enlarged by adjoining the set of rational points, then we have instead

$$\underline{\int}_N \phi \, df = \overline{\int}_N \phi \, df = \int_N \phi \, df = 0$$

for every set $N \in \mathcal{B}$.

EXAMPLE 14. The "inner measure" of subsets of the real unit interval $[0, 1]$ is a non-negative, superadditive set function; it is therefore absolutely integrable on the ring of all subsets of $[0, 1]$. For purposes of integration it is useless, however, as its additive part is identically zero.[1] "Outer measure", on the other hand, is not integrable; indeed, every interval is a disjoint union of infinitely many (non-measurable) sets of equal, positive outer measure, thus the integral of the outer measure on an interval is infinite.[2]

Functions of intervals.

Burkill[3] has developed a theory of non-additive integration for n-dimensional Euclidean space; it is the purpose of this section to compare and connect his work with ours. We shall not resort to formal definitions and proofs, since his results do not contribute significantly to the more general theory set forth in this paper.

An "interval" R may be defined as an n-tuple of pairs of real numbers:

$$R = \langle (a_1, b_1), \ \dots , \ (a_n, b_n) \rangle,$$

it being unimportant whether R be considered open or closed or

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1. This can be shown by means of the well known partition of the unit interval into two non-measurable sets of zero inner measure and a third set of measure 0. See, e.g., J. von Neumann, op. cit., pp. 38–41.

2. It seems likely that this holds for arbitrary sets of positive outer measure, but I know of no proof for non-measurable sets.

3. J. C. Burkill, Functions of intervals, Proc. Lon. Math. Soc. (2) 22 (1924), pp. 275–310.

neither. The "norm" of R is the maximum of $|a_i - b_i|$; the "norm" of a collection of intervals is the maximum of their norms. The collection of all intervals, $\mathcal{R}$, generates a ring $\mathcal{C} = \overline{\mathcal{R}}$. Let f be a function on $\mathcal{R}$. The Burkill integral of f on a set $N \in \mathcal{C}$ is defined to be

$$\operatorname*{limit}_{\mathcal{P}} \; J(f, \mathcal{P}),$$

with $\mathcal{P}$ restricted to partitions of N which consist of intervals only, and with the limit taken as the norm of $\mathcal{P}$ approaches zero.[1]

Before we can apply the directed-limit concept of integration to such a function f, we must extend it appropriately from $\mathcal{R}$ to $\mathcal{C}$. In the one-dimensional case, every set S in $\mathcal{C}$ can be represented uniquely as a union of disjoint, non-abutting intervals $R_1, \ldots , R_k$. We can therefore consistently define

$$f(S) = J(f, \{R_1, \ldots , R_k\}).$$

In higher dimensions, a canonical partition of non-intervals into intervals is not so quickly described — but is nevertheless possible. The accompanying figure indicates the nature of the difficulty that must be overcome. With a canonical partition

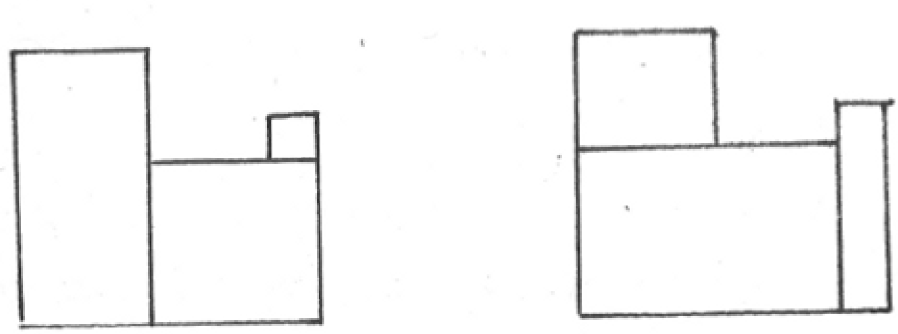

Figure 1.

1. This is his "extended integral". We have not followed his notation in all respects.

prescribed for non-intervals S, we then define $f(S)$ as above.

The Burkill definition of integral, just given, is evidently equivalent to the definition obtained by introducing the extended function and permitting arbitrary partitions $\mathcal{P}$. For, with each such $\mathcal{P}$, an interval partition $\mathcal{P}'$ can be associated by taking the union of the canonical partitions of the elements of $\mathcal{P}$; and we have

$$\text{norm } \mathcal{P}' \leq \text{norm } \mathcal{P}, \qquad \text{and} \qquad J(f, \mathcal{P}') = J(f, \mathcal{P}).$$

Using this equivalent definition, it is easy to see that if the Burkill integral of a function of intervals exists, then the directed-limit integral of this chapter exists for the extended function and is equal. (This is true because in any directed set of partitions the norm tends to zero.) However, the converse does not hold: there are functions of intervals whose extended functions are integrable in the sense of Definition 16 (page 16), yet whose Burkill integrals do not converge. An example is given below.

In all cases it can be shown that the directed-limit integral of the extension to $\mathcal{C}$ of a function on $\mathcal{R}$ is independent of the manner in which the extension is made — i. e., of the particular canonical partition prescribed for non-intervals.

Thus, the partial ordering of partitions by refinement, as contrasted with their simple ordering by decreasing norm, and the subsequent use of directed limits, strengthens and broadens the theory of integration of non-additive functions of intervals in Euclidean space of n dimensions. The new formulation is also more elegant, in that reference to the norm of intervals can be eliminated.

EXAMPLE 15.[1] The function g defined on the one-dimensional intervals by

$$g(\langle(-1/k,\ 1/k)\rangle) = (-1)^k, \qquad k = 1, 2, \ldots,$$

$$g(R) = 0 \qquad \text{all other } R \in \mathcal{R},$$

has no Burkill integral on the set $\langle(-1, 1)\rangle$, since the limit oscillates between 1 and -1. On the sets $\langle(-1, 0)\rangle$ and $\langle(0, 1)\rangle$, however, the Burkill integrals exist and are equal to 0. (Thus, the Burkill-integrable sets of a function do not form a ring.) On the other hand, the extension of g is integrable and absolutely integrable in the sense of this chapter, with $g^0 = 0$.

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1. Due to Burkill, *op. cit.*, p. 285.

5. Deviation of set functions.

In this chapter we invent an expression for the amount by which a function falls short of being additive. The "deviation" of a set function, as we shall call this quantity, is analogous in many formal respects to the <u>variation</u> of a function of a real variable. In this analogy, the additive set functions are the constants, the superadditive set functions are the monotonic increasing functions, etc. Our principal tool will be the partition trees introduced in Chapter 2 (page 8ff).

Let f be an arbitrary set function, let N be a non-empty set in $\mathcal{B}$, and let π be a partition function on $\mathcal{B}$ such that $\mathcal{T}(\pi, N)$ is a partition tree. Define the three quantities:

$$D_N^+(f, \pi) = \sum_{S \varepsilon \mathcal{T}(\pi,N)} \max \left[0, \ f(S) - J(f, \pi(S)) \right];$$

$$D_N^-(f, \pi) = \sum_{S \varepsilon \mathcal{T}(\pi,N)} -\min \left[0, \ f(S) - J(f, \pi(S)) \right];$$

$$D_N(f, \pi) = \sum_{S \varepsilon \mathcal{T}(\pi,N)} \mid f(S) - J(f, \pi(S)) \mid .$$

We then have the following theorem:

THEOREM 16. (a) We have

$$D_N^+(f, \pi) + D_N^-(f, \pi) = D_N(f, \pi).$$

(b) We have

$$D_N^+(f, \pi) - D_N^-(f, \pi) = f(N) - J(f, \pi^*(N)).$$

(c) If f is superadditive on N then $D_N^-(f, \pi) = 0$. If f is subadditive on N then $D_N^+(f, \pi) = 0$. If f is additive on N then $D_N(f, \pi) = 0$.

Proof. By the definitions.

THEOREM 17. If $\pi' \geq_N \pi$, then we have

$$D_N^+(f, \pi') \geq D_N^+(f, \pi),$$

$$D_N^-(f, \pi') \geq D_N^-(f, \pi).$$

Proof. As we noted in Definition 8, on page 9, the relation $\pi' \geq_N \pi$ implies that $(\pi', N) \supseteq (\pi, N)$. The inequalities of the theorem are now obvious.

DEFINITION 19. The <u>deviation</u> (resp. <u>upper deviation</u>, <u>lower deviation</u>) of a set function $f \varepsilon F$ on a set $N \varepsilon \mathcal{B}$, denoted by

$$\Delta_N(f) \qquad (\text{resp.} \quad \Delta_N^+(f), \quad \Delta_N^-(f)\ ),$$

is the least upper bound of $D_N(f, \pi)$ (resp. $D_N^+(f, \pi)$, $D_N^-(f, \pi)$), taken over all partition functions π for which $\tau(\pi, N)$ is a partition tree. If we admit the value $+\infty$ these quantities are well defined for every $f \varepsilon F$ and $N \varepsilon \mathcal{B}$.

In view of Theorem 17, these suprema have some of the quality of directed limits. However, not having the Moore-Smith property for "$\geq_N$", we are forced in the following proofs to base our convergence arguments on the properties given in Theorems 3 and 4 (pages 9 and 10) and in Theorem 17.

THEOREM 18. We have

$$\Delta_N(f) = \Delta_N^+(f) + \Delta_N^-(f).$$

Proof. The theorem asserts that the sum of the suprema is equal to the supremum of the sum. It will suffice to show that, given two trees $\tau(\pi^+, N)$ and $\tau(\pi^-, N)$, there is a tree $\tau(\pi, N)$ with the property:

$$(1) \qquad \begin{cases} D_N^+(f, \pi) \geq D_N^+(f, \pi^+), \\ D_N^-(f, \pi) \geq D_N^-(f, \pi^-). \end{cases}$$

Choose partition functions $\pi^{+\prime} \geq_N \pi^+$ and $\pi^{-\prime} \geq_N \pi^-$ according to Theorem 4 (page 10); we intend to show that at least one of these two functions has the property (1). Since the ground partitions $\pi^{+\prime *}(N)$ and $\pi^{-\prime *}(N)$ are equal, we have by Theorem 16 (b) the relation:

$$D_N^+(f, \pi^{+\prime}) - D_N^+(f, \pi^{-\prime}) = D_N^-(f, \pi^{+\prime}) - D_N^-(f, \pi^{-\prime}).$$

Call this quantity η; then $\eta \geq 0$ implies

$$D_N^-(f, \pi^{+\prime}) \geq D_N^-(f, \pi^{-\prime}),$$

and $\eta \leq 0$ implies

$$D_N^+(f, \pi^{-\prime}) \geq D_N^+(f, \pi^{+\prime}).$$

In the first case take $\pi = \pi^{+\prime}$; in the other, $\pi = \pi^{-\prime}$. Then (1) follows with the aid of Theorem 17.

THEOREM 19. If $\Delta_N(f)$ is finite, then $\Delta_S(f)$, $\Delta_S^+(f)$, and $\Delta_S^-(f)$ are superadditive set functions on $\mathcal{B}_N$.

Proof. Suppose $\Delta_N(f)$ finite, and let $\{N_1, N_2\}$ be a partition of N. Let $\tau(\pi_1, N_1)$ and $\tau(\pi_2, N_2)$ be partition trees. Define a partition function π:

$$\begin{cases} \pi(N) = \{N_1, N_2\}; \\ \pi(T) = \pi_i(T) \qquad \text{for } T \in \tau(\pi_i, N_i), \ i = 1, 2; \\ \pi(T) \text{ arbitrary} \quad \text{on other sets } T \in \mathcal{B}. \end{cases}$$

Then $\tau(\pi, N)$ is also a partition tree, consisting of the set N and the members of the disjoint collections $\tau(\pi_1, N_1)$ and $\tau(\pi_2, N_2)$. Hence,

$$D_N(f, \pi) \geq D_{N_1}(f, \pi_1) + D_{N_2}(f, \pi_2).$$

If we take the least upper bound of both sides with respect to all pairs π_1, π_2 of partition functions which generate trees on N_1 and N_2 respectively, then the right-hand side of the above inequality becomes the sum of the deviations of f on N_1 and N_2, while the left-hand side becomes something less than or equal to the deviation of f on N. Consequently we have

$$\Delta_N(f) \geq \Delta_{N_1}(f) + \Delta_{N_2}(f).$$

This result establishes the finiteness of $\Delta_S(f)$ for all $S \in \mathcal{B}_N$. A repetition of the whole argument, with "S" replacing "N", now establishes superadditivity. The proofs for Δ^+ and Δ^- are the same.

THEOREM 20. $\Delta_N(f) = 0$ if and only if f is additive on N.

Proof. If f is additive on N, then $\Delta_N(f)$ obviously must vanish. If f is not additive on N, then disjoint, non-empty sets S and T exist in $\mathcal{B}_N$ with

$$f(S) + f(T) \neq f(S \cup T).$$

The five sets N, $S \cup T$, $N - (S \cup T)$, S, and T constitute a partition tree $\mathcal{T} = \tau(\pi, N)$ for which the quantity $D_N(f, \pi)$ is positive (see Figure 2). Hence $\Delta_N(f) > 0$.

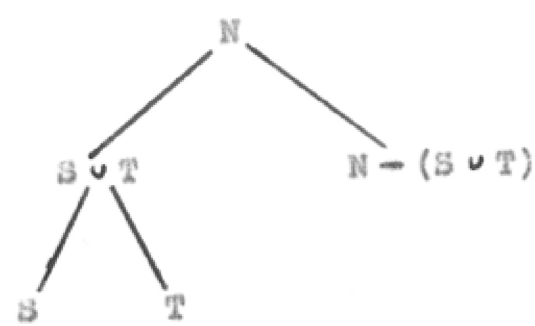

Figure 2.

Using this theorem and Theorem 12 (page 19) it is easy to show that the functional

$$\| f \| = \Delta_N(f) + \int_N |f|$$

is a norm on the linear space of absolutely integrable set functions with carrier N and with finite deviation on N. We shall not pursue this application of the deviation in the present paper.

<u>Canonical decomposition of functions of finite deviation.</u>

Any function whose deviation on all sets $S \in \mathcal{B}$ is finite can be expressed as the sum of three functions f^+, f^-, $f^\#$ —the first two defined by

$$f^{+}(S) = \Delta_S^{+}(f),$$

$$f^{-}(S) = -\Delta_S^{-}(f),$$

and the third being the remainder: $f^{\#} = f - f^{+} - f^{-}$.

DEFINITION 20. The functions f^{+} and f^{-} just defined are called the <u>superadditive</u> and <u>subadditive parts</u>, respectively, of the function f.

The next theorem asserts that $f^{\#}$ is the additive part of f, as defined earlier (Definition 17, page 18).

THEOREM 21. If the deviation on all sets of f is finite, then f is integrable and $f^{0} = f^{\#}$; hence

$$f = f^{+} + f^{-} + f^{0}.$$

Proof. Fix $N \varepsilon \mathcal{B}$, $\varepsilon > 0$. Since the deviation of f is finite, we can find partition functions π^{+}, π^{-} with

$$0 \leq \Delta_N^{+}(f) - D_N^{+}(f, \pi^{+}) < \varepsilon,$$

$$0 \leq \Delta_N^{-}(f) - D_N^{-}(f, \pi^{-}) < \varepsilon.$$

Exactly as in the proof of Theorem 18, we can find a single partition function π which satisfies both sets of inequalities at once. Moreover, by Theorem 17, if $\pi' \geq_N \pi$, then π' will also serve in place of π^{+} and π^{-} in the above inequalities. Inserting π', and subtracting, we obtain

$$\left| \Delta_N^{+}(f) - D_N^{+}(f, \pi') - \Delta_N^{-}(f) + D_N^{-}(f, \pi') \right| < \varepsilon.$$

Theorem 16 (b) (page 29) and the definition of $f^{\#}$ enable us to rewrite this as follows:

$$|J(f, \pi'^*(N)) - f^{\#}(N)| < \varepsilon.$$

By Theorem 3 (page 9) we see that the partition $\pi'^*(N)$ can be an arbitrary refinement of the partition $\pi^*(N)$. Taking the latter as the P_ε of our usual procedure, we see that the integral $\int_N f$ exists and is equal to $f^{\#}(N)$. This completes the proof.

It is obvious (Theorem 19) that f^+ is superadditive and f^- subadditive. We show next that their integrals are identically zero.

THEOREM 22. If the deviation on all sets of f is finite, then f^+ and f^- are integrable and their additive parts are identically 0.

Proof. Fix $N \in \mathcal{B}$, $\varepsilon > 0$, and choose π so that

$$\Delta_N^+(f) - D_N^+(f, \pi') < \varepsilon \tag{1}$$

holds for every $\pi' \geq_N \pi$. We assert that

$$J\big(f^+, \pi'^*(N)\big) \leq \varepsilon, \tag{2}$$

for every such π'. Suppose this is not the case for some $\pi' \geq_N \pi$. Let $R_1, \ldots, R_n$ be the elements of the ground partition of $\tau(\pi', N)$. Then we have

$$\sum_{i=1}^{n} f^+(R_i) = \sum_{i=1}^{n} \Delta_{R_i}^+(f) > \varepsilon.$$

The strict inequality means that we are able to construct trees $\mathcal{T}_i = \tau(\pi_i, R_i)$ on the sets R_i which satisfy

$$\sum_{i=1}^{n} D^{+}_{R_i}(f, \pi_i) > \varepsilon. \tag{3}$$

The collection consisting of the union of the $\mathcal{T}_i$ and the tree $\tau(\pi', N)$ is again a tree; let us call it $\tau(\pi'', N)$. (See Figure 3.)

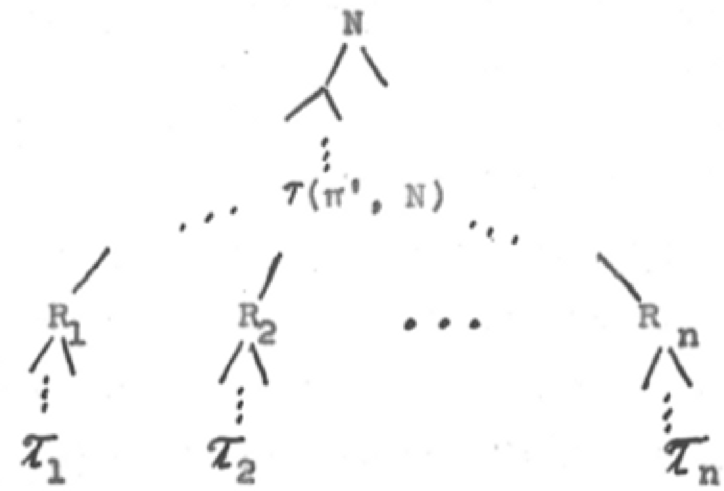

Figure 3.

The partition function π'' is described by saying that it agrees with π' on the non-minimal sets of $\tau(\pi', N)$ and with π_i on all sets of $\mathcal{T}_i$, $i = 1, 2, \ldots, n$. Therefore, by definition (page 29),

$$D^{+}_{N}(f, \pi'') = D^{+}_{N}(f, \pi') + \sum_{i=1}^{n} D^{+}_{R_i}(f, \pi_i).$$

This equation, together with the inequalities (1) and (3), yields the absurdity:

$$D^{+}_{N}(f, \pi'') > \Delta^{+}_{N}(f),$$

thereby establishing the assertion (2). To complete the proof, we simply note that the partition $\pi'{*}(N)$ can be taken to be any refinement of $\pi{*}(N)$ whatever, by Theorem 3 (page 9). Therefore the directed limit converges to zero — that is, $\int_N f^{+} = 0$.

By a similar argument, $\int_N f^- = 0$.

This theorem and some earlier results can be neatly expressed together by considering the superscript symbols "o", "+", "−", as operators in the space of functions of finite deviation.

THEOREM 23. The operators denoted by "o", "+", "−", applied to functions of finite deviation yield functions of finite deviation. They combine according to the rules:

$$f^{\mu\nu} = 0 \quad \text{if} \quad \mu \neq \nu,$$
$$f^{\mu\mu} = f^{\mu}.$$

Proof. A function f of finite deviation is integrable (Theorem 21), and Theorem 20 (page 32) shows that the deviation of f^0 is finite. Theorem 16 (b) (page 29) applied to the non-negative, superadditive function f^+ gives us

$$D_N^+(f^+, \pi) \leq f^+(N),$$

showing that the deviation of f^+ is finite — similarly for f^-. The first rule of combination is derived from Theorems 16 (c), 20, and 22 (pages 30, 32, 35); the second is derived from the first with the aid of Theorem 21, thus:

$$f^{++} = f^+ - f^{+-} - f^{+0} = f^+,$$
$$f^{--} = f^- - f^{-+} - f^{-0} = f^-,$$

— and from Theorem 9 (page 18).

Remark. Although these operators behave algebraically like orthogonal projections, they do not, of course, decompose the linear space of functions of finite deviation into three complementary, orthogonal subspaces. The functions f^+ and f^- lie in opposing cones which span the same subspace, orthogonal to the subspace of additive functions. Also, the operators "+" and "−" are not linear.

A function of finite deviation can be decomposed in many ways into an additive, a superadditive, and a subadditive function, the latter two with integral zero. For example, we have

$$f = (f^+ + g) + (f^- - g) + f^o,$$

where g is any superadditive function with $g^o = 0$. What distinguishes the decomposition of Theorem 21 is that it minimizes the ~~sum of the~~ deviations of the components. We can express this fact also by the statement, that no non-additive, superadditive function h exists such that $f^+ - h$ is superadditive <u>and</u> $f^- + h$ is subadditive.

<u>Characterization of the functions of finite deviation</u>.

The functions of finite deviation form an important class of integrable set functions. In the case of a finite ring of sets they obviously take in all set functions. The extent of this class in general may be judged by the following theorem, which is an easy consequence of our previous results.

THEOREM 24. The class of functions of finite deviation is precisely the subspace of F spanned by the integrable, superadditive functions.

Proof. For any superadditive function f, whether integrable or not, we have, by Theorem 16 (b) and (c) (pages 29–30),

$$D_N^+(f, \pi) = f(N) - J(f, \pi^*(N)).$$

By Theorem 10 (page 18), $\int_N f$ exists; hence we may pass to the limit (using Theorems 3 and 17 in the usual way) and obtain

$$\Delta_N^+(f) = f(N) - \int_N f.$$

Thus, integrability of f[1] assures that f has finite deviation. It follows that all functions in the subspace spanned by the integrable, superadditive functions also have finite deviation. Conversely, Theorem 21 (page 34) shows how every function of finite deviation can be built up out of superadditive, integrable functions.

Examples.

Example 11 (page 19) exhibits a subadditive function whose integral is infinite, and whose deviation is therefore infinite. This shows that the word "integrable" is essential in the statement of Theorem 24.

Example 12 (page 21) presents an additive function which is not absolutely integrable; Example 15 (below), on the other hand, presents an absolutely integrable function whose deviation is infinite. Hence, having finite deviation is neither necessary nor sufficient for absolute integrability.

EXAMPLE 16. Take $\mathcal{B}$ as the ring of Lebesgue-measurable subsets of the real line, and let $m \in FA$ be the Lebesgue measure. Define f by:

$$(1) \qquad f(S) = \phi(m(S))$$

where $\phi(\alpha) = 0$ for α rational and $\phi(\alpha) = \alpha^2$ for α irrational. Then f is integrable and absolutely integrable, and $f^o = 0$. However, the deviation of f on every set of

1. I.e., finiteness of $\int_N f$.

positive measure is infinite.

Proof (of the last statement). Let S have positive measure. Construct a partition tree $\tau(\pi_n, S)$ with the structure indicated in the figure. The measures of S_1, S_2, ... , S_n

Figure 4.

are made alternately rational and irrational. The measures of the R_i are made smaller than $2^{-(i+1)}m(S)$, for $i = 1, 2, ..., n$, so that their total measure is less than $m(S)/2$. Clearly we have

$$D_S(f, \pi_n) > n\,[m(S)]^2/4.$$

Letting n increase beyond all bounds, we see that the deviation of f on S is infinite.

The foregoing example promts an observation on the general behavior of set functions related to a measure by an equation of type (1): For such a function to be integrable it is necessary and sufficient that $\phi(\alpha)$ have a right-hand derivative at $\alpha = 0$. For the function to have finite deviation on sets of positive measure it is necessary (as the proof above shows) that ϕ be of bounded variation in every interval; and it is sufficient that ϕ be the difference of two convex functions having right-hand derivatives at $\alpha = 0$. (We are assuming in this discussion that ϕ is right-continuous at $\alpha = 0$, and that $\phi(0) = 0$.) Counterexamples exist showing that neither of these conditions is both necessary and sufficient.

A more systematic exploration of this zone of contact between the theories of set functions and functions of a real variable would lead us away from our main subject, and we therefore postpone further investigation of this interesting topic to another occasion.

6. Step-functions.

Our tools for probing into the nature of non-additive set functions have so far included the partitions — for the integral — and the partition trees — for the deviation. In Chapter 7, where the imputation operator is defined, we shall need also the finite subrings of $\mathcal{B}$. In the present chapter we use finite subrings to uncover the essential properties of a very special class of set functions, called "step-functions", whose deviation not only is finite, but is concentrated at a finite number of spots.[1] The main result is the inversion formula of Theorem 27 (page 44, below); this will be useful in Chapter 7, and is interesting in its own right as well.

DEFINITION 21. The <u>inclusion function</u> of a non-empty set $R \in \mathcal{B}$, denoted by e_R, is given by:

$$\begin{cases} e_R(S) = 1 & \text{if } S \supseteq R, \\ e_R(S) = 0 & \text{if } S \not\supseteq R. \end{cases}$$

We see immediately that if R is an atom of $\mathcal{B}$, then e_R is additive. If R is not an atom of $\mathcal{B}$, then e_R is superadditive, with integral $\int_S e_R = 0$ for every $S \in \mathcal{B}$ and deviation $\Delta_N(e_R) = 1$ for every $N \supseteq R$.

DEFINITION 22. A <u>step-function</u> is a finite linear combination of inclusion functions. The linear space of step-functions is denoted by E.

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1. However, on a finite ring all functions are step-functions.

THEOREM 25. The inclusion functions are linearly independent, and hence form a basis for E.

Proof. Suppose that there existed non-zero coefficients α_R such that

$$\sum_{R\varepsilon\mathcal{S}} \alpha_R e_R = 0,$$

$\mathcal{S}$ being a finite collection of sets. Let R' be a minimal element of $\mathcal{S}$. Then

$$\sum_{R\varepsilon\mathcal{S}} \alpha_R e_R(R') = \alpha_{R'} \neq 0.$$

This contradiction establishes the theorem.

It follows that every function $e \varepsilon E$ has a unique representation

(1) $$e = \sum_{R\varepsilon\sigma(e)} \alpha_R e_R, \qquad \text{all } \alpha_R \neq 0.$$

DEFINITION 23. The finite collection $\sigma(e)$ of sets which occur in the representation (1) of e is called the <u>spectrum</u> of e. The finite ring $\overline{\sigma(e)}$ generated by $\sigma(e)$ is called the <u>spectrum ring</u> of e.

THEOREM 26. The set $N \varepsilon \mathcal{B}$ is a carrier of $e \varepsilon E$ if and only if N contains the cover of $\sigma(e)$.

Proof. If N contains the cover of $\sigma(e)$, then N carries e as a consequence of Theorem 5 (c), page 14. Suppose that N does not contain the cover of $\sigma(e)$. Let R' [redacted] be minimal among the sets of $\sigma(e)$ which are not contained in N. We then verify that

$$e(N \cap R') = e(R') - \alpha_{R'}.$$

Therefore N does not carry e.

THEOREM 27. Let $\mathcal{a}$ be any finite subring of $\mathcal{B}$ which contains $\sigma(e)$, and let $j(T)$ denote the number of atoms of $\mathcal{a}$ contained in T. Then the coefficients α_R in the representation (1) are given by[1]

$$\alpha_R = \sum_{S \varepsilon \mathcal{a}_R} (-1)^{j(R-S)} e(S).$$

Proof. Given $e \varepsilon E$ and $\mathcal{a} \supseteq \sigma(e)$, the numbers α_R such that

$$e = \sum_{R \varepsilon \mathcal{a}} \alpha_R e_R$$

are uniquely defined for all R in $\mathcal{a}$ (some of them may of course be 0). Define the function e' by

$$e' = \sum_{R \varepsilon \mathcal{a}} \left[\sum_{S \varepsilon \mathcal{a}_R} (-1)^{j(R-S)} e(S) \right] e_R.$$

To prove the theorem we must show that $e' = e$. Let T be any set in $\mathcal{B}$. Then

$$e'(T) = \sum_{R \varepsilon \mathcal{a}_T} \sum_{S \varepsilon \mathcal{a}_R} (-1)^{j(R-S)} e(S).$$

Reversing order of summation gives

$$e'(T) = \sum_{S \varepsilon \mathcal{a}_T} \left[\sum_{\substack{R \varepsilon \mathcal{a} \\ S \subseteq R \subseteq T}} (-1)^{j(R)} \right] (-1)^{j(S)} e(S),$$

---

1. We recall that $\mathcal{a}_R$ denotes the subcollection of $\mathcal{a}$ consisting of subsets of R.

where we have made use of the additivity of j in splitting up the exponent. Let T' be the cover of $\mathcal{A}_T$; it is the largest set of $\mathcal{A}$ contained in T. If $S \in \mathcal{A}_T$ is a <u>proper</u> subset of T', then it is easy to show that the expression in brackets vanishes — it becomes an alternating sum of binomial coefficients. Therefore the only surviving term of the entire summation is the one with $S = R = T'$, giving us

$$e'(T) = e(T').$$

But, clearly, $e_R(T') = e_R(T)$ holds for every $R \in \mathcal{A}$, and $\mathcal{A}$ includes the spectrum of e, by hypothesis. Hence $e(T') = e(T)$. This completes the proof.[1]

The nature of the formula of Theorem 27 suggests that if α_R were suitably defined for sets R outside $\mathcal{A}$, it would be a sort of inversion of the function e. This is essentially what the next definition accomplishes, except that a factor of $(-1)^{j(R)}$ has been divided out for convenience.

DEFINITION 24. Let $\mathcal{A}$ be a finite subring of $\mathcal{B}$; let j be defined as in Theorem 27; let f be any set function. The <u>inversion</u> <u>in</u> $\mathcal{A}$ of f, denoted by $f^{\mathcal{A}}$, is the step-function defined by

$$f^{\mathcal{A}} = \sum_{S \in \mathcal{A}} (-1)^{j(S)} f(S)\, e_S.$$

THEOREM 28. If $e \in E$ and $\mathcal{A} \supseteq \sigma(e)$, then $e^{\mathcal{A}\mathcal{A}} = e$.

Proof. By Definition 24 and Theorem 27.

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1. The footnote on page 42 is a corollary to Theorem 27.

THEOREM 29. For any $f \in F$ and $T \in \mathcal{A}$,

$$f^{\mathcal{A}\mathcal{A}}(T) = f(T).$$

Proof. The argument is essentially a reproduction of the proof of Theorem 27, specialized to $T \in \mathcal{A}$.

It is a trivial observation that the directed limit

$$\lim_{\mathcal{A}} f^{\mathcal{A}\mathcal{A}}(S),$$

taken over the finite subrings $\mathcal{A}$ of $\mathcal{B}$, partially ordered by inclusion, exists and is equal to $f(S)$, for every $S \in \mathcal{B}$. It is only in this sense that the step-function $f^{\mathcal{A}\mathcal{A}}$ can be regarded as an approximation to f. The error $f - f^{\mathcal{A}\mathcal{A}}$ is quite uncontrolled on the sets of $\mathcal{B} - \mathcal{A}$, unless perhaps f satisfies some sort of continuity condition.

THEOREM 30. If f is additive, then $f^{\mathcal{A}}$ vanishes on $\mathcal{A} - \mathcal{A}^*$.

THEOREM 31. If e is a step-function, and if $\mathcal{A}$ is the spectrum ring of e, then $\mathcal{A}$ is also the spectrum ring of $e^{\mathcal{A}}$.

We omit the proofs.

<u>Example from number theory</u>.

EXAMPLE 17. Let $\mathcal{B}$ be the ring of square-free integers (Example 4, page 4). Let $j(T)$ denote the number of prime divisors of T. Let f be any set function on $\mathcal{B}$ with carrier N; then f is a step-function with $\sigma(f) \subseteq \mathcal{B}_N$.

We therefore have

$$f(S) = \sum_{R \varepsilon \mathcal{B}_N} \alpha_R e_R(S) = \sum_{R|S} \alpha_R$$

for all $S \varepsilon \mathcal{B}_N$ — i.e., for all $S|N$. Theorem 27 now tells us that

$$\alpha_S = \sum_{R|S} (-1)^{j(S \div R)} f(R)$$

for all $S|N$. This is the well-known Möbius inversion formula.[1] The carrier N can be increased indefinitely, or dispensed with entirely. The formula remains valid for functions defined over all integers if the convention

$$(-1)^{j(V)} = 0$$

is adopted for integers V with square divisors.

1. See T. Nagell, Introduction to Number Theory, New York & Stockholm, 1951, page 28.

7. Imputations.

Introduction.

This chapter introduces certain operators on set functions, which we call imputations. They take the form of linear projections onto the subspace of additive set function; in this property they resemble the integration operator of Chapter 4. However, an imputation leaves fixed a function on its carriers, adjusting the values on the smaller sets to achieve additivity. In contrast, the additive part (integral) of a set function depends only on how it behaves on the atoms and infinitesimal sets of the ring; it is in fact the additive function which best approximates the original function on those sets.

A few remarks are in order concerning the type of problem which these operators are designed to solve, and the origin of the term "imputation". In economics one is frequently led to consider "bundles" of heterogeneous, but related, goods (or services, production factors, etc.), and to assign to each bundle a price, cost, or other measure of value. A complete scheme of such values is evidently a set function. The non-additivity of these set functions has long been recognized by economists, though under a variety of names (e.g., "complementarity", "law of diminishing returns", "cheaper by the dozen") depending on the context, and the direction of the deviation from additivity.

The "problem of imputation" is the problem of how to apportion, or impute, the total value of a bundle among its component items.[1] A solution to the problem may be expressed as an additive set function, equal to the original on the full bundle, but in general different on the sub-bundles. One of our imputation operators can be viewed as a rule of apportionment which solves the imputation problem for a given Boolean ring of bundles of goods, whatever the initial assignment of values. However, before a particular operator can be singled out as the "natural" solution, the relative importances of the various items must be known, and expressed as a measure (or sequence of measures) on the ring.

In the case where the "bundles" are coalitions of individual firms or people engaged in competitive activity, the imputation problem becomes the problem of determining the value of an n-person game. We shall return to this subject in the final section of this chapter.

Preliminary definition of imputation.

We first define imputation for the inclusion functions; then extend the definition by linearity to the step-functions. It will be necessary for the time being to stick to functions which vanish on all sets of measure zero.

DEFINITION 25. A set function $f \varepsilon F$ is said to be absolutely continuous with respect to the measure m on $\mathcal{B}$ if $m(S) = 0$ implies $f(S) = 0$. The subspace of absolutely continuous functions is denoted by $F(m)$; the subspace of absolutely continuous step-functions by $E(m)$.

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1. An authoritative account of the imputation problem, though hardly in mathematical terms, may be found in the article Zurechnung by H. Mayer, pp. 1206-1228, Handwörterbuch der Staatswissenschaften, 4th ed., Vol. 8, Jena (1928).

THEOREM 32. A step-function is absolutely continuous if and only if the measure of every set in its spectrum is positive.

We omit the proof. Note that a function in $E(m)$ may have sets of measure zero in its spectrum ring.

DEFINITION 26. Given a measure m on $\mathcal{B}$ and a set R with $m(R) > 0$, then the imputation of the inclusion function e_R, written $e_R^{\,m}$, is the set function defined by

$$e_R^{\,m}(S) = \frac{m(R \cap S)}{m(R)}.$$

DEFINITION 27. Given a measure m on $\mathcal{B}$ and a step-function $e \in E(m)$:

$$e = \sum_{R \in \sigma(e)} \alpha_R e_R ,$$

then the imputation of e is the function

$$e^m = \sum_{R \in \sigma(e)} \alpha_R e_R^{\,m} .$$

We state some of the elementary properties of imputations on $E(m)$, omitting the proofs, which are routine in character. When we extend the definition of imputation to wider classes of functions we shall do so in such a way that these properties remain valid.

THEOREM 33. (a) If f is additive, then $f^m = f$. In any case, f^m is additive.

(b) If α and β are real numbers, then

$$(\alpha f + \beta g)^m = \alpha f^m + \beta g^m.$$

(c) If N carries f, then N also carries f^m, and $f^m(N) = f(N)$.

(d) If α is a positive number, then $f^{\alpha m} = f^m$.[1]

(e) If $m(S) = 0$, then $f^m(S) = 0$.

The fundamental formula for imputations.

We now proceed to derive an explicit expression for the function f^m, when f is in $E(m)$.

THEOREM 34. Let m be a measure on $\mathcal{B}$; let f be a function in $E(m)$; let $\mathcal{A}$ be a finite subring of $\mathcal{B}$ containing $\sigma(f)$; and let $j(S)$ denote the number of atoms of $\mathcal{A}$ contained in S. Define certain constants γ_S, for sets $S \in \mathcal{A}$ of positive measure, as follows:

$$\gamma_S = \sum_{\substack{Q \in \mathcal{A} \\ Q \cap S = 0}} \frac{(-1)^{j(Q)}}{m(S \cup Q)} .$$

Then we have

$$f^m(P) = m(P) \sum_{\substack{S \in \mathcal{A} \\ S \supseteq P}} \gamma_S \left[f(S) - f(S-P) \right],$$

for all atoms P of $\mathcal{A}$ of positive measure.

Proof. Write f in the form

$$f = \sum_{R \in \mathcal{A}} \alpha_R e_R .$$

1. Note that $F(\alpha m) = F(m)$.

Since $f \in E(m)$, we have $m(R) = 0$ only if $\alpha_R = 0$. By the definitions,

$$f^m(P) = \sum_{\substack{R \in \mathcal{a} \\ m(R)>0}} \gamma_R \frac{m(R \cap P)}{m(R)} = \sum_{\substack{R \in \mathcal{a} \\ R \supseteq P}} \alpha_R \frac{m(P)}{m(R)} .$$

Upon inserting the expression for α_R from Theorem 27 (page 44) and manipulating the result, we obtain the formula of the theorem. The details of the manipulation are as follows:

$$\begin{aligned}
f^m(P) &= \sum_{\substack{R \in \mathcal{a} \\ R \supseteq P}} \left\{ \sum_{S \in \mathcal{a}_R} (-1)^{j(R-S)} f(S) \right\} \frac{m(P)}{m(R)} \\
&= \sum_{\substack{R \in \mathcal{a} \\ R \supseteq P}} \left\{ \sum_{\substack{S \in \mathcal{a}_R \\ S \supseteq P}} (-1)^{j(R-S)} f(S) + \sum_{\substack{T \in \mathcal{a}_R \\ T \not\supseteq P}} (-1)^{j(R-T)} f(T) \right\} \frac{m(P)}{m(R)} \\
&= \sum_{\substack{R \in \mathcal{a} \\ R \supseteq P}} \left\{ \sum_{\substack{S \in \mathcal{a}_R \\ S \supseteq P}} (-1)^{j(R-S)} \left[f(S) - f(S-P) \right] \right\} \frac{m(P)}{m(R)} \\
&= m(P) \sum_{\substack{S \in \mathcal{a} \\ S \supseteq P}} \left\{ \sum_{\substack{R \in \mathcal{a} \\ R \supseteq S}} \frac{(-1)^{j(R-S)}}{m(R)} \right\} \left[f(S) - f(S-P) \right] \\
&= m(P) \sum_{\substack{S \in \mathcal{a} \\ S \supseteq P}} \gamma_S \left[f(S) - f(S-P) \right]. \qquad \text{Q. E. D.}
\end{aligned}$$

In order to establish an important property of the constants γ_S, we shall need the following lemma.[1]

LEMMA. If the real numbers $x_1, x_2, \ldots, x_n$, y, and λ are all positive, then the sum

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1. We only need the case $\lambda = 1$. However, the more general result is easier to prove.

$$\phi_n(y, \lambda) = \sum_{r=0}^{n} (-1)^r \sum_{\substack{1 \le i_1 < \ldots \\ \ldots < i_r \le n}} (y + x_{i_1} + \ldots + x_{i_r})^{-\lambda}$$

is positive.[1] If any $x_i = 0$, with the others non-negative and y positive, then $\phi_n(y, \lambda) = 0$.

Proof. The argument is an induction on n. The result is immediate for $n = 0$. Suppose that $\phi_{n-1}(y, \lambda)$ is positive for all positive values of y and λ. Then

$$\frac{\partial}{\partial y}\phi_{n-1}(y, \lambda) = -\lambda \phi_{n-1}(y, \lambda + 1) < 0.$$

Thus, ϕ_{n-1} is a strictly decreasing function of y. On the other hand, we have the recursive relationship

$$\phi_n(y, \lambda) = \phi_{n-1}(y, \lambda) - \phi_{n-1}(y + x_n, \lambda),$$

obtained by separating out those terms of the sum which involve x_n. Hence $\phi_n(y, \lambda)$ is positive, as was to be shown. The same relationship shows that $\phi_n = 0$ if $x_n = 0$. By the symmetry of ϕ_n, the same must be true for any x_i, $i = 1, 2, \ldots, n$. This completes the proof.

THEOREM 35. The constants γ_S of Theorem 34 are non-neagtive, and

$$\sum_{\substack{S \in \mathcal{A} \\ S \supseteq R}} \gamma_S = \frac{1}{m(R)}$$

for any set $R \in \mathcal{A}$ with $m(R) > 0$. Moreover, $\gamma_S = 0$ if and only if there is an atom of $\mathcal{A}$, not contained in S, having zero measure.

1. It is understood that the inner sum reduces to $y^{-\lambda}$ when $r = 0$.

Proof. To prove the first and last statements, we apply the lemma, setting $x_1, x_2, \ldots, x_n$ equal to the measures of the atoms of $\mathcal{a}$ not contained in S, and $y = m(S)$, $\lambda = 1$. To compute the sum of the γ_S, we first observe that the definition of γ_S is indifferent to the number of atoms inside S, or to their individual measures. Therefore we may as well assume that R is an atom of $\mathcal{a}$, since R is contained in all the sets S considered in the sum. Now consider the quantity $e_R{}^m(R)$. According to the original definition (page 50) it equals 1. According to Theorem 34 (page 51) it is equal to

$$m(R) \sum_{\substack{S \varepsilon \mathcal{a} \\ S \supseteq R}} \gamma_S [e_R(S) - e_R(S - R)],$$

which reduces to

$$m(R) \sum_{\substack{S \varepsilon \\ S \; R}} \gamma_S .$$

Equating these two amounts yields the desired result.

The last two theorems reveal that $f^m(P)$ is a weighted average of differences $f(T) - f(T - P)$, if P is an atom of a sufficiently fine, finite subring of $\mathcal{B}$. This observation leads quickly to two further properties of imputations.

THEOREM 36. If f is superadditive, then $f^m(S) \geq f^o(S)$ for all $S \varepsilon \mathcal{B}$; if f is subadditive, then $f^m(S) \leq f^o(S)$ for all $S \varepsilon \mathcal{B}$.

THEOREM 37. If f is monotone, then f^m is non-negative, and hence monotone.

The proofs are immediate. These properties, like those of Theorem 33, will remain valid for imputations defined on more extensive classes of functions (page 57ff. below).

The formula of Theorem 34 can be used to find $f^m(S)$ for an arbitrary set S in $\mathcal{B}$: one merely takes $\mathcal{a}$ to include S as well as the members of $\sigma(f)$, and adds up the values of f^m on the atoms of $\mathcal{a}$ contained in S. However, the next theorem enables us to avoid making $\mathcal{a}$ larger than the spectrum ring of f. Instead, to compute $f^m(S)$, we add up the values of f^m on all the atoms of $\mathcal{a}$, weighting each term according to the proportion of the atom which is contained in S.

THEOREM 38. If $f \in E(m)$, if P is an atom of a finite ring containing $\sigma(f)$, and if Q is any subset of P, then

$$f^m(Q) = \begin{cases} \frac{m(Q)}{m(P)} f^m(P) & \text{if } m(P) > 0, \\ 0 & \text{if } m(P) = 0. \end{cases}$$

Proof. If $m(P) = 0$, then also $m(Q) = 0$. Hence $f^m(Q) = 0$ by Theorem 33 (e). On the other hand, if $m(P) > 0$, then we have

$$\frac{m(R \cap Q)}{m(R)} = \frac{m(R \cap P)}{m(R)} \frac{m(Q)}{m(P)}$$

for every set R in $\sigma(f)$, because either (a) $R \cap P = P$ and $R \cap Q = Q$ or (b) $R \cap P = R \cap Q = 0$. The result now follows from Definitions 26 and 27 (page 50).

EXAMPLE 18. Consider the case of a finite ring $\mathcal{B}$ having n atoms. Let $j(S)$ denote the number of atoms in S.

Then j is a measure, and $E(j) = F$: all functions are absolutely continuous step-functions. If P is an atom, then we have the formula

$$f^j(P) = \sum_{\substack{S \in \mathcal{B} \\ S \supseteq P}} \frac{(j(S)-1)!\,(n-j(S))!}{n!} \left[f(S) - f(S-P)\right].$$

In this sum the total weight assigned to sets S of measure $j(S) = k$, for each $k = 1, 2, \ldots, n$, is exactly $1/n$. This formula has application to the theory of n-person games (see below).

Proof of formula. We have

$$\gamma_S = \sum_{Q \cap S = 0} \frac{(-1)^{j(Q)}}{j(S \cup Q)} = \sum_{q=0}^{m} \frac{(-1)^q}{s+q} \binom{m}{q} = \phi(m, s),$$

where we have put $q = j(Q)$, $s = j(S)$, and $m = n - s$. We must show that

(1) $$\phi(m, s) = \frac{(s-1)!\, m!}{(s+m)!}.$$

We proceed by induction on s. For $s = 1$ we have

$$(m+1)\phi(m, 1) = \sum_{q=0}^{m} (-1)^q \binom{m+1}{q+1} = 1 + \sum_{q=-1}^{m} (-1)^q \binom{m+1}{q+1} = 1,$$

as required. For the inductive step, we need the identity

$$\phi(m+1, s) = \phi(m, s) - \phi(m, s+1),$$

derived in the same way as its counterpart in the proof of the lemma on pages 52-53. Assuming (1) to have been established for a particular s and for all $m \geq 0$, we now obtain the result for $s + 1$:

$$\phi(m, s+1) = \phi(m, s) - \phi(m+1, s)$$

$$= \frac{(s-1)!\, m!}{(s+m)!} - \frac{(s-1)!\, (m+1)!}{(s+m+1)!}$$

$$= \frac{s!\, m!}{(s+m+1)!} .$$

Having thus evaluated γ_S, we obtain the desired formula from the formula of Theorem 34 (page 51).

Extended definition of imputation.

We now enlarge the domain of definition of the imputation operators. Our method will be to approximate the function $f \varepsilon F(m)$ by functions of the form f^{aa}. (We recall[1] that f^{aa} is the step-function with spectrum contained in $\mathcal{a}$ that agrees with f on all the sets of $\mathcal{a}$.) The limit of the imputations of f^{aa}, as $\mathcal{a}$ is enlarged and refined, is defined to be the imputation of f.

THEOREM 39. If f is absolutely continuous, and if $\mathcal{a}$ is a finite subring of $\mathcal{B}$, then f^{aa} is absolutely continuous (with respect to the same measure).

Proof. Let R be any set in $\mathcal{a}$ with $m(R) = 0$; we shall show that R is not in the spectrum of f^{aa}. The coefficient of e_R in the representation of f^{aa} is given by

$$\alpha_R = \sum_{S \varepsilon \mathcal{a}_R} (-1)^{J(R-S)} f^{aa}(S)$$

(Theorem 27, page 44). But $m(S) = 0$ for all $S \varepsilon \mathcal{a}_R$, by the monotonicity of m. Hence $f(S) = 0$. Hence $f^{aa}(S) = 0$. Hence

1. See Theorem 29, page 46.

$\alpha_R = 0$. Hence $R \notin \sigma(f^{\mathcal{a}\mathcal{a}})$.

DEFINITION 28. Let f be absolutely continuous with respect to the measure m on $\mathcal{B}$. Then f is said to be <u>imputable</u> if the directed limit

$$\lambda_S = \lim_{\mathcal{a}} (f^{\mathcal{a}\mathcal{a}})^m(S),$$

taken over the finite subrings $\mathcal{a}$ of $\mathcal{B}$ partially ordered by inclusion, exists for every $S \varepsilon \mathcal{B}$. The <u>imputation</u> f^m of f is then given by $f^m(S) = \lambda_S$.

THEOREM 40. The definition just given, applied to step-functions, is consistent with Definition 27 (page 50). Theorems 33, 36 and 37 (pages 50, 51 and 54) are valid for all imputable functions in $F(m)$.

We omit the proof.

No independent characterization of the class of imputable functions is known at present. The absolutely continuous step-functions are imputable; and it is easy to show that the absolutely continuous additive functions are all imputable. The example given below of a non-imputable function is non-integrable as well.[1] It may be conjectured that all (absolutely continuous) integrable functions are imputable.

In general one might expect that the imputability of a set function would depend on the choice of measure. We are unable to provide an example of this phenomenon.

EXAMPLE 19. Let f be identically 1 on all non-empty sets of $\mathcal{B}$. Then, if $\mathcal{B}$ is the ring generated by

1. See Example 11, page 19.

the half-open intervals $(a, b]$ in $(0, 1]$, and if m is the usual Lebesgue measure, then f is not imputable.

Proof. Let $\mathcal{a}$ be a finite subring of $\mathcal{B}$ and let P be an atom of $\mathcal{a}$. Then we have

$$(f^{aa})^m(P) = m(P)\gamma_P = m(P)\left[\frac{1}{m(P)} + \frac{1}{m(P)+m(Q)} + \dots \right].$$

If $m(P)$ is sufficiently small in comparison with the measures of the other atoms of $\mathcal{a}$, then this quantity will be as close as we please to 1. But $(f^{aa})^m$ is a non-negative, additive function, whose value on the cover of $\mathcal{a}$ is 1. (Theorem 33, pages 50–51.) Therefore, if S is any set in $\mathcal{B}$ other than O or I, and if $\mathcal{a}'$ is any fixed, finite subring of $\mathcal{B}$, we can choose $\mathcal{a}$ to include $\mathcal{a}'$ and such that $(f^{aa})^m(S)$ is as close to 0 as we please, by merely constructing a very small atom <u>outside</u> S. We can also make $(f^{aa})^m(S)$ as close to 1 as we please by constructing a very small atom <u>inside</u> S. It follows that the directed limit

$$\lim_{\mathcal{a}} (f^{aa})^m(S)$$

diverges, for all $S \varepsilon \mathcal{B}$, $S \neq O, I$. The proof has made use of the fact that every non-empty set in $\mathcal{B}$ possesses subsets of arbitrarily small measure; also the fact that there are no non-empty sets of measure zero.

In working with the directed limit of Definition 28, it is nearly always essential to discover how the constants γ_S (as defined in Theorem 34, page 51) change when $\mathcal{a}$ is replaced by a more inclusive, finite ring $\mathcal{a}'$. The next theorem fills this need. It should be pointed out that any finite ring that includes

$\mathcal{A}$ can be reached from $\mathcal{A}$ by a finite chain of extensions of the two elementary types considered in the theorem: (a) splitting an existing atom in two; and (b) introducing a new atom outside the cover of the existing set.

THEOREM 41. (a) Let $\mathcal{A}'$ be the ring obtained from the finite ring $\mathcal{A}$ by adjoining two sets P_1 and P_2 which partition an atom P of $\mathcal{A}$. Let S be a set of $\mathcal{A}$ not containing P, with $m(S) > 0$. Then

$$\gamma_S = \gamma'_S + \gamma'_{S \cup P_1} + \gamma'_{S \cup P_2}$$

and

$$\gamma_{S \cup P} = \gamma'_{S \cup P},$$

where γ and γ' are defined as in Theorem 34 with respect to the rings $\mathcal{A}$ and $\mathcal{A}'$, respectively.

(b) Let $\mathcal{A}'$ be the ring obtained from the finite ring $\mathcal{A}$ by adjoining a set P, not contained in the cover of $\mathcal{A}$. Let S be a set of $\mathcal{A}$ with $m(S) > 0$. Then

$$\gamma_S = \gamma'_S + \gamma'_{S \cup P},$$

where γ and γ' are defined as above.

Proof. Construct the step-function $d \in E(m)$ which is 1 on the set S, 0 on all other sets of $\mathcal{A}$, and whose spectrum is contained in $\mathcal{A}$. The behavior of d on $\mathcal{A}'$ is described by:

$$\begin{cases} d(S \cup P_1) = d(S \cup P_2) = d(S) = 1 \\ d(T) = 0 \qquad \text{(all other } T \in \mathcal{A}'). \end{cases}$$

Let Q be an atom of $\mathcal{A}_S$ of positive measure. Using the ring $\mathcal{A}$ in the formula of Theorem 34 (page 51), we find that

$$d^m(Q) = m(Q)\gamma_S .$$

Using the ring $\mathcal{A}'$, we find that

$$d^m(Q) = m(Q)\left[\gamma'_S + \gamma'_{S\cup P_1} + \gamma'_{S\cup P_2}\right] .$$

Equating the two gives us the first equation of (a). The second equation follows at once from the definition, since γ_S does not depend on what the atoms inside S are like. The proof of (b) follows exactly the same method.

<u>Generalized imputations</u>.

In this section we outline an extension of imputations to functions which are not absolutely continuous. The idea is to set up a secondary measure on the ~~ring~~ ideal of sets whose original measure was 0, and, if necessary, a third measure for the sets for which the first two measures were 0, and so on. In general we end up with a transfinite hierarchy of measure functions.[1]

DEFINITION 29. The <u>zero ring</u> $\mathcal{Z}(m)$ of a measure m is the ideal of sets S with $m(S) = 0$.

Let $\bar{m}$ denote a hierarchy of functions m_δ, where δ runs over the ordinals less than some fixed ordinal o, such that each m_δ is a measure defined on precisely the ideal $\mathcal{B}_\delta$:

$$\mathcal{B}_1 = \mathcal{B},$$

$$\mathcal{B}_\delta = \bigcap_{\delta' < \delta} \mathcal{Z}(m_{\delta'}),$$

1. The word "measure" is used in this section in a slightly modified sense from Definition 13 (page 13): it denotes a function (not necessarily in F) defined on an ideal of $\mathcal{B}$, and additive, non-

and such that $\mathcal{B}_0$ is the null ideal $\{0\}$. It is convenient to set $m_\delta(S)$ equal to ∞ for sets S outside of $\mathcal{B}_\delta$. Then there will be a unique ordinal δ_R, for each non-empty set R in $\mathcal{B}$, such that $0 < m_{\delta_R}(R) < \infty$.

DEFINITION 30. Let $e_R^{\overline{\mathfrak{m}}}$ be given by

$$e_R^{\overline{\mathfrak{m}}}(S) = \frac{m_{\delta_R}(R\ \ S)}{m_{\delta_R}(R)} ;$$

from this, define $e^{\overline{\mathfrak{m}}}$ for all $e \in E$ by linearity (compare Definition 27 (page 50)); finally, for all $f \in F$ for which the limits exist, let the <u>generalized imputation</u> $f^{\overline{\mathfrak{m}}}$ be defined by

$$f^{\overline{\mathfrak{m}}}(S) = \lim_{\mathcal{a}} (f^{\mathcal{a}\mathcal{a}})^{\overline{\mathfrak{m}}}(S),$$

the directed limit being taken over the finite subrings $\mathcal{a}$ of $\mathcal{B}$, partially ordered by inclusion.

The ordinary imputations correspond to the case $o = 2$ —that is, to the case where the hierarchy $\overline{\mathfrak{m}}$ consists of a single measure function m.

THEOREM 42. The counterparts of Theorems 33, 36, and 37 (pages 50, 51 and 54) hold for generalized imputations.

We omit the proofs. The generalized form of Theorem 33 (e) is vacuous (or meaningless). Next we describe the generalization of the formula of Theorem 34.

If f is a step-function, then its spectrum $\sigma(f)$ can be analyzed as follows:

$$\mathcal{S}_\delta = \sigma(f) \cap [\mathcal{B}_\delta - \mathcal{Z}(m_\delta)]$$

into a finite number of collections of sets having the same δ_S. Then f itself can be broken up into a finite sum:

$$f = \sum_\delta f_\delta,$$

where $\sigma(f_\delta) = \mathcal{S}_\delta$.

THEOREM 43. Let $\mathcal{A}$ be a finite subring of $\mathcal{B}$ containing $\sigma(f)$, $f \in E$, and let P be an atom of $\mathcal{A}$. Then we have

$$f^{\overline{m}}(P) = m_{\delta_P}(P) \sum_{\substack{S \in \mathcal{A} \\ S \supseteq P}} \gamma_S \left[f_{\delta_P}(S) - f_{\delta_P}(S-P)\right].$$

The constants γ_S are defined as in Theorem 34, using the measure m_{δ_P} with the convention $1/\infty = 0$. In particular, $\gamma_S = 0$ for S outside $\mathcal{B}_{\delta_P}$.

We omit the proof.

A measure hierarchy can be designed to produce a ranking of the atoms of $\mathcal{B}$, as described in the next example.

EXAMPLE 20. Let P be an atom of $\mathcal{B}$. Let m_{δ_P} be identical with the inclusion function e_P on the ideal $\mathcal{B}_{\delta_P}$. Then, if $f^{\overline{m}}$ exists, we have

$$(1) \qquad f^{\overline{m}}(P) = \lim_S \left[f(S) - f(S-P)\right],$$

the directed limit being taken over increasing $S \in \mathcal{B}_{\delta_P}$, partially ordered by inclusion. In particular, if the ideal

$\mathcal{B}_{\delta_P}$ has a unit — call it I_{δ_P} — then we have

$$f^{\overline{m}}(P) = f(I_{\delta_P}) - f(I_{\delta_P} - P).$$

If every $m_\delta \varepsilon \overline{m}$ is of this "atomic" form, then $\overline{m}$ well-orders the atoms of $\mathcal{B}$. (Such an $\overline{m}$ is possible if and only if every set in $\mathcal{B}$ contains at least one element of $\mathcal{B}^*$.) The generalized imputation $f^{\overline{m}}$ is then completely determined by (1), applied to all $P \varepsilon \mathcal{B}^*$.

Proof of (1). Let $\mathcal{A}$ be a finite subring of $\mathcal{B}$, containing P. The δ_P-component of $f^{\mathcal{A}\mathcal{A}}$ turns out to be

$$f^{\mathcal{A}\mathcal{A}}{}_{\delta_P}(T) = f(T) - f(T-P)$$

for sets T in $\mathcal{A} \cap \mathcal{B}_{\delta_P}$. (This calculation makes use of the inversion formula of Theorem 27, page 44.) Let S denote the unit of $\mathcal{A} \cap \mathcal{B}_{\delta_P}$. Applying Theorem 43, and noting that γ_T vanishes for all $T \varepsilon \mathcal{A}$ except S and that $m_{\delta_P}(P) = m_{\delta_P}(S) = 1$, we obtain

$$(f^{\mathcal{A}\mathcal{A}})^{\overline{m}}(P) = f^{\mathcal{A}\mathcal{A}}{}_{\delta_P}(S) - f^{\mathcal{A}\mathcal{A}}{}_{\delta_P}(S-P)$$

$$= f(S) - f(S-P).$$

As $\mathcal{A}$ is enlarged, the unit S of $\mathcal{A} \cap \mathcal{B}_{\delta_P}$ increases indefinitely within $\mathcal{B}_{\delta_P}$. Hence the directed limit of Definition 30 for $f^{\overline{m}}(P)$ reduces to the form (1).

Application to the theory of games.

The theory of von Neumann and Morgenstern represents an n-person game by a superadditive set function $v \varepsilon E$ with a finite carrier N containing n atoms, corresponding to the players. The number $v(S)$ represents the amount of money that the "coalition" S can win by best play, assuming that the other players act in direct opposition to the interests of the coalition.[1]

An outcome of a particular play of the game (including possibly payments made among the players outside the formal apparatus of the game) can be expressed by an n-vector, specifying the winnings of each player.[2] We prefer here to use the corresponding additive set function, which specifies the winnings of each coalition. Such an outcome function of course does not have to refer to a particular play of the game; it can also be used to represent a predicted outcome, or a recommended outcome. By a value of the game, we shall mean a set function interpreted in this way.

Let us suppose that there is a value operator, mapping each game v onto a value $\overline{v}$. It seems reasonable to demand the following properties of the operator:

(a) $\overline{v}(N) = v(N)$, and N carries $\overline{v}$.

(b) If v is additive, then $\overline{v} = v$.

(c) For all S, $\overline{v}(S) \geq v(S)$.

(d) If an automorphism of $\mathcal{B}$ maps v onto itself, then it also maps $\overline{v}$ onto itself.

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1. J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior, Princeton (1944, 1947).

2. This vector is an "imputation" in the sense of von Neumann and Morgenstern.

(e) If w is more favorable to the coalition T than v, in the sense that

$$\begin{cases} w(S) > v(S) & \text{implies} \quad S \supseteq T \\ w(S) < v(S) & \text{implies} \quad S \cap T = 0, \end{cases}$$

then $\overline{w}(T) \geq \overline{v}(T)$.

(f) $\overline{w + v} = \overline{w} + \overline{v}$.

However, one soon discovers that property (c) is too strong; it is not possible in general to satisfy all coalitions at once. We therefore weaken (c) to (c'):

(c') For all atoms P, $\overline{v}(P) \geq v(P)$.

We have shown elsewhere[1] that (a), (d) and (f) determine a unique value operator, namely the imputation

$$\overline{v} = v^j$$

based on the measure j, $j(S)$ being the number of atoms in S. Properties (b), (c') and (e) are then easily verified from Theorems 33 (a) and 36 (pages 50 and 54) and the formula in Example 18 (page 55–56), respectively.

It is easy to imagine games or game-like situations in which the symmetry assumption (d) is not appropriate, because of differences in the external characteristics of the players. (Internal differences are accounted for in the function v!) For example,

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1. L. S. Shapley, A value for n-person games, Annals of Mathematics Study No. 28: "Contributions to the Theory of Games, II", Princeton (1953). The paper contains a number of examples.

individuals might be competing with corporations, or governments, or there might be differences in "bargaining ability", or some other skill factor. These cases might be handled by means of imputation operators based on measures other than the symmetric measure j. The effect would be to calibrate the players according to their performance in the "pure bargaining" game

$$v = e_N .$$

However, the derivation of the value, v^m, would involve the further assumption that whenever two players meet in any pure bargaining game e_S, $S \subseteq N$, the ratio of the shares they obtain is the same. If one or more of the players received nothing in the game e_N, we would find it natural to replace the single measure by a finite hierarchy, and use generalized imputations. In the extreme case, the players would be arranged in a definite order $P_1, P_2, \ldots, P_n$, and the value would be given by

$$\bar{v}(P_i) = v(\{P_1, \ldots, P_i\}) - v(\{P_1, \ldots, P_{i-1}\}).$$

This solution is not necessarily advantageous to P_n; for example, in the simple majority voting game with $n \geq 3$ he would receive nothing.