Data fitting on manifolds by minimizing the mean square acceleration of a Bézier curve

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1 The problem

This problem of curve fitting is addressed on a Riemannian manifold $\mathcal{M}$. The data points $d_0, \dots, d_n \in \mathcal{M}$ are associated with real-valued time-parameters $t_0 < t_1 < \cdots < d_n$. Fitting a curve γ in this setting involves two main constrains: proximity to the data points, and the smoothness of γ.

A popular way to tackle this problem is to encapsulate those constraints into an optimization problem like

$$\min_{\gamma\in\Gamma} \int_{t_0}^{t_n} \left\| \frac{\mathrm{D}^2\gamma(t)}{\mathrm{d}t^2} \right\|_{\gamma(t)}^2 \mathrm{d}t + \frac{\lambda}{2}\sum_{i=0}^{n} \mathrm{d}^2\big(\gamma(t_i), d_i\big), \quad (1)$$

where Γ is an admissible space of curves $\gamma\colon [t_0, t_n] \to \mathcal{M}$, $\frac{\mathrm{D}^2}{\mathrm{d}t^2}$ is the Levi-Civita second covariant derivative, $\|\cdot\|_{\gamma(t)}$ is the Riemannian metric at $\gamma(t)$ and $\mathrm{d}(\cdot,\cdot)$ is the Riemannian distance. The proximity and regularity constraints are balanced via the parameter $\lambda \in \mathbb{R}$.

Several methods exist to perform numerical optimization on manifolds. We refer to the textbook [1] for details. The problem was tackled in different ways this last decade [4, 8, 6]. We consider here the case where Γ is a set of composite Bézier curves (see [5, 7] for details). Within this setting, the search space is drastically reduced to the so-called control points of the Bézier curve, and it becomes quite easy to impose C^1 time-differentiability to γ.

Solving (1) on $\mathcal{M}$ reduces to a highly nonlinear problem on the product manifold $\mathcal{M}^M$, where M is the number of control points of γ. We present here the results of [3]. We derive a closed-form of the gradient of Bézier curves with respect to their control points.

2 Gradient of the discretized regularizer

The regularizer $E = \int_{t_0}^{t_n} \left\| \frac{\mathrm{D}^2\gamma(t)}{\mathrm{d}t^2} \right\|_{\gamma(t)}^2 \mathrm{d}t$ is approximated by second order finite differences. A definition and the derivative of those finite differences are introduced in Bačák *et al.* [2]. The curve is discretized with $N+1$ equispaced points. We note $\tilde{E}$ this approximation. To evaluate the gradient of $\tilde{E}$, it remains to derive the manifold-valued gradient of a Bézier curve γ, w.r.t. its control points. As Bézier curves are obtained as a composition of geodesics (with the De Casteljau algorithm [5]), their derivative is given as a concatenation of adjoint Jacobi fields.

We finally solve problem (1) with a standard gradient descent. We show that our solution is computationally competitive with existing techniques and that it outperforms the current fitting methods using Bézier curves (as the one proposed in [7]) when the data points are not constrained to a given neighbourhood.

References

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