Interpolation on Riemannian manifolds with a $\mathcal{C}^1$ piecewise-Bézier curve

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More and more, manifolds (i.e. smooth spaces) are used to solve problems that would have been very time and ressources consuming if they were defined on the classical Euclidean space (because of non-linear constraints restricting the solutions to a certain subspace). In that purpose, interpolation [MSL04, PN07] and optimization [AMS08, BMAS14] tools are developed in several fields like big data mining [Bou14], image modeling and processing [Pey09] or invariant pattern recognition methods. We focus on interpolation.

In this work [GSA14], we propose a new framework to fit a smooth path to a set of $n+1$ data points on a Riemannian manifold. This path is composed of Bézier functions. Specifically, we seek a C^1 piecewise-Bézier interpolation curve for the data $(p_0, \ldots, p_n)$, such that the segment joining p_0 and p_1, as well as the segment joining p_{n-1} and p_n, are Bézier curves of order two (i.e., with one intermediate control point), while all the other segments are Bézier curves of order three (i.e., with two intermediate control points). In view of the endpoint velocity formula [PN07, (6)], fixing the curve velocity vector at each intermediate interpolation point fully determines all the control points, and thus the curve. The task therefore reduces to choosing those velocities. The velocity *directions* are chosen in an arbitrary way ; The optimal velocity *magnitudes* are solutions, in the Euclidean case, of a certain tridiagonal system of equations obtained by minimizing the mean squared acceleration of the curve. We generalize this linear system to Riemannian manifolds and use it to set the velocity magnitudes at the intermediate interpolation points. Even though the velocity directions are still suboptimal in general and the velocity magnitudes are suboptimal on nonlinear manifolds, the numerical experiments indicate that the method produces good-looking curves on several manifolds and for a wide range of data point positions.

The advantage of this method is that it is general on various Riemannian manifolds if only three objects of differential geometry are known on the manifold : the exponential and logarithmic maps and the inner scalar product. This method is generally neither time nor space consuming, and brings good quality visual results on the Euclidean space, the hypersphere, the "shape manifold" and the special orthogonal group $SO(3)$.

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