Piecewise-Bézier C^1 interpolation on Riemannian manifolds with application to 2D shape morphing

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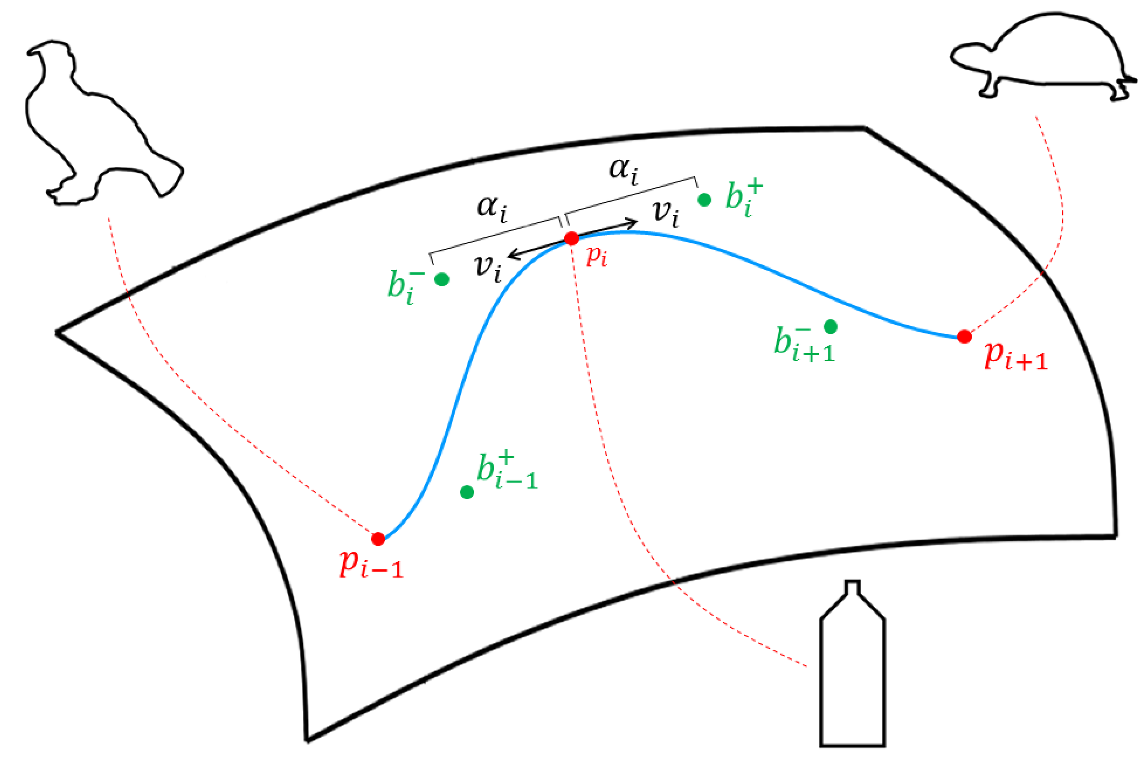
Context

We propose a new framework to

- fit $n+1$ data points $(p_i)_{0 \le i \le n}$ on a manifold $\mathcal{M}$;
- given $n-1$ velocity directions at internal data points $(v_i)_{1 \le i \le n-1}$ on the tangent space in p_i (noted $T_{p_i}\mathcal{M}$);
- with n Bézier functions $(\beta_i^k)_{0 \le i \le n-1} : [0,1] \to \mathcal{M}$
  - degree $k=2$: segments from p_0 to p_1 and from p_{n-1} to p_n
  - degree $k=3$: other segments.

The Bézier path is driven by its intermediate control points (b_i^-, b_i^+).
We ensure **low space and time complexity**.



Method on the Euclidean space

Optimize the norms $\alpha_i \ge 0$ of the velocity directions, which are independent of the manifold $\mathcal{M}$.

Constraint: the path is smooth at data points $\Rightarrow b_i^{\pm} = p_i \pm \alpha_i v_i$.

β_k: Bézier segment driven by p_{n-1}, b_{n-1}^+ and p_n

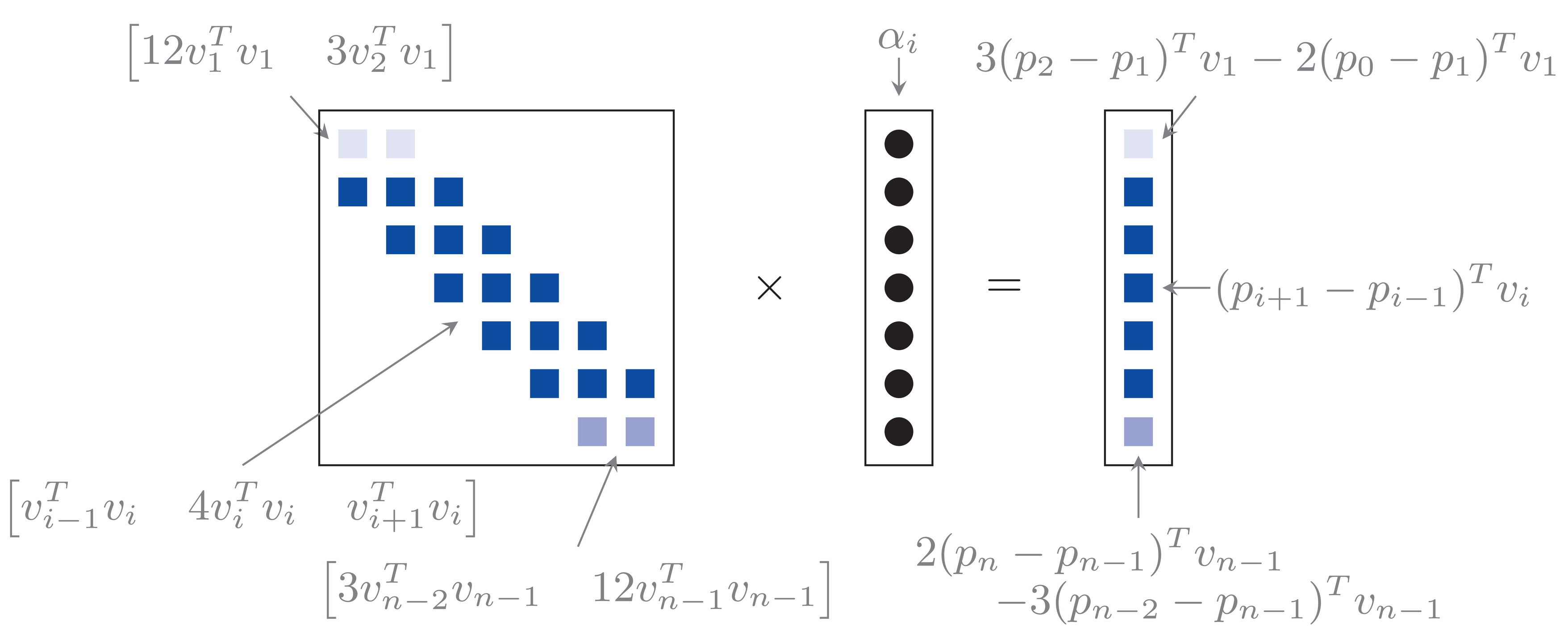
$$\min_{\alpha_1,\dots,\alpha_{n-1}} \int_0^1 \|\ddot{\beta}_2(t; p_0, b_1^-, p_1)\|^2 \mathrm{d}t + \sum_{i=1}^{n-2} \int_0^1 \|\ddot{\beta}_3(t; p_{i-1}, b_{i-1}^+, b_i^-, p_i)\|^2 \mathrm{d}t + \int_0^1 \|\ddot{\beta}_2(t; p_{n-1}, b_{n-1}^+, p_n)\|^2 \mathrm{d}t,$$

$b_1^- = p_1 - \alpha_1 v_1$ $\qquad$ $b_{i-1}^+ = p_{i-1} + \alpha_{i-1} v_{i-1}$

Quadratic polynomial $P(\alpha_i)$

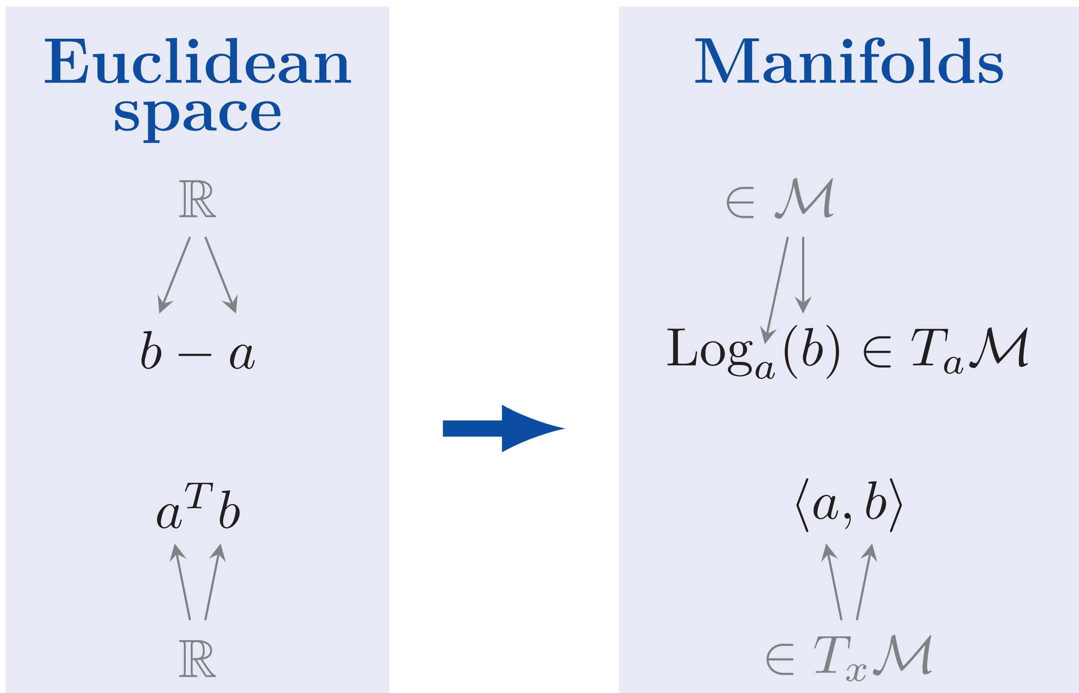
Solution on the Euclidean space

On $\mathbb{R}^m$, $\nabla P(\alpha_i) = 0 \Rightarrow$ tridiagonal linear system with unknowns α_i.

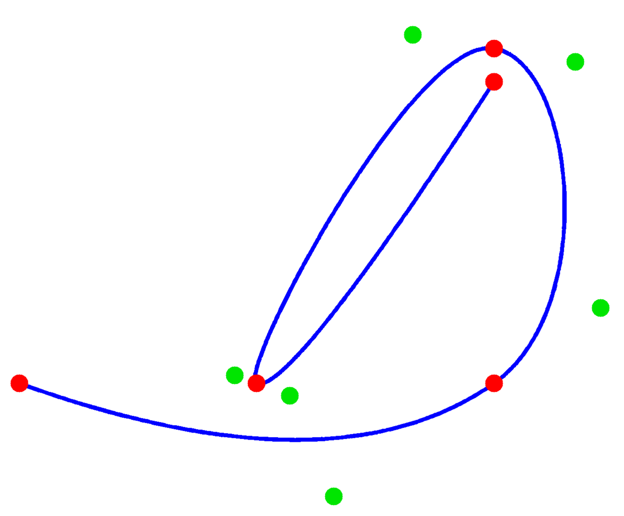


Generalization to manifold

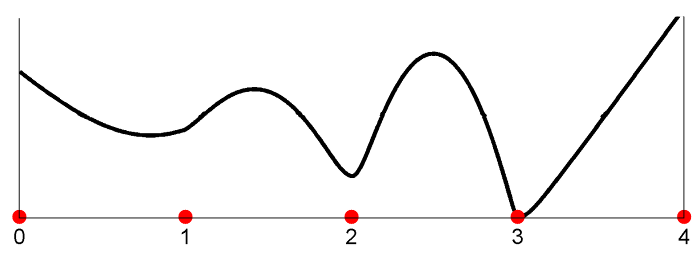
The tridiagonal system is generalized to manifolds using **tools of differential geometry**.



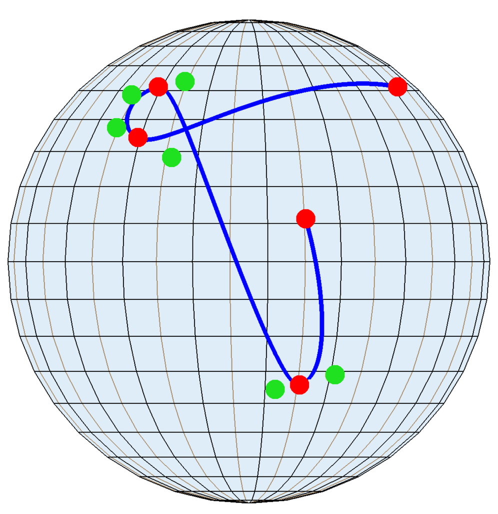
Results



Norm of the velocity of the Bézier curve



The optimization on a linear finite dimensional manifold produces a smooth path, as expected.



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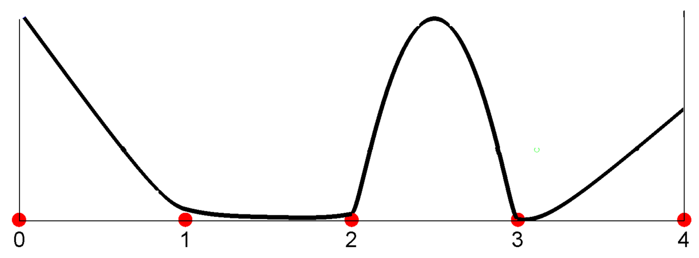
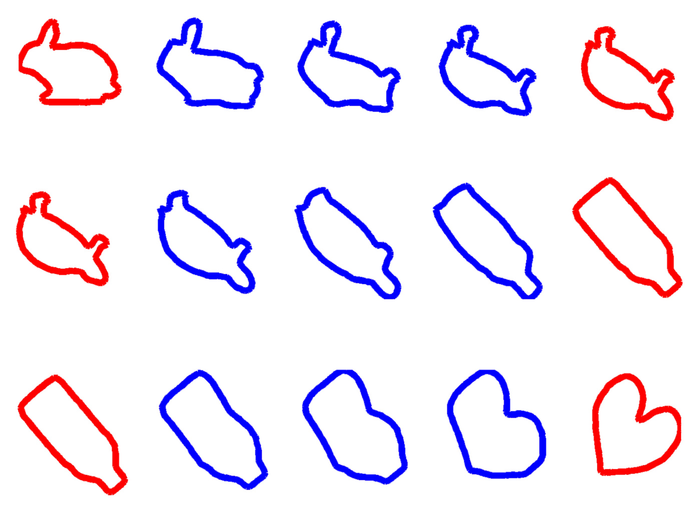
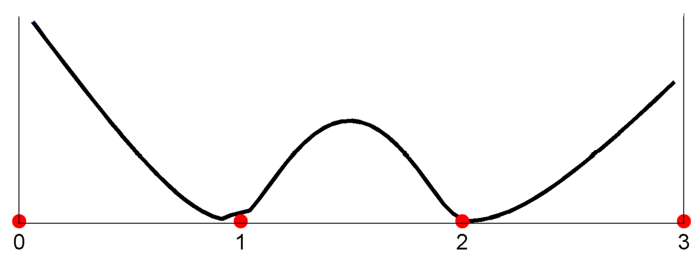


The generalization to a non-linear finite dimensional manifold leads to a smooth path.



Norm of the velocity of the Bézier curve



The generalization to a more general manifold leads to visually satisfying results.

Summary

Search the optimal intermediate control points $(b_i^+, b_i^-) \in \mathcal{M}$ driving an optimal $\mathcal{C}^1$ path.

↓

Constraint

$$b_i^+ = p_i + \alpha_i v_i$$
$$b_i^- = p_i - \alpha_i v_i$$

On the Euclidean space, minimize the mean square acceleration $P(\alpha_i)$ of the path.

↓

$\nabla P(\alpha_i) = 0$
Solve a tridiagonal system in α_i

↓

Manifold adaptation
the entries become:

$a^T b \to \langle a, b \rangle$
$b - a \to \mathrm{Log}_a(b)$
$a + vt \to \mathrm{Exp}_a(vt)$

↓

Compute back the intermediate control points:
$b_i^{\pm} = \mathrm{Exp}_{p_i}(\pm \alpha_i v_i)$

↓

Reconstruct the path with the *De Casteljau* algorithm generalized to $\mathcal{M}$ with geodesics.