Data fitting on manifolds by minimizing the mean squared acceleration of a Bézier curve

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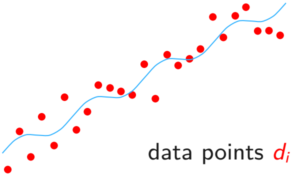
Benelux Meeting – March 19, 2019

What is the problem?

Given (t_i, d_i), find a $\mathcal{C}^1$ curve $\mathbf{B}(t)$, s.t.



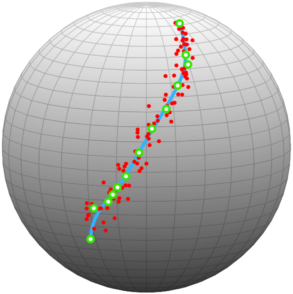
Bézier spline!

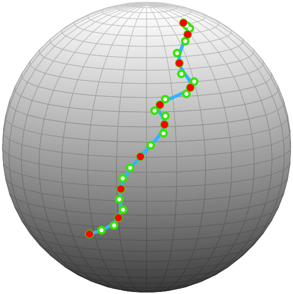
$$\operatorname*{argmin}_{\mathbf{B}\in\Gamma} E_\lambda(\mathbf{B}) := \int_{t_0}^{t_n} \left\| \frac{\mathrm{D}^2\mathbf{B}(t)}{\mathrm{d}t^2} \right\|^2_{\mathbf{B}(t)} \mathrm{d}t + \lambda \sum_{i=0}^{n} \mathrm{d}^2(\mathbf{B}(t_i), d_i),$$

regularizer

data attachment

Why is this important? – Sphere



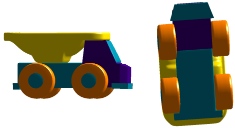


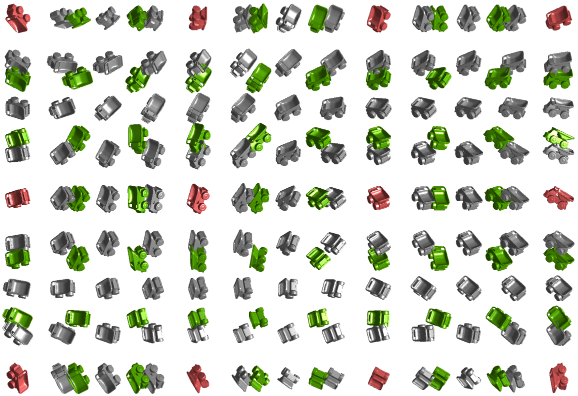
storm trajectories
birds migrations
distress planes roadmaps extrapolation

Data points $d_i \in \mathbb{S}^2$

curve **B** : $[0, n] \to \mathbb{S}^2$

Why is this important? – Orthogonal group





Rigid rotations of 3D objects
3D printing plannings

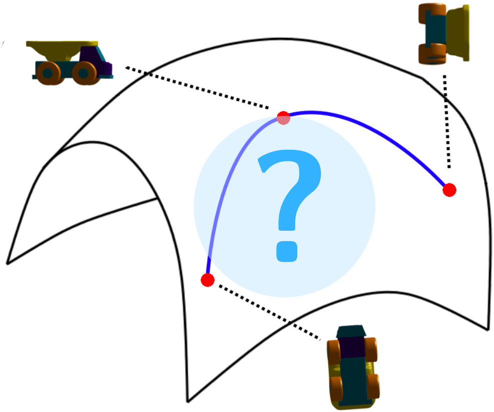
Computer vision, video games

Data points $d_i \in SO(3)$

curve $\mathbf{B} : [0, n] \to SO(3)$

What they have in common

$\mathbb{S}^2$, $\mathrm{SO}(3)$, $\mathcal{S}_+(p,r)$, $\mathcal{S}$,... are Riemannian manifolds.

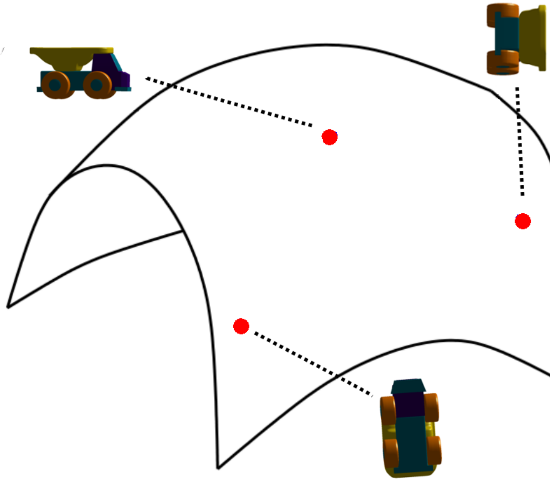


State of the art

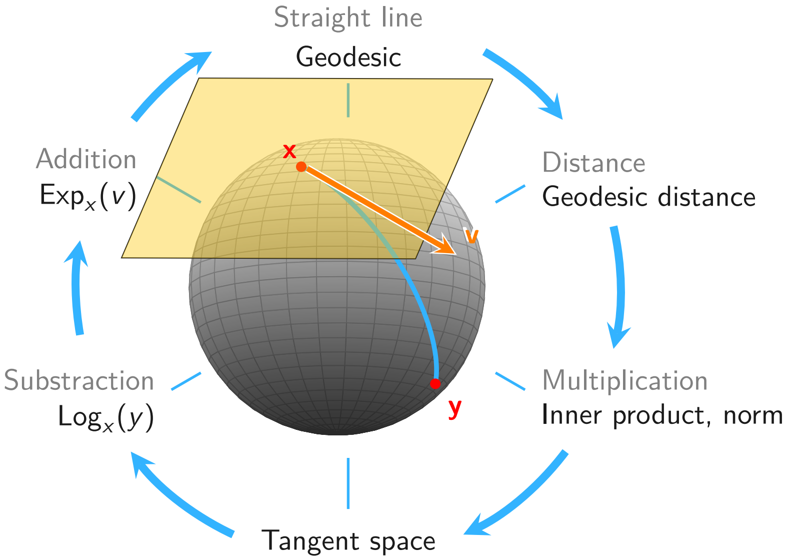
Given data points $d_0, \ldots, d_n$ on a Riemannian manifold $\mathcal{M}$ and associated to time parameters $t_0, \ldots, t_n \in \mathbb{R}$, we seek a curve $\mathbf{B}(t)$ such that $\mathbf{B}(t_i) = d_i$.

- Geodesic regression
  [Rentmeesters 2011; Fletcher 2013; Boumal 2013]
- Fitting in Sobolev space of curves
  [Samir *et al.* 2012]
- Interpolation and fitting with Bézier curves
  [Arnould *et al.* 2015; G. *et al.* 2018]
- Optimization on discretized curves
  [Boumal and Absil, 2011]
- Unrolling-unwrapping techniques, subdivision schemes
  [Kim 2018; Dyn 2008]

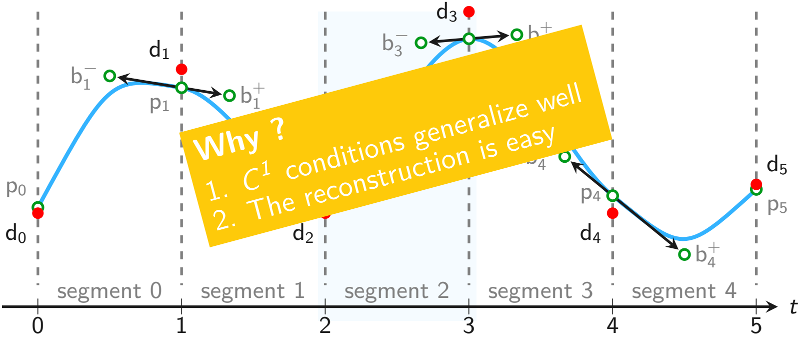
What is a manifold?



Tools of differential geometry: the sphere as an example



$\mathbf{B}(t)$ is a piecewise cubic Bézier curve

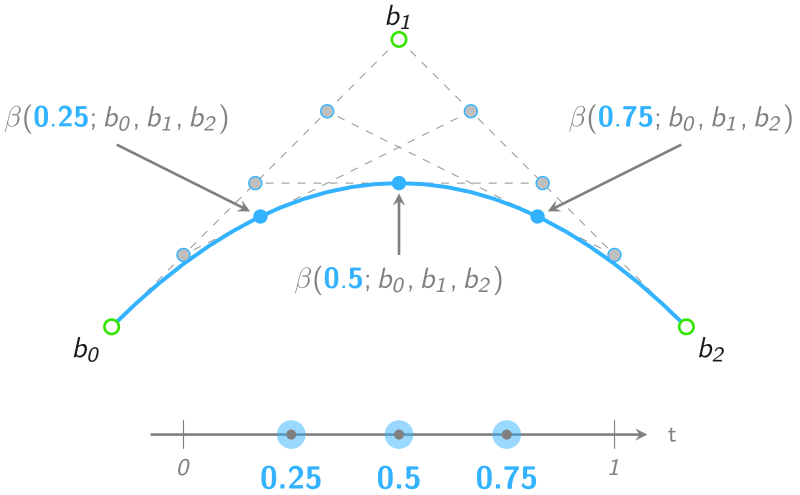


$$\mathbf{B} = \beta(t - m, \mathbf{b}^m) \text{ with } m = \lfloor t \rfloor$$

Each segment is a Bézier curve smoothly connected!
Unknowns: b_i^+, b_i^-, p_i.

C^1 conditions : $b_i^+ = g(2; b_i^-, p_i)$

Why Bézier? – De Casteljau Algorithm generalizes well

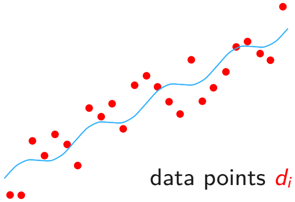


The best Bezier spline to fit the data points

This is a finite dimensionnal optimization problem in b_i^-, p_i.

The goal:

- Find the minimizer $\mathbf{B}$ (on $\mathcal{M} = \mathbb{R}^n$: natural cubic spline).
- What is the gradient ?



Fitting curve

$$\operatorname*{argmin}_{\mathbf{B}\in\Gamma} E_\lambda(\mathbf{B}) := \int_{t_0}^{t_n} \left\| \frac{\mathrm{D}^2\mathbf{B}(t)}{\mathrm{d}t^2} \right\|_{\mathbf{B}(t)}^2 \mathrm{d}t + \lambda \sum_{i=0}^{n} \mathrm{d}^2(\mathbf{B}(t_i), d_i),$$

this is just another geodesic...

???!!!

The second order derivative as finite differences

Replace the second covariant derivative by second order finite differences.

$$\int_{t_0}^{t_r} \left\| \frac{D^2 \mathbf{B}(t)}{\mathrm{d}t^2} \right\|_{\mathbf{B}(t)}^2 \mathrm{d}t \approx \sum_{k=1}^{N-1} \frac{\Delta_s \mathrm{d}_2^2[\mathbf{B}(s_{i-1}), \mathbf{B}(s_i), \mathbf{B}(s_{i+1})]}{\Delta_s^4}.$$

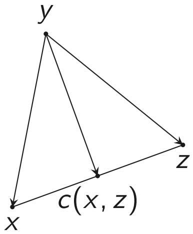
Discretize $[t_0, t_r]$ in $N + 1$ equispaced points $s_0, \dots, s_N$, with $\Delta_s = s_1 - s_0$.

The second order derivative as finite differences

The second order difference was studied by Bačák *et al.* (2016) as:

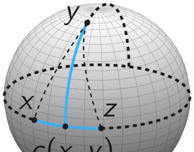
$$\mathrm{d}_2^2[x, y, z] := \min_{c \in \mathcal{C}_{x,z}} \mathrm{d}^2(c, y), \quad x, y, z \in \mathcal{M}$$

where $\mathcal{C}_{x,z}$ is the mid-point of the geodesic between x and z.



if $\mathcal{M} = \mathbb{R}^d$

$$\frac{1}{2}\|x - 2y + z\| = \left\|\frac{1}{2}(x + z) - y\right\|$$



if $\mathcal{M} = \mathbb{S}^2$

$$\min_{c \in \mathcal{C}_{x,z}} \mathrm{d}^2(c, y)$$

It's all a question of geodesics...

$$\underset{\mathbf{B}\in\Gamma}{\operatorname{argmin}}\, E_\lambda(\mathbf{B}) := \int_{t_0}^{t_n} \left\| \frac{\mathrm{D}^2\mathbf{B}(t)}{\mathrm{d}t^2} \right\|_{\mathbf{B}(t)}^2 \mathrm{d}t + \lambda \sum_{i=0}^{n} \mathrm{d}^2(\mathbf{B}(t_i), d_i),$$

The objective $E_\lambda(\mathbf{B})$ is only made of geodesics:

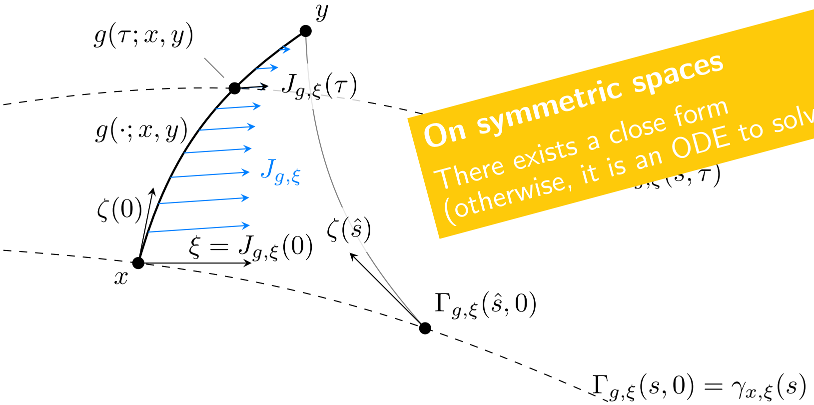
- $6(N+1)$ geodesics for the Bézier segment $\mathbf{B}(t)$;
- N geodesics for the midpoint evaluation $c(x,z)$;
- N geodesics for $\mathrm{d}^2(c,y)$.

Geodesic variation?

The geodesic variation

The variation of a geodesic $\mathbf{g}(t; x, y)$ with respect to its end-point x, in the direction $\xi \in T_x\mathcal{M}$ is called a Jacobi field.

$$D_x\mathbf{g}(t; \cdot, y)[\xi] = J_{\mathbf{g},\xi}(t)$$

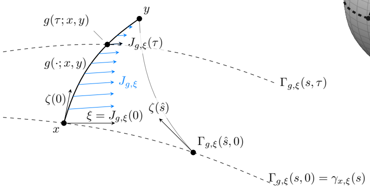


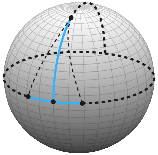
On symmetric spaces

There exists a close form
(otherwise, it is an ODE to solve)

Put that all together

$$\operatorname*{argmin}_{\mathbf{B}\in\Gamma} E_\lambda(\mathbf{B}) := \int_{t_0}^{t_n} \left\| \frac{\mathrm{D}^2\mathbf{B}(t)}{\mathrm{d}t^2} \right\|_{\mathbf{B}(t)}^2 \mathrm{d}t + \lambda \sum_{i=0}^{n} \mathrm{d}^2(\mathbf{B}(t_i), d_i),$$



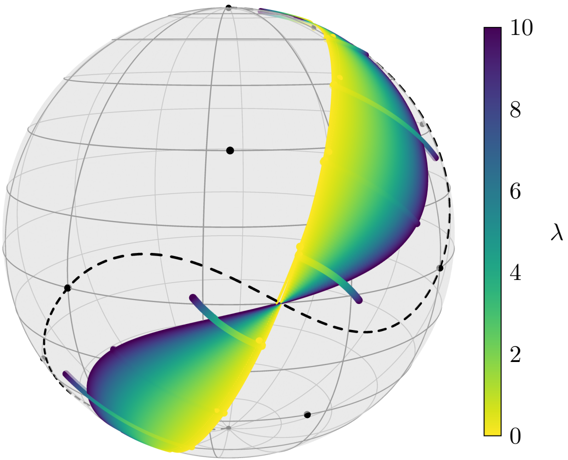


Chain rule:

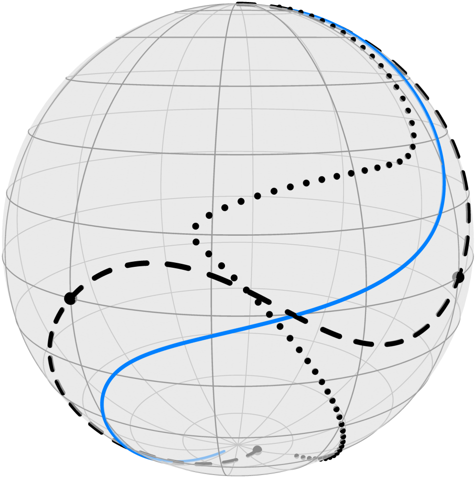
$$D_x(f \circ g)[\eta] = D_g f\left[D_x g[\eta]\right]$$

$$\nabla E_\lambda$$

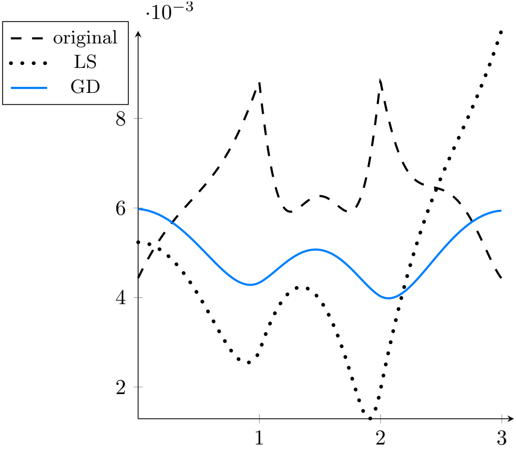
Results - Minimizer and influence of λ



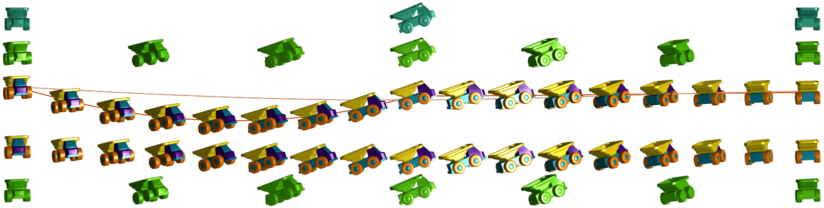
Results - Tangent space approach VS optimization



Results - Tangent space approach VS optimization



Results - SO(3)



Conclusions and future work

Take-home message:

- Recursive gradient of Bézier curves using Jacobi fields only ;
- Close form on symmetric spaces ;
- Tangent-space based methods are efficient for "local" data points ;
- Tangent-space based methods can be a good initializer on more "global" problems.

Future work:

- Generalization to **2D**, **3D**, is open.
- Application to real data is awaiting.

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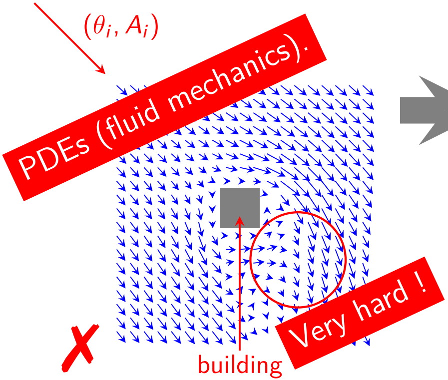
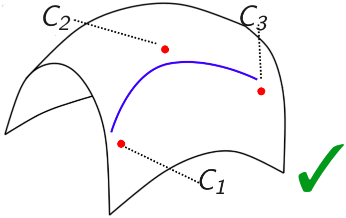
G. and Bergmann. *A variational model for data fitting on manifolds by minimizing the acceleration of a Bézier curve.* Frontiers in Applied Mathematics and Statistics, 4(59), 2018.

Code available soon on ronnybergmann.net/mvirt/

Why is this important? - SDP matrices of size p,rank r



$C_i \in \mathcal{S}_+(p, r)$

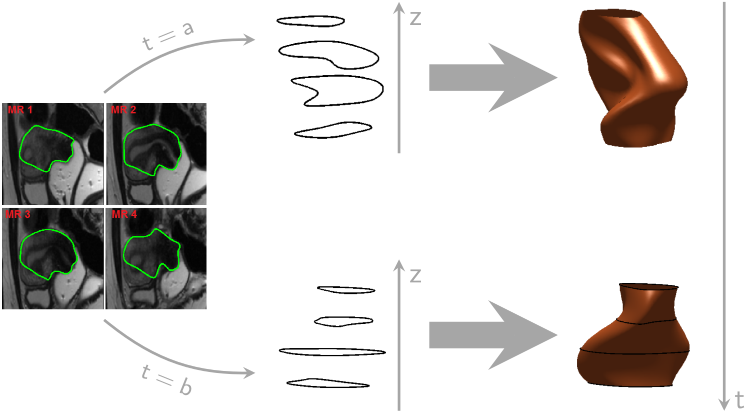


Wind field estimation for UAV

Data points $d_i \in \mathcal{S}_+(p, r)$

curve **B** : $[0, n] \to \mathcal{S}_+(p, r)$

Why is this important? - Shape space



medical imaging, harmed soldiers rehab'

Data points $d_i \in \mathcal{S}$

curve $\mathbf{B} : [0, n] \to \mathcal{S}$

The link between gradient and directional derivative

The gradient $\nabla f(x) \in T_x\mathcal{M}$ of f is given by

$$D_x f[\eta] = \langle \nabla f(x), \eta \rangle_x, \qquad x \in \mathcal{M}, \quad \eta \in T_x\mathcal{M}$$