Fast method to fit a $\mathcal{C}^1$ piecewise-Bézier function to manifold-valued data points: how suboptimal is the curve obtained on the sphere $\mathbb{S}^2$?

P.-Y. Gousenbourger, P.-A. Absil, L. Jacques
pierre-yves.gousenbourger@uclouvain.be

Université catholique de Louvain, ICTEAM

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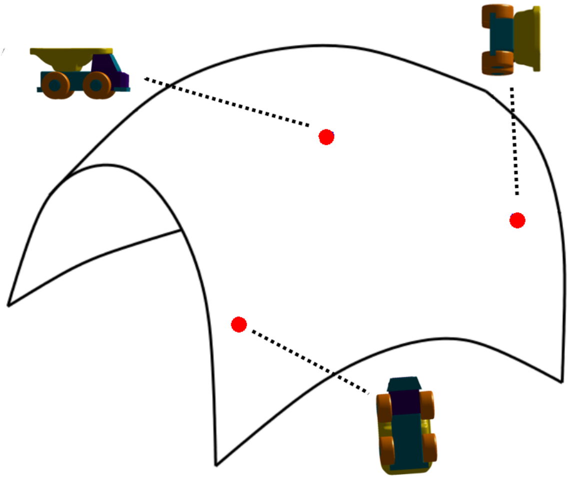
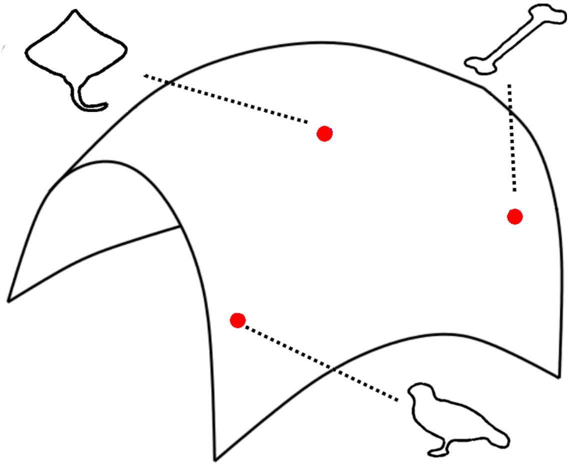
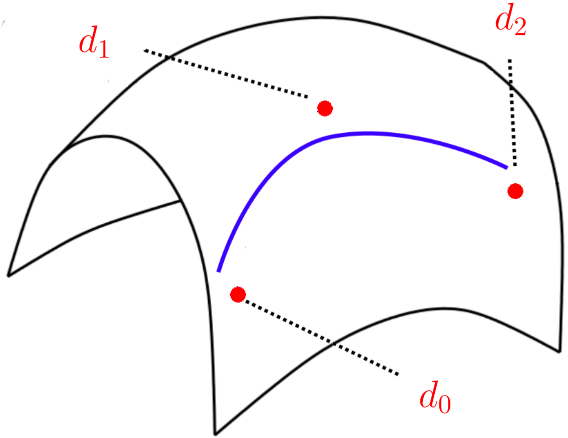
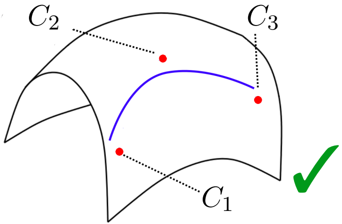
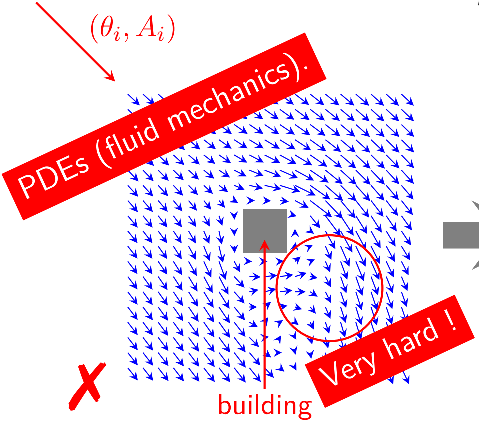





d_1
d_2
d_0

The wind field estimation





$C_i \in \mathcal{S}_+(r, n)$

Outline

Fast

(but suboptimal ?)

method

... and reminders on Bézier.

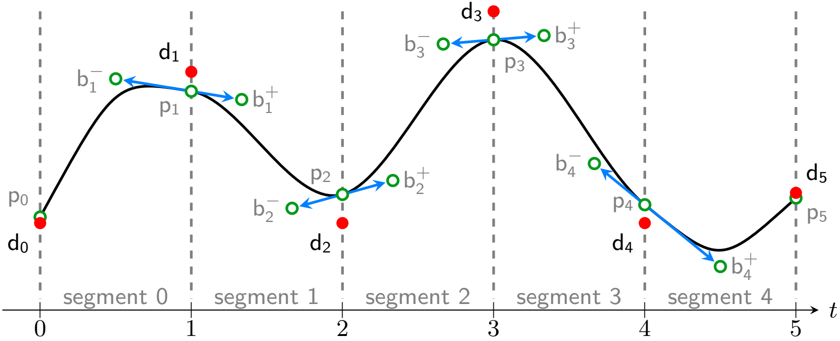
Accurate

(but less efficient)

method

... and a short comparison.

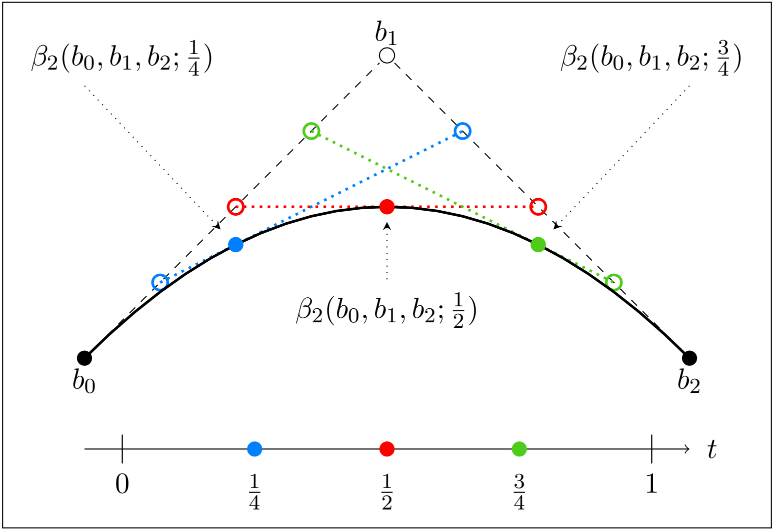
Smooth fitting with Bézier (in $\mathbb{R}^n$)



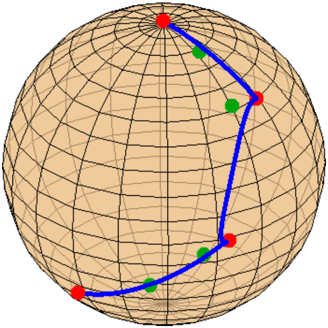
Each segment is a Bézier curve smoothly connected !

Unknowns : b_i^-, b_i^+, p_i.

Reconstruction : the De Casteljau algorithm



Example on the sphere



Where to place the control points?

Optimal control points on the Euclidean space

$$\min_{p_0, b_i^-, b_i^+, p_n} \int_0^n \|\ddot{\mathfrak{B}}(t; p_0, b_i^-, b_i^+, p_n)\|^2 \mathrm{d}t + \lambda \sum_{i=0}^{n} \|d_i - p_i\|_2^2$$

$$\text{s.t. } p_i = \frac{b_i^- + b_i^+}{2}$$

Quadratic polynomial in p_0, b_i^-, b_i^+ and p_n.

$$x_i = \sum_{j=0}^{n} q_{ij}(\lambda) d_j$$

Control points on manifolds

- The control points are given by :

$$x_i = \sum_{j=0}^{n} q_{i,j}(\lambda) d_j$$

- These points are invariant under translation, *i.e.* :

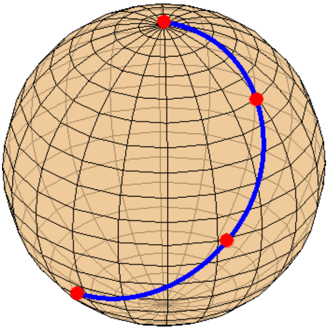
$$x_i - d^{ref} = \sum_{j=0}^{n} q_{i,j}(\lambda)(d_j - d^{ref})$$

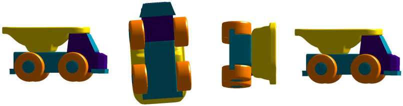
- On manifolds : representation of x_i in the **tangent space** of d^{ref} with the **Log**, as $a - b \Leftrightarrow \mathrm{Log}_b(a)$

$$v_i = \mathrm{Log}_{d^{ref}}(x_i) = \sum_{j=0}^{n} q_{i,j}(\lambda) \mathrm{Log}_{d^{ref}}(d_j)$$

- Back to the manifold with the **Exp** : $x_i = \mathrm{Exp}_{d^{ref}}(v_i)$, where $d^{ref} = d_i$ if x_i is b_i^-, p_i, b_i^+.

Example on the sphere and on SO(3)





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A discrete function to solve

$$\boxed{x^*}$$

$$f = \underbrace{\int_0^n \|\ddot{\mathfrak{B}}(t)\|_{\mathcal{M}}^2 \mathrm{d}t}_{\text{“mean square acceleration"}} + \underbrace{\lambda \sum_{i=0}^{n} \mathrm{d}^2(p_i, d_i)}_{\text{“fidelity"}}$$

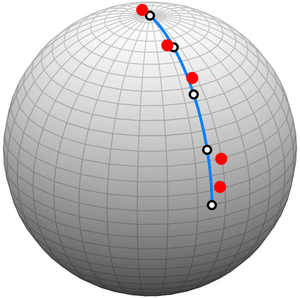
fast method

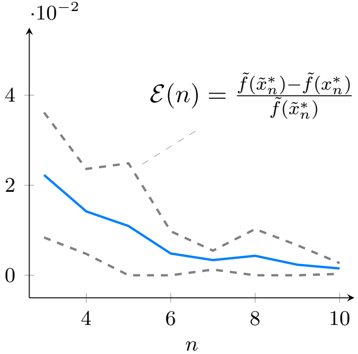
$$\tilde{f} = \sum_{k=1}^{M-1} \Delta\tau \underbrace{\mathrm{d}_{2,\Delta\tau}^2 \left(\mathfrak{B}(t_{k-1}), \mathfrak{B}(t_k), \mathfrak{B}(t_{k+1})\right)}_{\text{second order finite differences}} + \lambda \sum_{i=0}^{n} \mathrm{d}^2(p_i, d_i),$$

manopt

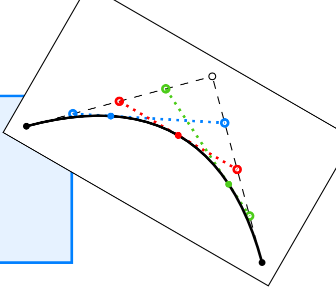
$$\boxed{\tilde{x}^*}$$

Optimisation without derivative (Manopt)





Future work : gradient descent ?



Core question :

$$\nabla_{b_i} d^2_{2,\Delta\tau}\left(\mathfrak{B}(t_{k-1}), \mathfrak{B}(t_k), \mathfrak{B}(t_{k+1})\right)$$

Chain rule of geodesics :

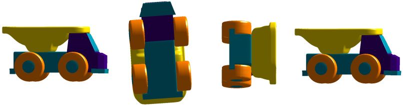
$$\begin{tikzcd}
(\textcolor{green}{\mathbf{b}}, t) \arrow[r, bend left, "g"] & g(\textcolor{green}{\mathbf{b}}, t) \arrow[r, bend left, "g"] & \cdots \arrow[r, bend left, "g"] & \mathfrak{B}(t) \arrow[r, bend left, "d_2^2"] & \mathrm{d}_2^2(x, y, z)
\end{tikzcd}$$

$\mathcal{C}^1$ constraints

- $\nabla \mathrm{d}_2^2(x, y, z)$: Bačák *et al.*[1]
- ∇g : Jacobi fields

1. Bacak, Bergmann, Steidl, and Weinmann, *A Second Order Nonsmooth Variational Model for Restoring Manifold-valued Images*, SIAM Journal on Computing 38 (2016), no. 1, 567–597.

Any questions ?



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