Poké-Collab: Kanto / 151 Pokemon by 151 Artists
July 22 - August 10 2013
A medical application

[Samir et al., 2015]
The wind field estimation

$C_i \in S_+(r, n)$
How to interpolate or fit points on $M$...

... in 1D and 2D?
Interpolation and fitting on manifolds with Bézier functions

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The path...

<table>
<thead>
<tr>
<th>Interpolation</th>
<th>Fitting</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D</td>
<td></td>
</tr>
<tr>
<td>2D</td>
<td>?</td>
</tr>
</tbody>
</table>
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<thead>
<tr>
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<th>Fitting</th>
</tr>
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<td></td>
</tr>
</tbody>
</table>

| 2D  | ![2D Interpolation](image) | ![2D Fitting](image) | ? |

1D Interpolation and Fitting examples:

- In 1D, Interpolation fits a smooth curve through the data points.
- In Fitting, a curve is drawn that passes through specific points.

2D Examples:

- Interpolation involves fitting a curve through data points in a 2D space.
- Fitting in 2D might represent a more complex scenario or a question mark for further exploration.
1D : Interpolative Bézier curves

Each segment between two consecutive points is a Bézier curve of degree $K$.

$$\beta_K(t, b) = \sum_{i=0}^{K} b_i \mathcal{B}_{iK}(t)$$

[G. et al. 2014, Arnould et al. 2015]
Reconstruction: the De Casteljau algorithm

\[ \beta_2(b_0, b_1, b_2; \frac{1}{4}) \]

\[ \beta_2(b_0, b_1, b_2; \frac{3}{4}) \]

\[ \beta_2(b_0, b_1, b_2; \frac{1}{2}) \]
Example on the sphere

It’s ugly. Make it smooth!
Smooth interpolation with Bézier (in $\mathbb{R}^n$)

Each segment is a Bézier curve smoothly connected!
Unknowns: $b_i^-, b_i^+$.
Differentiability

\[ b_i^+ = 2d_i - b_i^- \]
Optimal $C^1$-piecewise Bézier interpolation (in $\mathbb{R}^n$)

Minimization of the mean squared acceleration of the path

$$\min_{\beta_i^-} \int_0^1 \| \dot{\beta}_2^0 (b_i^-; t) \|^2 dt + \sum_{i=1}^{n-1} \int_0^1 \| \dot{\beta}_3^i (b_i^-; t) \|^2 dt + \int_0^1 \| \dot{\beta}_2^n (b_{n-1}^-; t) \|^2 dt$$

Second order polynomial $P(b_i^-)$

$$\min_{\beta_i^-} \int_0^1 \| \dot{\beta}_2^0 (b_i^-; t) \|^2 dt + \sum_{i=1}^{n-1} \int_0^1 \| \dot{\beta}_3^i (b_i^-; t) \|^2 dt + \int_0^1 \| \dot{\beta}_2^n (b_{n-1}^-; t) \|^2 dt$$

Second order polynomial $P(b_i^-)$
A result on $\mathbb{R}^2$
Optimal $C^1$-piecewise Bézier interpolation (on $\mathcal{M}$)

- The control points are given by:

$$b_i^- = \sum_{j=0}^{n} q_{i,j}d_j$$

- These points are invariant under translation, i.e.:

$$b_i^- - d^{\text{ref}} = \sum_{j=0}^{n} q_{i,j}(d_j - d^{\text{ref}})$$

- On manifolds: projection to the tangent space of $d^{\text{ref}}$ with the $\text{Log}$, as $a - b \Leftrightarrow \text{Log}_b(a)$

$$v_i = \text{Log}_{d^{\text{ref}}}(b_i^-) = \sum_{j=0}^{n} q_{i,j}\text{Log}_{d^{\text{ref}}}(d_j)$$

- Back to the manifold with the $\text{Exp}$: $b_i^- = \text{Exp}_{d^{\text{ref}}}(v_i)$. 
Application to MRI – the manifold of closed shapes
Interpolation with Bézier: pros and cons

✓ Optimality conditions are a closed form linear system.

✓ Method only needs exp and log maps.

✓ The curve is $C^1$.

✗ No guarantee on the optimality when $\mathcal{M}$ is not flat.

[G. et al., 2014] [Arnould et al., 2015] [Pyta et al., 2016]
The path...

Interpolation

Fitting

1D

2D
Smooth fitting with Bézier (in $\mathbb{R}^n$)

Now data points are **approached** but not interpolated!

Unknowns: $b_i^-, b_i^+, p_i$. 
Differentiability

\[ p_i = \frac{b_i^- + b_i^+}{2} \]
Optimal $C^1$-piecewise Bézier fitting (in $\mathbb{R}^n$)

Minimization of the mean squared acceleration of the path

$$\min_{p_0, b_i^-, b_i^+, p_n} \int_0^1 \|\dddot{\beta}_2\|^2 dt + \sum_{i=1}^{n-1} \int_0^1 \|\dddot{\beta}_3\|^2 dt + \int_0^1 \|\dddot{\beta}_2\|^2 dt + \lambda \sum_{i=0}^n \|d_i - p_i\|^2_2$$

Second order polynomial $P(p_0, b_i^-, b_i^+, p_n, \lambda)$

$$\nabla P(p_0, b_i^-, b_i^+, p_n, \lambda)$$
Optimal $C^1$-piecewise Bézier fitting (on $\mathcal{M}$)

- The control points are given by:

$$x_i = \sum_{j=0}^{n} q_{i,j}(\lambda)d_j$$

- These points are invariant under translation, i.e.:

$$x_i - d^{ref} = \sum_{j=0}^{n} q_{i,j}(\lambda)(d_j - d^{ref})$$

- On manifolds: projection to the tangent space of $d^{ref}$ with the Log, as $a - b \Leftrightarrow \text{Log}_b(a)$

$$v_i = \text{Log}_{d^{ref}}(x_i) = \sum_{j=0}^{n} q_{i,j}(\lambda)\text{Log}_{d^{ref}}(d_j)$$

- Back to the manifold with the Exp: $x_i = \text{Exp}_{d^{ref}}(v_i)$, where $d^{ref} = d_i$ if $x_i$ is $b_i^-$, $p_i$, $b_i^+$. 
Application: Wind field estimation

$(\theta_i, A_i)$

$C_i \in S_+(r, n)$
Application: Wind field estimation on $S_+(r,p)$.
No noise on data (joint work with E. Massart)
Application: Wind field estimation on $S_+(r, p)$. With artificial noise (8dB) on data (joint work with E. Massart)
Fitting with Bézier: pros and cons

✓ Optimality conditions are a closed form linear system.

✓ Method only needs exp and log maps.

✓ The curve is $C^1$.

✗ No guarantee on the optimality when $\mathcal{M}$ is not flat.

✓ We can do denoising.

Paper submitted at the ESANN conference, 2017. Joint work with MIT.
The path...

Interpolation

Fitting

1D

2D

1D

2D

M

?
2D : Interpolative Bézier surface

Each patch between four neighbour points is a **Bézier surface** of degree $K$.

\[
\beta_K(t_1, t_2, b) = \sum_{i=0}^{K} \sum_{j=0}^{K} b_{ij} B_{iK}(t_1) B_{jK}(t_2)
\]
Bézier surface on one patch

\[ \beta_K(t_1, t_2, \mathbf{b}) = \sum_{i=0}^{K} \sum_{j=0}^{K} b_{ij} B_{iK}(t_1) B_{jK}(t_2) = \sum_{i=0}^{K} \tilde{b}_j B_{jK}(t_2) \]

Two-curves
Bézier surface on one patch

\[
\beta_K(t_1, t_2, \mathbf{b}) = \sum_{i=0}^{K} \sum_{j=0}^{K} b_{ij} B_iK(t_1) B_jK(t_2) = \text{av}[[\mathbf{b}, w_{ij}]
\]

Two-curves  Karcher
Bézier surface on one patch

\[ \beta_K(t_1, t_2, b) = \sum_{i=0}^{K} \sum_{j=0}^{K} b_{ij} B_{iK}(t_1) B_j(t_2) \]
Continuity

\begin{align*}
    b_{m,n} &= b_{m,n-1} \\
    b_{0,j} &= b_{m-1,n}
\end{align*}
Differentiability

\[ b_{m,n}^{0,j} = \frac{b_{m,n}^{1,j} + b_{m,n}^{i,j}}{2} \]

\[ b_{m,n}^{i,0} = \frac{b_{m,n}^{i,-1} + b_{m,n}^{i,1}}{2} \]
Differentiability

\[ b_{0,j}^{m,n} = \text{av}[(b_{-1,j}^{m,n}, b_{1,j}^{m,n}), (\frac{1}{2}, \frac{1}{2})] \]

\[ b_{i,0}^{m,n} = \text{av}[(b_{i,-1}^{m,n}, b_{i,1}^{m,n}), (\frac{1}{2}, \frac{1}{2})] \]

not sufficient...
A new definition of Bézier surfaces in $\mathcal{M}$

$$\beta_3(t_1, t_2, b) = \text{av}[\tilde{b}, \tilde{w}_{ij}]$$

$C^1$ conditions!
Optimal $C^1$-piecewise Bézier surface (in $\mathbb{R}^n$)

Minimization of the mean squared acceleration of the surface

In the Euclidean space...

$$\min_{b_{ij}^{mn}} \sum_{m=0}^{M} \sum_{n=0}^{N} \hat{F}(\beta_{3}^{mn})$$

where

$$\hat{F}(\beta_{3}^{mn}) = \int_{[0,1] \times [0,1]} \left\| \frac{\partial^2 \beta_{3}^{mn}}{\partial (t_1, t_2)} \right\|_F^2 \, dt_1 dt_2 = \sum_{i,j,o,p=0}^{3} \alpha_{ijop} (b_{ij}^{mn} \cdot b_{op}^{mn})$$

Quadratic function, easy on the Euclidean space...

but not in $\mathcal{M}$. 
Optimal surface on $\mathcal{M}$: project on tangent spaces
Optimal surface on manifolds

Compute $v_{i,j}^{m,n}$ on the tangent space...
Optimal surface on manifolds

... and project back to the manifold.

$T_{pm,n} \quad v_{i,j} \quad \text{Exp}_{pm,n} (\cdot) \quad M$

$b_{i,j}^m$... well it’s a bit more complicated ;-)
A result on SO(3)
A cool result
The medical application
Interpolation with Bézier in 2D: pros and cons

✓ Optimality conditions are a closed form linear system.

✓ Method only needs exp and log maps and parallel transport.

✓ The surface is $C^1$.

✗ The control points generation might be very heavy.

Another method to generate the control points [Absil et al., 2016]

✗ No guarantee on the optimality when $\mathcal{M}$ is not flat.
The path...

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Conclusions

General $C^1$-interpolative/fitting methods on manifolds... with applications in medical imaging, wind estimation, model reduction,...

light • closed form • uses few elements in $\mathcal{M}$

Summary on interpolation:
“Differentiable Piecewise-Bézier Surfaces on Riemannian Manifolds”
Any questions?

Interpolation and fitting on manifolds with Bézier functions

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Optimal surface: prepare the manifold setting

\[
\hat{F}(\beta_{mn}^3) = \sum_{i,j,o,p=0}^{3} \frac{1}{4} \alpha_{ijop} \sum_{r,s \in \{0,1\}} (v_{ij}^{mn}(r, s) \cdot v_{op}^{mn}(r, s))
\]
Optimal surface: system reduction

\[ u_{m+1,n-1} - u_{m+1,n+1} = u_{0,1} - u_{0,0} \]

\[ u_{m-1,1} = u_{0,1} - u_{1,0} \]

\( C^0 \) and \( C^1 \) conditions
Optimal surface: constraints

\[
\begin{align*}
\tilde{T} \downarrow T + Z \\
\implies v_{2,1}^{m,n}(0,1) &= P_{p_{m,n+1} \leftarrow p_{m+1,n}}(u_{-1,-1}^{m+1,n}) \\
&\quad - \Log_{p_{m,n+1}}(p_{m+1,n})
\end{align*}
\]
Optimal surface: solution

The objective function

\[
L(X)_{ij} = \frac{1}{4} \sum_{o,p} \alpha_{ijop} x_{op}
\]

is solved through a linear system

\[
U_{opt} = - (S^* T^* LTS)^{-1} (S^* T^* LZ).
\]