Wind field estimation via C^1 Bézier smoothing on manifolds

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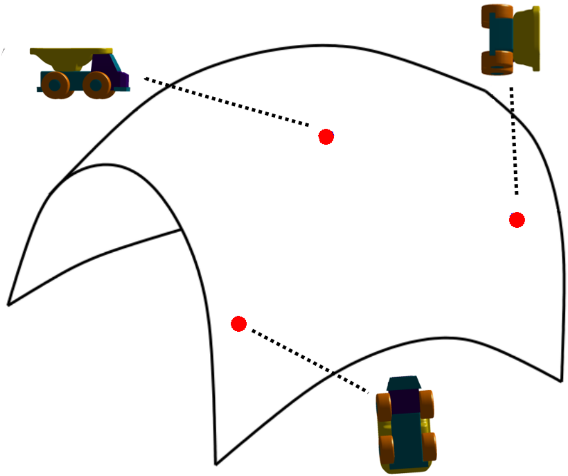
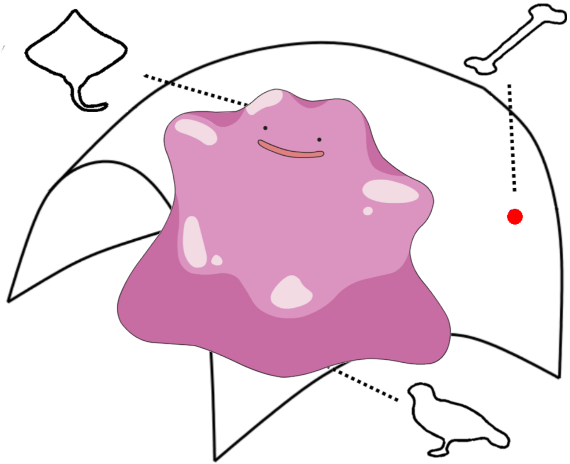
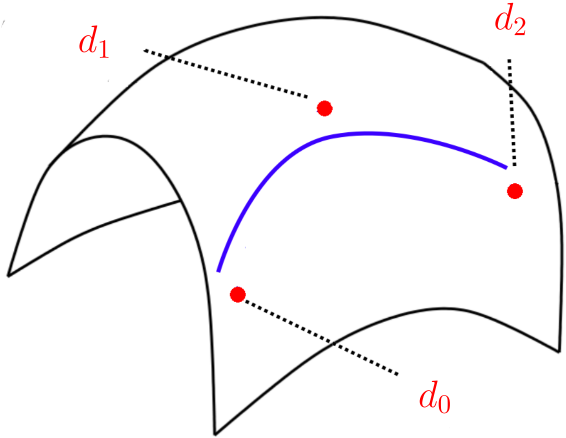
WIPS - August 30th, 2017

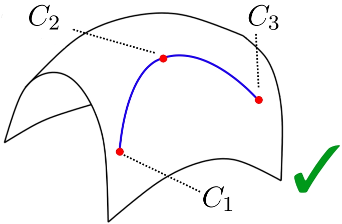


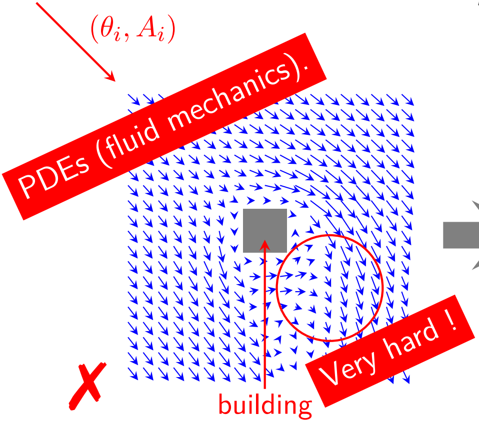


Poké-Collab: Kanto / 151 Pokemon by 151 Artists
July 22 - August 10 2013

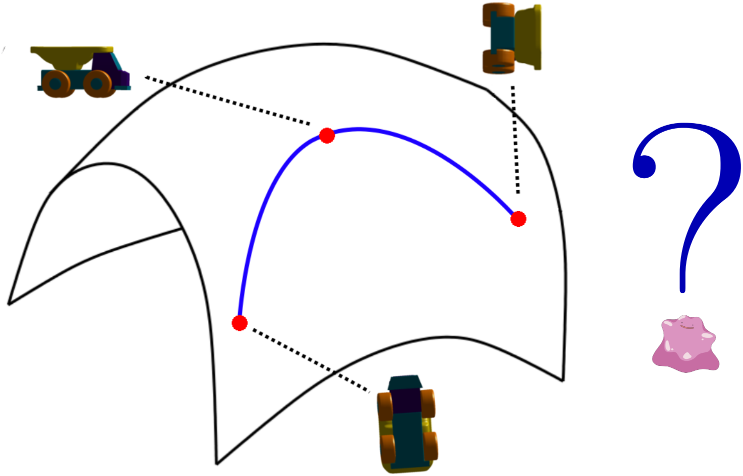

d_1
d_2
d_0

The wind field estimation





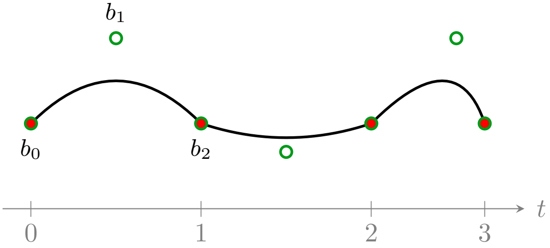
$C_i \in \mathcal{S}_+(r, n)$



How to fit a curve to points on $\mathcal{M}$?

1D : Interpolative Bézier curves

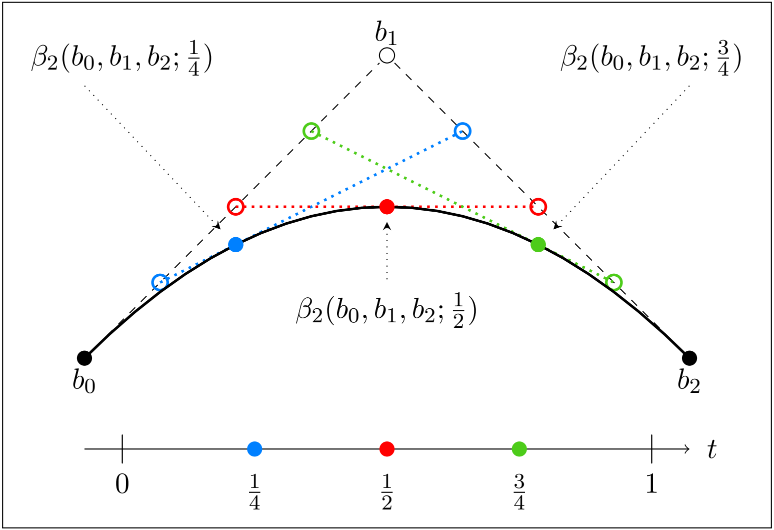
Each segment between two consecutive points is
a **Bézier curve** of degree K.



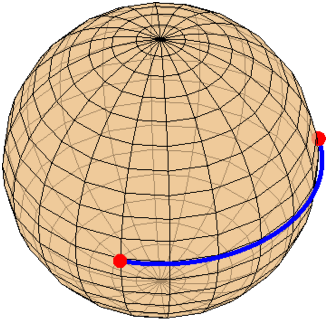
$$\beta_K(t, \mathbf{b}) = \sum_{i=0}^{K} b_i B_{iK}(t)$$

[G. *et al.* 2014, Arnould *et al.* 2015]

Reconstruction : the De Casteljau algorithm



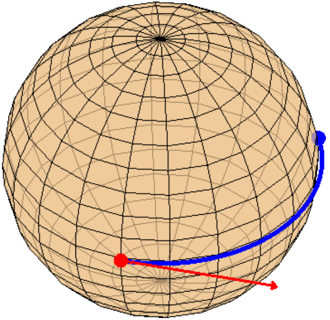
The straight line is a geodesic



$\Big($

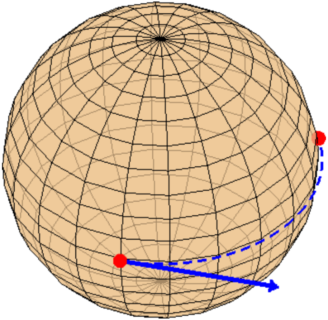
The exponential map to construct the geodesic

$$\gamma(t) = \mathrm{Exp}_x(t\xi_x)$$



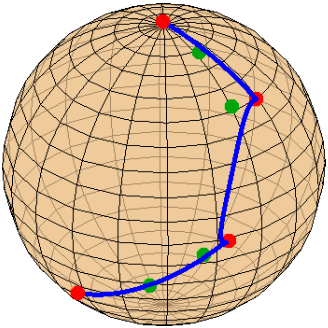
The logarithmic map to determine the starting velocity

$$\mathrm{Log}_{x}(y) = \xi_x$$



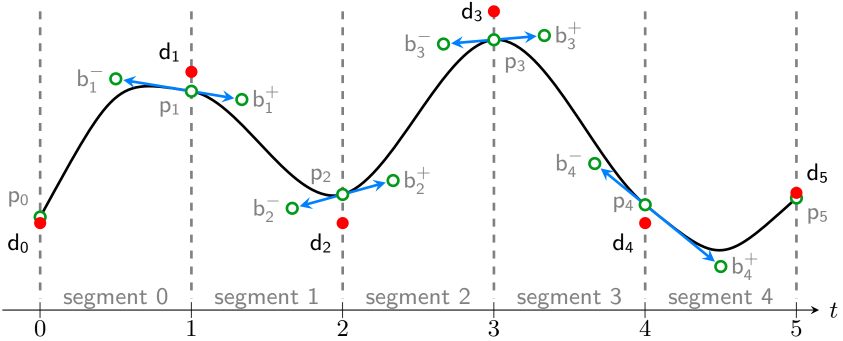
$$\Bigg)$$

Example on the sphere



Well ? ! What about fitting, now ?

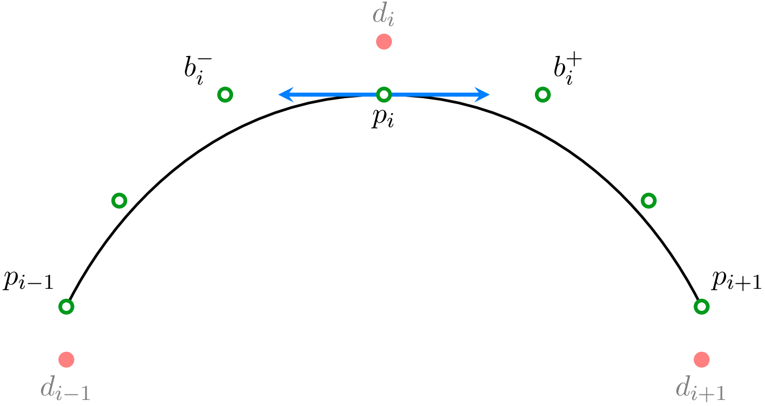
Smooth fitting with Bézier (in $\mathbb{R}^n$)



Each segment is a Bézier curve smoothly connected !

Unknowns : b_i^-, b_i^+, p_i.

Differentiability



$$p_i = \frac{b_i^- + b_i^+}{2}$$

Optimal $\mathcal{C}^1$-piecewise Bézier fitting (in $\mathbb{R}^n$)

Minimization of the mean squared acceleration of the path

$$\min_{p_0, b_i^-, b_i^+, p_n} \int_0^1 \|\ddot{\beta}_2^0\|^2 \mathrm{d}t + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i\|^2 \mathrm{d}t + \int_0^1 \|\ddot{\beta}_2^n\|^2 \mathrm{d}t + \lambda \sum_{i=0}^{n} \|d_i - p_i\|_2^2$$

$$\underbrace{\min_{p_0, b_i^-, b_i^+, p_n} \int_0^1 \|\ddot{\beta}_2^0\|^2 \mathrm{d}t + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i\|^2 \mathrm{d}t + \int_0^1 \|\ddot{\beta}_2^n\|^2 \mathrm{d}t + \lambda \sum_{i=0}^{n} \|d_i - p_i\|_2^2}_{\text{Second order polynomial } P(p_0, b_i^-, b_i^+, p_n, \lambda)}$$

$$\underbrace{\min_{p_0, b_i^-, b_i^+, p_n} \int_0^1 \|\ddot{\beta}_2^0\|^2 \mathrm{d}t + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i\|^2 \mathrm{d}t + \int_0^1 \|\ddot{\beta}_2^n\|^2 \mathrm{d}t + \lambda \sum_{i=0}^{n} \|d_i - p_i\|_2^2}_{\text{Second order polynomial } P(p_0, b_i^-, b_i^+, p_n, \lambda)}$$

$$\nabla P(p_0, b_i^-, b_i^+, p_n)\,!$$

Optimal $\mathcal{C}^1$-piecewise Bézier fitting (on $\mathcal{M}$)

- The control points are given by :

$$x_i = \sum_{j=0}^{n} q_{i,j}(\lambda) d_j$$

- These points are invariant under translation, *i.e.* :

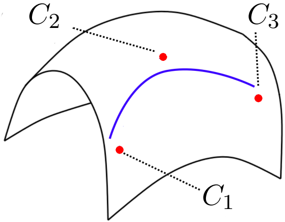
$$x_i - d^{ref} = \sum_{j=0}^{n} q_{i,j}(\lambda)(d_j - d^{ref})$$

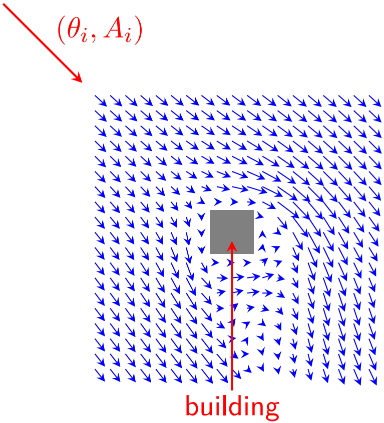
- On manifolds : representation of x_i in the **tangent space** of d^{ref} with the **Log**, as $a - b \Leftrightarrow \mathrm{Log}_b(a)$

$$v_i = \mathrm{Log}_{d^{ref}}(x_i) = \sum_{j=0}^{n} q_{i,j}(\lambda)\mathrm{Log}_{d^{ref}}(d_j)$$

- Back to the manifold with the **Exp** : $x_i = \mathrm{Exp}_{d^{ref}}(v_i)$, where $d^{ref} = d_i$ if x_i is b_i^-, p_i, b_i^+.

Application : Wind field estimation on $S_+(r, p)$.





$C_i \in \mathcal{S}_+(r, n)$

Application : Wind field estimation on $S_+(r, p)$.

How to estimate the error ?

- Training set : $\{C(\theta_i)\}_{i \in I_T}$ $\quad I_T = \{1, 3, \ldots, 33\}$;
  Validation set : $\{C(\theta_i)\}_{i \in I_V}$ $\quad I_V = \{2, 4, \ldots, 32\}$;

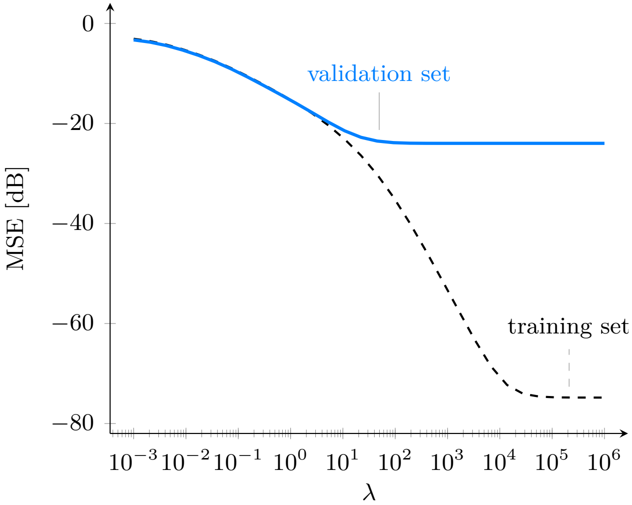
- Bézier spline $\mathbf{B}(\theta)$ with input data points from I_T

- Mean Squared Error :

$$\mathrm{MSE}(\mathbf{B}(\theta)) = 10 \log \left(\frac{\sum_{i \in I_\Omega} ||C(\theta_i) - \mathbf{B}(\theta_i)||_F^2}{\sum_{i \in I_\Omega} ||C(\theta_i)||_F^2} \right), \quad \Omega = \{I, V\}.$$

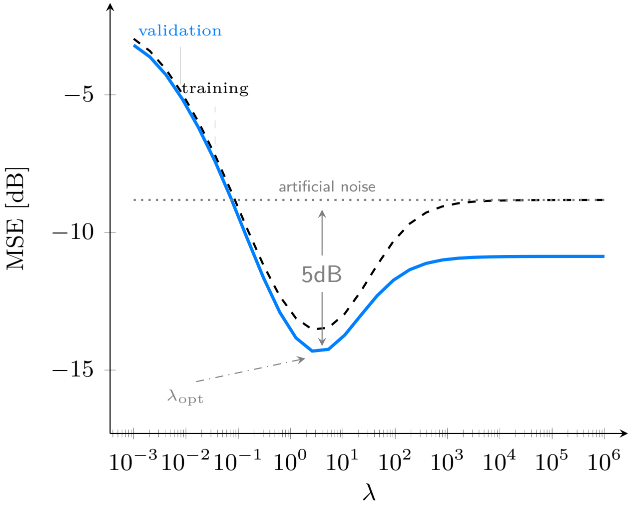
Application : Wind field estimation on $S_+(r, p)$.

No noise on data



Application : Wind field estimation on $S_+(r, p)$.

With artificial noise (8dB) on data



Fitting with Bézier : pros and cons

✓ Optimality conditions are a closed form linear system.

✓ Method only needs exp and log maps.

✓ The curve is $\mathcal{C}^1$.

✗ No guarantee on the optimality when $\mathcal{M}$ is not flat.

✓ We can do denoising.

ESANN, 2017. Joint work with MIT.

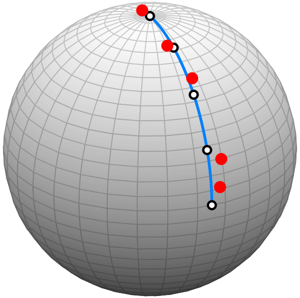
Some guarantees on the optimality

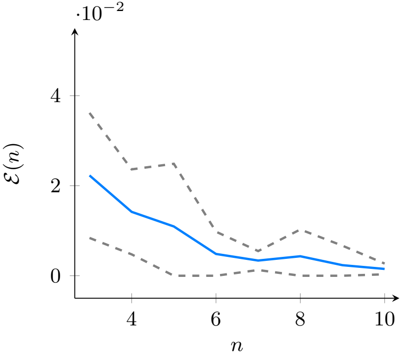
$$\underbrace{\sum_{i=0}^{n-1} \int_0^1 \|\ddot{\beta}^i(t)\|_{\mathcal{M}}^2 \mathrm{d}t}_{\text{“mean square acceleration"}} + \lambda \underbrace{\sum_{i=0}^{n} \mathrm{d}^2(p_i, d_i)}_{\text{“fidelity"}},$$

$$\downarrow$$

$$\sum_{k=1}^{M-1} \Delta\tau \underbrace{\mathrm{d}_2^2\left(\mathbf{B}(t_{k-1}), \mathbf{B}(t_k), \mathbf{B}(t_{k+1})\right)}_{\text{second order finite differences}} + \lambda \sum_{i=0}^{n} \mathrm{d}^2(p_i, d_i),$$

Optimisation without derivative (Manopt - Nelder-Mead)





- G., Jacques, Absil, GSI2017 (accepted)

Future work : evaluate a gradient of the objective

Core question : derivative with respect to b_i of

$$d_2^2\left(\mathbf{B}(t_{k-1}), \mathbf{B}(t_k), \mathbf{B}(t_{k+1})\right)$$

Deterministic chain rule of geodesics :

$$(\mathbf{b}, t) \xrightarrow{g} g(\mathbf{b}, t) \xrightarrow{g} \cdots \xrightarrow{g} \mathbf{B}(t) \xrightarrow{d_2^2} \mathrm{d}_2^2(x, y, z)$$

- $\nabla \mathrm{d}_2^2(x, y, z)$: Bačák *et al.*[1]
- ∇g : Jacobi fields

1. Miroslav Bacak, Ronny Bergmann, Gabriele Steidl, and Andreas Weinmann, A Second Order Nonsmooth Variational Model for Restoring Manifold-valued Images, SIAM Journal on Computing 38 (2016), no. 1, 567–597.

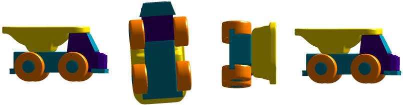
Conclusions

General C^1**-interpolative/fitting** methods on **manifolds**...
with applications in medical imaging, wind estimation, model reduction,...

light • closed form • uses few elements in $\mathcal{M}$

Summary on interpolation :
"Differentiable Piecewise-Bézier Surfaces on Riemannian Manifolds"
[Absil, Gousenbourger, Striewski, Wirth, *SIAM Journal on Imaging Sciences*, 2017].

Any questions ?



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