Scarcity, regulation and endogenous technical progress

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January 2011

Abstract

This paper studies to which extent a firm using a scarce resource input and facing environmental regulation can still manage to have a sustainable growth of output and profits. The firm has a vintage capital technology with two complementary factors, capital and a resource input subject to quota, the latter being increasingly scarce through an exogenously rising price. The firm can scrap obsolete capital and invest in adoptive and/or innovative R&D resource-saving activities. Within this realistic framework, we first characterize long-term growth regimes driven by scarcity (induced-innovation) vs long-term growth regimes driven by quota regulation (Porter-like innovation). More importantly, we study the interaction between scarcity and quota regulation. In particular, we show that there exists a threshold level for the growth rate of the resource price above which the Porter mechanism is killed while the scarcity-induced growth regime may emerge. Symmetrically, we also find that there must exist a threshold value for the environmental quota under which the growth regime induced by scarcity vanishes while the Porter-like growth regime may survive.

Keywords: Vintage capital, technological progress, dynamic optimization, Sustainability, scarcity, environmental regulation

JEL numbers: C61, D21, D92, O33, Q01

1 We would like to thank an anonymous referee of this journal, Francis Bloch, Thierry Bréchet, Lucas Bretschger, Paolo Brito, Edouard Challe, David Cuberes, David de la Croix, Frédéric Docquier, Johan Eyckmans, Gustav Feichtinger, Hans Gersbach, Peter Kort, Omar Licandro, Aude Pommeret, Jean-Pierre Ponsard, Francesco Ricci, Thomas Rutherford, Robert Tamura, Henry Tulkens, Cees Withagen and participants at seminars and conferences held at the Technical University of Vienna, University of Lille, ECTH-Zurich, Ecole Polytechnique of Paris, University of Glasgow and CORE, Louvain-la-Neuve, for invaluable comments on previous versions. The paper was partially written when Yatsenko visited CORE. Boucekkine acknowledges the financial support of the Belgian research programmes PAI P5/10 and ARC 08/14-018 on “Sustainability”. Hritonenko acknowledges the financial support of the NSF-DMS-1009197. The usual disclaimer applies.

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1. Introduction

A crucial issue repeatedly addressed in the ongoing debate on sustainable development is the possibility for the economies to keep on growing while confronted to physical limits and legal constraints such as those related to the limited availability or regenerative capacity of natural resources (fossil energy, fish, forest, etc.), to economic and ecological regulation (emission quotas, harvesting quotas, etc.), or to financial resource constraints at the firm or national economy level. One of the common ideas turns out to be that such a growth possibility is certainly widely open if the economies are able to maintain a permanent stream of innovations, assuring a long-term technological progress (see Arrow et al., 2004, for a comprehensive view of sustainability).

In terms of economic theory, the issue actually traces back to seminal studies on the relationship between resource scarcity and innovation. Scarcer resources are increasingly expensive, and this should in a way affect the behavior of consumers and firms and end up shaping the direction of technological progress. A related fundamental hypothesis, popularized by Hicks (1932), is the so-called induced-innovation hypothesis: a change of relative prices of production inputs stimulates innovation directed to save the production factor that becomes relatively expensive. In the context of the energy consumption debate, this hypothesis simply stipulates that in periods of rapidly rising energy prices (relatively to other inputs), economic agents will find it more profitable to develop alternative technologies, that is, energy-saving technologies. In their well-known work on the menu of home appliances available for sale in the US (between 1958 and 1993), Newell, Jaffee, and Stavins (1999) concluded that a large portion of energy efficiency improvements in US manufacturing seems to be autonomous, and therefore not driven by the Hicksian mechanism outlined above. However, they also concluded that a non-negligible part of the observed improvement could be attributed to price changes and to the emergence of new energy-efficiency standards, ultimately leading to the elimination of old models.

Indeed, just like scarcity, regulation can also be a decisive determinant of technological progress. As an immediate illustration of such a potential nexus, environmental economists are used to put forward the so-called Porter hypothesis (Porter, 1991) according to which a
carefully designed environmental regulation can increase firm competitiveness by encouraging innovation in environmental technologies. A considerable amount of studies has been devoted to the empirical substantiation of this hypothesis, reaching distinct and contrasted conclusions (see Parto and Herbert-Copley, 2007, for an excellent compilation of case studies).

In this paper, **we study optimal firm response to the simultaneous occurrence of environmental regulation constraints and to scarcity (of production inputs)**. This is definitely much more than an academic exercise: typically, firms have to deal with environmental regulation (like emission and extraction quotas and other environmental norms) and with the rising prices of some goods, usually natural resources prices. More importantly, the decisions of the firms, notably on the R&D investment side, do depend on both types of constraints in a far nontrivial way. A real-life example of such a situation is documented in Yarime (2007) on the Japanese chlor-alkali industry producing chlorine and caustic soda through electrolysis, therefore involving large energy consumption.\(^5\) Initially, the industry used a mercury-based electrolysis technique, which resulted in a major human and ecological damage.\(^6\) Environmental regulation pushed this industry to start the adoption of an alternative electrolysis technique, the US diaphragm technology. But this adoption was drastically slowed down during the first oil shock because it turned out that the US technology was much more expensive in terms of energy consumption than the traditional mercury-based technique. R&D programs have been therefore re-directed in the sake of alternative devises, less polluting than the latter and much less energy consuming than the former.

In this paper, we shall identify some theoretical mechanisms clarifying the interplay between environmental regulation and scarcity. In particular, we will be able to explain why and how a technological path which is considered optimal for given resource prices can turn to sub-optimal when resource prices experience a sharp increase. To this end, we consider a representative firm problem confronting both scarcity and environmental quotas. The firm uses capital and a natural resource, and to make things sharper, we assume that

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5 Yarime (2007) reports that about 3% of total industry electricity consumption in Japan can be attributed to the chlor-alkali industry in 1996, which also accounts for about one-fifth of total chemical industry in this year.

6 It was relatively quickly established that the mercury released by the chlor-alkali industry to the neighboring seas was the cause of the so-called Minimata disease, which caused about 700 victims in that time.
there is no substitution between the natural resource input and other production inputs. Scarcity is modelled through an exogenous increase of the price of the natural resource. The quota can be interpreted as an emission quota, in the tradition of Montgomery (1972), or as an extraction quota, or as an environmental norm (like the quantitative limits set on mercury released to the sea in the Japanese example above). In order to deeply understand how both scarcity and environmental quotas affect the optimal decisions and how they interact, we start by isolating the impact of scarcity (relaxing quotas) and the impact of quotas (relaxing scarcity) respectively, while allowing for both in the final stage of the paper. In all the cases studied, the main question is: given the constraint(s) assumed, could the firm experience a sustainable growth of profits?

Answering this question properly requires accounting for a comprehensive set of modernization instruments that the firm can use in response to the above constraints. At the first place, the role of innovation and technology adoption at the firm and/or industry level is a key. If the firms respond to the latter constraints and circumstances by doing more R&D and/or adopting better technologies, then the “sustainability problem”, stated in the beginning, can be at least partially solved. But firms cannot always push on this command button for many reasons. First of all, firms are subject to financial or liquidity constraints, as mentioned above. If the firms do not face any type of financial constraints, then they could finance R&D expenditures and/or technology adoption with no limit, which is certainly unrealistic. Second, technological complexity can be a decisive factor. It is very well known that the success of R&D and technology transfer programs depends, among others, on the complexity and sophistication of the technologies to be up-graded (see for example, Segerstrom, 2000). We shall account for it in our modeling.

In addition to innovative and/or adoptive R&D, firms may decide to scrap old and definitely non-sustainable technologies with their associated capital goods and to replace them (or not) with leading technologies and new equipment. If one aims to capture the mechanisms of modernization, the latter instruments are crucial. Typically, firms will respond by combining all these instruments and by choosing the optimal timing for each of them. We take this avenue here by considering vintage technologies at the firm level, allowing the firm to innovate, to scrap, and to invest.

We shall use vintage capital technologies in line with Malcomson (1975), Benhabib and Rustichini (1991), Boucekkine et al. (1997, 1998, 1999) and Hritonenko and Yatsenko
Beside realism, working with vintage capital production functions allows us to capture some key elements of the problem under consideration, which would be lost under the typical assumption of homogenous capital. For instance, facing an emission tax, firms are tempted to downsize. However, in the vintage capital framework where the firm also chooses the optimal age structure of capital, downsizing entails modernization: the oldest and, thus, the least efficient technologies are then removed.

**Main contributions**

Our paper makes three essential contributions:

i) Within a realistic (and, thus, sophisticated) firm framework, it characterizes finely the inducement mechanisms at work. More precisely, when only scarcity is present, we single out a long-term growth regime entirely driven by scarcity through a specific induced-innovation mechanism which we characterize. When only environmental regulation is active, we come out with a very distinct growth regime entirely driven by quota regulation, therefore illustrating the Porter hypothesis. While the long-term growth is entirely due to quota regulation, the associated growth rate does decrease when the quota is tightened; we therefore identify a kind of soft Porter-like case.

ii) In the second stage, we characterize the interplay between scarcity and quota regulation. In particular, we show that there exists a threshold level for the growth rate of the resource price above which the Porter mechanism is killed while the scarcity-induced growth regime may emerge. Symmetrically, we also find that there must exist a threshold value for the environmental quota under which the growth regime induced by scarcity vanishes while the Porter-like growth regime may survive. In this sense, our paper makes a contribution to the literature of growth under scarcity and regulation, which is an important component of the modern environmental economics literature (see for example, Tsur and Zemel, 2005).

iii) Last but not least, the contribution is technical. To our knowledge, this is the first paper with vintage capital, endogenous scrapping, and endogenous technological progress (see next paragraph for more details). The technical challenges are numerous but we manage to find a way to bring out a fine enough analytical characterization of optimal paths.
Relation to the literature

Our paper contributes to the literature of vintage capital models. Due to the analytical complexity of vintage models, very few papers rely on such specifications. A noticeable exception is Feichtinger et al. (2005) who introduced a proper specification of embodied technological progress underlying the considered vintage capital structure. They concluded that if learning costs are incorporated into the analysis (i.e., running new machines at their full productivity potential takes time), then the magnitude of modernization effect is reduced, and regulation has a markedly negative effect on industry profits (see also Feichtinger et al., 2006 and 2008, for more insight into vintage capital models on several economic issues). Our paper extends the latter result in two important directions: it endogenizes the optimal lifetime of technologies and associated equipment through endogenous scrapping decision and it endogenizes the pace of technological progress in the workplace by considering an optimal innovative or adoptive R&D decision. In such a context, the set of possible modernization strategies is much richer. On the other hand, our paper extends the more traditional vintage literature following Solow et al. (1966), like Boucekkine et al. (1997) or Hritonenko and Yatsenko (1996), by endogenizing technical progress, which definitely enriches the model in many directions as it will be explained along the way. Recently, Hart (2004) has built up a multisectoral endogenous growth model with an explicit vintage sector. Beside the macroeconomic approach taken, this paper differs from ours in many essential aspects: there are two types of R&D, one output-augmenting and the other, say, environmental-friendly, while in our model only resource-saving adoptive and/or innovative R&D is allowed. In addition, the model of Hart (2004) has no explicit scarcity feature, and the treatment of vintages is rather short (only two exogenously given vintages are considered in the end, no endogenous scrapping is incorporated).

On the other hand, our paper can be related to the empirical literature on technological progress under increasing energy prices and regulation, as surveyed by Jaffe et al. (2002). Indeed, it can be connected to the empirical findings in the field. In particular, the impact of resource prices and regulation on resource-saving technological progress, the pace of

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7 In Feichtinger et al. (2005, 2006, 2008), the technological progress is exogenous at the firm level.
capital accumulation and technological replacement depicted in this paper is largely compatible with the findings of Newell et al. (1999) or Popp (2002). Last but not least, our paper can be also directly connected to the theoretical literature on scarcity and growth originating in the limits to growth stream. Within this stream, we share the same objectives as Tsur and Zemel (2005), to cite an important recent contribution. Our vintage approach and the inclusion of environmental regulation allows for some more insight into the impact of scarcity on sustainable development.

The rest of the paper is organized as follows. Section 2 formally describes our firm optimization problem and outlines some of its peculiarities. Section 3 derives the optimality conditions and interprets them. Section 4 is concerned with the derivation of the optimal balanced growth paths induced by scarcity, by quota regulation, and finally by both. Section 5 concludes. Proofs are in Appendix (Section 6).

2. The firm problem

We shall consider the problem of a firm seeking to maximize the net profit that takes into account the consumption \( E(t) \) of a regulated resource, the investment \( R(t) \) to innovative and/or adoptive R&D, and the investment \( i(t) \) into new capital:

\[
\max_{i,a,R} \int_0^\infty e^{-\gamma t} \left[ (1-\theta)Q(t) - p(t)E(t) - R(t) - p_s(t)i(t) \right] dt
\]

where \( p_u(t) \) is the given unit capital price (per capacity unit), \( p(t) \) is the given price of the regulated resource, and \( e^{-\gamma t} \) is the discounting factor. We assume that \( p(t) = \overline{P}e^{\gamma t} \), \( \gamma \geq 0 \), \( \overline{P} \geq 0 \), and a positive \( \gamma \) reflects scarcity of the resource. Then, \( Q(t) \) is the total product output at \( t \),

\[
Q(t) = \int_a^{i(t)} i(\tau)d\tau,
\]

\[
c(t) = (1-\theta)Q(t) - p(t)E(t) - R(t) - p_s(t)i(t)
\]

is the net profit or cash flow, \( \theta \) is a tax rate on production or sales (which could be also interpreted as an emission tax in the environmental context, see Feichtinger et al., 2005) . We postulate a Leontief vintage capital production function as in Malcomson (1975), Boucekkine, Germain, and Licandro (1997, 1999) or Hritonenko and Yatsenko (1996, 2005). In equation (2), \( a(t) \) measures the vintage index of the oldest machine still in use at
time \( t \), or in other words, \( t-a(t) \) is the scrapping time at date \( t \). The complexity of the optimization problem considered in this paper comes from the fact that \( a \) is a control variable, which is quite unusual in economic theory. We shall come back to this point in detail later. For now, let us notice that we do not assume any output-augmenting (embodied or disembodied) technological progress: whatever the vintage \( \tau \) is, all machines produce one unit of output. In our framework, the technological progress is exclusively resource-saving, which is the key component of the debate around technological progress and environmental sustainability.

In contrast to the related literature (notably to Feichtinger et al., 2005, 2006, 2008), we assume that firms choose the optimal lifetime of their capital goods, and also invest in adoptive and/or innovative R&D. But in comparison to these papers, firms only invest in the new capital goods. In Feichtinger et al. (2006), for example, it might be optimal to invest in older and less efficient vintages because operating the new and more efficient vintages induce a larger learning cost. We have no learning costs in our model, consistently with the traditional framework à la Solow et al. (1966). Let us call \( \beta(t) \) the level of the resource-saving technological progress at date \( t \). We postulate that this level evolves endogenously according to:

\[
\frac{\beta(t)}{\beta(t)} = \frac{f(R(t))}{\beta'(t)}, \quad d > 0,
\]

where \( f \) is increasing and concave: \( df/dR > 0, d^2f/dR^2 < 0 \). Equation (4) deserves a few comments. It stipulates that the rate of resource-saving technical progress is an increasing (and concave) function of the R&D effort and a decreasing function of its level. The latter specification is designed to reflect the negative impact of technological complexity on R&D success. The parameter \( d \) measures the extent to which complexity impacts the rate of technological progress (see Segerstrom, 2000, for example). It will play an important role hereafter, consistently with the available evidence on the role of technological complexity in the adoption of new technologies.

We assume that the resource-saving technological progress is fully embodied in new capital goods, which implies, keeping the Leontief structure outlined above, that total resource consumption is given by

\[
E(t) = \int_{a(t)}^{t} \frac{i(\tau)}{\beta(\tau)} d\tau.
\]
We also consider a possible quota constraint on the resource:

\[ E(t) \leq E_{\text{max}}(t), \]  

where the regulation function \( E_{\text{max}}(t) \) is given. As mentioned in the introduction, we shall start with the “no quota” case where \( E_{\text{max}}(t) = \infty \), then we shall examine how the introduction of environmental quotas alters the structure of optimal (long term) solutions.

The firms are also possibly subject to a second type of constraint, financial constraints, which we do not model here in an extensive way. Feichtinger et al. (2008) have already modelled the problem of a firm subject to financial constraints and operating under a vintage capital technology. This typically involves a new state variable (the stock of debt) and a new control variable (dividends to shareholders). Therefore, introducing properly financial constraints into our already highly sophisticated setting has an unaffordable analytical cost.\(^8\) We therefore resort to the usual simplifying assumption that the firm cannot borrow, which obliges her to keep non-negative cash-flows at every moment, that is: \( c(t) \geq 0 \) for every \( t \). Specifically, and in order to focus on the complex technological mechanisms at work in the model, we will not devote too much space to this financial constraint: it will be not taken into account in the derivation of the maximum principle, it will be checked \textit{a posteriori}. Even more precisely, it will be exclusively used to overrule the balanced growth paths which violate it.\(^9\)

Let us now summarize the optimal control problem to tackle. The unknown functions are:

\begin{itemize}
  \item the investment \( i(t) \), \( i(t) \geq 0 \), into new capital (measured in the capacity units),
  \item the R&D investment \( R(t) \), \( R(t) \geq 0 \), and the technology \( \beta(t) \),
  \item the capital scrapping time \( t-a(t) \), \( a'(t) \geq 0 \), \( a(t) \leq t \),
  \item the output \( Q(t) \), cash-flow \( c(t) \), and resource consumption \( E(t) \), \( t \in [0,\infty) \).
\end{itemize}

The constraints are given by the following conditions:

\[ R(t) \geq 0, \quad i(t) \geq 0, \quad a'(t) \geq 0, \quad a(t) \leq t, \]  

the optional quota constraint (6), and the liquidity constraint \( c(t) \geq 0 \) for every \( t \). The constraint \( a'(t) \geq 0 \) is standard in vintage capital modeling (see for example, Malcomson,\(^8\) Recall that Feichtinger et al. (2005, 2008) have no scrapping and no R&D decisions.\(^9\) Boucekkine et al. (2008) have used numerical examples to illustrate the role of this financial constraint in short term dynamics.)
1975) and implies that scrapped machines cannot be reused by the firm.\textsuperscript{10} We shall also specify the initial conditions as follows:

\begin{equation}
\begin{align*}
a(0) &= a_0 < 0, \quad \beta(a_0) = \beta_0, \quad \iota(\tau) = \iota_0(\tau), \quad R(\tau) = R_0(\tau), \quad \tau \in [a_0, 0].
\end{align*}
\end{equation}

The optimal control problem (1)-(8) has several mathematical peculiarities. We come back to the technical part in the next Section 3 where the necessary optimality conditions are developed. Before this, let us emphasize that in our model, technological improvements affect only the new capital goods. This is crystal clear in equation (5) giving total resource consumption. Of course, this need not be a case in general. Part of resource-saving innovations is probably disembodied, and a more general formulation of the problem taking into account this aspect would, in particular, replace the ODE (4) for $\beta(t)$ by a PDE for $\beta(\tau, t)$. This extension is out of the scope of this paper. Second, one has to mention that the results obtained in this framework will remain qualitatively the same in a general equilibrium set-up with a linear utility function. With nonlinear utility functions, the (already extremely complicated) problem becomes even trickier due to the endogeneity of the interest rate. We, therefore, choose the firm problem setting, which is rather traditional for the questions raised in this paper (see Kamien and Schwartz, 1969).

3. Extremum conditions

Let us derive optimality conditions. For mathematical convenience, we change the unknown (decision) variable $i(t)$ to

\begin{equation}
m(t) = i(t) / \beta(t),
\end{equation}

which is also the investment into new capital (but measured in resource consumption units rather than in capacity units). In the variables $R, m$ and $a$, the optimization problem (OP) (1)-(8) becomes

\begin{equation}
I \equiv \int_{0}^{\infty} e^{-\tau} [(1 - \theta)Q(t) - p(t)E(t) - R(t) - p_k(t)\beta(t)m(t)] dt \rightarrow \max_{R, m, a}
\end{equation}

\begin{equation}
c(t) = (1 - \theta)Q(t) - p(t)E(t) - R(t) - p_k(t)\beta(t)m(t),
\end{equation}

\textsuperscript{10} Introducing the usual \textit{ad-hoc} salvage values for scrapped machines is unlikely to modify our results on the impact of scarcity Vs regulation on long-term growth. We do not consider here secondary markets where the firm can sell the scrapped machines, this is much beyond the scope of our firm problem. See Licandro et al. (2008) for such a model.

\textsuperscript{11} The computations for the optimal growth model with linear utility are available upon request.
\[ Q(t) = \int_{a(t)}^{t} \beta(\tau) m(\tau) d\tau, \quad (12) \]
\[ E(t) = \int_{a(t)}^{t} m(\tau) d\tau, \quad E(t) \leq E_{\text{max}}(t), \quad (13) \]
\[ R(t) \geq 0, \quad m(t) \geq 0, \quad a'(t) \geq 0, \quad a(t) \leq t, \quad (14) \]
\[ a(0) = a_0 < 0, \quad \beta(a_0) = \beta_0, \quad m(\tau) \equiv m_0(\tau), \quad R(\tau) = R_0(\tau), \quad \tau \in [a_0, 0]. \quad (15) \]

The substitution (9) removes \( \beta(t) \) from equation (5) and adds it to the last term in the functional (10). Equation (4) for the unknown \( \beta(t) \) remains the same and has a solution of the form: \[ \beta(\tau) = \left( \frac{1}{d} \int_{0}^{\tau} f(R(v)) dv + \frac{B}{d} \right)^{1/d}, \quad (16) \]
where the constant \( B = \beta(0) = \left( \frac{1}{d} \int_{a_0}^{0} f(R_0(v)) dv + \frac{\beta_0}{d} \right)^{1/d} \) is uniquely determined by the initial conditions (15). From now on, we work with the following explicit specification for the endogenous technological progress:
\[ f(R) = bR^n, \quad 0 < n < 1, \quad b > 0. \quad (17) \]

By (4), this implies that the elasticity of the rate of technological progress with respect to R&D expenditures is constant and equal to \( n \). The larger is \( n \), the bigger is the efficiency of investing in R&D.

The OP (10)-(17) includes seven unknown functions \( R, \beta, m, a, Q, c, \) and \( E \) connected by four equalities (11), (12), (13), and (16). Following Hritonenko and Yatsenko (1996) and Yatsenko (2004), we will choose \( R, m, \) and \( a' \) as the independent decision variables (controls) of the OP and consider the rest of the unknown functions \( \beta, m, a, Q, c, \) and \( E \) as the dependent (state) variables.

The majority of optimization models of mathematical economics are treated using first-order conditions for \textit{interior trajectories only}. In contrast, the nature of the OP (10)-(17) requires taking into account the inequalities \( E(t) \leq E_{\text{max}}(t), \quad R(t) \geq 0, \quad m(t) \geq 0, \quad a'(t) \geq 0, \quad a(t) \leq t, \) on unknown variables in the constraints (13) and (14). These inequalities have an

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\(^{12}\) This integral solution is obtained by solving the differential equation (4) by the technique of separation of variables. Indeed (4) can rewritten as: \( \beta^{d-1} d\beta = f(R(t)) dt \); integrating both sides of this equation between \( o \) and \( \tau \), and rearranging terms, gives the integral solution (16).
essential impact on extremum conditions and optimal dynamics and are treated differently in the below analysis. The inequalities \( R \geq 0 \) and \( m \geq 0 \) are standard constraints on control variables, which are common in the optimization theory. The non-standard constraints \( a'(t) \geq 0 \) and \( a(t) \leq t \) are handled following the technique developed by Hritonenko and Yatsenko in several papers already cited. The optional constraint \( E \leq E_{\text{max}} \) is considered in case B of Theorem 1 below.

Let the given functions \( p, p_k, \) and \( E_{\text{max}} \) be continuously differentiable, and \( m_0 \) and \( R_0 \) be continuous. To keep the OP statement correct, the smoothness of the unknown variables should be consistent. We will assume that \( R, m, \) and \( a' \) are measurable almost everywhere (a.e.) on \([0, \infty)\). Then, the unknown state variables \( a, c, Q, \) and \( E \) in (10)-(16) are a.e. continuous on \([0, \infty)\), as established in Hritonenko and Yatsenko (2006). We also assume a priori that the improper integral in (10) converges. The necessary condition for an extremum (NCE) in the OP (10)-(17) is given by the following statement

**Theorem 1.** Let \( R^*(t), m^*(t), a^*(t), \beta^*(t), Q^*(t), c^*(t), E^*(t), t \in [0, \infty) \), be a solution of the OP (10)-(17).

(A) If constraint (13) is absent or, at least, is not binding, \( E^*(t) < E_{\text{max}}(t) \) at \( t \in \Delta \) then

\[
I_{R'}(t) = 0 \text{ at } R^*(t) > 0, \quad I_{m'}(t) \leq 0 \text{ at } m^*(t) = 0, \quad I_{m'}(t) = 0 \text{ at } m^*(t) > 0, \tag{18}
\]

\[
I_{a'}(t) \leq 0 \text{ at } da^*(t)/dt = 0, \quad I_{a'}(t) = 0 \text{ at } da^*(t)/dt > 0, \quad t \in \Delta,
\]

where

\[
I_{R'}(t) = \int_{a^{-1}(t)}^{a(t)} \beta^{\tau-d}(\tau)m(\tau) \left[ \frac{e^{-r\tau} - e^{-ra^{-1}(\tau)}}{r} (1-\theta) - e^{-r\tau} p_\beta(\tau) \right] d\tau - e^{-rt}, \tag{19}
\]

\[
I_{m'}(t) = \int_{a^{-1}(t)}^{a(t)} e^{-r\tau} [\beta(t)(1-\theta) - p(\tau)] d\tau - e^{-rt} \beta(t) p_\beta(t), \tag{20}
\]

\[
I_{a'}(t) = \int_{a^{-1}(t)}^{a(t)} e^{-r\tau} [p(\tau) - (1-\theta)\beta(a(\tau))] m(a(\tau)) d\tau, \tag{21}
\]

\( a^{-1}(t) \) is the inverse function of \( a(t) \), and
\[ \beta(t) = \left( \frac{d b}{d} \int_{0}^{\tau} R^\gamma(\xi)d\xi + B^d \right)^\frac{1}{d}. \]  

(22)

(B) If \( E^*(t) = E_{\text{max}}(t) \) and \( E_{\text{max}}'(t) \leq 0 \) at \( t \in \Delta \subset [0, \infty) \), then

\[ I_{R'}(t) = 0 \text{ at } R^*(t) > 0, \]

\[ I_{m'}(t) \leq 0 \text{ at } m^*(t) = 0, \quad I_{m'}(t) = 0 \text{ at } m^*(t) > 0, \quad t \in \Delta, \]

(23, 24)

where

\[ I_m(t) = \int_{\lambda(t)} e^{-\tau t} (1 - \theta) \left[ \beta(t) - \beta(a(t)) \right] d\tau - e^{-\tau t} \beta(t) p_k(t), \]

(25)

the state variable \( a(t) \) is determined from (13), \( I_R(t) \) is as in (19), and \( \beta(t) \) is as in (22).

The proof is long and technical and we report all its details in Appendix. The expressions (19), (25), (20), and (21) are the Frechet derivatives of the functional \( I \) in variables \( R, m, \) and \( a' \). The derivative \( I_m'(t) \) has different forms (20) and (25) depending on whether the quota restriction (13) is active or not. Before giving the economic interpretation of the optimality conditions, some technical comments are in order.

In Theorem 1 and henceforth, the first case (A) is for the OP with no quota regulation while the second case (B) reflects the active quota restriction. If (13) is active (Case B), then the state variable \( a \) is determined from \( m(a(t))a'(t) = m(t) - E_{\text{max}}'(t) \) and the state restriction \( a' \geq 0 \) on the variable \( a \) in (14) is satisfied if \( E_{\text{max}}'(t) \leq 0, t \in [0, \infty) \). If the condition \( E_{\text{max}}'(t) \leq 0 \) fails for some \( t \in \Delta \subset [0, \infty) \), then Theorem 1 is still valid in Case A if we replace the differential constraint \( a'(t) \geq 0 \) in (14) with the stricter constraint \( m(t) \geq \max\{0, E_{\text{max}}'(t)\} \) on the control \( m \) (see Hritonenko and Yatsenko, 1996, 2006). Notice also that the corner case \( R^*(t) = 0 \) is impossible with our specification (17) of the function \( f(R) \). Finally, it is fair to acknowledge that the dimensionality of the problem (with three controls and three state equations), its strong nonlinearities (via the R&D equation (4)) and inequality constraints, have prevented us from extracting interpretable second-order conditions. The computation of the second variations of functional \( I \), mechanically replicating for example Hritonenko and Yatsenko (2005), pages 119-120, is of course never an issue. However, we have found no way to exploit the very long and very complicated formulas obtained to extract meaningful second-order conditions, and we have failed to identify special cases where the general formulas degenerate into interesting second-order conditions. We have therefore decided to
Economic Interpretation. Let us move now to some economic interpretations of the obtained first-order optimality conditions. In order to compare more easily with the existing literature, we start with Case (B), that is, when the quota constraint is binding. Indeed, in this case, the latter can be broadly viewed as an “equilibrium” condition in the resource market, where the quota plays the role of supply. Let us interpret the optimality conditions with respect to investment and R&D, the case of scrapping being trivially fixed by Remark 1 above. Using equation (25), the (interior) optimal investment rule may be rewritten as:

\[
(1-\theta) \int_t^{a(t)} e^{-r\tau} \left[ 1 - \frac{\beta(a(\tau))}{\beta(t)} \right] d\tau = e^{-r} p_k(t) .
\]

The interpretation of such a rule is quite natural having in mind the early vintage capital literature (notably, Solow et al., 1966, and Malcomson, 1975) as exploited in Boucekkine et al., 1997). In our model, one unit of capital at date \( t \) costs \( p_k(t) \) or \( e^{-r} p_k(t) \) in the present value. This is the right-hand side of the optimal rule above. The left-hand side should, therefore, give us the marginal benefit from investing. Effectively, it is the integral of discounted gains from investing over the lifetime of a machine bought at \( t \) (since \( a(t) \) is by construction the lifetime of such a machine). At any date between \( t \) and \( a(t) \), a machine bought at \( t \) will provide one unit of output but the firm has to pay the corresponding energy expenditures \( \frac{\beta(a(\tau))}{\beta(t)} \). Given our Leontief specifications, \( \frac{1}{\beta(t)} \) is the resource requirement of any machine bought at date \( t \). Therefore, \( \beta(a(\tau)) \) plays the role of the effective price of the input paid by the firm. How could this be rationalized? Simply by noticing that under a binding quota, the latter mimics a clearing market condition as in the early vintage macroeconomic literature (see for example, Solow et al., 1966).¹³ In such a framework, the marginal productivities of energy should be equalized across vintages, implying a tight connection between the effective price of resource and the resource requirement of the oldest machine still operated. More precisely, the latter price, which happens to be the Lagrange multiplier associated to the binding environmental constraint, is equal to the

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¹³ In Solow et al., the role of resource is played by labor.
inverse of the resource requirement of the oldest machine still in use, which is equal to 
\( \beta(a(\tau)) \) at any date \( \tau \) comprised between \( t \) and \( a^{-1}(t) \). Notice that in such a case, the 
effective price of resource \( \beta(a(\tau)) \) is not generally equal to \( p(t) \). The latter does not play 
any role since the total resource cost becomes predetermined equal to \( p(t)E_{\max}(t) \) in the 
constrained regime.

Things are quite different in the case where the quota is not binding (Case A of Theorem 
1). Then, the optimal interior investment rule becomes (following equation (20)):

\[
\int_{t}^{a^{-1}(t)} e^{-\tau t} \left[ 1 - \frac{p(\tau)}{(1-\theta)\beta(t)} \right] d\tau = e^{-\alpha} p_k(t),
\]

and \((1-\theta)\beta(a(t)) = p(t)\) by (21) as in the firm problem studied by Malcomson (1975), 
making a clear difference with respect to the restricted Case A. Our framework thus 
extends significantly the benchmark theory to allow for situations in which resource input 
markets do not necessarily clear due to institutional constraints.

Let us interpret now the R&D optimal rule, which is also new in the literature. Using (19), 
it is given by

\[
bnR_{n-1}(t) \int_{t}^{\infty} \beta^{1-d}(\tau)m(\tau) \left[ e^{-\tau t} - e^{-m^{-1}(\tau)} \right] \frac{(1-\theta)}{r} \left[ (1-\theta) - e^{-\tau t} p_k(\tau) \right] d\tau = e^{-\alpha}.
\]

The right-hand side is simply the present value of marginal investment in R&D. The 
marginal benefit is given by the left-hand side. Contrary to the optimal investment rule, the 
gains from doing R&D last forever: the R&D investment induces a knowledge 
accumulation process, which is not subjected to obsolescence in our case, in contrast to 
capital goods. The integrand can be understood if one has in mind the maximized function 
(10) in the form

\[
I = \int_{0}^{\infty} e^{-\tau t} [(1-\theta) \int_{a(t)}^{t} \beta(\tau)m(\tau)d\tau - p(t) \int_{a(t)}^{t} m(\tau)d\tau - R(t) - p_{k}(t)\beta(t)m(t)]dt
\]

and the given endogenous law (16),(17) of the motion of technological progress. It should 
be noticed that rewriting the problem in terms of \( m(t) \), rather than in terms of investment in 
physical units \( i(t) \), does not mean rewriting a problem with input-saving technical progress 
as a problem with output-augmenting technical progress. As one can see, at the fixed \( m(t) \), 
an increase of \( R(t) \) (and, therefore, \( \beta(t) \)) raises not only the output \( Q(t) \) but also the 
investment expenditures through the term \( p_{k}(t)\beta(t)m(t) \). The left-hand side of the optimal
R&D rule takes precisely into account this trade-off. On one hand, the marginal increase in $\beta(\tau), \tau \geq t$, following the marginal rise in $R(t)$, that is, $\frac{bnR^{\tau-1}(t)}{\beta^\tau(\tau)}$, impacts positively the output by improving the efficiency of all vintages after the date $t$. Let us notice that, since machines have a finite lifetime, this effect should be computed between $\tau$ and $a^{-1}(\tau)$ for each vintage $\tau$, which explains the factor $e^{-r\tau} - e^{-r a^{-1}(\tau)} = \int_{\tau}^{a^{-1}(\tau)} e^{-rs} ds$ in the integrand. On the other hand, the rising $\beta(t)$ increases investment expenditures (for a fixed $m(t)$), which explains the negative term, $e^{-r\tau} p_s(\tau)$, in the integrand.

Let us now move to the study of the system of the optimality conditions extracted above. We start by seeking for exponential solutions (for naturally growing variables like $R(t)$), so-called balanced growth paths, in order to address in a standard way the critical issue of sustainable growth under constraint, which is one of the main questions asked in this paper. We then interpret the obtained long-term dynamics as principal modernization routes.

### 4. Analysis of optimal long–term dynamics.

For the sake of clarity, we assume that

$$p_k(t) = p_k = \text{const} > 0, \quad E_{\text{max}}(t) = \bar{E} = \text{const} > 0, \quad p(t) = \frac{\bar{P} e^\gamma}{0 \leq \gamma < \rho}.$$  

(26)

In particular, the case $\bar{E} = \infty$ describes the OP without environmental quota regulation.

From now on, we also assume that $p_k < \frac{1-\theta}{r}$, which limits the value of the price of capital. Otherwise, no positive long-term growth can be generated because the new capital is too expensive compared to the economic value it can produce (formally, expressions (20) and (25) for the derivative $I_m(t)$ are always negative, so the optimal $m^* \equiv 0$)

Alternative trajectories for the exogenous variables $p_k(t)$ and $E_{\text{max}}(t)$ were studied in Boucekkine, Hritonenko, and Yatsenko (2008). By Theorem 1, the optimal long–term dynamics of the OP can involve interior regimes of, at least, following two types:

(A) The quota $E(t) < E_{\text{max}}(t)$ is not binding or absent at all, then the system to be solved is

14 Malcomson (1975) has the same kind of condition.
\[ I_R'(t) = 0, \quad I_m'(t) = 0, \quad I_a'(t) = 0, \quad t \in [t_l, \infty), \]  

(27)

where \( I_R'(t), I_m'(t) \) and \( I_a'(t) \) are determined by (19)-(21).

(B) The quota is active in the long run: \( E(t) = E_{\text{max}}(t) \) at \( t \in [t_l, \infty) \), \( t \geq 0 \). The corresponding long-term interior regime is determined by the system of three nonlinear integral equations

\[ I_R'(t) = 0, \quad I_m'(t) = 0, \]  

\[ \int_{a(t)}^{t} m(\tau) d\tau = E_{\text{max}}, \quad t \in [t_l, \infty), \]  

(28)

where \( I_R'(t) \) and \( I_m'(t) \) are determined by (19) and (25).

In Case (A) of the inactive quota, it is convenient to introduce the Frechet derivative

\[ I_a'(t) = e^{-rt} [\Phi e^x - (1-\theta)\beta(a(t))]m(a(t)) \]  

(30)

in \( a \) instead of the derivative (21) in \( a' \) and use it during the BGP analysis. Indeed, it is easy to see that if \( I_a'(t) \equiv 0 \) at \( t \in [t_l, \infty) \) for some \( t \geq 0 \), then \( I_a'(t) \equiv 0 \) at \( t \in [t_l, \infty) \).

We will explore the possibility of exponential solutions for \( R(t) \), while \( m(t) \) and \( t-a(t) = L \) are constant, to the systems (28)-(29) and (27). For this, we need the following preliminary result: if \( R(t) \) is exponential, then \( \beta(t) \) is almost exponential and practically undistinguishable from an exponent at large \( t \) in the sense of the following lemma:

**Lemma 1.** If \( R(t) = R_0 e^{gt} \) for some \( g > 0 \), then

\[ \beta(t) = R_0 \frac{b d}{g n} \left( e^{g t / d} \right)^{1/d} \]  

(31)

at large \( t \). In particular,

\[ \beta(t) = R_0 \frac{b d}{g n} \left( g n \right)^{1/d} \left( e^{g t / d} \right)^{1/d} \]  

if \( b d R_0^n = gnB^d \).

**Proof.** At \( R(t) = R_0 e^{gt} \), (16) becomes

\[ \beta(t) = \left( d \left[ b R_0^n e^{-\xi t} dv + B^d \right] \right)^{1/d} = \left( \frac{b d R_0^n}{g n} e^{C t} - \frac{b d R_0^n}{g n} + B^d \right)^{1/d}. \]

Dividing \( \beta(t) \) by \( \tilde{\beta}(t) = R_0 \left( \frac{b d}{g n} \right)^{1/d} e^{g t / d} \), we obtain

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15 For brevity, we will omit the expression “at large \( t \)” when using the notation \( f(t) = g(t) \).
\[
\beta(t) = \frac{1}{(bd/\lambda)^{1/d} e^{\alpha/d}} \left( \frac{bd}{gn} e^{\beta R t} + \frac{B^d}{R_0^n} - \frac{bd}{gn} \right)^{1/d} = \left[ 1 + \left( \frac{gnB^d}{bdR_0^n} - 1 \right) e^{-\gamma_{out}} \right]^{1/d}
\]

Expanding the function \((1+x)^e\) in (33) into the series, we obtain \(\frac{\beta(t)}{\beta_0} = 1 + \epsilon(t)\), where the small parameter \(\epsilon(t) = \frac{1}{d} \left( \frac{gnB^d}{bdR_0^n} - 1 \right) e^{-\gamma_{out}} + \frac{1}{2d} \left( \frac{1}{d} - 1 \right) \left( \frac{gnB^d}{bdR_0^n} - 1 \right)^2 e^{-2\gamma_{out}} + ...\) decreases as \(e^{-\gamma_{out}}\). The lemma is proved. \(\square\)

Now we can formally define the considered concept of balanced growth paths.

**Definition 1.** The Balanced Growth Path (BGP) is a solution \((R, m, a)\) to the nonlinear equations (27) or (28)-(29), where \(R(t)\) is exponential and \(m(t)\) and \(t-a(t)\) are positive constants, which satisfy constraints (14) and \(c(t) \geq 0\).

As shown below, the optimal long-term growth is possible only in the cases \(n=d\) and \(n>d\).

**4.1. Balanced growth in case \(n=d\).**

Let the parameter \(n\) of “R&D efficiency”, \(0<n<1\), be equal to the parameter \(d\) of “R&D complexity”, \(0<d<1\). In this case, the optimal long-term growth can involve the active regulation \(E(t) = E_{max}\) at natural conditions. In order to study in a clean and transparent way the specific role of scarcity and regulation in the existence of balanced growth paths, we shall proceed in three steps:

i) In the first step, we remove the quota regulation, and only examine the impact of scarcity on optimal decisions, that is, we isolate the impact of scarcity.

ii) In the second step, we keep the resource price constant, or, in other words, we remove scarcity. In exchange, we introduce the quota regulation and examine whether the firms can still experience a sustainable growth in such a case.

iii) In the last step, we study the interplay of both ingredients, the scarcity and quota regulation.
In the first step, we shall uncover the possible induced innovation mechanism at work in the model. In the second step, we investigate Porter-like mechanisms. The last step is devoted to the analysis of the interplay of both mechanisms.

**The “no quota” case**

In this case, the quota constraint is not active, and the analysis should be based on system (27). The proposition below delivers the characteristics of the balanced growth paths arising in this situation, we shall denote them BGP$^S$, the superscript S stands for “scarcity” since the growth is driven here by scarcity.

**Theorem 2.** Let $n=d$ and (26) hold with $E = \infty$ (no quota). At $0 < \gamma < r$ (resource scarcity), a unique interior optimal regime BGP$^S$ exists,

$$R^S(t) = R^S e^{\gamma t}, \quad Q^S(t), \beta^S(t), c^S(t) \sim e^{\gamma t}, \quad m^S(t) = \overline{M}^S = \text{const}, \quad t-a^S(t) = L^S = \text{const},$$

(34)

where $L^S > 0$ is uniquely determined by the nonlinear equation

$$\frac{1 - e^{-\gamma(t-a)} - e^{-\gamma L^S}}{r - \gamma} = \frac{p_1}{1 - \theta},$$

(35)

and

$$\overline{M}^S = r \left( \frac{\gamma}{b} \right)^{1/d} \frac{r - \gamma(1-d)}{\theta L} \left[ (1 - e^{-\gamma L^S})(1 - \theta) - p_1 r \right]^{1/d},$$

(36)

$$R^S = \frac{\overline{R}}{1 - \theta} \left( \frac{\gamma}{b} \right)^{1/d} e^{\gamma t}.$$  

(37)

The proof is in Appendix.

If no quota regulation is implemented, the resource scarcity pushes the firm to invest in less resource-consuming technologies (induced innovation). Two striking features of the extracted balanced growth paths should be initially commented here. First of all, the growth rate of BGP$^S$ is exactly equal to the exogenous growth rate $\gamma$ of the resource price: though technological progress is endogenous through R&D and/or adoption purposive decisions, the growth is exogenous and vanishes if $\gamma = 0$. Second, and related to the first point, BGP$^S$ is fully determined in levels, in the sense that all the long-term levels, notably $\overline{M}^S$ and $\overline{R}^S$, can be identified by (36)-(37) once exogenous prices are given and the lifetime $L^S$ is
derived from (35). This is a fundamental characteristic of exogenous growth models, and we will see later that the Porter-driven BGP does not have this property.

Indeed, the BGP driven by scarcity, derived just below, is somewhat classical: despite the endogenous innovation side incorporated, it shares most of the characteristics of the existing vintage capital growth models with exogenous embodied technical progress. In the latter, the long-term lifetime of machines is usually derived from equations similar to (35), and the comparative statics are therefore similar in many dimensions (see for example Boucekkine et al., 1998). In particular, the optimal lifetime generally decreases with the rate of resource-saving technical progress, here equal to the exogenous growth rate of the resource price. This said, BGP$_S$ has some specific features, which cannot be found in the class of vintage models mentioned just above. These features come, of course, from the original endogenous innovation side of the model. The following corollary gives an accurate idea of the comparative statics along BGP$_S$.

**Corollary 1. If the resource quota is absent (or not active along the BGP), then an increase of the resource price rate $\gamma$ raises the BGP$_S$ rate and R&D investment and decreases the long-term lifetime of capital goods. The comparative statics are not straightforward for other variables. In particular, if $d>0.5$, the energy consumption $E^S = \bar{M}^S L^S$ decreases in $\gamma$ and $\bar{M}^S L^S \to \infty$ as $\gamma \to 0$, while $\bar{M}^S L^S$ increases in $\gamma$ and $\bar{M}^S L^S \to 0$ as $\gamma \to 0$ when $d<0.5$.

The proof of the crucial last claim is in Appendix.

The ambiguity in the response of energy consumption to an increase in $\gamma$ relies on the ambiguity in the response of investment (in level), the machine’s lifetime going unambiguously downward. The interpretation is not trivial because we are working under the assumption $n=d$ (with the subsequent substitutions), so that $d$ takes also the role of $n$, which measures the R&D efficiency (rather than R&D complexity as the original $d$). This said, one can deduce from the previous corollary that all in all, the case $d>0.5$ delivers the normal parametric case, while the case $d<0.5$ is rather paradoxical: in the latter, as the resource becomes scarcer, energy consumption increases! We shall rely on the normal case in the following. Now we move to the Porter-driven growth paths.
**The “no scarcity” case**

We now fix $\gamma = 0$, and introduce a quota on the resource use. The next theorem gives a BGP existence result.

**Theorem 3.** Let (26) hold at a given $0 < \bar{E} < \infty$ (resource quota). Then:

i) A balanced growth path $BGP^R$ is characterized by

$$R^R(t) = \bar{R}^R e^{gt}, \quad \bar{Q}^R(t), \bar{B}^R(t), \bar{C}^R(t) \sim e^{gt}, \quad \bar{m}^R(t) = \bar{M}^R = \text{const}, \quad a^R(t) = t - \bar{E} / \bar{M}^R,$$

with the active quota $E^R(t) = \bar{E}$, where the positive constants $g$ and $\bar{M}^R$ are found from the nonlinear system

$$g^{1/d} [r/l + d - 1] = db^{1/d} \left[ \frac{1 - e^{-E/M}}{r} - \frac{p_k}{1 - \theta} \right] (1 - \theta), \quad (39)$$

$$\frac{1 - e^{-E/M}}{r} - \frac{e^{-E/M} - e^{-E/M}}{r - g} = \frac{p_k}{1 - \theta}, \quad (40)$$

ii) In the normal case, that is when $d > 0.5$, the system (39)-(40) has a positive solution such that $g \to 0$ and the corresponding $L^R = t - d^R(t) \to \infty$ as $\bar{E} \to \infty$.

iii) If $r \bar{E} < 1$ and

$$r^{1/d} < \bar{E} b^{1/d} [1 - 2p_k r/(1 - \theta)], \quad (41)$$

then the solution $(g, \bar{M}^R)$, $0 < g < r$, of the nonlinear system (39)-(40) exists and is determined by the nonlinear equation

$$g^{(1+d)/d} [r - g(1 - d)] = db^{1/d} \left[ \frac{p_k}{\sqrt{2(1 - \theta)}) \left( \frac{r}{\sqrt{g}} + \sqrt{g} \right) \right] + o(r), \quad (42)$$

and $\bar{M}^R = \bar{E} \sqrt{g / 2p_k (1 - \theta)} + o(r)$. The uniqueness of $g$ is guaranteed if

$$r^{1/d - 1/2} < \frac{b^{1/d} \bar{E} d^2}{4(1 - d)} \sqrt{2p_k / 1 - \theta}. \quad (43)$$

The proof is in Appendix.

The theorem is long and deserves some comments. The simultaneous system (39)-(40) characterizes $BGP^R$, where the superscript “R” stands for regulation. The single equation (42) for $g$ is obtained by means of approximations of (39)-(40) under the assumption that $r \bar{E}$ is small enough. This covers, in particular, the case where the interest rate $r$ is small.
enough and \( \bar{E} < < 1/r \), which determines a quite large range of admissible values for the quota. The existence and uniqueness conditions (41) and (43) for \( BGP^R \) also hold under \( r \bar{E} < < 1 \). Notice that (41) and (43) trivially hold for the R&D productivity parameter \( b \) large enough, which is a typical requirement in R&D-based endogenous growth models. Also notice that these conditions are violated when the quota \( \bar{E} \) goes to zero: in this limit case, the firm does not produce at all!

Now, it is crucial to mention that \( BGP^R \) is truly induced by regulation and, therefore, is a perfect illustration of a Porter-like mechanism. This is clearly established in Theorem 3, ii): in the normal case, removing the quota regulation leads to zero long-run growth, and \( BGP^R \) does indeed vanish. The quota regulation makes an intensive and permanent R&D activity optimal, which in turn leaves room for a BGP to arise. Also, importantly enough, the Porter-induced long term growth is endogenous in contrast to the scarcity case studied above. The growth rate \( g \) is indeed a complex implicit function of many parameters of the model (the comparative statics are given later). Actually, the picture is even more complicated. Namely, equation (42) has another solution \( g_2, r < g_2 < r/(1-d) \). However, the larger solution \( g_2 \) has no sense, since at \( g > r \) the value of (1) is infinite. Finally, the endogenous nature of growth in our Porter-like experiment shows that the \( BGP^R \) scale parameter \( \bar{R} \) is actually undetermined. Such indeterminacy is characteristic in endogenous growth. A typical example is the Lucas-Uzawa model (Boucekkine and Ruiz-Tamarit, 2008).

It is now the time to examine the comparative statics corresponding to \( BGP^R \).

**Corollary 2.** Along the \( BGP^R \) (38)-(40), at \( \bar{E} < < 1/r \) and \( r < < 1 \), a decrease of \( \bar{E} \) leads to the decrease of both optimal parameters \( g \) and \( \bar{M}^R \), and leaves the long-term lifetime of capital goods unaltered since \( \bar{M}^R \sim \bar{E} \). A decrease of capital price \( k \) and/or of the tax \( \theta \) increases the optimal \( g \) and \( \bar{M}^R \) and diminishes the long-term lifetime of capital goods.

While \( BGP^R \) is undoubtedly driven by quota regulation, a more stringent regulation through a decrease in \( \bar{E} \) is bad for the growth rate of firms’ output and profit. Even though the firms can respond to tighter quotas by more innovations, such an instrument does not allow completely circumventing the impact of more severe regulation. So, our model naturally
yields a kind of soft Porter-like outcome, not a strong form where a more stringent regulation is associated with a larger growth. Our model clearly displays a non-monotonic relationship between the growth and quota regulation along $\text{BGP}^R$: the growth rate goes to zero either when $\bar{E}$ goes to zero (extreme regulation) or when it goes to infinity (no regulation). Therefore, there must exist an optimal level of regulation, that is a (long term) profit-maximizing quota level $\bar{E}$.

The corollary has further interesting results. Lower capital prices are good for investment (in resource consumptions units) and also prove to be beneficial for the growth rate of firms’ output and profit. Lower equipment prices make firms wealthier and such a positive wealth effect boosts the investment either in capital or in R&D. For the same reasons, a lower tax rate $\theta$ raises the optimal growth rate $g$, and stimulates the two latter forms of investment. Another interesting result concerns the optimal long-term lifetime of capital goods. Since $d^R(t) = -\bar{M}^R/\bar{E}$, and $\bar{M}^R \sim \bar{E}$, it follows that a tighter environmental regulation leaves the optimal lifetime $L^R$ of capital goods unaltered. While a lower $\bar{E}$ does reduce the optimal lifetime of machines, such a tighter regulation also pushes the investment downward, which forces the maximizing firm to use fewer machines longer. These two effects appear to offset each other in our framework. Under decreasing prices for capital goods (or a falling tax rate), the firm invests more and uses the machines for a shorter time. This is somehow consistent with the recent literature on embodied technological progress observing that a more rapid investment-specific technological progress (like the one conveyed by the information and communication technologies) reduces the relative price of capital goods and decreases their lifetime due to rising obsolescence costs (see, for example, Krusell, 1998).

**Existence of $\text{BGP}^S$ and $\text{BGP}^R$ for any $\gamma$ and $\bar{E}$**

We now consider the case where both scarcity and quota are present, which mimics the case of the Japanese industry outlined in the introduction. Recall that environmental regulation (quotas on mercury released to the seas) led the industry to adopt a cleaner US technology but then this adoption was drastically slowed down during the first oil shock because it turned out to be too expensive in terms of energy consumption. What could our theory tell us in such a case? To answer this question, we study the outcomes of the simultaneous presence of resource scarcity (that is, $\gamma > 0$) and environmental quota ($\bar{E} > 0$).
Two remarks should be made at this stage to easily get the results. First of all, and by construction, the characteristics of $BGP^R$ are independent of $\gamma$: because $BGP^R$ is constructed with the quota constraint binding, the optimal interior solutions of the firm are independent of the price of the resource, and therefore of $\gamma$. However, the latter price still matters in the level of the cash-flows, $c(t)$, which are required to be positive. Without scarcity, that is when $\gamma=0$, this is not an issue, but as soon as $\gamma>0$, the $BGP^R$ depicted in Theorem 3 may cease to exist. Second, the scarcity-based long-term growth paths, that is, $BGP^S$, can be considered even at $\bar{E}<\infty$, provided that the energy consumption (denoted $E^S$ in Theorem 2) is strictly lower than the quota, that is, $E^S<\bar{E}$. Symmetrically to the first remark made just above, here $BGP^S$ may also cease to exist for certain values of the quota. In fact, these two remarks represent the final analytical result of this paper settling the case where both scarcity and quota are present.

**Theorem 4** Let $\gamma<r<1$ and $d>0.5$ (normal case), and consider the growth rate $g$ found in Theorem 3. Then, if $g>\gamma$, the $BGP^S$ (34)-(37) with the growth rate $\gamma$ is not possible because its associated energy consumption $E^S>\bar{E}$. If the calculated $g<\gamma$, then the $BGP^S$ (34)-(37) with the growth rate $\gamma$ exists, but the $BGP^R$ does not exist because it violates the condition $c(t)\geq 0$.

The proof is in Appendix.

Theorem 4 brings interesting results on the interplay of the scarcity and regulation. Without scarcity ($\gamma=0$), a sustainable regime with a positive growth rate $g$ is possible due to the presence of the input quota regulation (Theorem 3). And such a property actually holds up to a certain level of the resource price growth rate $\gamma>0$ (or of the scarcity degree). Even with the scarcity ($\gamma>0$), the quota regulation can lead to a $BGP^R$ (38) with a higher rate $g$ than the resource price rate $\gamma$. Here the Porter mechanism works in such a strong way that the long-term growth rate is larger than the scarcity degree. However when the resource becomes increasingly scarcer, $BGP^R$ ceases to exist: this happens exactly when $g$ goes below $\gamma$. In other words, increasing scarcity kills the Porter mechanism, which is one of the main outcomes of our study. Coming back to our Japanese example and under the necessary provisos (since we are using analytical results on long-term optimal paths), we

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16 Again notice that by construction, the $BRP^R$ growth rate $g$ does not depend on the energy price.
can explain why a technological path which is considered optimal for given resource prices can turn to sub-optimal when the resource prices experience a sharp increase.

Also, Theorem 4 addresses the issue of the BGP uniqueness. In the normal case, that is when $d > 0.5$, and under small $\gamma < 1$, only one BGP (BGP$^R$ or BGP$^S$) exists. The economic reasoning behind this is the following. The strictness of the quota regulation is reflected by the value $\overline{E}$. If the calculated value $g$ is such that $g < \gamma$ then the BGP$^S$ with the growth rate $\gamma$ exists only if its associated energy consumption $E^S$ satisfies $E^S \leq \overline{E}$. So, for small enough values of $\overline{E}$, the BGP$^S$ does not exist and the only possibility is the Porter-like BGP$^R$. On the other hand, the BGP$^R$ exists only if its rate $g$ is higher than the resource price rate $\gamma$. Otherwise, the only possibility is the above ‘scarcity-generated’ BGP$^S$.

4.2. Cases $n < d$ and $n > d$.

In these cases, no BGP in the sense of Definition 1 exists. However, a long-term regime with exponentially growing $R$ and decreasing $m$ appears to be possible at $n > d$ and $\gamma > 0$ (see also Yatsenko, Boucekkine and Hritonenko 2009, for other related dynamics).

**Theorem 5.** Let (26) hold. Then:

(a) If $n < d$, then no interior optimal regime with an exponentially growing $R$ exists.

(b) If $n > d$ and $0 < \gamma < r$, then an interior optimal regime $(R^*_A, m^*_A, a^*_A)$ exists such that $E(t) < E_{\text{max}}$ and $R^*_A(t) \sim e^{\gamma t}$, $m^*_A(t) \rightarrow 0$, $a^*_A(t) \rightarrow t$ as $t \rightarrow \infty$.

The proof is provided in Boucekkine, Hritonenko, and Yatsenko (2008). When $n > d$, the efficiency of the R&D investment appears to be higher as compared with the investment into the new capital. Theorem 5 concludes that, in the optimal long-time regime, almost all the output goes to R&D investment and the part of capital investments (exponentially) decreases in the total distribution of the output. Also, the *quota constraint needs not to be binding* and we can keep a larger amount of older assets (since we buy an increasingly smaller amount of new capital).

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17 Because of the complexity of the obtained expressions, we cannot analytically rule out the possibility of an interval of $\overline{E}$, where no BGP exist or both BGPs are present (for example when $d < 0.5$).
However, even in this high efficiency case $n > d$, the prices still matter. By (19), the restriction $p_k < (1 - \theta)/r$ on the given capital price is necessary for the existence of any positive optimal regime. The resource price $P e^\gamma$ plays a decisive role in the case $n > d$, in particular, an interior optimal path with an exponential $R_A$ is impossible if $\gamma = 0$ (no resource scarcity, the resource price does not increase). Only when the resource price increases at a rate $\gamma > 0$, an interior regime with exponentially increasing $R_A$ and decreasing $m_A$ is possible. The increase of $P e^\gamma$ raises $a_A(t)$, that is, decreases the lifetime of capital goods. In other words, a kind of induced-innovation mechanism is active in the case $n > d$, that is, when the R&D activity is highly efficient, so efficient that the investment into equipment goes to zero. In such a case, the firm is in perpetual sharp modernization, and is not suffering at all from any regulation. This regime is not a BGP in the sense of Definition 1, because $m_A(t)$ asymptotically tends to zero.

5. Concluding remarks

In this paper, we have studied in depth the optimal behavior of a firm subject to environmental-based quotas and resource scarcity. Within a realistic framework, we have characterized the inducement mechanisms at work. In particular, we have finely characterized long-term growth regimes driven by scarcity vs long-term growth regime driven by quota regulation. More importantly, we have studied in depth the interaction between scarcity and quota regulation. In particular, we have shown that there exists a threshold level for the growth rate of the resource price above which the Porter mechanism is killed. Symmetrically, we have also found that there must exist a threshold value for the environmental quota under which the growth regime induced by scarcity vanishes while the Porter-like growth regime may survive.

A few remarks are in order. Of course, our results are based on price-taking firms and our modeling of liquidity-constraints is probably too simple. Adding market power is no problem, although it is not likely that our results would be dramatically altered. Modelling and treating the liquidity constraints more accurately is a much more complicated task, both mathematically and conceptually. We believe that allowing firms to incur into debt to fasten its modernization and compliance to legal standards is a quite decisive issue that should be considered in more comprehensive research. This is our next step.
6. Appendix

Proof of Theorem 1: The proof is based on general perturbation techniques of the optimization theory. An analogous approach has been earlier used in Hritonenko and Yatsenko (1996, 2005, 2006) and Yatsenko (2004) for vintage models with exogenous technological change.

Case (A). If the restriction (13) is inactive, \( E^*(t) < E_{\text{max}}(t) \) at \( t \in \Delta \), then we choose \( R, m, \) and \( v = a' \) as the independent unknown variables of the OP. Then, the differential restriction \( a'(t) \geq 0 \) in (14) takes the standard form \( v(t) \geq 0 \). We assume that \( R, m, \) and \( v \) are measurable and \( R(t)e^{rt}, m(t)e^{rt}, v(t)e^{rt} \) are bounded \( a.e. \) on \([0, \infty)\). Substituting (17) to (16), we obtain expression (22) for \( \beta(t) \).

We refer to measurable functions \( \delta R, \delta m, \) and \( \delta v \) as admissible variations, if \( R, m, v, R+\delta R, m+\delta m, \)

and \( v+\delta v, \) satisfy constraints (14)-(15). Let us give small admissible variations \( \delta R(t), \delta m(t), \) and \( \delta v(t), \) \( t \in (0, \infty), \) to \( a, m, \) and \( R \) and find the corresponding variation \( \delta I = I(R + \delta R, m + \delta m, v + \delta v) - I(R, m, v) \) of the objective functional \( I \). Using (10)-(13), we obtain that

\[
\delta I = \int_0^\infty e^{-rt} \left[ \int_0^t \left\{ dB \left[ (R(\xi) + \delta R(\xi))^n d\xi + B^d \right] \right\}^\frac{1}{2} \left( m(\tau) + \delta m(\tau) \right) d\tau \\
- \int_0^t \left( m(\tau) + \delta m(\tau) \right) d\tau - (R(t) + \delta R(t)) \right] d\tau \\
- k(t)(m(t) + \delta m(t)) \left[ \int_0^t \left( \left( R(\xi) + \delta R(\xi) \right)^n d\xi + B^d \right) \right] \frac{1}{2} dt \\
- \int_0^\infty e^{-rt} \left[ \int_0^t \left( dB \left[ R^n(\xi) d\xi + B^d \right] \right)^\frac{1}{2} m(\tau) d\tau - p(t) \right] \left[ m(\tau) d\tau - R(t) \right] dt \\
- p(t)m(t) \left[ \int_0^t R^n(\xi) d\xi + B^d \right] \frac{1}{2} dt \\
\delta a(t) = \int_0^t \delta v(\xi) d\xi. \]

To prove the theorem, we shall transform the expression (A1) to the form

\[
\delta I = \int_0^\infty \left( \frac{\partial I_R(\tau)}{\partial R} \cdot \delta R(\tau) + \frac{\partial I_m(\tau)}{\partial m} \cdot \delta m(\tau) + \frac{\partial I_v(\tau)}{\partial v} \cdot \delta v(\tau) \right) dt + o(\|\delta R\|,\|\delta m\|,\|\delta v\|). \]  

(A2)
where the norm is \( \| f \| = \text{ess sup}_{[0,\infty)} | e^{-\tau} f(t) | \). This transformation involves several steps. First, using the Taylor expansion \( f(x + \delta x) = f(x) + f'(x) \delta x + o(\delta x) \) twice, we have that

\[
\left( db \int_0^T (R(\xi) + \delta R(\xi))^n d\xi + B^d \right)^{1/2}
\]

\[
\left( db \int_0^T (R^*(\xi) + nR^{n-1}(\xi)\delta R(\xi) + o(\delta R(\xi)))^n d\xi + B^d \right)^{1/2}
\]

\[
= \beta(t) + bn\beta^{1-d}(t) \int_0^T R^{n-1}(\xi)\delta R(\xi) d\xi + \int_0^T o(\delta R(\xi)) d\xi.
\]

Next, using (A3) and the elementary property \( \int_t a(t) d\tau \), we transform (A1) to

\[
\delta t = \int_0^T e^{-\tau} \left[ \int_a(t) m(t) \beta^{1-d}(t) \int \frac{R^{n-1}(\xi)\delta R(\xi) d\xi d\tau}{\max\{a(t),0\}} + \int_0^T (\beta(t) - p(t)) \delta m(t) d\tau \right]
\]

\[
+ \int_0^T e^{-\tau} \left[ (p(t) - \beta(t)) m(t) d\tau + \int_0^T [\delta R(t) + p_\delta(t) \beta(t) \delta m(t)] dt \right]
\]

\[
- \int_0^T e^{-\tau} p_\delta(t) m(t) \beta^{1-d}(t) \left[ bnR^{n-1}(\xi)\delta R(\xi) d\xi + \int_0^T o(\delta R(\xi)) d\xi \right],
\]

where \( \max\{a(t),0\} \) appears because the variations \( \delta R(t), \delta m(t) \) are zero on the interval \( [a_0,0] \).

Next, we interchange the limits of integration in the second term of (A4) as

\[
\int_0^T e^{-\tau} \left[ \int_a(t) (\beta(t) - p(t)) \delta m(t) d\tau \right]
\]

in the first term as

\[
\int_0^T e^{-\tau} \left[ \int \frac{m(t) \beta^{1-d}(t) R^{n-1}(\xi)\delta R(\xi) d\xi d\tau}{\max\{a(t),0\}} \right]
\]

\[
= bn \int_0^T e^{-\tau} \int_0^T \frac{m(t) \beta^{1-d}(t) R^{n-1}(\xi)\delta R(\xi) d\xi d\tau}{\max\{a(t),0\}} dt
\]

and similarly in the fifth term. To transform the third term, we use the Taylor expansion \( \int_a(t) f(t,\tau) d\tau = f(t,a(t)) + o(\delta a(t)) \). Then, collecting coefficients of \( \delta R, \delta m, \) and \( \delta a \), we rewrite (A4) as:
Finally, recalling $\delta a(t) = \int_0^t \delta \tilde{a}(t) \, d\xi$, we convert the last expression to

$$
\mathcal{I} = \int_0^\infty \left[ -e^{-\gamma t} + bn \int_0^t \left( \int_\tau^t e^{-\gamma \xi} \, d\xi - e^{-\gamma \tau} \, p_1(\tau) \right) \cdot m(\tau) \beta^{1-d}(\tau) \, d\tau \cdot R^{n-1}(\tau) \right] \cdot \delta R(t) \, dt
$$

$$
+ \int_0^\infty \left[ \int_\tau^t e^{-\gamma \tau} \left( \beta(t) - p(\tau) \right) d\tau - e^{-\gamma t} \, p_1(t) \beta(t) \right] \cdot \delta m(t) \, dt
$$

$$
+ \int_0^\infty e^{-\gamma t} (p(t) - \beta(a(t)) m(a(t))) \cdot \delta a(t) \, dt + \int_0^\infty e^{-\gamma t} o(\delta R(t), \delta m(t), \delta a(t)) \, dt
$$

Formally (A5) in notations (25), (20), (21) provides the required expression (A2). The domain (14) of admissible controls $R, m, v$ has the simple standard form $R \geq 0, m \geq 0, v \geq 0$. So, the NCE (18) follows from the obvious necessary condition that the variation $\delta I$ of functional $I$ can not be positive for any admissible variations $\delta R(t), \delta m(t), \delta a(t), t \in [0, \infty)$.

**Case (B).** If the restriction of (13) is active: $E(t) = E_{\text{max}}(t)$ at $t \in \Delta \subset [0, \infty)$, then we choose $R$ and $m$ as the independent unknowns of the OP. The dependent (state) variable $a$ is uniquely determined from the initial problem

$$
m(a(t)) a'(t) = m(t) - E'_{\text{max}}(t), \quad a(0) = a_0,
$$

obtained after differentiating (13). As shown in Hr. and Yatsenko (2006), if $E_{\text{max}}'(t) \leq 0$, then for any measurable $m(t) \geq 0$, a unique a.e. continuous function $a(t) < t$ exists and a.e. has $a'(t) \geq 0$ (see Remark 1 about the possible case $E_{\text{max}}'(t) > 0$). Therefore, the state restrictions $a'(t) \geq 0$ and $a(t) < t$ in (14) are satisfied automatically, so we can exclude $a$ from the extremum condition.

Similarly to the previous case, let us give small admissible variations $\delta R(t)$ and $\delta m(t), t \in [0, \infty)$, to $R$ and $m$ and find the corresponding variation $\delta I = I(R + \delta R, m + \delta m) - I(R, m)$ of the functional $I$.

In this case, the variation $\delta I$ is determined by $\delta m$. To find their connection, let us present (13) as

$$
E_{\text{max}}(t) = \int_{a(t)}^{a(t) + \delta a(t)} m(\tau) \, d\tau = \int_{a(t)}^{a(t) + \delta a(t)} (m(\tau) + \delta m(\tau)) \, d\tau
$$
then
\[ \int_{\max[a(t),0]}^{a(t)+\delta a(t)} \delta n(t) d\tau = \int_{a(t)}^{a(t)+\delta n(t)} m(t) d\tau + o(\|\delta n\|, \|\delta a\|). \] (A6)

We will use the above formula (A4) for the variation $\delta I$ as a function of $\delta R$, $\delta n$, and $\delta a$ and eliminate $\delta a$ from (A4) using (A6). To do that, we rewrite the third term of (A4) by adding
\[ \pm \int_0^\infty e^{-\tau} \beta(a(t)) \int_{a(t)}^{a(t)+\delta n(t)} m(t) d\tau d\tau \] and applying (A6) as
\[ \int_0^\infty e^{-\tau} \int_{a(t)}^{a(t)+\delta n(t)} (p(t) - \beta(\tau)) m(t) d\tau d\tau \]
\[ = \int_0^\infty e^{-\tau} (p(t) - \beta(a(t))) \int_{a(t)}^{a(t)+\delta n(t)} m(t) d\tau d\tau + \int_0^\infty e^{-\tau} \int_{a(t)}^{a(t)+\delta n(t)} (\beta(a(t)) - \beta(\tau)) m(t) d\tau d\tau \]
\[ = \int_0^\infty e^{-\tau} (p(t) - \beta(a(t))) \int_{a(t)}^{a(t)+\delta n(t)} m(t) d\tau d\tau + \int_0^\infty e^{-\tau} \int_{a(t)}^{a(t)+\delta n(t)} m(t) d\tau d\tau \] (A7)

The integral $\int_{a(t)}^{a(t)+\delta n(t)} (\beta(a) - \beta(\tau)) m(t) d\tau$ in (A7) has the order $o(\delta a)$ because $\beta(\tau)$ is continuous.

Substituting (A7) into (A4) and collecting the coefficients of $\delta n$ and $\delta R$, we obtain the expression
\[ \delta I = \int_0^\infty (I'_R(t) \cdot \delta R(t) + I'_m(t) \cdot \delta m(t)) dt + o(\|\delta R\|, \|\delta n\|) \] (A8)
in the notations (19) and (25). The rest of the proof is identical to Case B.

The Theorem has been proven. \( \square \)

**Proof of Theorem 2:** By Lemma 1, at $n=d$
\[ \beta_\lambda(t) = \overline{R}(b, g) \frac{1}{a} e^{\theta t}. \] (A9)

The extremum condition of Part A of Theorem 1 is written for interior solutions as (27).

Substituting (30), (34), and (A9) to the equation $I'_a(t) = 0$ of (27) gives
\[ (1-\theta)\overline{R}(b, g) \frac{1}{a} e^{-\lambda t} e^{\theta t} = \overline{R} e^{\theta t} \] (A10)

which can hold for any $t$ only if $g=\chi$ At $g=\chi$, (A10) gives (37). Next, substituting the latter formula, (25), and (A9) to the equation $I'_m(t) = 0$ of (27) gives
that after integration becomes the equation (35) for $L$. Equation (35) appeared earlier in the vintage models with exogenous technological change (Hritonenko & Yatsenko 1996, Boucekkine et al. 1997), where it was shown to have a unique positive solution $L$ for any given $\gamma, r, 0 < \gamma < r$.

Finally, the substitution of (19), (34), and (A9) into equation $I_g'(t) = 0$ of (27) leads to

$$bd\bar{M}(\bar{R}e^{\eta})^{d-1} \int_{t}^{t+t_L} \left( \frac{b}{g} \right)^{\frac{1}{d}} e^{\eta t} \left[ e^{-\eta t} - \frac{1 - e^{-\eta (t+L)}}{r} (1 - \theta) - p_k e^{-\eta t} \right] d\tau = e^{-\eta t},$$

and, after integration, to

$$\frac{d\bar{M}b^d g^{-d}}{g(1-d)-r} \left[ \frac{1 - e^{-\eta t}}{r} (1 - \theta) - p_k \right] e^{-\eta t} = e^{-\eta t},$$

that produces (36) and completes the proof. □

**Proof of Corollary 1:** Let us check whether the value $\bar{M}^S L^S$ increases or decreases in $\gamma$. Because of the complexity of the nonlinear equation (35), we restrict ourselves with the asymptotic case of small $\gamma < r < 1$. Then, at $\gamma \to 0$ we have $L \to \infty$ and, therefore, (35) leads to

$$\frac{1}{r} - \frac{e^{-\eta t}}{r} \to \frac{p_k r}{1 - \theta}$$

or $\gamma L \to -\ln \frac{p_k r}{1 - \theta}$. Therefore, by (35) and (36),

$$\bar{M} L = L \left( \frac{\gamma}{b} \right)^{1/d} \frac{r - \gamma (1-d)}{\gamma L (1-d)} \left[ \frac{1 - e^{-\eta t}}{r} (1 - \theta) - \frac{p_k r}{1 - \theta} \right]^{-1} \to -\frac{r^{1/d-2}}{b^{1/d-2}} \frac{r}{d(1 - \theta)} \ln \frac{p_k r}{1 - \theta}.$$

So, $\bar{M}^S L^S$ decreases in $\gamma$ (and $\bar{M}^S L^S \to \infty$ as $\gamma \to 0$) at $d > 0.5$, and increases at $d < 0.5$.

The corollary has been proven. □

**Proof of Theorem 3:** Let us check whether an interior BGP is possible under the active quota constraint $E(t) = \bar{E}$. The extremum condition for this case is provided by equations (28) for interior solutions. The substitution of (19), (38), and (A9) into the first equation (28) leads to

$$bd\bar{M}(\bar{R}e^{\eta})^{d-1} \int_{t}^{t+t_L} \left( \frac{b}{g} \right)^{\frac{1}{d}} e^{\eta t} \left[ e^{-\eta t} - \frac{1 - e^{-\eta (t+L)}}{r} (1 - \theta) - p_k e^{-\eta t} \right] d\tau = e^{-\eta t},$$

at $t \in [t_L, \infty)$ and, after integration, to
that can be rewritten as (39). Substituting (38) and (A9) to the second equation (28) gives

\((1-\theta) \int_0^{E/E} \left[1 - e^{g(t-E/E)}\right] e^{-\tau} d\tau = p_k e^{-rt} ,\)

which becomes (40) after integration. It proves the statement (i) of the Theorem.

Let us now show that the equations (39)-(40) possess a solution in the asymptotic case of large \(E\). Let \(E \to \infty\). The value of \(g\) is always limited by \(r\), so the value \(\bar{M}\) cannot tend to \(\infty\) in the general case when \(p_k r / 1-\theta < 1\). At \(E \to \infty\), the system (39)-(40) becomes

\[
rg^{(1-d)/d} - (1-d)g^{1/d} = db^{1/d} \bar{M} \left[1 - \frac{p_k}{1-\theta}\right] (1-\theta) ,
\]

Finding \(\bar{M} = -gE \ln \left( \frac{1}{r} - \frac{p_k}{1-\theta} (r-g) \right) = -gE \ln \left( \frac{1}{r} - \frac{p_k (r-g) / 1-\theta}{1-\theta} \right) > 0 \) from the last equation and substituting it into the first one, we obtain a single equation for \(g\):

\[
gr^{(1-2d)/d} - (1-d)g^{(1-2d)/d} = -E db^{1/d} \left[1 - \frac{p_k}{1-\theta}\right] (1-\theta) \ln \left( \frac{1}{r} - \frac{p_k (r-g) / 1-\theta}{1-\theta} \right)
\]

Next, we will analyze the existence of a solution for this equation at large enough \(E\). Let us assume for a second that a solution \(g>0\) exists. If \(E \to \infty\), then its increase in the RHS of the equation needs to be compensated by a corresponding change of \(g\). The logarithm in the RHS takes a finite negative value for any positive \(g\). So, the only way to compensate the \(E\) increase is to have \(rg^{(1-2d)/d} - (1-d)g^{(1-2d)/d} \to \infty\). If \(d>0.5\), then the exponent \((1-2d)/d\) is negative, therefore, \(rg^{(1-2d)/d} - (1-d)g^{(1-2d)/d} \to \infty\) if and only if \(g \to 0\)\(^{18}\). Thus, if the solution \(g>0\) exists, it necessarily tends to zero when \(E \to \infty\). At small \(g\), the term \(-(1-d)g^{(1-2d)/d}\) is small and

\[
\ln \left( \frac{1}{r} - \frac{p_k (r-g) / 1-\theta}{1-\theta} \right) \approx \ln \left(1 - \frac{p_k r}{1-\theta}\right) , \text{ so, the equation becomes}
\]

\(^{18}\) If \(d<0.5\), then the exponent \((1-2d)/d\) is positive and this condition can be satisfied only if \(g \to \infty\) (which makes no economic sense)
\[ r g^{(1-2d)/\theta} = -E db^{1/d} (1 - \theta) \left[ 1 - \frac{p_k}{r - \theta} \right] \ln \left( 1 - \frac{p_k r}{1 - \theta} \right) \]

It is easy to see that the solution \( g \) of the last equation exists and \( g \to 0 \) as \( E \to \infty \). It justifies the statement (ii) of the Theorem.

Finally, let us prove the statement (iii) that equations (39) and (40) have a positive solution \( g \) and \( M \) at \( rE << 1 \). Then, presenting the exponents in (40) as the Taylor series, we obtain

\[
\left[ \frac{1}{r} \left[ 1 - 1 + \frac{rE}{M} - \frac{1}{2} \left( \frac{rE}{M} \right)^2 + o(r^2) \right] \right]
\]

or

\[
\left[ \frac{1}{r - g} \left[ 1 - \frac{gE}{M} + \frac{1}{2} \left( \frac{gE}{M} \right)^2 + o(g^2) - 1 + \frac{rE}{M} - \frac{1}{2} \left( \frac{rE}{M} \right)^2 + o(r^2) \right] \right] = \frac{p_k}{1 - \theta}
\]

or

\[
(\frac{E}{M})^2 \left[ (r + g) - r \right] + o(r) = 2 p_k / (1 - \theta), \tag{A11}
\]

which has the solution \( M = E \sqrt{g / 2 p_k / (1 - \theta) + o(r)} \).

Now, expressing the exponent in (39) as the Taylor series, we obtain

\[
g^{(1-d)/d} [r - g(1 - d)] = \frac{E}{M} \left[ \frac{r}{2} \left( \frac{E}{M} \right)^2 + o(r) - k \sqrt{1 - \theta} \right]. \tag{A12}
\]

Substituting (A11) into (A12) leads to one equation (42) for \( g \). To analyze this equation, we introduce the new variable \( x = \sqrt{g} \) and rewrite (42) as

\[
F_1(x) = F_2(x), \tag{A13}
\]

where \( F_1(x) = x^{2/d-2} (r + x^2 (d - 1)) \), \( F_2(x) = db^{1/d} E \left[ 1 - \frac{p_k}{2(1 - \theta)} \left( \frac{r}{x} + 1 \right) \right] \).

These functions are shown in Figure 1 and are such that \( F_1(0) = 0, F_1'(x) > 0 \) at \( 0 < x < \sqrt{r} \), \( F_1'(x) = 0 \) at \( x = \sqrt{r} \), \( F_2(x) > 0 \) at \( 0 < x < \sqrt{r} \), \( F_2'(x) = 0 \) at \( x = \sqrt{r} \). Also, \( F_2(x) < 0 < F_1(x) \) at small \( 0 < x << 1 \).

Therefore, to have a solution \( 0 < x < \sqrt{r} \) to equation (A13), it is necessary and sufficient that \( F_2(\sqrt{r}) > F_1(\sqrt{r}) \), which leads to the inequality (41). The sufficient condition for the uniqueness of \( x \) is \( F_1'(x) < F_2'(x) \) at \( 0 < x < \sqrt{r} \), which leads to (43).

The theorem has been proven. \( \square \)
Proof of Theorem 4: Let the value \( \bar{E} < \infty \) be fixed and given. Let also the system (39)-(40) have a solution \( g \), \( 0 < g < r \) (\( g \) does not depend on \( \gamma \)), then, the BGP\( ^R \) \( (R^R, Q^R, \beta^R; c^R, m^R, a^R) \) exists. It is true, at least, in cases (ii) and (iii) of the previous theorem.

First, we will prove that, if \( g > \gamma \) then \( c^R(t) > 0 \) at large \( t \). By (12),

\[
Q^R(t) = \frac{b}{g} \left( 1 - e^{-\frac{t}{\bar{E}}/M} \right) e^{\gamma t}. \]

Therefore,

\[
c^R(t) = (1 - \theta) Q^R(t) - p_k \beta^R(t)m^R(t) - R^R(t) - \bar{E} p(t)
\]

\[
= \frac{b}{g} \left( 1 - e^{-\frac{t}{\bar{E}}/M} \right) (1 - \theta) \left( 1 - \frac{\bar{E}}{M} - k \right) e^{\gamma t} - \bar{E} Pe^{\gamma t}. \tag{A14}
\]

Expressing the exponent in (A14) as the Taylor series and using (A12), we obtain

\[
c^R(t) = \frac{b}{g} \left( 1 - e^{-\frac{t}{\bar{E}}/M} \right) (1 - \theta) \left( 1 - \frac{\bar{E}}{M} - k \right) e^{\gamma t} - \bar{E} Pe^{\gamma t}.
\]

Thus, \( c^R(t) > 0 \) at large enough \( t \) for any positive value \( \bar{E} \) if \( g > \gamma \) and at \( \bar{E} > \frac{\bar{E}P gd}{(r - g)(1 - \theta)} \) if \( \gamma = g \).

If the calculated \( g < \gamma \) then \( c^R(t) < 0 \) and the BGP\( ^R \) has no economic meaning.

Now, let us discuss the existence of BGP\( ^S \). We assume that the given rate \( \gamma \) increases from 0 to \( r \). At \( \gamma = 0 \), there is no BGP\( ^S \) and the only BGP\( ^R \) exists. Let us consider the critical value \( \gamma_c = g \) of \( \gamma \). One can easily see that the nonlinear system (35)-(36) is equivalent to the expressions (39)-(40) if we denote \( M^R L^R = \bar{E} \). Therefore, we obtain \( M^S = M^R \) and \( L^S = L^R = \bar{E} / M^S \) at \( \gamma = g \). So, at \( \gamma = g \) we have a critical case of BGP\( ^S \) in the sense that its \( E_{\max} = M^S L^S \) is on the edge \( \bar{E} \). Let us estimate whether the value \( M^S L^S \) increases or decreases in \( \gamma \). By Theorem 2, \( M^S L^S \) decreases in \( \gamma \) at \( d > 0.5 \). Therefore, \( M^S L^S < \bar{E} \) at \( \gamma = g \), so, BGP\( ^S \) exists. Correspondingly, BGP\( ^S \) does not exist at \( \gamma = g \).

The theorem has been proven. \( \square \)

References


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Figure 1. Solving the nonlinear equation (A13) with respect to the unknown $x = \sqrt{C}$. 