Sustainable economic growth under physical constraints: optimal R&D, investment and replacement policies[1]

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Abstract

We consider the optimal control of nonlinear integral equations with state constraints, which describes the endogenous growth of an economy subjected to exogenous physical constraints. The economy has three instruments to reach sustainable growth: R&D to develop new more efficient technologies, investment in new capital goods, and scrapping of obsolete capital. The R&D technology depends negatively on a complexity component and positively on the R&D investment at a constant elasticity. First, we characterize exponential steady state trajectories (balanced growth paths) for different parameterizations of the R&D technology. Second, we study transitional dynamics to the balanced growth. We prove that regardless of how relaxed the physical constraint is, the transition dynamics always leads to the balanced growth with the active constraint in a finite time.

Keywords: Optimal control, state constraints, sustainable growth, age-structured populations, endogenous R&D

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1. Introduction

The paper introduces and investigates a nonlinear optimization problem with state constraints, which is of great importance to the economic growth theory. Identifying sustainable growth paths is becoming a central question of this theory. The issue has many challenging normative, demographic, and technological aspects (Arrow et al., 2004). On the technological side, many research avenues have been taken so far resulting in a quite dense literature. In particular, *research and development* (R&D) programs have been designed to meet the environmental and physical constraints required for sustainability. These constraints include, but are not limited to extraction quotas, pollution quotas, and technical feasibility. A central question of the latter literature turns out to be whether such constraints can ultimately deliver a win-win situation when economies facing them will have strong incentives to innovate resulting in new R&D-based growth regimes. This mechanism, often referred to as the *Porter hypothesis*, has been studied in numerous papers, some empirical (like the seminal paper by Newell et al., 1999) and others more theoretically-oriented (see Acemoglu et al., 2011, for one of the most recent contribution on the induced-innovation hypothesis under environmental constraints). This paper is in line with the latter approach. More precisely, we consider an optimal growth model with R&D expenditures under physical constraints. Technological progress is, therefore, endogenous; it is also specified as embodied in capital goods: thanks to R&D efforts, new capital goods use less and less resource (say, energy). The view of technological progress as an embodied, endogenous, and energy-saving phenomenon is documented and commented in a substantial literature. In particular, Ayres (2005) declares that “technical progress is essentially equivalent to increasing efficiency of converting raw resources, such as coal, into useful work”. More specifically, substantial economic evidence supports the direct impact of the *R&D spending* on the *industry-level capital-embodied technological change*. Wilson (2002) used industrial data to confirm that the cross-industry variation in estimates of embodied technological change matches the cross-industry variation in embodied R&D and concluded that “the technological change, or innovation, embodied in an industry’s capital is proportional to the R&D that is done (upstream) by the economy as a whole”.

Our paper has three salient and distinctive features. First of all, it explicitly uses a vintage capital framework in the tradition of Solow et al. (1966) with capital and raw resources as production factors. Raw materials include fossil energy and minerals. Capital and resources are complementary (Leontief technology), and new vintages consume less resources over time (resource-saving technical progress). Second, we explicitly account for physical constraints on the extraction and use of resources, which we formalize as **exogenous** upper-bounds on resource consumption. The presence of these constraints together with capital-resource complementarity induces an obsolescence mechanism, which in turn opens the door to endogenous scrapping: as in Boucekkine et al. (1997) and Hritonenko and Yatsenko (1996), the oldest vintages will be removed from the workplace and replaced by less resource consuming new vintages. Third, after characterizing possible steady states of the dynamic system, we shall study transitional dynamics. In particular, we aim at identifying clearly the different routes to sustainable growth, which is not so frequent in the endogenous growth literature, usually restricted to a balanced growth analysis.

The resulting optimization problem is novel and describes a nonlinear optimal control of the age-structured population of heterogeneous capital assets. The mathematical complexity of the problem comes from non-traditional nonlinear relations among variables and the presence of state constraints. In order to achieve clear-cut analytical results, we consider a linear utility function: strictly concave utility functions render the analytical work intractable, even for steady state analysis. With this simplification, we are able to derive an analytical characterization of steady states and display the optimal transition to the steady states. We provide a complete dynamic analysis of this optimization problem, which determined the mathematical novelty of the paper.

Our framework extends (Boucekkine et al., 2011) that addressed a related firm problem, while here we will solve an optimal growth model in the Ramsey sense. To the best of our knowledge, and with exception of (Jovanovic, 2009) and (Boucekkine et al., 2011), no other paper has considered R&D decisions and vintage capital with endogenous scrapping at the same time. Jovanovic (2009) suggested and analyzed a general-equilibrium model of endogenous growth with human capital, technology markets, and more detailed mechanism of innovation accumulation and diffusion that ours. His model also delivers a sustainable balanced growth and our model complements his findings.

Using the central planner framework and more aggregate modeling of R&D, we are not only able to bring out a thorough analysis of the induced innovation hypothesis in the steady state, but also, and more importantly, to deliver the transitional dynamics of the endogenous growth model under scrutiny and the underlying dynamic processes. In the aggregate description of the R&D sector, we follow the well-known endogenous growth models of Romer (1990) and Jones (1995) adding the vintage capital structure and endogenous capital scrapping into them. The major remaining difference between these models and ours is that our R&D block uses a part of the endogenous output as an input while they use a part of a given (labor) resource. A vintage version of (Jones, 1995) model was briefly analyzed in (Yatsenko et al, 1999).

Feichtinger et al. (2005, 2006) have developed an alternative vintage framework balancing the efficiency gains of running new vintages with the learning costs associated, which opens the door to optimal investment in old vintages, in contrast to our modelling where such a possibility does not exist (no learning costs). Having said this, Feichtinger et al. (2005, 2006) have not integrated R&D decisions in their setting, nor have they endogenized scrapping. Hart (2004) has constructed a multi-sector endogenous growth model with an explicit vintage structure. But his paper differs from ours in at least two aspects: it is built on two types of R&D, one output-augmenting and the other, say, environmental-friendly, while in our model only resource-saving adoptive and/or innovative R&D is allowed. Also, the vintage structure of Hart (2004) includes a fixed number of vintages and, therefore, there is no way to uncover a comprehensive modernization policy optimally combining the scrapping of the dirtiest technologies and the development of new clean technologies.

A systematic exposure into the mathematical treatment and properties of such optimal control problems was done in Hritonenko and Yatsenko (2005), where the extremum conditions and qualitative analysis were provided for several problems of a simpler structure. In particular, a prototype vintage model with endogenous R&D investment was considered, which possessed no interior steady states. Veliov (2008) derived optimality conditions in the form of a maximum principle for more general and abstract models of heterogeneous distributed systems with nonlocal dynamics.

Finally, the modeling philosophy and outcomes of this paper are close to some age-structured models studied in the monograph of Boucekkine et al (2010).

This paper enhances both optimization and economic theories. It has two major contributions. First, it significantly extends the steady state analysis of (Boucekkine et al., 2011). In the latter, the R&D technology is taken “balanced” in the sense that the standard (negative) complexity component *à la* Segerstrom (2000) compensates the (positive) return to R&D investment component in the parameterization considered. In this paper, we shall explore all the cases: when the negative component dominates and when it is dominated. We believe that this extension is worth doing because the R&D technology is not the same across countries: some countries (like certain Scandinavian countries) are historically more sensitive to the development of resource-saving technologies than others, and are likely to be more efficient at this. Others are lagging clearly behind. We show that they should experience different balanced growth paths if any. For example, the countries with under-performing R&D sector would need the physical constraints to be more and more relaxed over time (in a very accurate sense to be given) to ensure a sustainable growth.

The second major contribution of this paper is the complete dynamic analysis of the formulated optimization problem, which demonstrates the convergence of optimal trajectories to the steady state. Namely, regardless how lenient physical constraint is, the transition dynamics in the model leads in a finite time to a balanced growth with the active physical constraint. The derived transition dynamics indicates several possible short-term regimes, among them, an intensive growth (sustained investment in new capital and R&D with scrapping the oldest capital goods), and an extensive growth (sustained investment in new capital and R&D without scrapping the oldest capital). Our paper is the first one to disentangle the latter regime as a short-term optimal transition regime. Namely, if the country is not initially rich in capital (the resource consumption is lower than the upper bound), then it should initially use more new capital without scrapping the old one, so the country experiences an extensive economic growth.

The paper is organized as follows. Section 2 formulates the optimal control problem and derives the optimality conditions in the form of a maximum principle. Section 3 is devoted to the steady state analysis of the problem and its applied interpretation as the existence and properties of a long-term economic growth. Section 4 characterizes optimal transitional dynamics of the problem. Section 5 concludes.

2. The optimal growth problem

We consider a benevolent social planner of a national economy who maximizes the discounted utility from the consumption over the infinite horizon. The corresponding optimal control problem can be formulated as

$$\max_{i,R,a} I = \max_{i,R,a} \int_0^\infty u(y(t) - i(t) - R(t)) e^{-rt} dt\,, \tag{1}$$

where $u(.)$ is the utility function, r is the social discount rate, $i(t)$ is the investment into new capital, $R(t)$ is the investment into R&D,

$$y(t) = \int_{a(t)}^{t} i(\tau) d\tau \tag{2}$$

is the production output at time t, $a(t)$ is the capital scrapping time, subject to the following constraints

$$0 \le i(t) \le y(t) – R(t), \quad R(t) \ge 0, \quad a'(t) \ge 0, \quad a(t) \le t. \tag{3}$$

The total resource consumption is

$$E(t) = \int_{a(t)}^{t} \frac{i(\tau)}{\beta(\tau)} d\tau \tag{4}$$

To address capital modernization, the model (1)-(4) departs from the concept of homogeneous capital and assumes that newer capital vintages consume less energy (and, therefore, are environmentally friendlier). In (4), the resource consumption by one machine of vintage t (i.e., installed at time t) is equal to $1/\beta(t)$. The variable $\beta(t)$ is endogenous and reflects a broadly defined resource-saving embodied technological level, which may be implemented in new resource–efficient machines. For clarity, our model does not involve any output-augmenting embodied or disembodied technological change: each machine (old or new) produces exactly one unit of output. Needless to say, the output not invested (either in R&D or in new capital) is consumed, that is: $c(t) = y(t) – i(t) – R(t)$.

We assume that the economy is committed to various physical feasibility and/or regulation constraints on current resource consumption (and/or extraction), which can be formulated as the following constraint:

$$E(t) \le E_{\max}(t). \tag{5}$$

Next, we assume that the level of the technological progress $\beta(\tau)$ depends on the R&D investment $R(t)$ as

$$\frac{\beta'(\tau)}{\beta(\tau)} = \frac{f(R(\tau))}{\beta^d(\tau)}, \qquad 0 < d < 1, \tag{6}$$

$f'(R)>0$, $f''(R)<0$. By (6), the rate β'/β of technological progress is a concave increasing function $f(R)$ in R and a decreasing function of the level β itself. This specification reflects a negative "fishing-out" impact of technological complexity on R&D success (see Jones, 1995; Segerstrom, 2000; Jovanovic, 2009). The parameter d measures the impact of the R&D complexity on the technological progress rate. It is consistent with the available evidence on the role of technological complexity in the adoption of new technologies. Also, we restrict ourselves to the case

$$f(R)=bR^n, \quad 0<n<1, \quad b>0,$$

which means that the elasticity n of the rate of technological progress with respect to R&D expenditures is constant. The R&D investment is more efficient for larger n.

Assuming a nonlinear utility would not allow for a full analytical characterization of possible steady state regimes, so, we stick to the linear utility function u in (1). In parallel with the investment $i(t)$ in output units, we will use investment $m(t)=i(t)/\beta(t)$ in the resource consumption units. In the variables (m, R, a), the optimization problem (1)-(6) becomes

$$\max_{m,R,a} I = \max_{m,R,a} \int_0^{\infty} [y(t) - \beta(t)m(t) - R(t)]e^{-rt} dt \ , \quad r > 0, \tag{7}$$

$$y(t) = \int_{a(t)}^{t} \beta(\tau)m(\tau)d\tau, \tag{8}$$

$$\frac{\beta'(\tau)}{\beta(\tau)} = b\frac{R^n(\tau)}{\beta^d(\tau)}, \qquad 0 < d < 1, \tag{9}$$

$$E(t)=\int_{a(t)}^{t} m(\tau)d\tau \le E_{\max}(t)\,, \tag{10}$$

$$0 \le \beta(t)m(t) \le y(t)-R(t),\quad R(t)\ge 0,\quad a'(t)\ge 0,\ \ a(t)\le t, \tag{11}$$

with the given initial conditions on the prehistory:

$$a(0) = a_0<0,\quad \beta(a_0)=\beta_0,\quad m(\tau)\equiv m_0(\tau),\ \ R(\tau)\equiv R_0(\tau),\ \ \tau\in[a_0, 0]. \tag{12}$$

The optimization problem (7)-(12) includes six unknown functions $m(t)$, $R(t)$, $a(t)$, $y(t)$, $E(t)$, and $\beta(t)$, $t\in[0,\infty)$, connected by three equalities (8)-(10). We choose R, m, and $v=a'$ as *independent* controls and consider y, a, E, and β as *dependent state* variables. Let R, m, v belong to the space $L^{\infty}_{\mathrm{loc}}[0,\infty)$ of measurable on $[0,\infty)$ functions bounded almost everywhere (*a.e.*) on any finite subinterval of $[0,\infty)$ (Corduneanu, 1997). We also assume a priori that the integral in (7) converges (it will be true in all subsequent theorems). Solving the differential equation (9), we obtain the explicit formula for the productivity $\beta(\tau)$ through the previous R&D investment R on $[a_0,\tau]$:

$$\beta(\tau)=\left(bd\int_0^{\tau} R^n(v)dv+B^d\right)^{1/d},\qquad B=\left(bd\int_{a_0}^{0} R_0{}^n(v)dv+\beta_0{}^d\right)^{1/d}. \tag{13}$$

The problem (7)-(12) is an optimal control problem with state constraints. To analyze its complete dynamics, we need optimality conditions that will include all possible combinations of the state constraints-inequalities $E(t)\le E_{\max}(t)$ and $\beta(t)m(t)\le y(t)-R(t)$. Notice that the latter is equivalent to the non-negativity of consumption. Clearly, having a concave utility function satisfying Inada conditions would rule out the corner regime $c(t)=0$ associated with this condition. For mathematical consistency, we shall consider here all the possible cases allowed by linear utility. The optimality conditions are given by Theorem 1 below. As we shall see, all combinations can appear during the long-term dynamics (Section 3) or the transition dynamics (Section 4).

Theorem 1 (*necessary condition for an extremum*). *Let (R^*, m^*, a^*, β^*, y^*, E^*) be a solution of the optimization problem (7)-(12). Then:*

(A) *If $E^*(t)=E_{\max}(t)$ and $\beta^*(t)m^*(t)<y^*(t)-R^*(t)$ at $t\in\Delta\subset[0,\infty)$, and $E_{\max}'(t)\le 0$, then*

$$I_R'(t)\le 0 \ \ at\ \ R^*(t)=0,\qquad I_R'(t)=0\ \ at\ \ R^*(t)>0, \tag{14}$$

$$I_m'(t)\le 0 \text{ at } m^*(t)=0, \qquad I_m'(t)=0 \text{ at } m^*(t)>0, \quad t\in\Delta, \tag{15}$$

where

$$I_R'(t)=bnR^{n-1}(t)\int_t^{\infty}\beta^{1-d}(\tau)m(\tau)\left[\frac{e^{-r\tau}-e^{-ra^{-1}(\tau)}}{r}-e^{-r\tau}\right]d\tau-e^{-rt}, \tag{16}$$

$$I_m'(t)=\int_t^{a^{-1}(t)}e^{-r\tau}\left[\beta(t)-\beta(a(\tau))\right]d\tau-e^{-rt}\beta(t), \tag{17}$$

the state variable $a(t)$ *is determined from (10),* $a^{-1}(t)$ *is the inverse function of* $a(t)$*, and* $\beta(t)$ *is given by (13).*

(B) *If* $E^*(t)<E_{\max}(t)$ *and* $\beta^*(t)m^*(t)<y^*(t)-R^*(t)$ *at* $t\in\Delta$*, then*

$$I_R'(t)\le 0 \text{ at } R^*(t)=0, \qquad I_R'(t)=0 \text{ at } R^*(t)>0,$$

$$I_m'(t)\le 0 \text{ at } m^*(t)=0, \qquad I_m'(t)=0 \text{ at } m^*(t)>0, \tag{18}$$

$$I_{a'}'(t)\le 0 \text{ at } da^*(t)/dt=0, \qquad I_{a'}'(t)=0 \text{ at } da^*(t)/dt>0, \quad t\in\Delta,$$

where

$$I_m'(t)=\beta(t)\left[\int_t^{a^{-1}(t)}e^{-r\tau}d\tau-e^{-rt}\right], \tag{19}$$

$$I_{a'}'(t)=-\int_t^{\infty}e^{-r\tau}\beta(a(\tau))m(a(\tau))d\tau, \tag{20}$$

$I_R'(t)$ *is as in (16), and* $\beta(t)$ *is as in (13).*

(C) *If* $E^*(t)<E_{\max}(t)$ *and* $\beta^*(t)m^*(t)=y^*(t)-R^*(t)$ *at* $t\in\Delta\subset[0,\infty)$*, then*

$$I_R'(t)\le 0 \text{ at } R^*(t)=0, \qquad I_R'(t)=0 \text{ at } R^*(t)>0, \tag{21}$$

$$I_{a'}'(t)\le 0 \text{ at } da^*(t)/dt=0, \qquad I_{a'}'(t)=0 \text{ at } da^*(t)/dt>0, \quad t\in\Delta,$$

where $I_{a'}'(t)$ *is as in (20),* $m^*(t)$ *and* $y^*(t)$ *are determined from equation (8),*

$$\widehat{I}_R'(t)=bnR^{n-1}(t)\int_t^{\infty}\beta^{1-d}(\tau)m(\tau)\left[\int_\tau^{a^{-1}(\tau)}\chi(\xi)d\xi-\chi(\tau)\right]d\tau-\chi(t), \tag{22}$$

$\chi(t)$ *is found from the integral equation*

$$\chi(t) = \int_{t}^{a^{-1}(t)} \chi(\tau) d\tau \quad at \quad t \in \Delta \tag{23}$$

and $\chi(t) = e^{-rt}$ *at* $t \in [0,\infty) - \Delta$.

(D) *If* $E^*(t) = E_{\max}(t)$ *and* $\beta^*(t)m^*(t) = y^*(t) - R^*(t)$ *at* $t \in \Delta \subset [0,\infty)$, *then*

$$I_R'(t) \le 0 \ at \ R^*(t)=0, \qquad I_R'(t)=0 \ at \ R^*(t)>0, \tag{24}$$

where $I_R'(t)$ *is given by (22),* $\chi(t)$ *is found from*

$$\chi(t) = \int_{t_k}^{a^{-1}(t)} \left[1 - \frac{\beta(a(\tau))}{\beta(t)} \right] \chi(\tau) d\tau \quad at \quad t \in \Delta, \tag{25}$$

and $\chi(t) = e^{-rt}$ *at* $t \in [0,\infty) - \Delta$, *and* $m^*(t)$, $a^*(t)$, *and* $y^*(t)$ *are determined from nonlinear equations (8) and (10).*

The proof is provided in Appendix. Theorem 1 is a significant extension of Theorem 1 in (Boucekkine et al., 2011) that explored only Cases A and B. For those two cases, some differences also show up. An essential difference emerges in the derivation of optimal scrapping when the physical constraint is inactive. It is easy to see from (20) that $I_{a'}'(t)<0$. Hence, $a^* \equiv a_0$ is corner and the optimal regime is boundary in a. This might not be the case in the counterpart firm problem.[5] This outcome is natural from the economic point of view: in our central planner problem, the unique reason to shorten the lifetime of capital goods is the active physical constraints, while exogenously increasing energy prices (for example, reflecting scarcity) is an additional motive to scrap in the corresponding firm problem. Mathematically speaking, the problem (7)-(12) with a linear utility does not possess exogenous resource and capital prices. So, the firm model has a room for a non-boundary control in the scrapping age even when the physical constraint is inactive.

More differences emerge in the expression of the optimal interior investment rules depicted in Theorem 1. Let us briefly interpret the optimal interior investment rules in Case A, which will turn out to be the important one in the long-run as demonstrated in Section 3. As to the optimal investment rule in new capital, it can be reformulated as:

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[5] See equation (21) in (Boucekkine et al., 2011).

$$\int_{t}^{a^{-1}(t)} e^{-r\tau}\left[1-\frac{\beta(a(\tau))}{\beta(t)}\right]d\tau = e^{-rt}$$

The rule is very close to the counterpart in Boucekkine et al. (1997): it equalizes the present value of marginal investment cost (the right-hand side) and the discounted value of the quasi-rents generated by one unit of capital bought at *t* along its lifetime (the left-hand side). Here costs and benefits are expressed in terms of marginal utility, but since utility is linear, marginal utility terms do not show up. The quasi-rent at τ generated by a machine of vintage *t* is the difference between the unit of consumption forgone to buy one unit of new capital and the operation cost at τ, which is the product of the amount of energy consumed to operate any machine of vintage *t*, that is $\frac{1}{\beta(t)}$, and the shadow price of energy $\beta(a(\tau))$ at the date $a(\tau)$.

The optimal investment rule in R&D in Case A may be rewritten as:

$$bnR^{n-1}(t)\int_{t}^{\infty} \beta^{1-d}(\tau)m(\tau)\left[\frac{e^{-r\tau}-e^{-ra^{-1}(\tau)}}{r}-e^{-r\tau}\right]d\tau = e^{-rt}.$$

As for investment in capital, this rule equalizes the marginal cost of R&D (right-hand side) and its marginal benefit (the left-hand side). As before, the marginal utility terms do not show up due to the linear utility. Now note that in contrast to a unit of capital, which is necessarily scrapped at finite time, the benefit of R&D investment is everlasting through R&D cumulative technology, which explains integration from *t* to infinity in the left-hand side. Other than this, the left-hand side of the rule can be interpreted as the marginal increase in $\beta(\tau)$, $\tau \geq t$, following the marginal rise in $R(t)$, that is $\frac{bnR^{n-1}(t)}{\beta^{d}(\tau)}$, increases profitability by improving the efficiency of all vintages after the date *t*, but since machines have a finite lifetime, this effect should be computed between τ and $a^{-1}(\tau)$ for each vintage τ, which explains the factor $\frac{e^{-r\tau}-e^{-ra^{-1}(\tau)}}{r} = \int_{\tau}^{a^{-1}(\tau)} e^{-rs}ds$ in the integrand.

In the next section, we analyze the long-term dynamics of the optimization problem (7)-(12) and look for possible exponential balanced growth regimes. After such interior regimes are indentified, the next step will be the analysis of the short-term transition dynamics of the problem provided in Section 4.

3. Optimal long-term dynamics.

In this section, we identify interior optimal trajectories over a "long–term" interval $[t_l,\infty)$ starting with some finite instant $t_l \geq 0$ and examine what kinds of long–term interior regimes are possible in (7)-(12). The necessary extremum condition of Theorem 1 specifies four possible Cases A-D. We can immediately rule out Cases C and D in the long run because then the integrand of the objective function (7) is zero over $[t_l,\infty)$ and it is straightforward to show that these cases cannot be optimal in the sense that they are dominated by other solution paths.

Next, Case B with non-binding physical constraint $E<E_{\max}$ appears to be also impossible. Indeed, then an interior solution should be found from the system

$$I_R'(t)=0, \qquad I_m'(t)=0, \qquad I_{a'}'(t)=0, \quad t\in[t_l,\infty),$$

where $I_R'(t)$, $I_m'(t)$ and $I_{a'}'(t)$ are determined by (16), (19), and (20). As we explained before, this case implies an optimal regime which is boundary in a. Therefore, no long-run interior regime with inactive physical constraint $E<E_{\max}$ is possible. We shall see in Section 4 that such a regime (extensive growth) can arise in the short-term dynamics and it leads to Case A with binging constraint $E=E_{\max}$ in a finite time.

So, the only possible long-run solution is Case A with the binding physical constraint (10): $E(t)=E_{\max}(t)$ at $t\in[t_l,\infty)$. Then the optimal long–term dynamics can involve an interior regime (R,m,a) determined by the system of three nonlinear equations

$$I_R'(t)=0, \qquad I_m'(t)=0,$$

$$\int_{a(t)}^{t} m(\tau)d\tau = E_{\max}(t), \quad t\in[t_l,\infty), \tag{26}$$

where $I_R'(t)$ and $I_m'(t)$ are determined by (16) and (17). Let $r<1$ here and thereafter, otherwise, $I_R'(t)<0$ and $I_m'(t)<0$ by (16),(17). The equations $I_R'(t)=0$ and $I_m'(t)=0$ lead to

$$bnR^{n-1}(t)\int_t^{\infty}\left[bd\int_0^{\tau}R^n(\xi)d\xi+B^d\right]^{1/d-1}m(\tau)\left[\frac{e^{-r\tau}-e^{-ra^{-1}(\tau)}}{r}-e^{-r\tau}\right]d\tau=e^{-rt}, \qquad (27)$$

$$\int_t^{a^{-1}(t)}\left[1-\left[bd\int_0^{a(\tau)}R^n(\xi)d\xi+B^d\right]^{1/d}/\left[bd\int_0^{t}R^n(\xi)d\xi+B^d\right]^{1/d}\right]e^{-r\tau}d\tau=e^{-rt} \qquad (28)$$

at $t\in[t_l,\infty)$.

So, the optimal long-term growth in our model necessarily involves the active physical constraint (Case A of Theorem 1). We can summarize this as the following theorem.

Theorem 2. *Long-term interior optimal regimes are possible in the problem (7)-(12) only under the binding physical constraint* $E=E_{\max}$.

We are interested in *exponential* interior solutions to the problem (7)-(12). The following lemma is helpful in this context.

Lemma 1 (Boucekkine et al., 2011). *If* $R(t)=\overline{R}\,e^{Ct}$ *for some* $C>0$, *then the productivity* $\beta(t)$ *is almost exponential:*

$$\beta(t)\approx\overline{R}^{\,n/d}\left(\frac{bd}{Cn}\right)^{1/d}e^{Cnt/d} \quad \textit{at large } t. \qquad (29)$$

The productivity is the exact exponential function $\beta(t)=Be^{\hat{C}nt/d}=\overline{R}^{\,n/d}\left(bd/\hat{C}n\right)^{1/d}e^{\hat{C}nt/d}$ *at the specially chosen rate* $\hat{C}=nB^d/(bd\overline{R}^{\,n})$.

For brevity, we will later omit the expression "at large t" in the notation $f(t)\approx g(t)$. Now we can formalize the concept of a balanced growth path in problem (7)-(12).

Definition 1. *The Balanced Growth Path (BGP) is a solution* $(R_\Lambda, m_\Lambda, a_\Lambda)$ *to the system of three nonlinear equations (26), (27) and (28), such that* $R_\Lambda(t)$ *grows exponentially,* $m_\Lambda(t)$ *is exponential or constant,* $t-a_\Lambda(t)$ *is a positive constant, and the constraints (11) hold.*

We will explore the possibility of the BGP under the binding physical constraint separately in the cases $n=d$, $n<d$, and $n>d$. We start with the inequality cases $n<d$ and $n>d$, which were not covered in the firm problem of (Boucekkine et al., 2011). We believe that this analysis is important as explained in Introduction: the R&D technology is different for various countries.

3.1. Case *n<d*.

Let us start with the situation where the complexity parameter d is larger than the efficiency parameter, n, which is the case of national economies where the R&D technology is not likely to ensure a balanced growth in the long-run on its own. We show that in this case the physical constraints should be more and more relaxed over time (in a precise sense to be given) for the economy to have balanced and sustainable growth.

Theorem 3. *Let $n<d$. If $E_{\max}(t)$ does not increase exponentially, then there is no interior BGP in the problem (7)-(12). However, if*

$$E_{\max}(t)=\overline{E}\, e^{gt}, \qquad 0<g<\min\{rd/n,\ r(d-n)/n\}, \tag{30}$$

then the problem (7)-(12) has an interior exponential solution

$$R_\Lambda(t)\approx \overline{R}\, e^{Ct},\quad y_\Lambda(t)\sim e^{Ct},\quad \beta_\Lambda(t)\sim e^{Cnt/d},\quad m_\Lambda(t)=\overline{M}\, e^{gt}\quad a_\Lambda(t)=t-T, \tag{31}$$

where

$$C=\frac{gd}{d-n},\qquad \overline{M}=\overline{E}g/\left(1-e^{-gT}\right), \tag{32}$$

$$\overline{R}^{\,d-n}=bn^{2d-1}d^{1-d}\overline{M}^{\,d}\,\frac{C^{d-1}}{r-C(1-n)}\left(\frac{1-e^{-rT}}{r}-1\right)^{d}, \tag{33}$$

and the positive constant T is found from the nonlinear equation

$$\frac{1-e^{-rT}}{r}-\frac{e^{-CnT/d}-e^{-rT}}{r-Cn/d}=1. \tag{34}$$

The solution $(R_\Lambda, m_\Lambda, a_\Lambda)$ is a BGP, at least, when

$$n>1-\frac{1-e^{-CT}}{C}. \tag{35}$$

If $g>\min\{rd/n,\ r(d-n)/n\}$ in (30), then the problem does not possess a finite solution because $E_{max}(t)$ increases too fast.

Proof. Let us substitute

$$R(t) = \bar{R}\, e^{Ct} \qquad \text{and} \qquad t-a(t) = T = \text{const} > 0 \tag{36}$$

into (26), (27) and (28) and estimate the growth order of $m(t)$ at large t. By (26), $m(t)$ satisfies

$$m(t) = m(t-T) + E_{\max}{}'(t)\ . \tag{37}$$

Applying Lemma 1 and Theorems 1 and 2, we find that $\beta(t) \approx R_0{}^{n/d}\left(\dfrac{bd}{Cn}\right)^{1/d} e^{Cnt/d}$,

$$bn\bar{R}^{n-1}e^{C(n-1)t}\int_t^{\infty}\left[\frac{bd}{Cn}\bar{R}^n e^{Cn\tau}\right]^{(1-d)/d} m(\tau)\left[\frac{1}{r}-\frac{e^{-rT}}{r}-1\right]e^{-r\tau}d\tau - e^{-rt} \approx 0\,, \tag{38}$$

$$\bar{R}^{n/d}\left(\frac{bd}{Cn}\right)^{1/d}\left\{\int_t^{t+T}\left[e^{Cnt/d}-e^{Cn(\tau-T)/d}\right]e^{-r\tau}d\tau - e^{Cnt/d}e^{-rt}\right\} \approx 0 \tag{39}$$

at large t. To keep (38), we need an exponentially growing $m(t)$ with the rate $C(1-n/d)>0$. By (37), it is possible only if $E_{\max}(t)$ increases exponentially, i.e., (30) holds. Otherwise, no BGP exists.

Let (30) hold, then m is found from (10) as $m_\Lambda(t)=\bar{M}\, e^{gt}$, where $\bar{M}>0$ is determined by (32). Substituting it into (38), we have $I_R'(t)=0$ only if $g=C(1-n/d)$ and the constant $\bar{R}$ satisfies (33).

The integral equation (39) with respect to T has appeared before in the vintage models with exogenous technological change (Boucekkine et al, 1998; Hritonenko and Yatsenko, 1996). After evaluating the integrals, it leads to the nonlinear (but not integral) equation (34), which has a unique positive solution T if $C<rd/n$ (Hritonenko and Yatsenko, 1996).

To prove that the path (32)-(34) is a BGP indeed, we need to show that the state constant $\beta_\Lambda(t)m_\Lambda(t) - R_\Lambda(t) < y_\Lambda(t)$ holds, at least, at large t. By (8) and (38), $y_\Lambda(t) \approx \bar{R}^{n/d}\bar{M}\left(\dfrac{bd}{Cn}\right)^{1/d}\dfrac{1-e^{-CT}}{C}e^{Ct}$. Therefore,

$$y_\Lambda(t) - \beta_\Lambda(t)m_\Lambda(t) - R_\Lambda(t) \approx \bar{R}^{n/d}\left\{\bar{M}\left(\frac{bd}{Cn}\right)^{1/d}\left[\frac{1-e^{-CT}}{C}-1\right]-\bar{R}^{1-n/d}\right\}e^{Ct}.$$

$$= \bar{R}^{n/d}e^{Ct}\left\{\bar{M}\left(\frac{bd}{Cn}\right)^{1/d}\left[\frac{1-e^{-CT}}{C}-1\right]-\frac{\bar{M}bn}{(r-C+Cn)}\left(\frac{bd}{Cn}\right)^{1/d-1}\left[\frac{1-e^{-rT}}{r}-1\right]\right\}$$

$$= \overline{R}^{n/d} e^{Ct} \overline{M} \left(\frac{bd}{Cn} \right)^{1/d} \frac{Cn}{(r - C + Cn)} \left\{ \left(1 + \frac{r - C}{Cn} \right) \left[\frac{1 - e^{-CT}}{C} - 1 \right] - \frac{n}{d} \left[\frac{1 - e^{-rT}}{r} - 1 \right] \right\}$$

Next, substituting e^{-rT} from (34) into this formula and combining similar terms, we obtain

$$y_\Lambda(t) - \beta_\Lambda(t) m_\Lambda(t) - R_\Lambda(t) \approx \left(\frac{bd}{Cn} \right)^{1/d} \frac{n \overline{R}^{n/d} \overline{M} e^{Ct}}{(r - C + Cn)} \times \left\{ \frac{r - C}{Cn} \left(1 - C(1 - n) - e^{-CT} \right) + e^{-\frac{CnT}{d}} - e^{-CT} \right\}. \quad (40)$$

The first term in brackets is positive at (35) and the second term is positive at $n<d$.

The theorem is proven. □

Some comments are in order here. First, one has to observe that the sufficient condition (35) for the existence of BGPs involves endogenous magnitudes, C and T. It is challenging to express this condition in terms of given model parameters. Nevertheless, it appears to be valid for all economically reasonable ranges of the parameters n, C, and T, for example, if $C<0.1$ and $0.05<n<1$, then (35) holds at $T>1$ year, which is definitely reassuring. Second, it is important to notice that the balanced growth is compatible with a substantial interval (30) of the growth rate g of $E_{\max}$. An arbitrarily small g is enough to ensure a balanced growth, which is a non-trivial and remarkable property. In contrast, too large values of g lead the economy to explosive growth, which is economically straightforward. Third, in this case, the growth rate C of the economy is proportional to the growth rate of E_{max}: clearly, the R&D sector and the associated induced-innovation mechanism are too weak to ensure a balanced growth in this case of under-performing R&D sector. Thus, relaxing physical constraints over time is a necessary accompanying condition. A final crucial remark is worth doing: the innovation rate is equal to $Cn/d=gn/(d-n)$, while the growth rate of production is $C=gd/(d-n)$. Consistently, if $n=0$, then the growth rate of innovation is zero while the growth rate of production C is g. That is to say, the growth generated in this case is *semi-endogenous*: there are two interdependent engines of growth, one exogenous coming from the physical constraint and the other is endogenous reflecting the Porter mechanism. The R&D sector is not necessary for the existence of (exogenous) balanced growth paths; however, operating it allows reaching higher values of growth and welfare.

Last but not least, it is worthwhile to comment on the constraint relaxation condition obtained for the economies with inefficient R&D sector to reach balanced growth paths in Theorem 3. In particular, it is interesting to give it an economic interpretation and to reflect on its feasability. How could an economy relax the physical constraints? If the latter are interpreted as extraction quotas imposed for ecological reasons (that's to prevent ecological catastrophes), then relaxing them does not make sense, and the unique remaining route to sustainable growth is to upgrade the R&D technology (that's to increase *n* and/or decrease *d*) and/or to develop backstop technologies. New discoveries of mineral and other natural resources may help relaxing the physical constraints but it's hard to think of this as a sustainable solution in the sense of the condition required by Theorem 3. If the economy were open, the constraint relaxation condition would call for a more straightforward interpretation and implementation: the economies with inefficient R&D technologies could "relax" their physical constraints by resorting to international markets for raw materials. If the physical constraints are interpreted as environmental regulation constraints like pollution quotas, then resorting to international market of pollution permits is a way to relax these constraints. In the absence of international pollution permits, as in our model, countries with different R&D technologies will converge to different long-term states.

3.2 The case *n>d*

This case is formally symmetrical to the previous one, so we state it briefly. The theorem below gives the technical details for this case.

Theorem 4. *Let n>d. If $E_{\max}(t)$ does not decrease exponentially, then no BGP is possible in the problem (7)-(12). If*

$$E_{\max}(t)=\overline{E}\,e^{-gt},\quad 0<g<<1-d/n,\quad r<<1, \tag{41}$$

then a unique BGP (R_A, m_A, a_A) exists,

$$R_A(t)\approx\overline{R}\,e^{Ct},\quad y_A(t)\sim e^{Ct},\quad \beta_A(t)\sim e^{Cnt/d},\quad m_A(t)=\overline{M}\,e^{-gt},\quad a_A(t)=t-T, \tag{42}$$

where $C=\dfrac{gd}{n-d}$, $\overline{M}=\overline{E}g/\left(1-e^{-gT}\right)$, *and the positive constants* $\overline{R}$ *and T are found from formulas (33) and (34).*

Proof essentially follows the proof of Theorem 3 and leads to similar expressions with the exception that now $m(t)$ decreases rather than increases with the rate g. Formulas (36)-(39) remain valid. To keep $I_R'(t)=0$ by (38), we need an increasing $R(t)\sim e^{Ct}$ and a decreasing $m(t)\sim e^{-gt}$ with $g=C(1-d/n)>0$. If $m(t)$ decreases exponentially, then by (10) $E_{\max}(t)$ also must decrease exponentially with the same rate g to have a BGP. The main difference in the proof is that $\beta_\Lambda(t)m_\Lambda(t) - R_\Lambda(t) < y_\Lambda(t)$ at large t, because the second term in brackets in (40) is negative at $n<d$. So, we assume that r is small, $r<<1$.

By (45), $Cn/d<r<<1$ is also small. Then, as shown in (Hritonenko and Yatsenko, 1996), the unique solution T of equation (34) is large and such that $T \sim (Cn/d)^{-0.5}$. Therefore, $nT/d<<1$ and $CT<<1$. Expressing the exponents in (40) as the Taylor series, we have

$$y_\Lambda(t) - \beta_\Lambda(t)m_\Lambda(t) - R_\Lambda(t) \approx \left(\frac{bd}{Cn}\right)^{1/d} \frac{n\overline{R}^{\,n/d}\,\overline{M}e^{Ct}}{(r-C+Cn)} \left\{\frac{r-C}{n}(T-1+n) - \frac{Cn}{d}T + CT\right\}$$

Finally, because T is large, the last equality leads to

$$y_\Lambda(t) - \beta_\Lambda(t)m_\Lambda(t) - R_\Lambda(t) \approx \left(\frac{bd}{Cn}\right)^{1/d} \frac{n\overline{R}^{\,n/d}\,\overline{M}e^{Ct}}{(r-C+Cn)} CT\left\{\frac{r}{Cn} - \frac{1}{n} + 1 - \frac{n}{d} + o\left(T^{-1}\right)\right\}$$
$$> \left(\frac{bd}{Cn}\right)^{1/d} \frac{n\overline{R}^{\,n/d}\,\overline{M}e^{Ct}}{(r-C+Cn)} CT\left\{\left(\frac{n}{d}-1\right)\left(\frac{1}{n}-1\right) + o\left(T^{-1}\right)\right\} > 0$$

The theorem is proven. □

Therefore, countries with a highly efficient R&D sector should necessarily consume less and less resource to have a long-term growth with a constant capital lifetime.

3.3. Balanced growth at *n=d*

Let us address the situation when the parameter of "R&D efficiency" n equals the parameter of "R&D complexity" d. Then, an interior BGP regime is possible only if the physical resource consumption limit $E_{\max}(t)$ is constant.

Theorem 5. *If $n=d$ and $E_{max}(t)$ is not constant at large t, then no BGP with positive growth exists.*

Proof. By Theorems 1 and 2, any interior regime (R, m, a) has to satisfy the nonlinear system (26)-(28). Let $R(t)=\overline{R}\,e^{Ct}$ and $t-a(t)=T$=const>0. Then, (26) leads to (37). Under the assumption that $E_{max}(t)$ varies in time, $m(t)$ cannot be constant by (37). On the other side, in our case $\beta(t) \approx \bar{R}\left(\frac{b}{C}\right)^{1/n} e^{Ct}$ and equality (27) is

$$bn\bar{R}^{n-1}e^{C(n-1)t}\int_t^\infty \left[\frac{b}{C}\bar{\mathrm{R}}\mathrm{e}^{C\tau}\right]^{(1-n)} m(\tau)\left[\frac{1}{r}-\frac{e^{-rT}}{r}-1\right]e^{-r\tau}d\tau \approx e^{-rt} \tag{43}$$

Differentiating (43), we have

$$bn\left[\frac{b}{C}\right]^{(1-n)} m(t)\left[\frac{1}{r}-\frac{e^{-rT}}{r}-1\right]e^{-rt}e^{C(1-n)t} = d\left(e^{-rt}e^{C(1-n)t}\right)/\,dt$$

It means that $m(t)$ must be constant to satisfy (43). Hence, no BGP exists.

The theorem is proven. □

We now move to the case of constant exogenous environment, which is the case where BGPs typically arise. The findings are summarized in the following theorem.

Theorem 6. *If $n=d$ and $E_{max}(t)=\bar{E}$ =const, then an interior solution of problem (7)-(12)*

$$R_\Lambda(t) \approx \bar{R}\, e^{Ct},\quad \beta_\Lambda(t) \sim e^{Ct},\quad y_\Lambda(t) \sim e^{Ct},\quad m_\Lambda(t)=\bar{M}=\text{const},\quad a_\Lambda(t)=t-\bar{E}/\bar{M}, \tag{44}$$

is possible, where constants C and $\bar{M}$ are determined by the nonlinear system

$$C^{1/d}[r/C+d-1] = d\bar{M}b^{1/d}\left[\frac{1-e^{-r\bar{E}/\bar{M}}}{r}-1\right], \tag{45}$$

$$\frac{1-e^{-r\bar{E}/\bar{M}}}{r} - \frac{e^{-C\bar{E}/\bar{M}} - e^{-r\bar{E}/\bar{M}}}{r-C} = 1. \tag{46}$$

The solution $(R_\Lambda, m_\Lambda, a_\Lambda)$ exists and represents a BGP, at least, in the following cases:

(i) $d>0.5$ *and large enough* $\bar{E}$ *; then the optimal* $C\to 0$ *and* $t-a_\Lambda(t)\to\infty$ *as* $\bar{E}\to\infty$.

(ii) $r\bar{E} << 1,\qquad r^{1/d} < \bar{E}b^{1/d}[1-\sqrt{2r}],$ (47)

then C, $0<C<r$, is a solution of the nonlinear equation

$$C^{(1-d)/d}[r-C(1-d)] = d\bar{E}b^{1/d}\left[1-\sqrt{\frac{1}{2}\left(\frac{r}{\sqrt{C}}+\sqrt{C}\right)}\right]+o(r) \tag{48}$$

and $\bar{M} = \bar{E}\sqrt{C/2}+o(r)$. *The uniqueness of the solution is guaranteed if*

$$r^{1/d-1/2} < \frac{d^2}{4(1-d)} \bar{E} b^{1/d} \sqrt{2} \; . \qquad (49)$$

Proof. Formulas (44)-(49) are obtained in Theorem 3 of (Boucekkine et al., 2011), where the system (45)-(46) is also shown to have a solution C>0 and $\bar{M}$ >0 in the cases (i) and (ii).

To prove that the path (44) is a BGP indeed, we need to show that the state constant $y_A(t) - \beta_A(t)m_A(t) - R_A(t)$>0 holds along (44), at least, at large t. By (8), (29), and (32), $y_\Lambda(t) \approx \bar{R}\bar{M}\left(\frac{b}{C}\right)^{1/d} \frac{1-e^{-C\bar{E}/\bar{M}}}{C} e^{Ct}$. Therefore,

$$y_A(t) - \beta_A(t)m_A(t) - R_A(t) \approx \bar{\mathrm{R}}\mathrm{e}^{Ct}\left\{\bar{M}\left(\frac{b}{C}\right)^{1/d}\left[\frac{1-e^{-C\bar{E}/\bar{M}}}{C}-1\right]-1\right\}.$$

Expressing the exponent above as the Taylor series, we obtain

$$y_\Lambda(t) - \beta_\Lambda(t)m_\Lambda(t) - R_\Lambda(t) = \bar{R}\left\{\bar{M}\left(\frac{b}{C}\right)^{1/d}\left[\bar{E}/\bar{M} - \frac{C}{2}\left(\bar{E}/\bar{M}\right)^2 + o(r) - 1\right] - 1\right\}e^{Ct}.$$

On the other side, expressing the exponent in (33) as the Taylor series, we have

$$C^{(1-d)/d}[r - C(1-d)] = d\bar{M}b^{1/d}\left[\bar{E}/\bar{M} - \frac{r}{2}\left(\bar{E}/\bar{M}\right)^2 + o(r) - 1\right].$$

Combining the last two formulas, we obtain

$$y_\Lambda(t) - \beta_\Lambda(t)m_\Lambda(t) - R_\Lambda(t) \approx \bar{R}\left\{\frac{r - C(1-d)}{Cd} - 1\right\}e^{Ct} = \frac{r-C}{Cd}\bar{\mathrm{R}}\mathrm{e}^{Ct} > 0.$$

The theorem is proven. □

The conditions (i) and (ii) are sufficient for the existence of the BGP. The BGP can also exist when these conditions do not hold. The uniqueness condition (49) is also sufficient. The only possible case of non-uniqueness when we need condition (49) is when the optimal C is close to r.

It is clear that the BGP in case $n=d$ is also induced by the R&D sector of the economy and illustrates a Porter-like mechanism. Indeed, as statement (i) of Theorem 6 indicates, the growth rate tends to zero when the constraint level $\bar{E}$ goes to zero. The long term growth is endogenous and is determined by the model parameters r and d and the constraint level $\overline{E}$. It can readily shown that a further decrease of $\overline{E}$ leads to the decrease of both optimal growth rate C and optimal investment in efficiency units $\overline{M}$. In other words, while an induced-innovation mechanism is at work, tightening physical constraints negatively affects the rate of innovation and growth of the economy. Thus, we uncover a kind of *mild Porter-like mechanism* in the balanced case $n=d$: physical constraints are necessary for R&D to get launched but too strict constraints kill the growth. By (44), the growth rate C of β (the *innovation rate*) is equal to the growth rate of production y and investments i and R along the BGP.[6]

4. Transition Dynamics

We can show that the short-term dynamics will remain qualitatively the same for any bounded constraint level $E_{\max}(t)$, provided that a long-term interior regime exists. However, as shown in Section 3, essential difficulties arise in finding such regimes. For this reason and for clarity sake, we restrict ourselves in this section with the case of $n=d$ and a constant function $E_{\max}(t)=\overline{E}$. The long-term interior regime in this case is the BGP (R_Λ, m_Λ, a_Λ) determined by Theorem 6.

As proven in Theorem 2, the long-term dynamics necessarily involves the active physical constraint (10). In this section, we will show that:

(1) All Cases A-D from Theorem 1 are possible in short–term dynamics. The optimal trajectories during the transition period appear to be qualitatively different depending on whether the physical constraint (10) is initially active, $E(0)=E_{\max}$ (Cases B and C), or inactive, $E(0)<E_{\max}$ (Cases A and D).

(2) The short-term transition dynamics always leads to the long-term interior regime with the active physical constraint.

The solution $R^*(t)$, $m^*(t)$, and $a^*(t)$, $t\in[0,\infty)$, of the optimization problem (7)-(12) must satisfy the initial conditions (12). The initial condition $a(0)=a_0$ is essential because of the continuity of the unknown a. If $a_0\neq a_\Lambda(0)$, then the dynamics of (R^*, m^*, a^*) involves a

[6] We don't detail here the computation of the BGP. It goes without saying that given that growth is endogenous, we also face a problem of indeterminacy in levels. This technical point is made precisely in (Boucekkine et al., 2011).

transition from the initial state $a(0)=a_0$ to the long-term interior trajectory $a_\Lambda(t)$ from Theorem 6. Also, the given model functions shall satisfy the inequality

$$R_0(0)+Bm_0(0)\le \int_{-a_0}^{0}\left[\beta_0+\int_{-a_0}^{\tau}R_0(v)dv\right]m_0(\tau)d\tau \ . \tag{50}$$

Otherwise, the constraint (11) is violated at t=0 and the economic system is not possible.

4.1. Optimal intensive growth at active physical constraint.

Let $E(t)=E_{\max}$ starting from the initial time t=0. Then the optimal dynamics are subjected to Case A or D of Theorem 1 (with the active restriction $E(t)=E_{\max}$ on $[0,\infty)$). This regime is *a* ***growth with intensive capital renovation*** induced by technical progress. In order to make a new capital investment $m(t)$ at $t\ge t_k$, some obsolete capital should be removed, following the equality (10) under the given $E(t)=E_{\max}$ or

$$\int_{a(t)}^{t} m(\tau)d\tau = E_{\max}.$$

In the long-term dynamics considered in Section 3, the optimal R&D innovation $R^*(t)$ is the interior trajectory $R_\Lambda(t)$ determined from $I_R'(t)=0$, where $I_R'(t)$ is given by (16). The optimal $R^*(t)$ reaches the trajectory $R_\Lambda(t)$ immediately at t=0. The long-term dynamics has the interior turnpike trajectory a_Λ for the capital lifetime, determined from $I_m'(t)=0$ or

$$\int_{t}^{a^{-1}(t)} e^{-r\tau}\left[\beta(t)-\beta(a(\tau))\right]d\tau = e^{-rt}\beta(t)\,.$$

If $a_0=a_\Lambda(0)$, then the optimal capital lifetime $a^*\equiv a_\Lambda$, that is, no transition dynamics at all. If $a_0\ne a_\Lambda(0)$, then we can show that the optimal $a^*(t)$ will reach $a_\Lambda(t)$ at some time $t_l>0$. If $a_0>a_\Lambda(0)$, then the optimal investment $m^*(t)=0$ is minimal at $0<t\le t_l$ (Case A). If $a_0<a_\Lambda(0)$, then the optimal investment $m^*(t)=(y^*(t)-R^*(t))/\beta^*(t)$ is maximal at $0<t\le t_l$ (Case D).

After the transition, at $t>t_l$, the optimal long-term trajectory $m^*(t)$ possesses a repetitive pattern in a general case (Hritonenko and Yatsenko, 1996; Boucekkine et al., 1997) determined by the dynamics of $m(t)$ on the interval $[a_0, t_l]$. These replacement echoes are absent at the "perfect" initial condition $a_0=a_\Lambda(0)$, $m_0(\tau)=\overline{M}$, $\tau\in[a_0,0]$. To illustrate them, we provide a numeric example shown in Figure 1 that will be also used and developed in the next section.

Example 1. Let

$$r=0.05, \quad d=n=0.5, \quad b=0.005, \quad E_{\max}(t)=\overline{E}=22,$$

$$a_0=-22, \quad \beta_0=1, \quad R_0(\tau)=0, \quad m_0(\tau)=1, \quad \tau\in[-22,0].$$

Then, $B=\beta(0)=1$ by (13) and the BGP exists:

$$R_A(t)=R_0e^{Ct}, \; C=0.00225, \quad m_A(t)=M_0=0.55, \quad a_A(t)=t-40, \; t\in[0,\infty).$$

The BGP is indicated by the dotted lines in Figure 1. In this case, $E(0)=m_0a_0=22$ is equal to $E_{\max}(0)=\overline{E}$, hence, the physical constraint (10) is active starting t=0. Since $a_A(0)=-40 < a_0=-22$, then the optimal $a^*(t)=-22$ and $m^*(t)=0$ at $0<t\leq t_l=18$ (Case A). After t_l, the optimal $a^*(t)$ coincides with $a_A(t)$ and $m^*(t)=m^*(t-40)$ exhibits *replacement echoes* (shown with dotted lines).

4.2. Optimal extensive growth

Let the resource consumption $E(t)$ be lower than the limit $E_{\max}$ at time t=0. We assume that $E(t)<E_{\max}$ over a finite interval $0\leq t<t_k$, where the moment t_k is to be determined. Then, we have Case B or C of Theorem 1, at least, at the beginning of the planning horizon. Since $I_{a'}'(t)<0$ by (20), the boundary regime $a^*(t)\equiv a_0$ is always optimal while $E(t)<E_{\max}$.

First, let $m(t)<(y(t)-R(t))/\beta(t)$ (Case B), then $I_m'(t)\leq 0$, otherwise the optimal investment m^* is maximal possible and we immediately switch to Case C. By (19), Case B is highly unlikely in economic practice. It means an extremely underfunded initial capital (determined by the length a_0 of the prehistory) combined with a high impatience (a high discount rate r). Indeed, simple calculations show that for the discount rates $10\%<r<50\%$, Case B occurs if the initial prehistory length a_0 is less than 1.05 - 1.4 years. For such values of a_0, the constraint (50) imposes extremely severe restrictions on the initial functions m_0 and R_0 and value β. In Case B, the optimal investment m^* is zero and no capital scrapping occurs, which corresponds to the trivial solution $R^0\equiv 0$, $m^0\equiv 0$ of the problem (7)-(12). In this case, the non-trivial long-run solution with investing into new capital and R&D is not possible.

For economically reasonable values of the discount r<10%/year and the initial capital lifetime a_0 more than one year, $I_m'(t)>0$ by (19). Hence, the optimal investment m^* is maximal possible and we have Case C. Then, the country can use more new capital and there is no need to remove the old one, which can be classified as ***an extensive economic growth***. The upper bound for $m(t)$ is given by the constraint (11) and the optimal $m^*(t)$ jumps to this bound immediately after t=0. In this case, the inequality-constraint $m(t) \leq (y(t) - R(t))/\beta(t)$ limits both optimal controls R^* and m^*. Therefore, the transition dynamics on some initial period $[0, t_k]$ is determined by the restriction

$$R^*(t) + \beta^*(t)m^*(t) = y^*(t) \tag{51}$$

until $E(t_k)=E_{max}$. Resource consumption $E(t) = \int_{a_0}^{t} m^*(\tau)d\tau$ is increased fast and the limit E_{max} will be reached shortly, which will mean the end of the extensive growth phase. Following Case C of Theorem 1, the optimal $R^*(t)$, $m^*(t)$ and $y^*(t)$ over $[0,t_k]$ are determined from the system of three nonlinear equations (10), (51), and $I_R'(t)=0$.

The end t_k of the "extensive-growth" transition period $[0, t_k]$ is determined from the condition $E(t_k)=E_{max}$. After the transition period $[0, t_k]$, the optimal dynamics will switch to the scenario of Section 4.1 with the active constraint (10).

If $a^*(t_k) \neq a_A(t_k)$, then the "extensive-growth" transition on $[0, t_k]$ is followed by one of the intensive growth transition scenarios on $[t_k,t_l]$, $t_l>t_k$, described in Section 4.1. If $a^*(t_k)>a_A(t_k)$, then the optimal investment $m^*(t)=0$ is minimal on $[t_k,t_l]$ (Case A). If $a^*(t_k)<a_A(t_k)$, then the optimal investment $m^*(t)=(y^*(t) - R^*(t))/\beta^*(t)$ is maximal on $[t_k,t_l]$ (Case D).

Example 2. Let all given parameters be as in Example 1 except for $m_0(\tau)$ =0.5, $\tau \in [-22,0]$. Then the BGP is the same as in Example 1 but the transition dynamics is different and is shown in Figure 2.

In this case, $E(0)=m_0a_0=0.5*22=11$ is less than $E_{max}(0)=22$, hence, the physical constraint (10) is inactive on an initial interval $[0, t_k]$ at the beginning of the planning horizon. The dynamics of the optimal $m^*(t)$ and $R^*(t)$ on $[0, t_k]$ follows the restriction $R^*(t)+\beta^*(t)m^*(t)=y^*(t)$ (Case C of Theorem 1). The optimal $R^*(t)$ over $[0, t_k]$ is found from (27) as

$$R^{0.5}(t) = 2b\chi^{-1}(t)\int_{t_k}^{\infty} \beta^{0.5}(\tau)m(\tau)\left[\left(e^{-r\tau} - e^{-ra^{-1}(\tau)}\right)/r - e^{-r\tau}\right]d\tau$$

where $\chi(t) = \int_{t}^{t_k} \chi(\tau)d\tau + \int_{t_k}^{a^{-1}(t)} e^{-r\tau}d\tau$ over [0, t_k] is found from (23) and $\chi(t)= e^{-rt}$ on [t_k,∞).

Finding an approximate solution of the arising equations, we obtain that $R^*(t)\approx 0.2$ at $0\le t\le t_k$. Then, the optimal $m^*(t)\approx 10.8$ at $t\approx 0$ and $m^*(t)\approx 21.8$ at $t\approx t_k$. The corresponding $E^*(t)$ increases fast and reaches the limit value $E_{max}=22$ at $t_k\approx 0.75$. The corresponding $y^*(t)$ also increases fast from $y^*(0)\approx 11$ to $y^*(t_k)\approx 22$.

The further optimal dynamics on [t_k,∞) is similar to Example 1 and follows Case A. It is shown in Figure 2 with black curves.

As opposed to the "intensive-growth" scenario of Example 2, the optimal trajectory $m^*(t)$ *always* possesses the replacement echoes after the transition. Indeed, no "perfect" initial condition is possible in this case. If $m_0(\tau)=\overline{M}$ on [a_0,0], then $a_0>a_\Lambda(0)$ by $E(0)<E_{max}$. Alternatively, if $a_0=a_\Lambda(0)$, then $\int_{a_0}^{0} m_0(\tau)d\tau < \int_{a_0}^{0} \overline{M}d\tau$. The optimal short-term trajectory $m^*(t)$ is different from $\overline{M}$ on the "extensive-growth" transition period [0, t_k], and the optimal trajectory $m^*(t)$ will repeat the dynamics of $m(t)$ on [a_0, t_k].

We can summarize the above reasoning in the following statement.

Theorem 7. *In the case n=d and a constant E_{max}, the transition dynamics of the problem (7)-(12) leads to the BGP with active physical constraint (described by Theorem 6) in a finite time $t_k\ge 0$, regardless how large the value E_{max} is. The transition dynamics is absent (t_k=0) only if*

$$a_0=a_\Lambda(0) \quad and \quad E(0)=E_{max}. \tag{52}$$

If (52) holds and $m_0\equiv\overline{M}$, then the optimal $m^*\equiv\overline{M}$, otherwise, the optimal trajectory m possesses everlasting replacement echoes that repeat the dynamics of m* on the interval* [a_0, t_k].

This theorem describes the complete dynamics of the central planner problem (7)-(12) in case $d=n$. The dynamics will be qualitatively similar for any values of n and d and any bounded function E_{max}.

5. Concluding remarks

In this paper, we have studied the optimal investment and capital replacement policies in an economy with R&D sector under physical constraints. The corresponding optimal control problem includes nonlinear integral equations with endogenous integration limits and the state constraints active along the optimal trajectories, which justifies the mathematical novelty. We have provided a systematic qualitative analysis of this problem and extended significantly previous results by characterizing all possible balanced growth paths for any parameterizations of the R&D technology. In particular, we demonstrate that the presence of the physical constraint is essential for getting a meaningful optimal dynamics in the central planner problem (7)-(12) with linear utility. It complements and clarifies the result of (Yatsenko et al., 2009) that the model (8)-(12) without physical constraints has only *blow-up solutions* that strive to the infinity in a finite time for any R&D parameterization and the corresponding objective functional (7) is always infinite.

Next, we have studied transitional dynamics to balanced growth, a task not undertaken so far. The optimal dynamics obtained in this paper is quite new in the related economic literature (see for example, Boucekkine et al., 1997). We have uncovered two optimal transition regimes: an intensive growth (sustained investment in new capital and R&D with scrapping the oldest capital goods), and an extensive growth (sustained investment in new capital and R&D without scrapping the oldest capital). In the short run, the modernization policy can consist of increasing investment in new capital and R&D without scrapping the older and more resource consuming capital. The long-run modernization policy encompasses scrapping the oldest capital goods following the intensive growth scenario described above. The reason behind this is quite elementary: a country with a low enough initial capital stock (and so, with low enough initial resource consumption) has no incentive to scrap its old capital assets as long as its resource constraint is not binding. In contrary, at a binding constraint, investing in new assets is not possible without scrapping some obsolete older assets because of market clearing conditions or binding regulation or technological constraints. In other words, our model predicts that historically poor countries may find it optimal to massively invest and, therefore, over-exploit their resources during the early stage of their development process. Such transition growth regime comes to the end when the physical constraint upper-bound is reached and is followed by an intensive balanced growth with scrapping of old capital under active constraints. After the transition dynamics ends, the optimal capital investment possesses everlasting replacement echoes that repeat the investment dynamics during the prehistory and transition periods. In general, the modernization policy is similar to simpler vintage models with exogenous technological change (Boucekkine et al., 1997, 1998; Hritonenko and Yatsenko, 2005, 2008).

In contrast, the optimal R&D policy is more robust, exponential, and not sensitive to the initial structure of capital distribution. This outcome is in a good agreement with the celebrated non-vintage model of endogenous growth under restricted non-renewable resource (Romer, 1990), which produces a sustainable exponential balanced growth for *any* R&D efficiency. In our model, the rate of the endogenous growth is determined only by the R&D parameterization, the physical constraint, and the discount rate.

Appendix

Proof of Theorem 1: The proof is based on perturbation techniques of the optimization theory. It extends the approach earlier applied by Hritonenko and Yatsenko (2005, 2008) to vintage models with exogenous technological change and state constraints.

Case B. Let the restrictions (10),(11) be inactive on a certain subset Δ of the interval $[0,\infty)$: $E^*(t)<E_{max}(t)$ and $R(t)+\beta(t)m(t)<y(t)$ at $t\in\Delta\subset[0,\infty)$. We choose R, m, and $v=a'$ to be the *independent* unknown variables of the OP (7)-(12). Then, the differential restriction $a'(t)\geq 0$ in (11) takes the standard form $v(t)\geq 0$. The dependent variables $y(t)$, $E(t)$ and $\beta(t)$ can be found from (8), (10), and (13). We assume that $R, m, v\in L^{\infty}_{loc}[0,\infty)$.

We refer to measurable functions δR, δm, and δv as *admissible variations*, if R, m, v, $R+\delta R$, $m+\delta m$, and $v+\delta v$, satisfy (8)-(11). Let us give small admissible variations $\delta R(t)$, $\delta m(t)$, and $\delta v(t)$, $t\in(0,\infty)$, to R, a, m, and find the corresponding variation $\delta I = I(R+\delta R, m+\delta m, v+\delta v) - I(R,m,v)$ of the objective functional I. Using (7)-(10) and (13), we obtain that

$$\delta I = \int_0^\infty e^{-rt}[\int_{a(t)+\delta a(t)}^{t} \left(db\int_0^\tau (R(\xi)+\delta R(\xi))^n d\xi + B^d \right)^{\frac{1}{d}} (m(\tau)+\delta m(\tau))d\tau$$

$$-(R(t)+\delta R(t)) - \left(db\int_0^t (R(\xi)+\delta R(\xi))^n d\xi + B^d \right)^{\frac{1}{d}} (m(t)+\delta m(t))]dt \tag{A1}$$

$$-\int_0^\infty e^{-rt}[\int_{a(t)}^{t} \left(db\int_0^\tau R^n(\xi)d\xi + B^d \right)^{\frac{1}{d}} m(\tau)d\tau - m(t)\left(db\int_0^t R^n(\xi)d\xi + B^d \right)^{\frac{1}{d}} + R(t)]dt$$

where $\delta a(t) = \int_0^t \delta v(\xi)d\xi$. To prove the theorem, we shall transform (A1) to the form

$$\delta I = \int_0^\infty (I'_R(t)\cdot\delta R(t) + I'_m(t)\cdot\delta m(t) + I'_v(t)\cdot\delta v(t))dt + o(\|\delta R\|,\|\delta m\|,\|\delta v\|)\,, \tag{A2}$$

where the norm is $\|f\| = ess\sup_{[0,\infty)} |e^{-rt} f(t)|$. This transformation involves several steps. First, applying the Taylor expansion, we have

$$\begin{aligned}
&\left(db\int_0^\tau (R(\xi)+\delta R(\xi))^n d\xi + B^d \right)^{\frac{1}{d}} \\
&= \left(db\int_0^\tau (R^n(\xi) + nR^{n-1}(\xi)\delta R(\xi) + o(\delta R(\xi)))d\xi + B^d \right)^{\frac{1}{d}} \\
&= \beta(\tau) + bn\beta^{1-d}(\tau)\int_0^\tau R^{n-1}(\xi)\delta R(\xi)d\xi + \int_0^\tau o(\delta R(\xi))d\xi.
\end{aligned} \tag{A3}$$

Next, using (A3) and properties of integrals, (A1) can be rewritten as

$$\begin{aligned}
\delta I = &\int_0^\infty e^{-rt}[bn\int_{\max\{a(t),0\}}^{t} m(\tau)\beta^{1-d}(\tau)\int_0^\tau R^{n-1}(\xi)\delta R(\xi)d\xi d\tau dt \\
&-\int_0^\infty e^{-rt}\int_{\max\{a(t),0\}}^{t} \beta(\tau)\delta m(\tau)d\tau dt \\
&-\int_0^\infty e^{-rt}\int_{a(t)}^{a(t)+\delta a(t)} \beta(\tau)m(\tau)d\tau dt - \int_0^\infty e^{-rt}[\delta R(t) + \beta(t)\delta m(t)]dt \\
&-\int_0^\infty e^{-rt}m(t)\beta^{1-d}(t)\int_0^t bnR^{n-1}(\xi)\delta R(\xi)d\xi dt + \int_0^\infty e^{-rt}o(\delta R(t),\delta m(t))dt\,,
\end{aligned} \tag{A4}$$

where max{$a(t)$,0} appears because the variations $\delta R(t)$, $\delta m(t)$ are zero on the interval [a_0,0]. Interchanging limits of integration in the second term of (A4)

$$\int_0^\infty e^{-rt}[\int_{a(t)}^{t} \beta(\tau)\delta m(\tau)d\tau dt = \int_0^\infty \int_t^{a^{-1}(t)} e^{-r\tau}d\tau \cdot \beta(t)\delta m(t)dt,$$

and the first term

$$\int_0^\infty e^{-rt}[\int_{\max\{a(t),0\}}^{t} m(\tau)\beta^{1-d}(\tau)\int_0^\tau bnR^{n-1}(\xi)\delta R(\xi)d\xi d\tau]dt$$

$$= bn\int_0^\infty \int_t^\infty \int_\tau^{a^{-1}(\tau)} e^{-r\xi}d\xi \cdot m(\tau)\beta(\tau)^{1-d}\, d\tau \cdot R^{n-1}(t)\delta R(t)dt,$$

and applying the Taylor expansion, (A4) can be rewritten as:

$$\delta I = \int_0^\infty [-e^{-rt} + bn\int_t^\infty (\int_\tau^{a^{-1}(\tau)} e^{-r\xi}d\xi - e^{-r\tau})\cdot m(\tau)\beta^{1-d}(\tau)d\tau \cdot R^{n-1}(t)]\cdot \delta R(t)dt$$

$$+\int_0^\infty [\int_t^{a^{-1}(t)} e^{-r\tau}d\tau - e^{-rt}]\cdot \beta(t)\delta m(t)dt$$

$$+-\int_0^\infty e^{-rt}\beta(a(t))m(a(t))\cdot \delta a(t)dt + \int_0^\infty e^{-rt}o(\delta R(t),\delta m(t),\delta a(t))dt\,.$$

Finally, recalling that $\delta a(t) = \int_0^t \delta v(\xi)d\xi$, we convert the last expression to

$$\delta I = \int_0^\infty [-e^{-rr} + bn\int_t^\infty (\int_\tau^{a^{-1}(\tau)} e^{-r\xi}d\xi - e^{-r\tau})\cdot m(\tau)\beta^{1-d}(\tau)d\tau \cdot R^{n-1}(t)]\cdot \delta R(t)dt$$

$$+\int_0^\infty [\int_t^{a^{-1}(t)} e^{-r\tau}d\tau - e^{-rt}]\cdot \beta(t)\delta m(t)dt \tag{A5}$$

$$-\int_0^\infty \int_t^\infty e^{-r\tau}\beta(a(\tau))m(a(\tau))d\tau \cdot \delta v(t)dt + \int_0^\infty e^{-rt}o(\delta R(t),\delta m(t),\delta v(t))dt$$

The combination of (A5), (17), (19), and (20) leads to (A2). The domain (11) of admissible controls R, m, v has the simple standard form $R\geq 0$, $m\geq 0$, $v\geq 0$. So, the optimality condition (18) follows from the obvious necessary condition that the variation δI of the functional I cannot be positive for any admissible variations $\delta R(t)$, $\delta m(t)$, $\delta v(t)$, $t\in[0,\infty)$.

Case A. If the constraint $R(t)+\beta(t)m(t)<y(t)$ is inactive and the restriction (10) is active: $E(t)=E_{\max}(t)$ at $t\in\Delta\subset[0,\infty)$, then we choose R and m to be *independent unknowns* of the OP. The dependent (*state*) variable a is uniquely determined from the initial problem

$$m(a(t))a'(t) = m(t) - E_{\max}'(t), \quad a(0)= a_0,$$

obtained after differentiating (10). As shown in Hritonenko and Yatsenko (2008), if $E_{\max}'(t)\le 0$, then for any measurable $m(t)\ge 0$, a unique a.e. continuous function $a(t)<t$ exists and a.e. has $a'(t)\ge 0$[7]. Therefore, the state restrictions $a'(t)\ge 0$ and $a(t)<t$ in (11) are satisfied automatically, so we can exclude the dependent variable a from the optimality condition.

Similarly to the previous case, let us give small admissible variations $\delta R(t)$ and $\delta m(t)$, $t\in[0,\infty)$, to R and m and find the corresponding variation $\delta I = I(R+\delta R, m+\delta m) - I(R,m)$ of the functional I. In this case, the variation δa is determined by δm. To find their connection, let us present (10) as

$$E_{\max}(t) = \int_{a(t)}^{t} m(\tau)d\tau = \int_{a(t)+\delta a(t)}^{t} (m(\tau)+\delta m(\tau))d\tau$$

then

$$\int_{\max\{a(t),0\}}^{t} \delta m(\tau)d\tau = \int_{a(t)}^{a(t)+\delta a(t)} m(\tau)d\tau + o(\|\delta m\|,\|\delta a\|)\ . \tag{A6}$$

Next, we use (A4) for the variation δI and eliminate δa using (A6). To do that, we rewrite the third term of (A4) by adding $\pm\int_0^\infty e^{-rt}\beta(a(t))\int_{a(t)}^{a(t)+\delta a(t)} m(\tau)d\tau dt$ and applying (A6) as

$$\begin{aligned}
&-\int_0^\infty e^{-rt}\int_{a(t)}^{a(t)+\delta a(t)} \beta(\tau)m(\tau)d\tau dt \\
&= -\int_0^\infty e^{-r}\beta(a(t))\int_{a(t)}^{a(t)+\delta a(t)} m(\tau)d\tau dt + \int_0^\infty e^{-rt}\int_{a(t)}^{a(t)+\delta a(t)} (\beta(a(t))-\beta(\tau))m(\tau)d\tau dt \\
&= -\int_0^\infty e^{-rt}\beta(a(t))\int_{\max\{a(t),0\}}^{t} \delta m(\tau)d\tau dt + \int_0^\infty e^{-rt}o(\delta a(t),\delta m(t))dt \\
&= -\int_0^\infty \int_t^{a^{-1}(\tau)} e^{-r\tau}\beta(a(\tau))d\tau\cdot\delta m(t)dt + \int_0^\infty e^{-rt}o(\delta a(t),\delta m(t))dt
\end{aligned} \tag{A7}$$

The integral $\int_a^{a+\delta a} (\beta(a)-\beta(\tau))m(\tau)d\tau$ in (A7) has the order $o(\delta a)$ because $\beta(\tau)$ is continuous.

Substituting (A7) into (A4) and collecting the coefficients of δm and δR, we obtain the expression

$$\delta I = \int_0^\infty (I_R'(t)\cdot\delta R(t) + I_m'(t)\cdot\delta m(t))dt + o(\|\delta R\|,\|\delta m\|) \tag{A8}$$

in the notations (16) and (17). The rest of the proof is similar to Case A.

[7] For brevity, the theorem omits the possible case $E_{\max}'(t)>0$ treated in Hritonenko and Yatsenko (2005, 2008).

Case C. Now the active constraint $R(t)$ + $\beta(t)m(t)$ = $y(t)$ on Δ involves four unknown variables So, we cannot handle this constraint as easy as the constraint $E(t)$=$E_{max}(t)$ in Case B. We shall apply the method of Lagrange multipliers and take into account the equality-constraint $R(t)$ + $\beta(t)m(t)$ = $y(t)$, $t\in\Delta$.

Let us introduce the Lagrange multiplier $\lambda(t)$, $t\in[0,\infty)$, for the equality $R(t)$ + $\beta(t)m(t)$ = $y(t)$ on Δ and make the usual assumption that $\lambda(t)$=0 at $t\in[0,\infty)$–Δ because of the complementary slackness condition. Now we minimize the Lagrangian

$$L = I + \int_0^\infty (y(t) - R(t) - \beta(t)m(t))\lambda(t)dt \tag{A9}$$

instead of the functional I (7). As in previous cases, we give small admissible variations to R, m, and a and find the corresponding variation $\delta L = L(R+\delta R, m+\delta m, v+\delta v) - L(R,m,v)$ of (A9). Providing all necessary transformations as above, we arrive to the following expression

$$\delta L = \int_0^\infty (\hat{I}'_R(t)\cdot\delta R(t) + \hat{I}'_m(t)\cdot\delta m(t) + I'_v(t)\cdot\delta v(t))dt\,,$$

where

$$\hat{I}_R{}'(t) = bnR^{n-1}(t)\int_t^\infty \beta^{1-d}(\tau)m(\tau)\left[\int_\tau^{a^{-1}(\tau)} e^{-r\xi}[1-\lambda(\xi)]d\xi - e^{-r\tau}[1-\lambda(\tau)]k(\tau)\right]d\tau - e^{-rt}[1-\lambda(t)]$$

$$\hat{I}_m{}'(t) = \beta(t)\int_t^{a^{-1}(t)} e^{-r\tau}[1-\lambda(\tau)]d\tau - e^{-rt}[1-\lambda(t)]\beta(t)\,,$$

and $I_v{}'(t)$ is given by the same formula (20).

As usually in the method of Lagrange multipliers, we choose $\lambda(t)$ from the condition $\hat{I}'_m(t)$=0 at $t\in\Delta$ which after introducing the new variable $\chi(t)$=[1-$\lambda(t)$]e^{-rt} leads to (23). The expression for $\hat{I}'_R(t)$ in the variable χ is (22).

Case D is the combination of Cases C and A. It is proven by combining reasoning and transformations of Cases A and C. The theorem is proven. □

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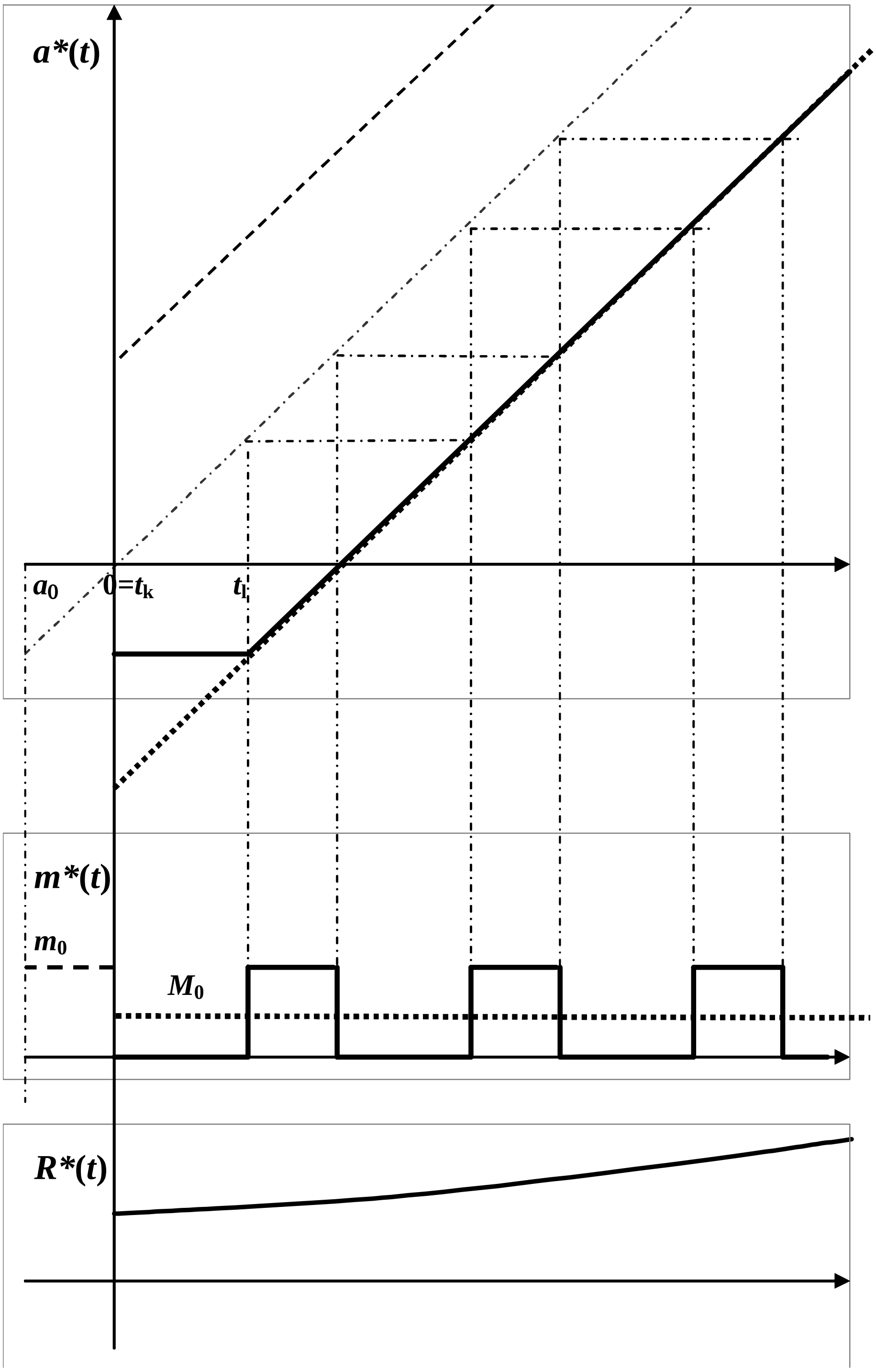

Figure 1. Transition and long-term dynamics under active environment regulation from Example 1. The dashed line shows the inverse function a^{-1}. The dotted lines indicate the BGP regime.

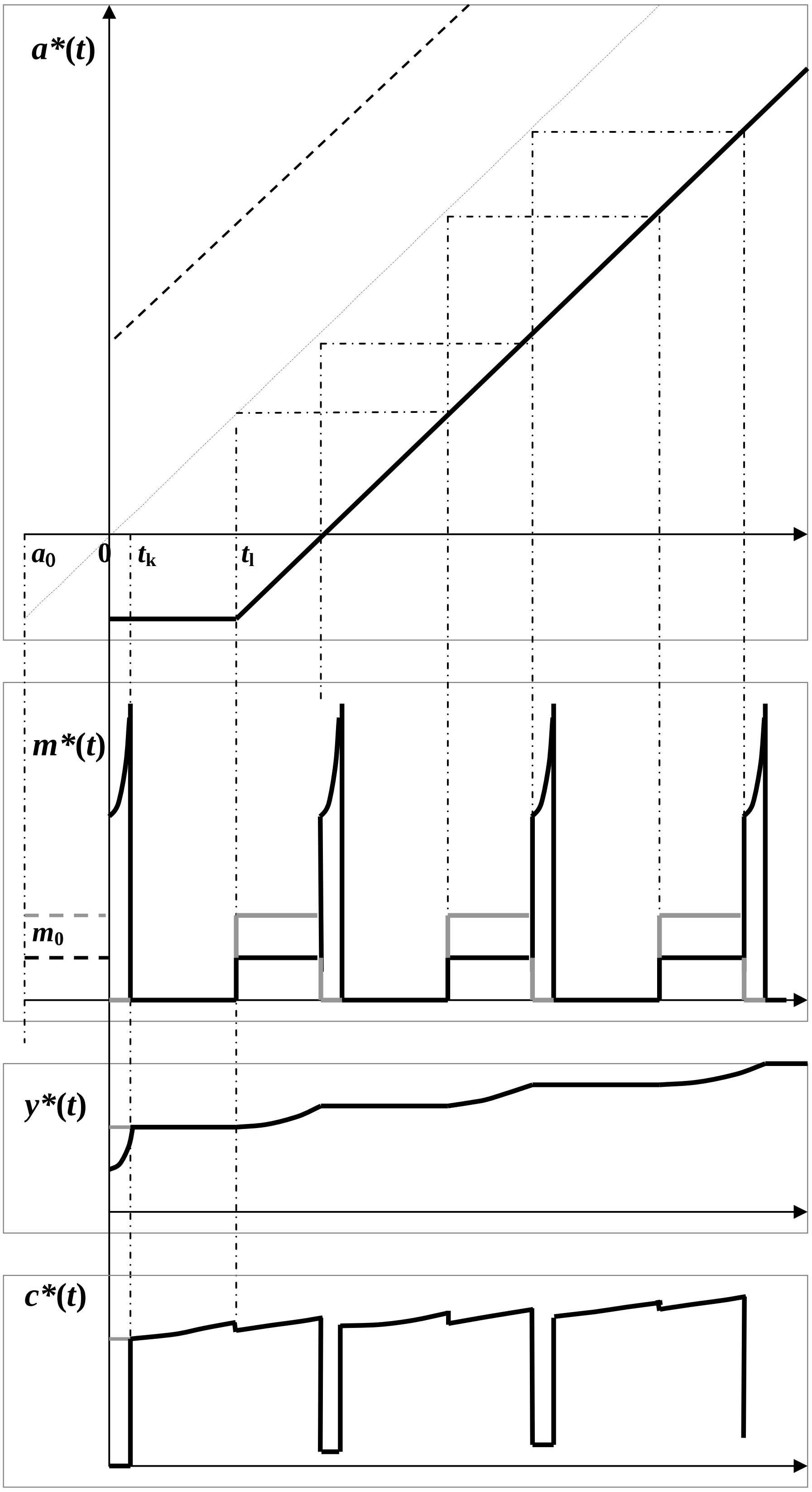

Figure 2. Transition and long-term dynamics under initial inactive environment regulation from Example 3. The optimal dynamics at active regulation from Example 2 is shown in grey color.