SIAM J. Matrix Anal. Appl.
Vol. 0, No. 0, pp. 000–000

COUNTEREXAMPLES TO THE COMPLEX POLYTOPE EXTREMALITY CONJECTURE*

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Abstract. We disprove a recent conjecture of Guglielmi, Wirth, and Zennaro, stating that any nondefective set of matrices having the finiteness property has an extremal complex polytope norm. We give two counterexamples that show that the conjecture is false even if the set of matrices is supposed to admit the positive orthant as an invariant cone, or even if the set of matrices is assumed to be irreducible.

Key words. joint spectral radius, finiteness property, counterexamples

AMS subject classifications. 93D09, 93D20, 15A99

DOI. 10.1137/080730652

1. Introduction.

The joint spectral radius of a finite set of matrices $\Sigma \subset \mathbb{R}^{n\times n}$ characterizes the maximal asymptotic rate of growth of products of matrices taken from Σ. Let $\|\cdot\|$ be any submultiplicative matrix norm. For a given natural t, we denote $\hat\rho_t(\Sigma) = \max\{\|A_1\dots A_t\|^{1/t} : A_i \in \Sigma\}$. The *joint spectral radius* of the set Σ is defined as

$$\rho(\Sigma) \triangleq \lim_{t\to\infty} \hat\rho_t(\Sigma). \tag{1.1}$$

It is well known that the limit exists and that this quantity does not depend on the norm chosen. Moreover, it can also be defined as the asymptotic rate of growth of the maximal spectral radius of products of matrices from Σ: Define $\rho_t(\Sigma) = \max\{\rho(A_1\dots A_t)^{1/t} : A_i \in \Sigma\}$, where $\rho(A) = \lim_{t\to\infty}\|A^t\|^{1/t}$ is the spectral radius of the matrix A, that is, the largest modulus of its eigenvalues, one has an alternative definition of the joint spectral radius:

$$\rho(\Sigma) \triangleq \limsup_{t\to\infty} \rho_t(\Sigma) \tag{1.2}$$

(see [2] for the proof). The joint spectral radius has many applications in functional analysis, probability, approximation theory, discrete mathematics, linear switching systems, etc. (see [13, 21] for many references). The importance of this notion can be illustrated, for instance, by the following theorem from the theory of linear dynamical systems.

Theorem 1 (see [13]). *Let $\Sigma \subset \mathbb{R}^{n\times n}$ be a set of matrices. The dynamical system*

$$\begin{aligned} &x_{t+1} = A_t x_t : \quad A_t \in \Sigma,\\ &Sx_0 \in \mathbb{R}^n \end{aligned} \tag{1.3}$$

is stable if and only if $\rho(\Sigma) < 1$.

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*Received by the editors July 21, 2008, accepted for publication (in revised form) by N. Mastronardi December 23, 2008; published electronically DATE.
http://www.siam.org/journals/simax/x-x/73065.html

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The joint spectral radius is notoriously hard to compute. It is known that the problem of determining, given a set of matrices Σ, if $\rho(\Sigma) \leq 1$ is Turing-undecidable, and this result remains true even if Σ contains only nonnegative rational entries [3,7]. Moreover, unless $P = NP$, there is no algorithm for approximating the joint spectral radius of arbitrary sets of matrices Σ, whose execution time is polynomial in the size of Σ and in the required accuracy [6].

In some cases, however, a set of matrices can have special properties that might make easier the joint spectral radius computation. If, for a set Σ, the function $\rho_t(\Sigma)$ converges in finite time, we say that Σ has the *finiteness property.* In what follows, we denote by Σ^t the set of products of length t of matrices from Σ.

Definition 1. *A set of matrices is said to possess the finiteness property if there exists $t \in \mathbb{N}$ and a product $A \in \Sigma^t$ such that $\rho(\Sigma) = \rho(A)^{1/t}$.*

It has been shown [5,8,16] that, unfortunately, not all families of matrices possess this property. As an example, we cite the following result.

Theorem 2 (see [5]). *Let*

$$\Sigma(\alpha) = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ , \ \alpha \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}. \tag{1.4}$$

There is an uncountable number of values of the parameter $\alpha \in (0, 1)$ such that $\Sigma(\alpha)$ does not satisfy the finiteness property.

In other words, define $\rho_\alpha = \rho(\Sigma(\alpha))$, and

$$B_0 = \frac{1}{\rho_\alpha} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \frac{\alpha}{\rho_\alpha} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{1.5}$$

Then for an uncountable number of values $\alpha \in (0, 1)$, the set $\Sigma = \{B_0, B_1\}$ has joint spectral radius equal to one, even though all finite products of matrices in Σ have spectral radius smaller than one.

An efficient way to prove that $\rho(\Sigma) \leq r$ for a real number r is to find a convex body P (convex compact with a nonempty interior), centrally symmetric around the origin such that

$$\forall A \in \Sigma \quad \forall x \in P \qquad Ax \in r\,P. \tag{1.6}$$

Indeed, such a convex body represents the unit ball of the corresponding Minkowski norm $\|x\|_P = \inf\{\lambda > 0 \mid \lambda^{-1} x \in P\}$. For the corresponding induced matrix norm, one has

$$\forall A \in \Sigma \qquad \|A\| \leq r. \tag{1.7}$$

This, in turn, implies that $\rho(\Sigma) \leq r$ by submultiplicativity [1, 18, 19]. If, moreover, $\rho(\Sigma) = r$, we say that P is the unit ball of an extremal norm, since r is the smallest value such that (1.6) is possible. Finally, if P is a complex polytope, we say that Σ admits a *complex polytope extremal norm.* Recall that a complex polytope with vertices $p_1, \dots, p_k \in \mathbb{C}^n$ is defined as

$$P = \left\{ x = \sum_{i=1}^{k} \lambda_i p_i : \lambda_i \in \mathbb{C}, \sum_{i=1}^{k} |\lambda_i| \leq 1 \right\}.$$

In what follows, we assume that any complex polytope has at least two different vertices, i.e., contains more than one point. We call a polytope nondegenerate if it possesses a nonempty interior, or, which is the same, it is not contained in a hyperplane. We have the following result.

THEOREM 3 (see [9]). *If a set of matrices Σ admits a complex polytope extremal norm, then it satisfies the finiteness property.*

Let us remark that this result for real polytope extremal norms was proved earlier in [12]. We say that a set of matrices Σ is *nondefective* if $(1/\rho)\Sigma$ generates a bounded semigroup, that is, if there exists a real number K such that

$$\forall t \in \mathbb{N}, \forall A \in \Sigma^t \qquad \|A\| \leq K\rho^t. \tag{1.8}$$

It is known that if a family is irreducible (i.e., the matrices do not have a common nontrivial invariant subspace), then it is nondefective [2,18]. In particular, the family $\Sigma(\alpha)$ defined in (1.4) is nondefective for any $\alpha \in (0,1)$. Moreover, reducible families can be nondefective as well (if the so-called *valency* equals to one [21]). Clearly, if Σ has an extremal norm, it is nondefective. Indeed, the induced matrix norm is such that for all $A \in \Sigma$ one has $\|A\| \leq \rho$, and since all induced norms are submultiplicative, condition (1.8) is satisfied. In [9] the authors conjecture that under the nondefectiveness assumption the converse to Theorem 3 holds.

CONJECTURE 1 (see [9]). *Every nondefective finite family of complex matrices that possesses the finiteness property has a complex polytope extremal norm.*

It is known (see [14]) that for sets of matrices satisfying the finiteness property, the question $\rho < 1$ is decidable. This question is of high importance in practice, and, for that reason, it is needed to have a good understanding of the finiteness property and to characterize sets of matrices that have this property. If the conjecture was true, we would have a characterization of such (nondefective) sets in terms of the existence of a complex polytope extremal norm.

It is not difficult to show that Conjecture 1 holds for families consisting of one matrix: if a matrix A is nondefective (its powers A^k are bounded uniformly over all $k \in \mathbb{N}$), then it has an invariant complex polytope. Furthermore, the authors of [9] introduce the notion of *asymptotic simplicity* of a set of matrices and prove that Conjecture 1 is true for asymptotically simple sets. Roughly speaking, asymptotic simplicity means that there is only one product reaching the joint spectral radius (up to cyclic permutations and taking powers of that product) and that maximizing product has only one leading eigenvector. Algorithmic aspects of that result for computing the joint spectral radius were studied in [10]. More recently, these results were partially generalized to the case where the maximizing product has two leading eigenvectors [11].

In this paper we show that Conjecture 1 is false, in general, if one relaxes the asymptotic simplicity hypothesis.

2. The counterexamples. The following theorem disproving Conjecture 1 is the main result of this paper.

THEOREM 4. *There exists an irreducible pair of* 3×3 *orthogonal matrices* A_0, A_1 *for which there is no complex polytope* $P \subset \mathbb{C}^3$ *such that* $A_iP \subset P\,,\ i = 0,1$.

Also there exists a nondefective pair of 3×3 *matrices* A_0, A_1 *with nonnegative entries for which* $\rho(\{A_0, A_1\}) = \rho(A_0) = \rho(A_1) = 1$, *but there is no nondegenerate complex polytope* $P \subset \mathbb{C}^3$ *such that* $A_iP \subset P\,,\ i = 0,1$.

Note that for orthogonal matrices $\rho(\{A_0, A_1\}) = \rho(A_0) = \rho(A_1) = 1$, and this pair of matrices is nondefective. So, the pairs of matrices from Theorem 4 are both nondefective and satisfy the finiteness property already for $t = 1$.

Proof. The corresponding examples are given in Propositions 1 and 2. ∎

Let us first introduce the example with irreducible matrices.

PROPOSITION 1. *Let $0 \le \alpha_i \le 2\pi, \frac{\alpha_i}{\pi} \notin \mathbb{Q}, i = 0, 1$, and denote by A_i the operator of rotation of the space $\mathbb{R}^3$ by the angle α_i with respect to an axis $a_i, i = 0, 1$. Assume the axes a_0 and a_1 are orthogonal. Then the family $\{A_0, A_1\}$ does not have an invariant complex polytope.*

Since these matrices are orthogonal, the family $\{A_0, A_1\}$ is clearly nondefective, and its joint spectral radius, which is equal to 1, is attained at every product of these matrices. Therefore, Theorem 4 disproves Conjecture 1. To give a proof we first describe the structure of real parts of complex polytopes. For a given set $M \subset \mathbb{C}^d$, we denote by $\mathrm{Re}M = \{\mathrm{Re}\, z \mid z \in M\}$ the real part of M. By an ellipse we mean two-dimensional ellipse centered at the origin, including the degenerate case of a segment. This is an easy exercise to show that a set $E \subset \mathbb{R}^d$ is an ellipse if and only if there is a vector $z \in \mathbb{C}^d$ such that $E = \{\mathrm{Re}(e^{i\alpha} z) | \, \alpha \in \mathbb{T}\}$. We denote this ellipse by E_z.

LEMMA 1 (see [17]). *A subset of $\mathbb{R}^d$ is the real part of a complex polytope if and only if it is the convex hull of several ellipses. If all the ellipses degenerate to segments, then it is a convex real polytope.*

Proof. Let $P = \{\sum_{k=1}^N \lambda_k z_k | \sum_{k=1}^N |\lambda_k| \le 1\}$ be a complex polytope $z_k \in \mathbb{C}^d, k = 1, \dots, N$. Using the short notation $E_{z_k} = E_k, |\lambda_k| = t_k$, we get

$$\begin{aligned} \mathrm{Re}\, P &= \left\{ \textstyle\sum_{k=1}^N t_k x_k \quad \Big| \quad x_k \in E_k\,,\ t_k \ge 0,\ \sum_{k=1}^N t_k \le 1 \right\} \\ &= \mathrm{Conv}\, \{E_k,\ k = 1, \dots, N\}\,. \end{aligned}$$

Now the convex hull of the ellipses $\{E_k\}$ is the real part of a complex polytope with the corresponding vertices $\{z_k\}$. ☐

We are now able to prove Proposition 1.

Proof of Proposition 1. Let us show that the pair $\{A_o, A_1\}$ has no invariant complex polytope. Assume the contrary: there is a complex polytope P such that $A_i P \subset P, i = 0, 1$. Then its real part $P' = \mathrm{Re}\, P$ possesses the same property: $A_i P' \subset P', i = 0, 1$. Let $b \in P'$ be the most distant point from the origin. We show that P' is actually a Euclidean ball of radius $|b|$: $P' = \{x \in \mathbb{R}^3 \mid |x| \le |b|\}$. Since the angle α_0 is irrational, it follows that the points $\{A_0^k\, b\,,\ k \in \mathbb{N}\}$ fill everywhere densely the circle γ centered on the line a_0 contained on the plane orthogonal to a_0 and passing through the point b. Hence, $\gamma \subset P'$. Taking an element $c \in \gamma$, which is orthogonal to a_1, we see that the points $\{A_1^k\, c\,,\ k \in \mathbb{N}\}$ fill everywhere densely the circle of radius $|b|$ centered at the origin and contained on the plane orthogonal to a_1. Applying iterations of the matrix A_0 to this circle, we fill everywhere densely the sphere of radius $|b|$ centered at the origin. Since P' is convex and closed, we see that it contains the corresponding ball. However, by the assumption, P' does not have points outside this ball. So, P' is a ball. This contradicts Lemma 1, because a Euclidean ball in $\mathbb{R}^3$ is not a convex hull of finitely many ellipses. ☐

One could assume that Conjecture 1, while false in general, is true for nonnegative matrices. Indeed, these matrices admit an invariant cone (the positive orthant), and, for this reason among others, sets of nonnegative matrices allow one to derive much stronger results (see, for instance, [4, 15, 20]). Our second counterexample shows that Conjecture 1 does not hold, even if the matrices are supposed to have nonnegative entries. We start with the following simple observation.

LEMMA 2. *Let $\Sigma \subset \mathbb{R}^{n \times n}$ be a set of m block diagonal matrices:*

$$\Sigma = \left\{ \begin{pmatrix} A_i & 0 \\ 0 & B_i \end{pmatrix}\ , \qquad 1 \le i \le m \right\},$$

where $\Sigma' = \{B_i\} \subset \mathbb{R}^{n' \times n'}, n' < n$. *If* Σ *admits a nondegenerate invariant complex polytope, so does* Σ'.

Proof. Let

$$P = \left\{ x = \sum_{i=1}^{k} \lambda_i \, p_i \ , \qquad \sum_{i=1}^{k} |\lambda_i| \leq 1 \right\}$$

be a complex polytope generated by k points, p_i that is invariant under the matrices in Σ. Denoting p_i' the projection of the vectors p_i on their last n' coordinates, for all p_i' and for all $B \in \Sigma'$ there exist $\lambda_1 \dots \lambda_k \ : \ \sum |\lambda_i| \leq 1$ such that

$$B \, p_i' = \sum_{i=1}^{k} \lambda_i \, p_i' \, .$$

Hence, the complex polytope generated by p_i' is clearly nondegenerate and is invariant with respect to Σ'. ∎

Note that the set of projected vectors p_i' might well not be an *essential system of vertices* for the complex polytope (see [11]). That is, some p_i' might be redundant in a description of the polytope.

PROPOSITION 2. *Let* B_0, B_1 *be matrices defined in* (1.5) *for which the finiteness property does not hold. Let* $\tilde{\Sigma} = \{\tilde{B_0}, \tilde{B_1}\}$, *where*

$$\tilde{B_0} = \begin{pmatrix} 1 & 0 \\ 0 & B_0 \end{pmatrix}, \quad \tilde{B_1} = \begin{pmatrix} 1 & 0 \\ 0 & B_1 \end{pmatrix}.$$

The set $\tilde{\Sigma}$ *has the finiteness property and is nondefective, but it does not admit an extremal complex polytope norm, thus, violating Conjecture* 1.

Proof. The joint spectral radius of $\tilde{\Sigma}$ is equal to one. Indeed, any product $\tilde{B} \in \tilde{\Sigma}^t$ can be written as

$$\tilde{B} = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix},$$

with $B \in \Sigma^t$ and

$$\limsup_{t \to \infty} \max_{\tilde{B} \in \tilde{\Sigma}^t} \rho(\tilde{B}) = 1.$$

Now $\rho(B_0) = 1$, and so $\tilde{\Sigma}$ has the finiteness property. Furthermore, since the family Σ is irreducible, it is nondefective, and therefore, $\tilde{\Sigma}$ is nondefective either. We now show that $\tilde{\Sigma}$ does not admit an extremal complex polytope norm. Let us suppose by contradiction that there exists a nondegenerate invariant polytope. Lemma 2 implies that $\Sigma = \{B_0, B_1\}$ also admits a nondegenerate invariant polytope. Hence, Σ has a complex polytope extremal norm, which is in contradiction with Theorem 3. ∎

3. Conclusion. We have provided two counterexamples to the complex polytope extremality (CPE) conjecture: the existence of an extremal complex polytope norm is not a criterion for recognizing sets of matrices satisfying the finiteness property. We leave the following question, raised in [13], open: does there exist an algorithm that recognizes sets of matrices satisfying the finiteness property? Also, even though this is not a criterion for the finiteness property, it would be interesting to recognize sets of matrices that admit an extremal complex polytope norm, since for these sets, it is possible to compute the joint spectral radius exactly [9]. To the best of the authors' knowledge, no such algorithms are known thus far.

Acknowledgment. The authors are grateful to both anonymous referees for their valuable remarks and comments.

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