Path-complete Lyapunov techniques

Raphaël Jungers (UCLouvain, Belgium)

Dysco'17
Leuven, Nov 2017

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Outline

- **Switching systems**
- Path-complete methods for switching systems stability
- Further results and open problems
- Conclusion and perspectives

Applications of Wireless Control Networks

Industrial automation

Maurice Heemels (TU/e)

Physical Security and Control

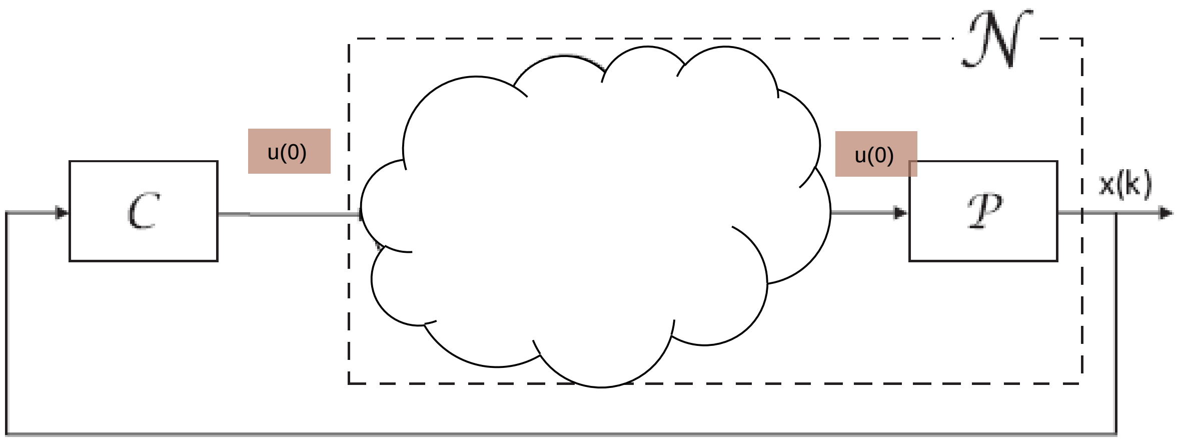
Supply Chain and Asset Management

Environmental Monitoring, Disaster Recovery and Preventive Conservation

Controllability with packet dropouts

The delay is constant, but some packets are dropped

$$x(1) = Ax(0) + Bu(0)$$



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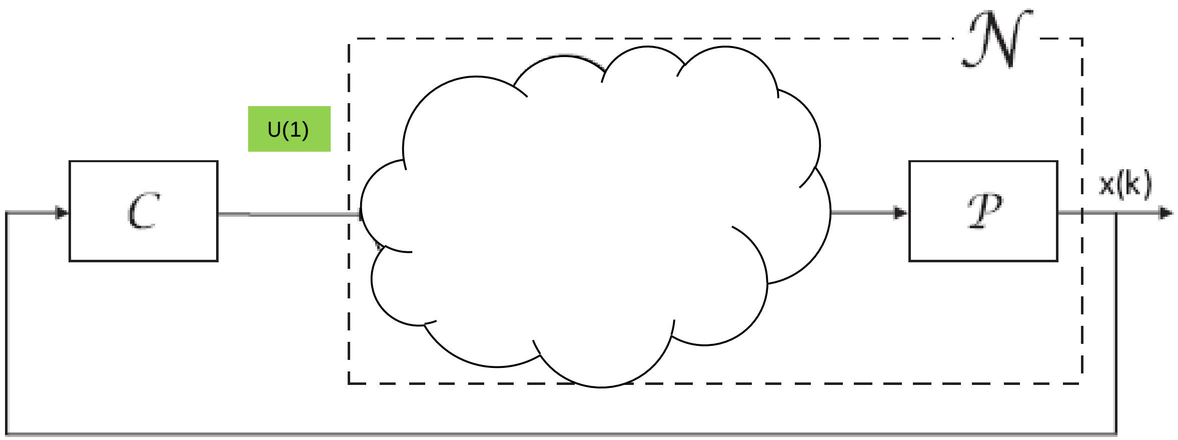


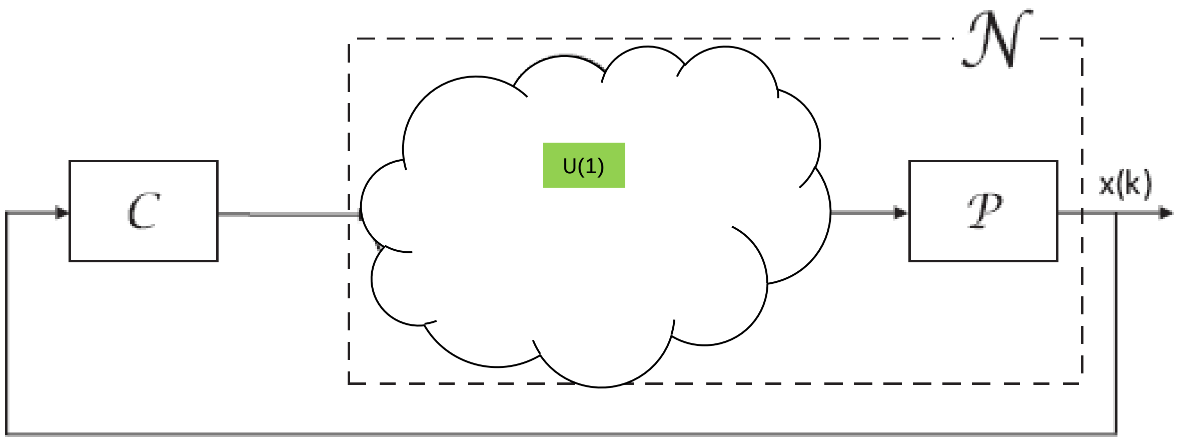
Controllability with packet dropouts

The delay is constant, but some packets are dropped

$\boxed{\sigma = 1001\ldots}$

$$\sigma(0) = 1$$
$$\sigma(1) = 0$$

$$x(1) = Ax(0) + Bu(0)$$
$$x(2) = A^2x(0) + ABu(0)$$



A data loss signal determines the packet dropouts $\sigma(t) =$ 1 or 0

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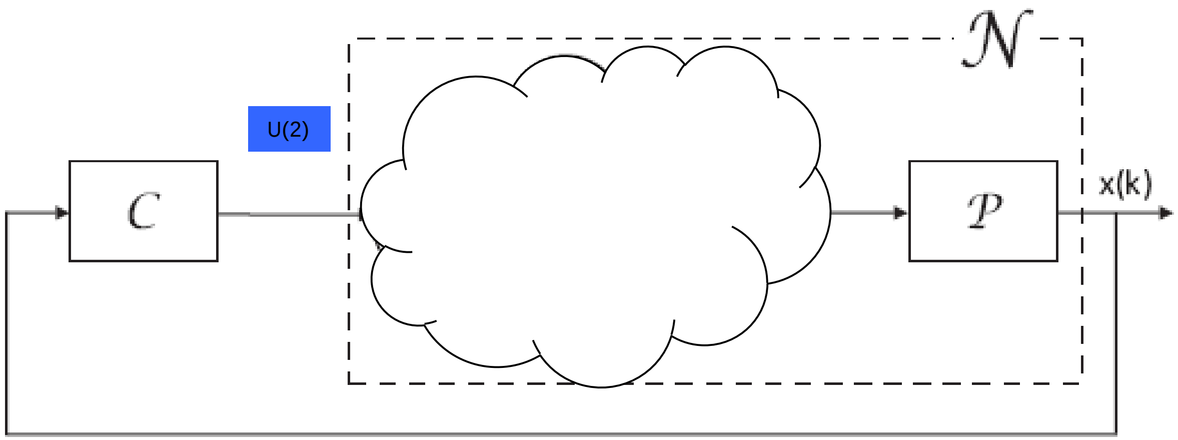

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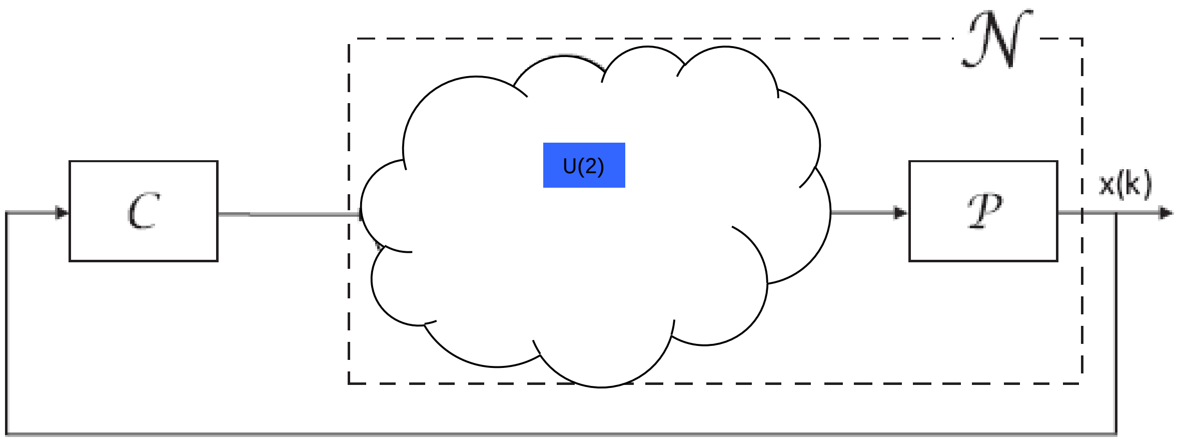

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Controllability with packet dropouts

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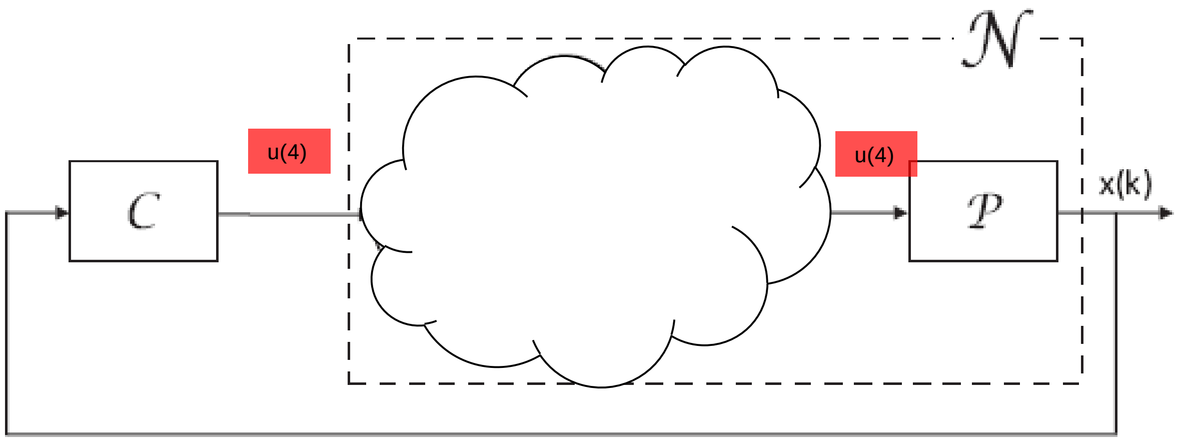
$$x(1) = Ax(0) + Bu(0)$$
$$x(2) = A^2x(0) + ABu(0)$$
$$x(3) = A^3x(0) + A^2Bu(0)$$
$$x(4) = A^4x(0) + A^3Bu(0) + Bu(3)$$



A data loss signal determines the packet dropouts $\sigma(t) =$ 1 or 0

...this is a switching system! $x(t+1) = \begin{cases} Ax(t) + bu(t), & \text{if } \sigma(t) = 1, \\ Ax(t), & \text{if } \sigma(t) = 0 \end{cases}$

The switching signal

We are interested in the controllability of such a system

$$\boxed{\sigma = 1001\ldots}$$

$$\sigma(0) = 1$$
$$\sigma(1) = 0$$
$$\sigma(2) = 0$$

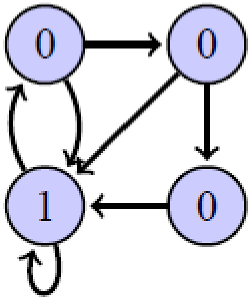
$$\begin{aligned} x(1) &= Ax(0) + Bu(0) \\ x(2) &= A^2x(0) + ABu(0) \\ x(3) &= A^3x(0) + A^2Bu(0) \\ x(4) &= A^4x(0) + A^3Bu(0) + Bu(3) \end{aligned}$$

Of course we need an assumption on the switching signal

The switching signal is constrained by an automaton

Example:

Bounded number of consecutive dropouts (here, 3)



Switching systems

$$x(t+1) = A_0 x(t)$$

or

$$x(t+1) = A_1 x(t)$$

Global convergence to the origin Do all products of the type $A_0\ A_0\ A_1\ A_0\ \ldots\ A_1$ converge to zero?

[Rota, Strang, 1960]

Outline

- Joint spectral characteristics
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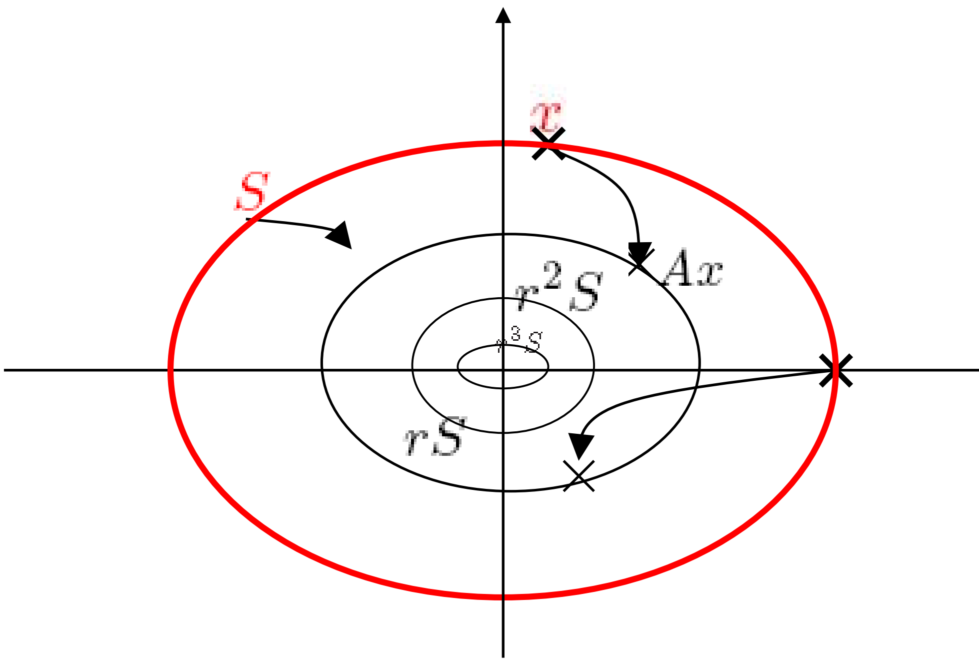
Switching systems stability (a.k.a. JSR computation)

The CQLF method (Common Quadratic Lyapunov Function)

$$\begin{array}{llll} \inf_{r\in\mathbb{R}^+} & & r & \\ \text{s.t.} & & & \\ A^TPA & \preceq & r^2P, & \forall A \in \Sigma \\ P & \succeq & 0. & \end{array}$$

$\Leftrightarrow$ Every x in S is mapped in the scaled ellipsoid rS:

$$\Leftrightarrow \frac{|Ax|_P}{|x|_P} \leq r$$



Stability!

Yet another LMI method

- A strange semidefinite program

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P & \succeq & 0.
\end{array}
$$

Stability!

[Goebel, Hu, Teel 06]

- But also...

[Daafouz Bernussou 01]

[Bliman Ferrari-Trecate 03]

[Lee and Dullerud 06] ...

[Ahmadi, J., Parrilo, Roozbehani10]

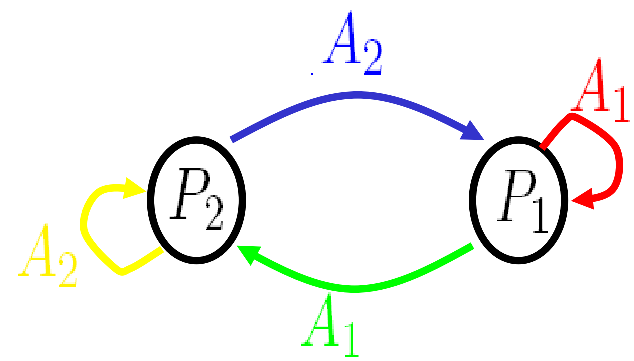
Yet another LMI method

- Questions:
  - Can we characterize all the LMIs that work, in a unified framework?
  - Which LMIs are better than others?
  - How to prove that an LMI works?
  - Can we provide converse Lyapunov theorems for more methods?

A. Ahmadi (Princeton),
P. Parrilo, M. Roozbehani (MIT)

From LMIs to an automaton

$$
\begin{array}{lcl}
\min_{r\in\mathbb{R}+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P_i & \succeq & 0.
\end{array}
$$



Sufficient condition for stability

Path complete (generates all the possible words)

Theorem

G is path-complete IFF the LMIs are a sufficient condition for stability.

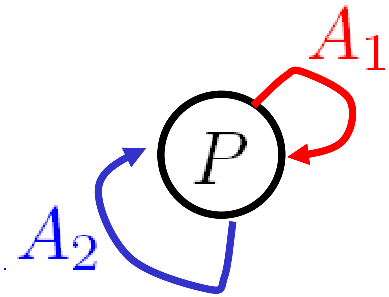
[Ahmadi J. Parrilo Roozbehani 14]

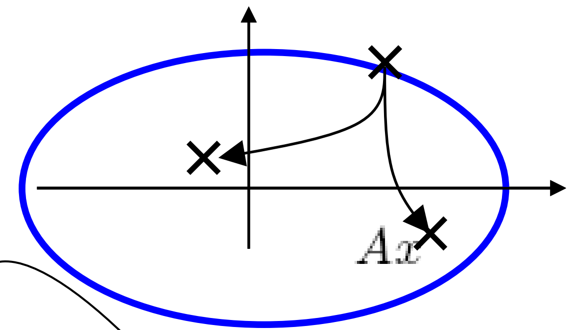
[J. Ahmadi Parrilo Roozbehani 17]

Results valid beyond the LMI framework

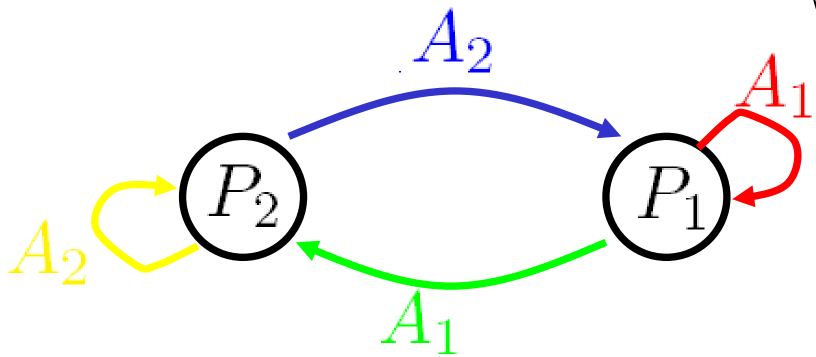
Some examples

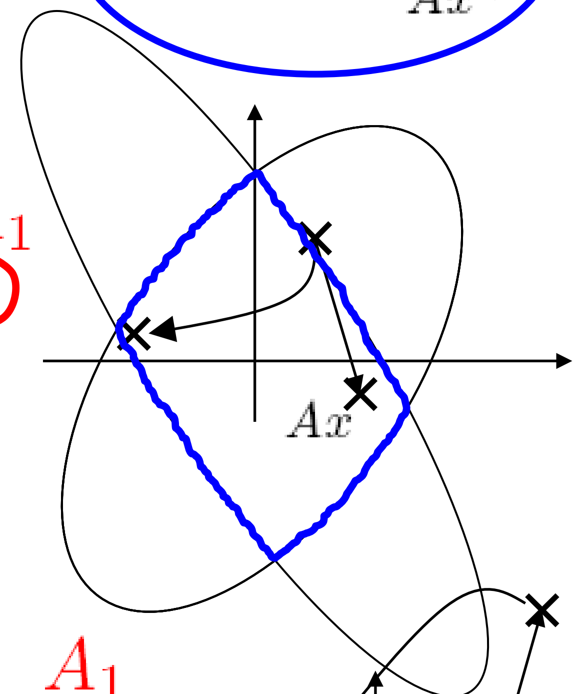
- Examples:
  - CQLF





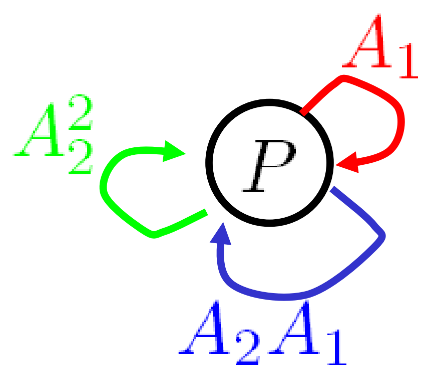
  - Example 1

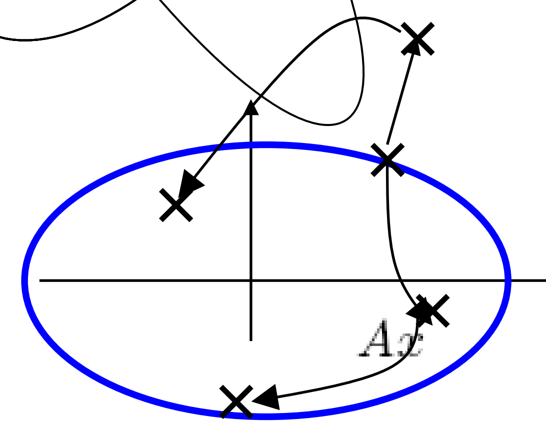




This type of graph gives a max-of-quadratics Lyapunov function (i.e. intersection of ellipsoids)

  - Example 2





Invariant set unclear…

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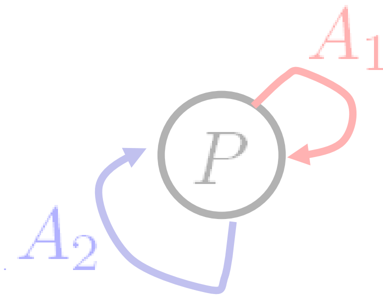
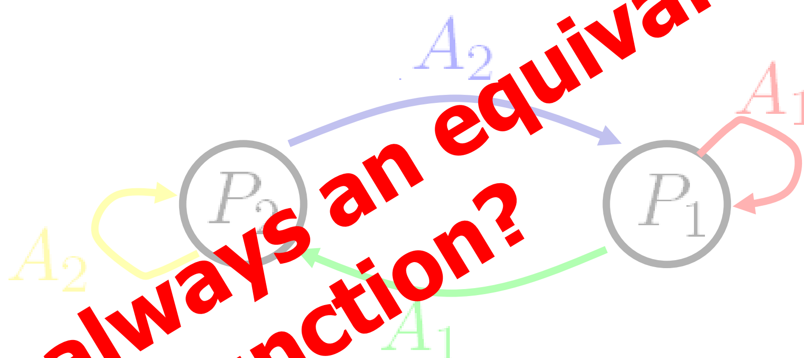
Some examples

- Examples:
  - CQLF



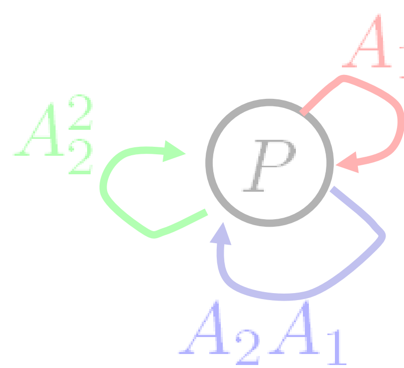
  - Example 1

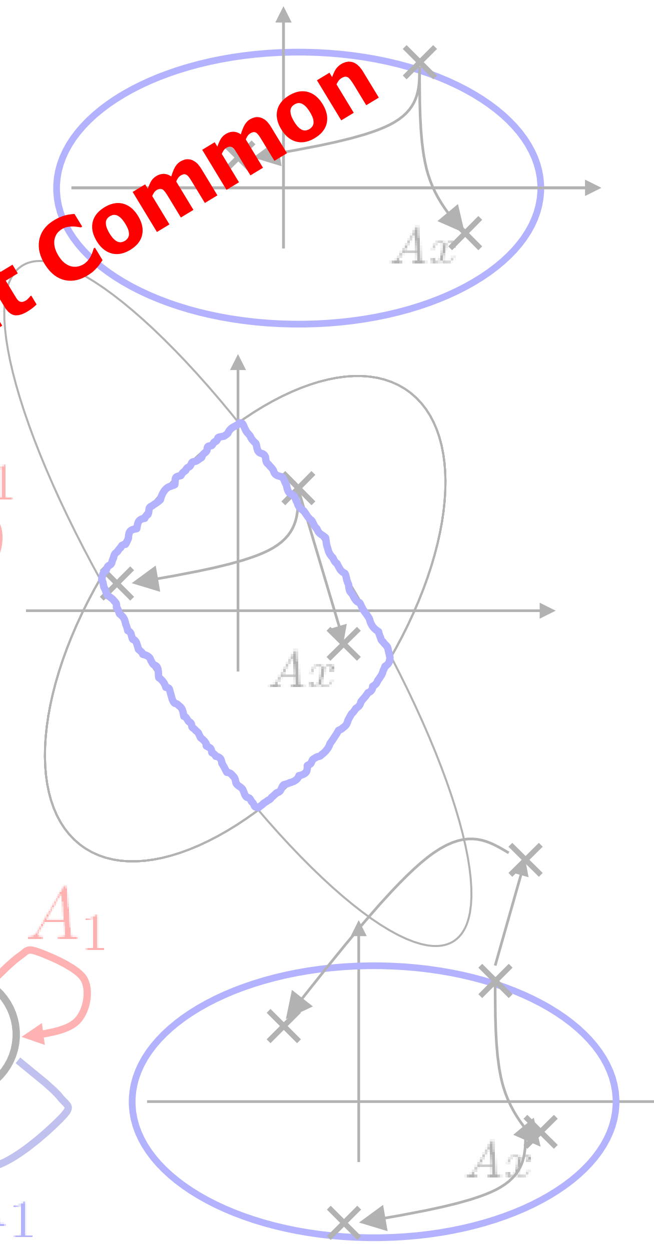


This type of graph gives a max-of-quadratics Lyapunov function (i.e. intersection of ellipsoids)

  - Example 2

This type of graph gives a common Lyapunov function for a generating set of words





Is there always an equivalent Common Lyapunov Function?

Is there always an equivalent Common Lyapunov Function?

- Theorem Every path-complete criterion implies the existence of a Common Lyapunov function. This Lyapunov function can be expressed analytically as the minimum of maxima of the quadratic functions.

[Angeli Athanasopoulos Philippe J., 2017]

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
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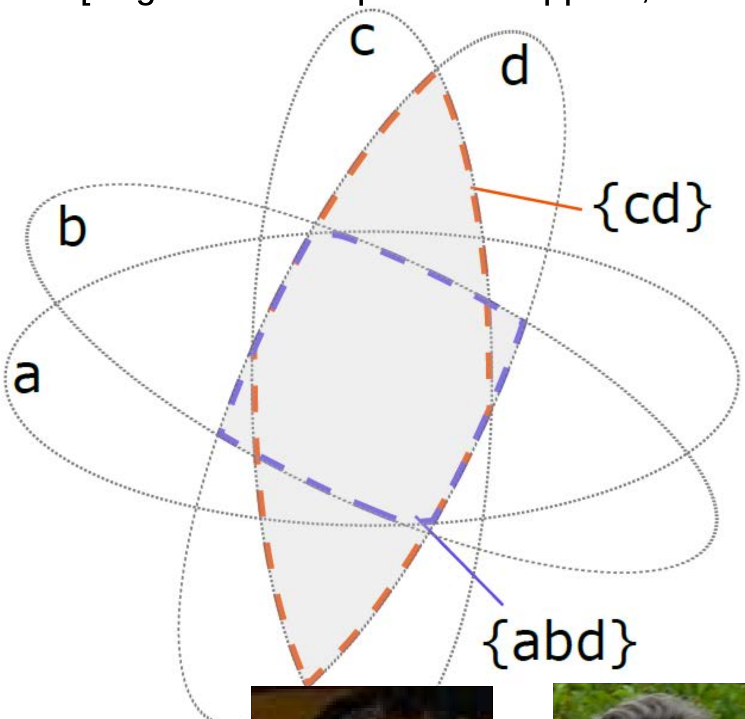
$$



David Angeli (Imperial)

Philippe, Athanasopoulos

Further results and open problems

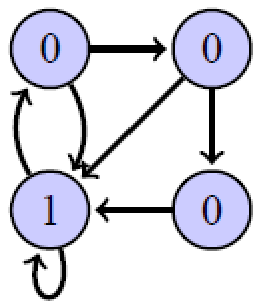
This approach naturally generalizes to other problems

$$x(t+1) = A_0 x(t)$$

or

$$x(t+1) = A_1 x(t)$$

$$\boxed{\sigma = 1001\ldots}$$



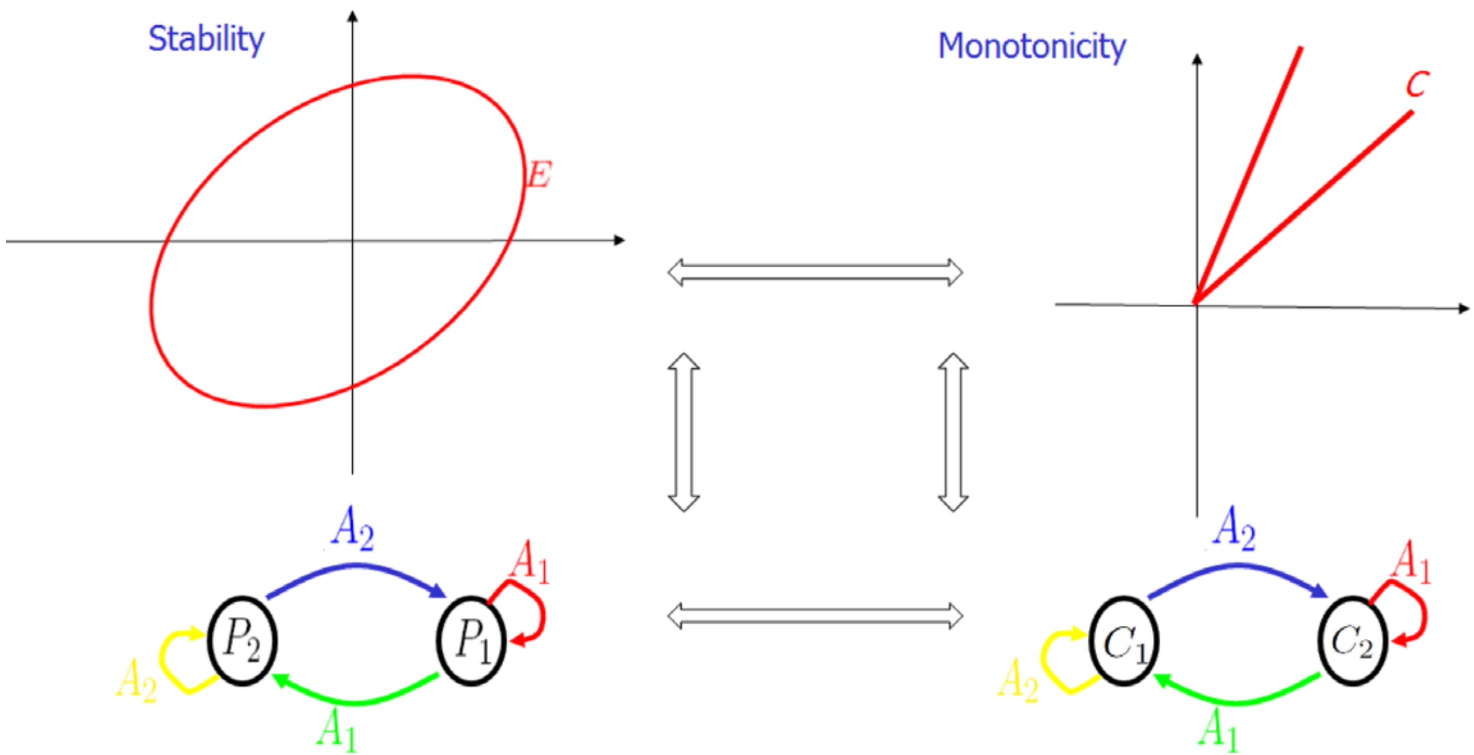
Geir Dullerud
(UIUC)

- **Constrained switching systems**
- Path-complete monotonicity
- Automatically optimized abstractions of cyber-physical systems
-

Further results and open problems

Replace invariant compact sets by invariant cones

F. Forni and R. Sepulchre (Cambridge)



- Constrained switching systems
- **Path-complete monotonicity**
- Automatically optimized abstractions of cyber-physical systems

Further results and open problems

Refining the Control Structure of Loops using Static Analysis

Gogul Balakrishnan
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agupta@nec-labs.com

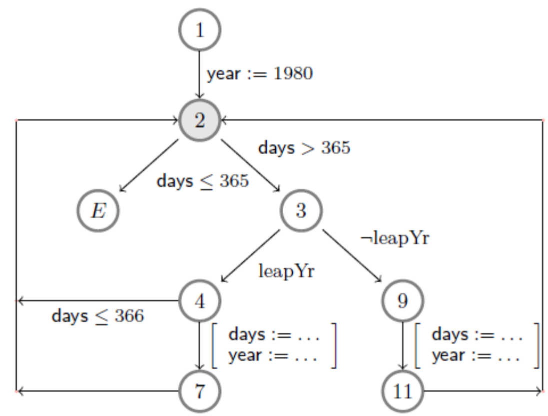
Paulo Tabuada (UCLA)

Loop analysis refinement by 'lifting' the initial automaton

Abstracting the 'dynamics'

This impossible 'fragment' can be removed from the language

```
year := 1980 ;
while (days > 365){
    if (IsLeapYear(year)) {
        if (days > 366) {
            days −= 366;
            year += 1;
        }
    } else {
        days −= 365;
        year += 1;
    }
}
E:
```



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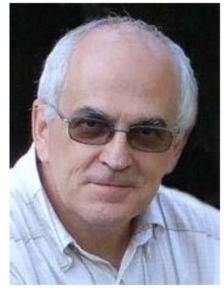
Conclusion: a perspective on switching systems

[Furstenberg Kesten, 1960]

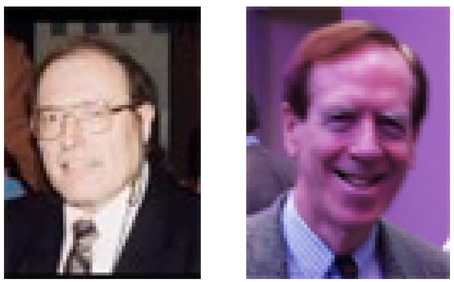
[Gurvits, 1995]



[Kozyakin, 1990]

[Daafouz Bernussou, 2002]

[Rantzer Johansson 1998]



[Rota, Strang, 1960]

[Blondel Tsitsiklis, 98+]

[Lee Dullerud 2006]

[Parrilo Jadbabaie 2008]

(sensor) networks

Software analysis

Bisimulation design

consensus problems

Social/big data control

…

60s 70s

Mathematical properties

90s

TCS inspired
Negative
Complexity results

2000s

Lyapunov/LMI
Techniques
(S-procedure)

now

CPS applic.
Ad hoc techniques

Thanks!

Questions?

Ads

The JSR Toolbox:
http://www.mathworks.com/matlabcentral/fileexchange/33202-the-jsr-toolbox
[Van Keerberghen, Hendrickx, J. HSCC 2014]
The CSS toolbox, 2015

Several open positions:
raphael.jungers@uclouvain.be

References:
http://perso.uclouvain.be/raphael.jungers/

Joint work with

A.A. Ahmadi (Princeton), D. Angeli (Imperial), N. Athanasopoulos (UCLouvain), V. Blondel (UCL), G. Dullerud (UIUC), F. Forni (Cambridge) B. Legat (UCLouvain), P. Parrilo (MIT), M. Philiippe (UCLouvain), V. Protasov (Moscow), M. Roozbehani (MIT), R. Sepulchre (Cambridge)...