

# JOINT SPECTRAL RADIUS AND PATH-COMPLETE GRAPH LYAPUNOV FUNCTIONS\*

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**Abstract.** We introduce the framework of path-complete graph Lyapunov functions for approximation of the joint spectral radius. The approach is based on the analysis of the underlying switched system via inequalities imposed among multiple Lyapunov functions associated to a labeled directed graph. Inspired by concepts in automata theory and symbolic dynamics, we define a class of graphs called path-complete graphs, and show that any such graph gives rise to a method for proving stability of the switched system. This enables us to derive several asymptotically tight hierarchies of semidefinite programming relaxations that unify and generalize many existing techniques such as common quadratic, common sum of squares, and maximum/minimum-of-quadratics Lyapunov functions. We compare the quality of approximation obtained by certain classes of path-complete graphs including a family of dual graphs and all path-complete graphs with two nodes on an alphabet of two matrices. We provide approximation guarantees for several families of path-complete graphs, such as the De Bruijn graphs, establishing as a byproduct a constructive converse Lyapunov theorem for maximum/minimum-of-quadratics Lyapunov functions.

**Key words.** joint spectral radius, stability of switched systems, linear difference inclusions, finite automata, Lyapunov methods, semidefinite programming.

**1. Introduction.** Given a finite set of square matrices  $\mathcal{A} := \{A_1, \dots, A_m\}$ , their *joint spectral radius*  $\rho(\mathcal{A})$  is defined as

$$\rho(\mathcal{A}) = \lim_{k \rightarrow \infty} \max_{\sigma \in \{1, \dots, m\}^k} \|A_{\sigma_k} \dots A_{\sigma_2} A_{\sigma_1}\|^{1/k}, \quad (1.1)$$

where the quantity  $\rho(\mathcal{A})$  is independent of the norm used in (1.1). The joint spectral radius (JSR) is a natural generalization of the spectral radius of a single square matrix and it characterizes the maximal growth rate that can be obtained by taking products, of arbitrary length, of all possible permutations of  $A_1, \dots, A_m$ . This concept was introduced by Rota and Strang [28] in the early 60s and has since been the subject of extensive research within the engineering and the mathematics communities alike. Aside from a wealth of fascinating mathematical questions that arise from the JSR, the notion emerges in many areas of application such as stability of switched linear dynamical systems, Leontief input-output model of the economy with uncertain data, computation of the capacity of codes, continuity of wavelet functions, convergence of consensus algorithms, trackability of graphs, and many others. See [19] and references therein for a recent survey of the theory and applications of the JSR.

Motivated by the abundance of applications, there has been much work on efficient computation of the joint spectral radius; see e.g. [6], [5], [24], and references therein. Unfortunately, the negative results in the literature certainly restrict the horizon of possibilities. In [7], Blondel and Tsitsiklis prove that even when the set  $\mathcal{A}$  consists of only two matrices, the question of testing whether  $\rho(\mathcal{A}) \leq 1$  is undecidable. They also show that unless P=NP, one cannot compute an approximation  $\hat{\rho}$  of  $\rho$  that satisfies  $|\hat{\rho} - \rho| \leq \epsilon\rho$ , in a number of steps polynomial in the bit size of  $\mathcal{A}$  and the bit size

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of  $\epsilon$  [29]. It is easy to show that the spectral radius of any finite product of length  $k$  raised to the power of  $1/k$  gives a lower bound on  $\rho$  [19]. However, for reasons that we explain next, our focus will be on computing upper bounds for  $\rho$ .

There is an attractive connection between the joint spectral radius and the stability properties of an arbitrarily switched linear system; i.e., dynamical systems of the form

$$x_{k+1} = A_{\sigma(k)} x_k, \quad (1.2)$$

where  $\sigma : \mathbb{Z} \rightarrow \{1, \dots, m\}$  is a map from the set of integers to the set of indices. It is well-known that  $\rho < 1$  if and only if system (1.2) is *absolutely asymptotically stable* (AAS), that is, (globally) asymptotically stable for all switching sequences. Moreover, it is known [21] that absolute asymptotic stability of (1.2) is equivalent to absolute asymptotic stability of the linear difference inclusion

$$x_{k+1} \in \text{co}\mathcal{A} x_k, \quad (1.3)$$

where  $\text{co}\mathcal{A}$  here denotes the convex hull of the set  $\mathcal{A}$ . Therefore, any method for obtaining upper bounds on the joint spectral radius provides sufficient conditions for stability of systems of type (1.2) or (1.3). Conversely, if we can prove absolute asymptotic stability of (1.2) or (1.3) for the set  $\mathcal{A}_\gamma := \{\gamma A_1, \dots, \gamma A_m\}$  for some positive scalar  $\gamma$ , then we get an upper bound of  $\frac{1}{\gamma}$  on  $\rho(\mathcal{A})$ . (This follows from the scaling property of the JSR:  $\rho(\mathcal{A}_\gamma) = \gamma\rho(\mathcal{A})$ .) One advantage of working with the notion of the joint spectral radius is that it gives a way of rigorously quantifying the performance guarantee of different techniques for stability analysis of systems (1.2) or (1.3).

Perhaps the most well-established technique for proving stability of switched systems is the use of a *common (or simultaneous) Lyapunov function*. The idea here is that if there is a continuous, positive, and homogeneous (Lyapunov) function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  that for some  $\gamma > 1$  satisfies

$$V(\gamma A_i x) \leq V(x) \quad \forall i = 1, \dots, m, \quad \forall x \in \mathbb{R}^n, \quad (1.4)$$

(i.e.,  $V(x)$  decreases no matter which matrix is applied), then the system in (1.2) (or in (1.3)) is AAS. Conversely, it is known that if the system is AAS, then there exists a *convex* common Lyapunov function (in fact a norm); see e.g. [19, p. 24]. However, this function is not in general finitely constructable. A popular approach has been to try to approximate this function by a class of functions that we can efficiently search for using convex optimization and in particular semidefinite programming. Semidefinite programs (SDPs) can be solved with arbitrary accuracy in polynomial time and lead to efficient computational methods for approximation of the JSR. As an example, if we take the Lyapunov function to be quadratic (i.e.,  $V(x) = x^T P x$ ), then the search for such a Lyapunov function can be formulated as the following SDP:

$$\begin{array}{ccl} P & \succ & 0 \\ \gamma^2 A_i^T P A_i & \preceq & P \quad \forall i = 1, \dots, m. \end{array} \quad (1.5)$$

The quality of approximation of common quadratic Lyapunov functions is a well-studied topic. In particular, it is known [6] that the estimate  $\hat{\rho}_{\mathcal{V}^2}$  obtained by this method<sup>1</sup> satisfies

$$\frac{1}{\sqrt{n}} \hat{\rho}_{\mathcal{V}^2}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}_{\mathcal{V}^2}(\mathcal{A}), \quad (1.6)$$

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<sup>1</sup>The estimate  $\hat{\rho}_{\mathcal{V}^2}$  is the reciprocal of the largest  $\gamma$  that satisfies (1.5) and can be found by bisection.

where  $n$  is the dimension of the matrices. This bound is a direct consequence of John's ellipsoid theorem and is known to be tight [4].

In [24], the use of sum of squares (SOS) polynomial Lyapunov functions of degree  $2d$  was proposed as a common Lyapunov function for the switched system in (1.2). The search for such a Lyapunov function can again be formulated as a semidefinite program. This method does considerably better than a common quadratic Lyapunov function in practice and its estimate  $\hat{\rho}_{\mathcal{V}^{SOS,2d}}$  satisfies the bound

$$\frac{1}{\sqrt[2d]{\eta}} \hat{\rho}_{\mathcal{V}^{SOS,2d}}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}_{\mathcal{V}^{SOS,2d}}(\mathcal{A}), \quad (1.7)$$

where  $\eta = \min\{m, \binom{n+d-1}{d}\}$ . Furthermore, as the degree  $2d$  goes to infinity, the estimate  $\hat{\rho}_{\mathcal{V}^{SOS,2d}}$  converges to the true value of  $\rho$  [24]. The semidefinite programming based methods for approximation of the JSR have been recently generalized and put in the framework of conic programming [25].

**1.1. Contributions and organization.** It is natural to ask whether one can develop better approximation schemes for the joint spectral radius by using multiple Lyapunov functions as opposed to requiring simultaneous contractibility of a single Lyapunov function with respect to all the matrices. More concretely, our goal is to understand how we can write inequalities among, say,  $k$  different Lyapunov functions  $V_1(x), \dots, V_k(x)$  that imply absolute asymptotic stability of (1.2) and can be checked via semidefinite programming.

The general idea of using several Lyapunov functions for analysis of switched systems is a very natural one and has already appeared in the literature (although to our knowledge not in the context of the approximation of the JSR); see e.g. [18], [8], [16], [15], [12]. Perhaps one of the earliest references is the work on “piecewise quadratic Lyapunov functions” in [18]. However, this work is in the different framework of state dependent switching, where the dynamics switches depending on which region of the space the trajectory is traversing (as opposed to arbitrary switching). In this setting, there is a natural way of using several Lyapunov functions: assign one Lyapunov function per region and “glue them together”. Closer to our setting, there is a body of work in the literature that gives sufficient conditions for existence of piecewise Lyapunov functions of the type  $\max\{x^T P_1 x, \dots, x^T P_k x\}$ ,  $\min\{x^T P_1 x, \dots, x^T P_k x\}$ , and  $\text{conv}\{x^T P_1 x, \dots, x^T P_k x\}$ , i.e., the pointwise maximum, the pointwise minimum, and the convex envelope of a set of quadratic functions [16], [15], [12], [17]. These works are mostly concerned with analysis of linear differential inclusions in continuous time, but they have obvious discrete time counterparts. The main drawback of these methods is that in their greatest generality, they involve solving bilinear matrix inequalities, which are non-convex and in general NP-hard. One therefore has to turn to heuristics, which have no performance guarantees and their computation time quickly becomes prohibitive when the dimension of the system increases. Moreover, all of these methods solely provide sufficient conditions for stability with no performance guarantees.

There are several unanswered questions that in our view deserve a more thorough study: (i) With a focus on conditions that are amenable to convex optimization, what are the different ways to write a set of inequalities among  $k$  Lyapunov functions that imply absolute asymptotic stability of (1.2)? Can we give a unifying framework that includes the previously proposed Lyapunov functions and perhaps also introduces new ones? (ii) Among the different sets of inequalities that imply stability, can we identify some that are more powerful than some other? (iii) The available methods

on piecewise Lyapunov functions solely provide sufficient conditions for stability with no guarantee on their performance. Can we give converse theorems that guarantee the existence of a feasible solution to our search for a given accuracy?

The contributions of this paper to these questions are as follows. We propose a unifying framework based on a representation of Lyapunov inequalities with labeled graphs and by making some connections with basic concepts in automata theory. This is done in Section 2, where we define the notion of a path-complete graph (Definition 2.2) and prove that any such graph provides an approximation scheme for the JSR (Theorem 2.4). In Section 3, we give examples of families of path-complete graphs and show that many of the previously proposed techniques come from particular classes of simple path-complete graphs (e.g., Corollary 3.4, Corollary 3.5, and Remark 3.2). In Section 4, we characterize all the path-complete graphs with two nodes for the analysis of the JSR of two matrices. We determine how the approximations obtained from all of these graphs compare (Proposition 4.2). In Section 5, we study in more depth the approximation properties of a particular pair of “dual” path-complete graphs that seem to perform very well in practice. Subsection 5.1 contains more general results about duality within path-complete graphs and its connection to transposition of matrices (Theorem 5.1). Subsection 5.2 gives an approximation guarantee for the graphs studied in Section 5 (Theorem 5.4), and Subsection 5.3 contains some numerical examples. In Section 6, we prove a converse theorem for the method of max-of-quadratics Lyapunov functions (Theorem 6.1) and an approximation guarantee for a new class of methods for proving stability of switched systems (Theorem 6.2). Finally, our conclusions and some future directions are presented in Section 7.

**2. Path-complete graphs and the joint spectral radius.** In what follows, we will think of the set of matrices  $\mathcal{A} := \{A_1, \dots, A_m\}$  as a finite alphabet and we will often refer to a finite product of matrices from this set as a *word*. We denote the set of all words  $A_{i_t} \dots A_{i_1}$  of length  $t$  by  $\mathcal{A}^t$ . Contrary to the standard convention in automata theory, our convention is to read a word from right to left. This is in accordance with the order of matrix multiplication. The set of all finite words is denoted by  $\mathcal{A}^*$ ; i.e.,  $\mathcal{A}^* = \bigcup_{t \in \mathbb{Z}^+} \mathcal{A}^t$ .

The basic idea behind our framework is to represent through a graph all the possible occurrences of products that can appear in a run of the dynamical system in (1.2), and assert via some Lyapunov inequalities that no matter what occurrence appears, the product must remain stable. A convenient way of representing these Lyapunov inequalities is via a directed labeled graph  $G(N, E)$ . Each node of this graph is associated with a (continuous, positive definite, and homogeneous) Lyapunov function  $V_i : \mathbb{R}^n \rightarrow \mathbb{R}$ , and each edge is labeled by a finite product of matrices, i.e., by a word from the set  $\mathcal{A}^*$ . As illustrated in Figure 2.1, given two nodes with Lyapunov functions  $V_i(x)$  and  $V_j(x)$  and an edge going from node  $i$  to node  $j$  labeled with the matrix  $A_l$ , we write the Lyapunov inequality:

$$V_j(A_l x) \leq V_i(x) \quad \forall x \in \mathbb{R}^n. \quad (2.1)$$

The problem that we are interested in is to understand which sets of Lyapunov inequalities imply stability of the switched system in (1.2). We will answer this question based on the corresponding graph.

For reasons that will become clear shortly, we would like to reduce graphs whose edges have arbitrary labels from the set  $\mathcal{A}^*$  to graphs whose edges have labels from the set  $\mathcal{A}$ , i.e., labels of length one. This is explained next.

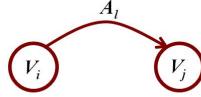


FIG. 2.1. *Graphical representation of Lyapunov inequalities.* The edge in the graph above corresponds to the Lyapunov inequality  $V_j(A_l x) \leq V_i(x)$ . Here,  $A_l$  can be a single matrix from  $\mathcal{A}$  or a finite product of matrices from  $\mathcal{A}$ .

**DEFINITION 2.1.** Given a labeled directed graph  $G(N, E)$ , we define its expanded graph  $G^e(N^e, E^e)$  as the outcome of the following procedure. For every edge  $(i, j) \in E$  with label  $A_{i1} \dots A_{ik} \in \mathcal{A}^k$ , where  $k > 1$ , we remove the edge  $(i, j)$  and replace it with  $k$  new edges  $(s_q, s_{q+1}) \in E^e \setminus E : q \in \{0, \dots, k-1\}$ , where  $s_0 = i$  and  $s_k = j$ .<sup>2</sup> (These new edges go from node  $i$  through  $k-1$  newly added nodes  $s_1, \dots, s_{k-1}$  and then to node  $j$ .) We then label the new edges  $(i, s_1), \dots, (s_q, s_{q+1}), \dots, (s_{k-1}, j)$  with  $A_{i1}, \dots, A_{ik}$  respectively.

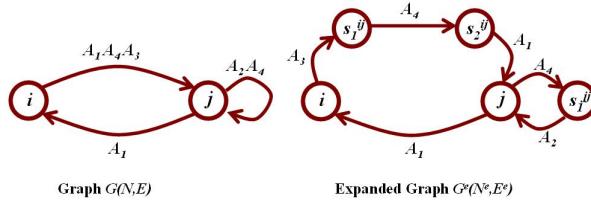


FIG. 2.2. *Graph expansion:* edges with labels of length more than one are broken into new edges with labels of length one.

An example of a graph and its expansion is given in Figure 2.2. Note that if a graph has only labels of length one, then its expanded graph equals itself. The next definition is central to our development.

**DEFINITION 2.2.** Given a directed graph  $G(N, E)$  whose edges are labeled with words from the set  $\mathcal{A}^*$ , we say that the graph is path-complete, if for all finite words  $A_{\sigma_k} \dots A_{\sigma_1}$  of any length  $k$  (i.e., for all words in  $\mathcal{A}^*$ ), there is a directed path in its expanded graph  $G^e(N^e, E^e)$  such that the labels on the edges of this path are the labels  $A_{\sigma_1}$  up to  $A_{\sigma_k}$ .

In Figure 2.3, we present seven path-complete graphs on the alphabet  $\mathcal{A} = \{A_1, A_2\}$ . The fact that these graphs are path-complete is easy to see for graphs  $H_1, H_2, G_3$ , and  $G_4$ , but perhaps not so obvious for graphs  $H_3, G_1$ , and  $G_2$ . One way to check if a graph is path-complete is to think of it as a finite automaton by introducing an auxiliary start node (state) with free transitions to every node and by making all the other nodes be accepting states. Then, there are well-known algorithms (see e.g. [14, Chap. 4]) that check whether the language accepted by an automaton is  $\mathcal{A}^*$ , which is equivalent to the graph being path-complete. Similar algorithms exist in the symbolic dynamics literature; see e.g. [22, Chap. 3]. Our interest in path-complete graphs stems from Theorem 2.4 below that establishes that any such graph gives a method for approximation of the JSR. We introduce one last definition before we state

<sup>2</sup>It is understood that the node index  $s_q$  depends on the original nodes  $i$  and  $j$ . To keep the notation simple we write  $s_q$  instead of  $s_q^{ij}$ .

this theorem.

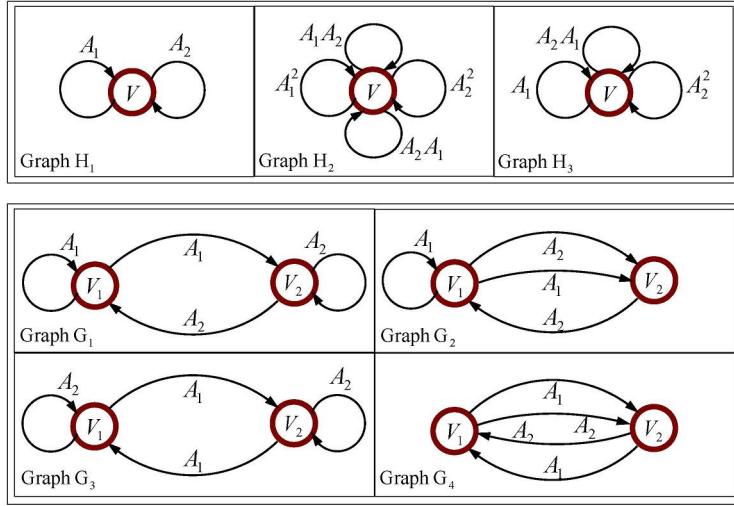


FIG. 2.3. Examples of path-complete graphs for the alphabet  $\{A_1, A_2\}$ . If Lyapunov functions satisfying the inequalities associated with any of these graphs are found, then we get an upper bound of unity on  $\rho(A_1, A_2)$ .

DEFINITION 2.3. Let  $\mathcal{A} = \{A_1, \dots, A_m\}$  be a set of matrices. Given a path-complete graph  $G(N, E)$  and  $|N|$  functions  $V_i(x)$ , we say that  $\{V_i(x) \mid i = 1, \dots, |N|\}$  is a graph Lyapunov function (GLF) associated with  $G(N, E)$  if

$$V_j(L((i, j))x) \leq V_i(x) \quad \forall x \in \mathbb{R}^n, \quad \forall (i, j) \in E,$$

where  $L((i, j)) \in \mathcal{A}^*$  is the label associated with edge  $(i, j) \in E$  going from node  $i$  to node  $j$ .

THEOREM 2.4. Consider a finite set of matrices  $\mathcal{A} = \{A_1, \dots, A_m\}$ . For a scalar  $\gamma > 0$ , let  $\mathcal{A}_\gamma := \{\gamma A_1, \dots, \gamma A_m\}$ . Let  $G(N, E)$  be a path-complete graph whose edges are labeled with words from  $\mathcal{A}_\gamma^*$ . If there exist positive, continuous, and homogeneous<sup>3</sup> functions  $V_i(x)$ , one per node of the graph, such that  $\{V_i(x) \mid i = 1, \dots, |N|\}$  is a graph Lyapunov function associated with  $G(N, E)$ , then  $\rho(\mathcal{A}) \leq \frac{1}{\gamma}$ .

*Proof.* We will first prove the claim for the special case where the edge labels of  $G(N, E)$  belong to  $\mathcal{A}_\gamma$  and therefore  $G(N, E) = G^e(N^e, E^e)$ . The general case will be reduced to this case afterwards. Let  $d$  be the degree of homogeneity of the Lyapunov functions  $V_i(x)$ , i.e.,  $V_i(\lambda x) = \lambda^d V_i(x)$  for all  $\lambda \in \mathbb{R}$ . (The actual value of  $d$  is irrelevant.) By positivity, continuity, and homogeneity of  $V_i(x)$ , there exist scalars  $\alpha_i$  and  $\beta_i$  with  $0 < \alpha_i \leq \beta_i$  for  $i = 1, \dots, |N|$ , such that

$$\alpha_i \|x\|^d \leq V_i(x) \leq \beta_i \|x\|^d, \tag{2.2}$$

for all  $x \in \mathbb{R}^n$  and for all  $i = 1, \dots, |N|$ , where  $\|x\|$  here denotes the Euclidean norm of  $x$ . Let

$$\xi = \max_{i, j \in \{1, \dots, |N|\}^2} \frac{\beta_i}{\alpha_j}. \tag{2.3}$$

<sup>3</sup>The requirement of homogeneity can be replaced by radial unboundedness which is implied by homogeneity and positivity. However, since the dynamical system in (1.2) is homogeneous, there is no conservatism in asking  $V_i(x)$  to be homogeneous.

Now consider an arbitrary product  $A_{\sigma_k} \dots A_{\sigma_1}$  of length  $k$ . Because the graph is path-complete, there will be a directed path corresponding to this product that consists of  $k$  edges, and goes from some node  $i$  to some node  $j$ . If we write the chain of  $k$  Lyapunov inequalities associated with these edges (cf. Figure 2.1), then we get

$$V_j(\gamma^k A_{\sigma_k} \dots A_{\sigma_1} x) \leq V_i(x),$$

which by homogeneity of the Lyapunov functions can be rearranged to

$$\left( \frac{V_j(A_{\sigma_k} \dots A_{\sigma_1} x)}{V_i(x)} \right)^{\frac{1}{d}} \leq \frac{1}{\gamma^k}. \quad (2.4)$$

We can now bound the spectral norm of  $A_{\sigma_k} \dots A_{\sigma_1}$  as follows:

$$\begin{aligned} \|A_{\sigma_k} \dots A_{\sigma_1}\| &\leq \max_x \frac{\|A_{\sigma_k} \dots A_{\sigma_1} x\|}{\|x\|} \\ &\leq \left( \frac{\beta_i}{\alpha_j} \right)^{\frac{1}{d}} \max_x \frac{V_j^{\frac{1}{d}}(A_{\sigma_k} \dots A_{\sigma_1} x)}{V_i^{\frac{1}{d}}(x)} \\ &\leq \left( \frac{\beta_i}{\alpha_j} \right)^{\frac{1}{d}} \frac{1}{\gamma^k} \\ &\leq \xi^{\frac{1}{d}} \frac{1}{\gamma^k}, \end{aligned}$$

where the last three inequalities follow from (2.2), (2.4), and (2.3) respectively. From the definition of the JSR in (1.1), after taking the  $k$ -th root and the limit  $k \rightarrow \infty$ , we get that  $\rho(\mathcal{A}) \leq \frac{1}{\gamma}$  and the claim is established.

Now consider the case where at least one edge of  $G(N, E)$  has a label of length more than one and hence  $G^e(N^e, E^e) \neq G(N, E)$ . We will start with the Lyapunov functions  $V_i(x)$  assigned to the nodes of  $G(N, E)$  and from them we will explicitly construct  $|N^e|$  Lyapunov functions for the nodes of  $G^e(N^e, E^e)$  that satisfy the Lyapunov inequalities associated to the edges in  $E^e$ . Once this is done, in view of our preceding argument and the fact that the edges of  $G^e(N^e, E^e)$  have labels of length one by definition, the proof will be completed.

For  $j \in N^e$ , let us denote the new Lyapunov functions by  $V_j^e(x)$ . We give the construction for the case where  $|N^e| = |N| + 1$ . The result for the general case follows by iterating this simple construction. Let  $s \in N^e \setminus N$  be the added node in the expanded graph, and  $q, r \in N$  be such that  $(s, q) \in E^e$  and  $(r, s) \in E^e$  with  $A_{sq}$  and  $A_{rs}$  as the corresponding labels respectively. Define

$$V_j^e(x) = \begin{cases} V_j(x), & \text{if } j \in N \\ V_q(A_{sq}x), & \text{if } j = s. \end{cases} \quad (2.5)$$

By construction,  $r$  and  $q$ , and subsequently,  $A_{sq}$  and  $A_{rs}$  are uniquely defined and hence,  $\{V_j^e(x) \mid j \in N^e\}$  is well defined. We only need to show that

$$V_q(A_{sq}x) \leq V_s^e(x) \quad (2.6)$$

$$V_s^e(A_{rs}x) \leq V_r(x). \quad (2.7)$$

Inequality (2.6) follows trivially from (2.5). Furthermore, it follows from (2.5) that

$$\begin{aligned} V_s^e(A_{rs}x) &= V_q(A_{sq}A_{rs}x) \\ &\leq V_r(x), \end{aligned}$$

where the inequality follows from the fact that for  $i \in N$ , the functions  $V_i(x)$  satisfy the Lyapunov inequalities of the edges of  $G(N, E)$ .  $\square$

**REMARK 2.1.** *If the matrix  $A_{sq}$  is not invertible, the extended function  $V_j^e(x)$  as defined in (2.5) will only be positive semidefinite. However, since our goal is to approximate the JSR, we will never be concerned with invertibility of the matrices in  $\mathcal{A}$ . Indeed, since the JSR is continuous in the entries of the matrices [19, p. 18], we can always perturb the matrices slightly to make them invertible without changing the JSR by much. In particular, for any  $\alpha > 0$ , there exist  $0 < \varepsilon, \delta < \alpha$  such that*

$$\hat{A}_{sq} = \frac{A_{sq} + \delta I}{1 + \varepsilon}$$

*is invertible and (2.5)–(2.7) are satisfied with  $A_{sq} = \hat{A}_{sq}$ .*

To understand the generality of the framework of “path-complete graph Lyapunov functions” more clearly, let us revisit the path-complete graphs in Figure 2.3 for the study of the case where the set  $\mathcal{A} = \{A_1, A_2\}$  consists of only two matrices. For all of these graphs if our choice for the Lyapunov functions  $V(x)$  or  $V_1(x)$  and  $V_2(x)$  are quadratic functions or sum of squares polynomial functions, then we can formulate the well-established semidefinite programs that search for these candidate Lyapunov functions.

Graph  $H_1$ , which is clearly the simplest possible one, corresponds to the well-known common Lyapunov function approach. Graph  $H_2$  is a common Lyapunov function applied to all products of length two. This graph also obviously implies stability.<sup>4</sup> But graph  $H_3$  tells us that if we find a Lyapunov function that decreases whenever  $A_1$ ,  $A_2^2$ , and  $A_2A_1$  are applied (but with no requirement when  $A_1A_2$  is applied), then we still get stability. This is a priori not obvious and we believe this approach has not appeared in the literature before. Graph  $H_3$  is also an example that explains why we needed the expansion process. Note that for the unexpanded graph, there is no path for any word of the form  $(A_1A_2)^k$  or of the form  $A_2^{2k-1}$ , for any  $k \in \mathbb{N}$ . However, one can check that in the expanded graph of graph  $H_3$ , there is a path for every finite word, and this in turn allows us to conclude stability from the Lyapunov inequalities of graph  $H_3$ .

The remaining graphs in Figure 2.3 which all have two nodes and four edges have a connection to the method of min-of-quadratics or max-of-quadratics Lyapunov functions [16], [15], [12], [17]. If Lyapunov inequalities associated with any of these four graphs are satisfied, then either  $\min\{V_1(x), V_2(x)\}$  or  $\max\{V_1(x), V_2(x)\}$  or both serve as a common Lyapunov function for the switched system. In the next section, we assert these facts in a more general setting (Corollaries 3.4 and 3.5) and show that these graphs in some sense belong to “simplest” families of path-complete graphs.

**3. Duality and examples of families of path-complete graphs.** Now that we have shown that *any* path-complete graph yields a method for proving stability of switched systems, our next focus is naturally on showing how one can produce graphs

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<sup>4</sup>By slight abuse of terminology, we say that a graph implies stability meaning that the associated Lyapunov inequalities imply stability.

that are path-complete. Before we proceed to some basic constructions of such graphs, let us define a notion of *duality* among graphs which essentially doubles the number of path-complete graphs that we can generate.

**DEFINITION 3.1.** *Given a directed graph  $G(N, E)$  whose edges are labeled with words in  $\mathcal{A}^*$ , we define its dual graph  $G'(N, E')$  to be the graph obtained by reversing the direction of the edges of  $G$ , and changing the labels  $A_{\sigma_k} \dots A_{\sigma_1}$  of every edge of  $G$  to its reversed version  $A_{\sigma_1} \dots A_{\sigma_k}$ .*

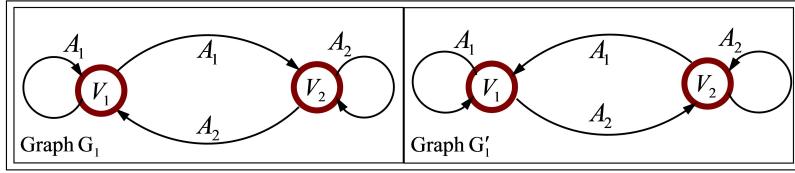


FIG. 3.1. An example of a pair of dual graphs.

An example of a pair of dual graphs with labels of length one is given in Figure 3.1. The following theorem relates dual graphs and path-completeness.

**THEOREM 3.2.** *If a graph  $G(N, E)$  is path-complete, then its dual graph  $G'(N, E')$  is also path-complete.*

*Proof.* Consider an arbitrary finite word  $A_{i_k} \dots A_{i_1}$ . By definition of path-completeness, our task is to show that there exists a path corresponding to this word in the expanded graph of the dual graph  $G'$ . It is easy to see that the expanded graph of the dual graph of  $G$  is the same as the dual graph of the expanded graph of  $G$ ; i.e.,  $G'^e(N^e, E'^e) = G^{e'}(N^e, E^{e'})$ . Therefore, we show a path for  $A_{i_k} \dots A_{i_1}$  in  $G^{e'}$ . Consider the reversed word  $A_{i_1} \dots A_{i_k}$ . Since  $G$  is path-complete, there is a path corresponding to this reversed word in  $G^e$ . Now if we just trace this path backwards, we get exactly a path for the original word  $A_{i_k} \dots A_{i_1}$  in  $G^{e'}$ . This completes the proof.  $\square$

The next proposition offers a very simple construction for obtaining a large family of path-complete graphs with labels of length one.

**PROPOSITION 3.3.** *A graph having any of the two properties below is path-complete.*

*Property (i): every node has outgoing edges with all the labels in  $\mathcal{A}$ .*

*Property (ii): every node has incoming edges with all the labels in  $\mathcal{A}$ .*

*Proof.* If a graph has Property (i), then it is obviously path-complete. If a graph has Property (ii), then its dual has Property (i) and therefore by Theorem 3.2 it is path-complete.  $\square$

Examples of path-complete graphs that fall in the category of this proposition include graphs  $G_1, G_2, G_3$ , and  $G_4$  in Figure 2.3 and all of their dual graphs. By combining the previous proposition with Theorem 2.4, we obtain the following two simple corollaries which unify several linear matrix inequalities (LMIs) that have been previously proposed in the literature. These corollaries also provide a link to min/max-of-quadratics Lyapunov functions. Different special cases of these LMIs have appeared in [16], [15], [12], [17], [20], [9]. Note that the framework of path-complete graph Lyapunov functions makes the proof of the fact that these LMIs imply stability immediate.

**COROLLARY 3.4.** *Consider the set  $\mathcal{A} = \{A_1, \dots, A_m\}$  and the associated switched*

linear system in (1.2) or (1.3). If there exist  $k$  positive definite matrices  $P_j$  such that

$$\begin{aligned} \forall(i, k) \in \{1, \dots, m\}^2, \exists j \in \{1, \dots, m\} \\ \text{such that } \gamma^2 A_i^T P_j A_i \preceq P_k, \end{aligned} \quad (3.1)$$

for some  $\gamma > 1$ , then the system is absolutely asymptotically stable, i.e.,  $\rho(\mathcal{A}) < 1$ . Moreover, the pointwise minimum

$$\min\{x^T P_1 x, \dots, x^T P_k x\}$$

of the quadratic functions serves as a common Lyapunov function.

*Proof.* The inequalities in (3.1) imply that every node of the associated graph has outgoing edges labeled with all the different  $m$  matrices. Therefore, by Proposition 3.3 the graph is path-complete, and by Theorem 2.4 this implies absolute asymptotic stability. The proof that the pointwise minimum of the quadratics is a common Lyapunov function is easy and left to the reader.  $\square$

**COROLLARY 3.5.** Consider the set  $\mathcal{A} = \{A_1, \dots, A_m\}$  and the associated switched linear system in (1.2) or (1.3). If there exist  $k$  positive definite matrices  $P_j$  such that

$$\begin{aligned} \forall(i, j) \in \{1, \dots, m\}^2, \exists k \in \{1, \dots, m\} \\ \text{such that } \gamma^2 A_i^T P_j A_i \preceq P_k, \end{aligned} \quad (3.2)$$

for some  $\gamma > 1$ , then the system is absolutely asymptotically stable, i.e.,  $\rho(\mathcal{A}) < 1$ . Moreover, the pointwise maximum

$$\max\{x^T P_1 x, \dots, x^T P_k x\}$$

of the quadratic functions serves as a common Lyapunov function.

*Proof.* The inequalities in (3.2) imply that every node of the associated graph has incoming edges labeled with all the different  $m$  matrices. Therefore, by Proposition 3.3 the graph is path-complete and the proof of absolute asymptotic stability then follows. The proof that the pointwise maximum of the quadratics is a common Lyapunov function is again left to the reader.  $\square$

**REMARK 3.1.** The linear matrix inequalities in (3.1) and (3.2) are (convex) sufficient conditions for existence of min-of-quadratics or max-of-quadratics Lyapunov functions. The converse is not true. The works in [16], [15], [12], [17] have additional multipliers in (3.1) and (3.2) that make the inequalities non-convex but when solved with a heuristic method contain a larger family of min-of-quadratics and max-of-quadratics Lyapunov functions. Even if the non-convex inequalities with multipliers could be solved exactly, except for special cases where the  $\mathcal{S}$ -procedure is exact (e.g., the case of two quadratic functions), these methods still do not completely characterize min-of-quadratics and max-of-quadratics functions.

**REMARK 3.2.** The work in [20] on “path-dependent quadratic Lyapunov functions” and the work in [9] on “parameter dependent Lyapunov functions”—when specialized to the analysis of arbitrarily switched linear systems—are special cases of Corollary 3.4 and 3.5 respectively. This observation makes a connection between these techniques and min/max-of-quadratics Lyapunov functions which is not established in [20], [9]. It is also interesting to note that the path-complete graph corresponding to the LMIs proposed in [20] (see Theorem 9 there) is the well-known De Bruijn graph [13]. We will analyze the bound on the JSR obtained by analysis via this path-complete graph in the later sections.

The set of path-complete graphs is much broader than the simple family of graphs constructed in Proposition 3.3. Indeed, there are many graphs that are path-complete without having outgoing (or incoming) edges with all the labels on every node; see e.g. graph  $H_4^e$  in Figure 3.2. This in turn means that there are several interesting and unexplored Lyapunov inequalities that we can impose for proving stability of switched systems. Below, we give one particular example of such “non-obvious” inequalities for the case of switching between two matrices.

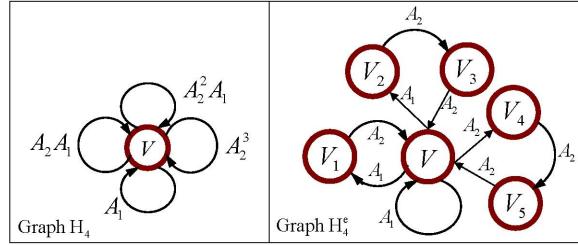


FIG. 3.2. The path-complete graphs corresponding to Proposition 3.6.

**PROPOSITION 3.6.** Consider the set  $\mathcal{A} = \{A_1, A_2\}$  and the switched linear system in (1.2) or (1.3). If there exist a positive definite matrix  $P$  such that

$$\begin{aligned}\gamma^2 A_1^T P A_1 &\preceq P, \\ \gamma^4 (A_2 A_1)^T P (A_2 A_1) &\preceq P, \\ \gamma^6 (A_2^2 A_1)^T P (A_2^2 A_1) &\preceq P, \\ \gamma^6 A_2^{3^T} P A_2^3 &\preceq P,\end{aligned}$$

for some  $\gamma > 1$ , then the system is absolutely asymptotically stable, i.e.,  $\rho(\mathcal{A}) < 1$ .

*Proof.* The graph  $H_4$  associated with the LMIs above and its expanded version  $H_4^e$  are drawn in Figure 3.2. We leave it as an exercise for the reader to show (e.g. by induction on the length of the word) that there is path for every finite word in  $H_4^e$ . Therefore,  $H_4$  is path-complete and in view of Theorem 2.4 the claim is established.  $\square$

**REMARK 3.3.** Proposition 3.6 can be generalized as follows: If a single Lyapunov function decreases with respect to the matrix products

$$\{A_1, A_2 A_1, A_2^2 A_1, \dots, A_2^{k-1} A_1, A_2^k\}$$

for some integer  $k \geq 1$ , then  $\rho(A_1, A_2) < 1$ . We omit the proof of this generalization due to space limitations. We will later prove (Theorem 6.2) a bound for the quality of approximation of path-complete graphs of this type, where a common Lyapunov function is required to decrease with respect to products of different lengths.

When we have so many different ways of imposing conditions for stability, it is natural to ask which ones are more powerful. The answer clearly depends on the combinatorial structure of the graphs and does not seem to be easy in general. Nevertheless, in the next section, we compare the performance of all path-complete graphs with two nodes for analysis of switched systems with two matrices. Some interesting connections between the bounds obtained from these graphs will arise. For example, we will see that the graphs  $H_1$ ,  $G_3$ , and  $G_4$  always give the same bound on the joint spectral radius; i.e., one graph will succeed in proving stability if and

only if the other two will. So, there is no point in increasing the number of decision variables and the number of constraints and impose  $G_3$  or  $G_4$  in place of  $H_1$ . The same is true for the graphs in  $H_3$  and  $G_2$ , which makes graph  $H_3$  preferable to graph  $G_2$ . (See Proposition 4.2.)

**4. Path-complete graphs with two nodes.** In this section, we characterize the set of all path-complete graphs consisting of two nodes, an alphabet set  $\mathcal{A} = \{A_1, A_2\}$ , and edge labels of unit length. We will elaborate on the set of all admissible topologies arising in this setup and compare the performance—in the sense of conservatism of the ensuing analysis—of different path-complete graph topologies.

Before we proceed, we introduce a notation that will prove to be convenient in Subsection 4.2: Given a labeled graph  $G(N, E)$  associated with two matrices  $A_1$  and  $A_2$ , we denote by  $\bar{G}(N, E)$ , the graph obtained by swapping of  $A_1$  and  $A_2$  in all the labels on every edge.

**4.1. The set of path-complete graphs.** The next lemma establishes that for thorough analysis of the case of two matrices and two nodes, we only need to examine graphs with four or fewer edges.

LEMMA 4.1. *Let  $G(\{1, 2\}, E)$  be a path-complete graph with labels of length one for  $\mathcal{A} = \{A_1, A_2\}$ . Let  $\{V_1, V_2\}$  be a graph Lyapunov function for  $G$ . If  $|E| > 4$ , then, either*

- (i) *there exists  $\hat{e} \in E$  such that  $G(\{1, 2\}, E \setminus \hat{e})$  is a path-complete graph,*  
*or*
- (ii) *either  $V_1$  or  $V_2$  or both are common Lyapunov functions for  $\mathcal{A}$ .*

*Proof.* If  $|E| > 4$ , then at least one node has three or more outgoing edges. Without loss of generality let node 1 be a node with exactly three outgoing edges  $e_1, e_2, e_3$ , and let  $L(e_1) = L(e_2) = A_1$ . Let  $D(e)$  denote the destination node of an edge  $e \in E$ . If  $D(e_1) = D(e_2)$ , then  $e_1$  (or  $e_2$ ) can be removed without changing the output set of words. If  $D(e_1) \neq D(e_2)$ , assume, without loss of generality, that  $D(e_1) = 1$  and  $D(e_2) = 2$ . Now, if  $L(e_3) = A_1$ , then regardless of its destination node,  $e_3$  can be removed. If  $L(e_3) = A_2$  and  $D(e_3) = 1$ , then  $V_1$  is a common Lyapunov function for  $\mathcal{A}$ . The only remaining possibility is that  $L(e_3) = A_2$  and  $D(e_3) = 2$ . Note that there must be an edge  $e_4 \in E$  from node 2 to node 1, otherwise either node 2 would have two self-edges with the same label or  $V_2$  would be a common Lyapunov function for  $\mathcal{A}$ . If  $L(e_4) = A_2$  then it can be verified that  $G(\{1, 2\}, \{e_1, e_2, e_3, e_4\})$  is path-complete and thus all other edge can be removed. If there is no edge from node 2 to node 1 with label  $A_2$  then  $L(e_4) = A_1$  and node 2 must have a self-edge  $e_5 \in E$  with label  $L(e_5) = A_2$ , otherwise the graph would not be path-complete. In this case, it can be verified that  $e_2$  can be removed without affecting the output set of words.  $\square$

One can easily verify that a path-complete graph with two nodes and fewer than four edges must necessarily place two self-loops with different labels on one node, which necessitates existence of a common Lyapunov function for the underlying switched system. Since we are interested in exploiting the favorable properties of graph Lyapunov functions in approximation of the JSR, we will focus on graphs with four edges.

**4.2. Comparison of performance.** It can be verified that for path-complete graphs with two nodes, four edges, and two matrices, and without multiple self-loops on a single node, there are a total of nine distinct graph topologies to consider. Of the nine graphs, six have the property that every node has two incoming edges with different labels. These are graphs  $G_1$ ,  $G_2$ ,  $\bar{G}_2$ ,  $G_3$ ,  $\bar{G}_3$ , and  $G_4$  (Figure 2.3). Note

that  $\overline{G}_1 = G_1$  and  $\overline{G}_4 = G_4$ . The duals of these six graphs, i.e.,  $G'_1$ ,  $G'_2$ ,  $\overline{G}'_2$ ,  $G'_3 = G_3$ ,  $\overline{G}'_3 = \overline{G}_3$ , and  $G'_4 = G_4$  have the property that every node has two outgoing edges with different labels. Evidently,  $G_3$ ,  $\overline{G}_3$ , and  $G_4$  are *self-dual graphs*, i.e., they are isomorphic to their dual graphs. The self-dual graphs are least interesting to us since, as we will show, they necessitate existence of a common Lyapunov function for  $\mathcal{A}$  (cf. Proposition 4.2, equation (4.2)).

Note that all of these graphs perform at least as well as a common Lyapunov function because we can always take  $V_1(x) = V_2(x)$ . Furthermore, we know from Corollaries 3.5 and 3.4 that if Lyapunov inequalities associated with  $G_1$ ,  $G_2$ ,  $\overline{G}_2$ ,  $G_3$ ,  $\overline{G}_3$ , and  $G_4$  are satisfied, then  $\max\{V_1(x), V_2(x)\}$  is a common Lyapunov function, whereas, in the case of graphs  $G'_1$ ,  $G'_2$ ,  $\overline{G}'_2$ ,  $G'_3$ ,  $\overline{G}'_3$ , and  $G'_4$ , the function  $\min\{V_1(x), V_2(x)\}$  would serve as a common Lyapunov function. Clearly, for the self-dual graphs  $G_3$ ,  $\overline{G}_3$ , and  $G_4$  both  $\max\{V_1(x), V_2(x)\}$  and  $\min\{V_1(x), V_2(x)\}$  are common Lyapunov functions.

**Notation:** Given a set of matrices  $\mathcal{A} = \{A_1, \dots, A_m\}$ , a path-complete graph  $G(N, E)$ , and a class of functions  $\mathcal{V}$ , we denote by  $\hat{\rho}_{\mathcal{V}, G}(\mathcal{A})$ , the upper bound on the JSR of  $\mathcal{A}$  that can be obtained by numerical optimization of GLFs  $V_i \in \mathcal{V}$ ,  $i \in N$ , defined over  $G$ . With a slight abuse of notation, we denote by  $\hat{\rho}_{\mathcal{V}}(\mathcal{A})$ , the upper bound that is obtained by using a common Lyapunov function  $V \in \mathcal{V}$ .

**PROPOSITION 4.2.** *Consider the set  $\mathcal{A} = \{A_1, A_2\}$ , and let  $G_1$ ,  $G_2$ ,  $G_3$ ,  $G_4$ , and  $H_3$  be the path-complete graphs shown in Figure 2.3. Then, the upper bounds on the JSR of  $\mathcal{A}$  obtained via the associated GLFs satisfy the following relations:*

$$\hat{\rho}_{\mathcal{V}, G_1}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}, G'_1}(\mathcal{A}) \quad (4.1)$$

and

$$\hat{\rho}_{\mathcal{V}}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}, G_3}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}, \overline{G}_3}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}, G_4}(\mathcal{A}) \quad (4.2)$$

and

$$\hat{\rho}_{\mathcal{V}, G_2}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}, H_3}(\mathcal{A}), \quad \hat{\rho}_{\mathcal{V}, \overline{G}_2}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}, \overline{H}_3}(\mathcal{A}) \quad (4.3)$$

and

$$\hat{\rho}_{\mathcal{V}, G'_2}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}, H'_3}(\mathcal{A}), \quad \hat{\rho}_{\mathcal{V}, \overline{G}'_2}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}, \overline{H}'_3}(\mathcal{A}). \quad (4.4)$$

*Proof.* A proof of (4.1) in more generality is provided in Section 5 (cf. Corollary 5.3). The proof of (4.2) is based on symmetry arguments. Let  $\{V_1, V_2\}$  be a GLF associated with  $G_3$  ( $V_1$  is associated with node 1 and  $V_2$  is associated with node 2). Then, by symmetry,  $\{V_2, V_1\}$  is also a GLF for  $G_3$  (where  $V_1$  is associated with node 2 and  $V_2$  is associated with node 1). Therefore, letting  $V = V_1 + V_2$ , we have that  $\{V, V\}$  is a GLF for  $G_3$  and thus,  $V = V_1 + V_2$  is also a common Lyapunov function for  $\mathcal{A}$ , which implies that  $\hat{\rho}_{\mathcal{V}, G_3}(\mathcal{A}) \geq \hat{\rho}_{\mathcal{V}}(\mathcal{A})$ . The other direction is trivial: If  $V \in \mathcal{V}$  is a common Lyapunov function for  $\mathcal{A}$ , then  $\{V_1, V_2 \mid V_1 = V_2 = V\}$  is a GLF associated with  $G_3$ , and hence,  $\hat{\rho}_{\mathcal{V}, G_3}(\mathcal{A}) \leq \hat{\rho}_{\mathcal{V}}(\mathcal{A})$ . Identical arguments based on symmetry hold for  $\overline{G}_3$  and  $G_4$ . We now prove the left equality in (4.3), the proofs for the remaining equalities in (4.3) and (4.4) are analogous. The equivalence between  $G_2$  and  $H_3$  is a special case of the relation between a graph and its *reduced model*, obtained by removing a node without any self-loops, adding a new edge per each pair of incoming

and outgoing edges to that node, and then labeling the new edges by taking the composition of the labels of the corresponding incoming and outgoing edges in the original graph; see [27], [26, Chap. 5]. Note that  $H_3$  is an offspring of  $G_2$  in this sense. This intuition helps construct a proof. Let  $\{V_1, V_2\}$  be a GLF associated with  $G_2$ . It can be verified that  $V_1$  is a Lyapunov function associated with  $H_3$ , and therefore,  $\hat{\rho}_{V,H_3}(\mathcal{A}) \leq \hat{\rho}_{V,G_2}(\mathcal{A})$ . Similarly, if  $V \in \mathcal{V}$  is a Lyapunov function associated with  $H_3$ , then one can check that  $\{V_1, V_2 \mid V_1(x) = V(x), V_2(x) = V(A_2x)\}$  is a GLF associated with  $G_2$ , and hence,  $\hat{\rho}_{V,H_3}(\mathcal{A}) \geq \hat{\rho}_{V,G_2}(\mathcal{A})$ .  $\square$

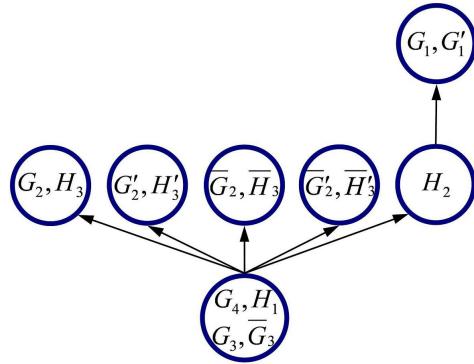


FIG. 4.1. A diagram describing the relative performance of the path-complete graphs of Figure 2.3 together with their duals and label permutations. The graphs placed in the same circle always give the same approximation of the JSR. A graph at the end of an arrow results in an approximation of the JSR that is always at least as good as that of the graph at the start of the arrow. When there is no directed path between two graphs in this diagram, either graph can outperform the other depending on the set of matrices  $\mathcal{A}$ .

REMARK 4.1. Proposition 4.2 (equation 4.1) establishes the equivalence of the bounds obtained from the pair of dual graphs  $G_1$  and  $G'_1$ . This, however, is not true for graphs  $G_2$  and  $\bar{G}_2$  as there exist examples for which

$$\begin{aligned}\hat{\rho}_{V,G_2}(\mathcal{A}) &\neq \hat{\rho}_{V,G'_2}(\mathcal{A}), \\ \hat{\rho}_{V,\bar{G}_2}(\mathcal{A}) &\neq \hat{\rho}_{V,\bar{G}'_2}(\mathcal{A}).\end{aligned}$$

The diagram in Figure 4.1 summarizes the results of this section. We remark that no relations other than the ones given in Figure 4.1 can be established among these path-complete graphs. Indeed, whenever there are no relations between two graphs in Figure 4.1, we have examples of matrices  $A_1, A_2$  (not presented here) for which one graph can outperform the other.

Based on our (incomprehensive) numerical experiments, the graphs  $G_1$  and  $G'_1$  seem to statistically perform better than all other graphs in Figure 4.1. For example, we ran experiments on a set of 100 random  $5 \times 5$  matrices  $\{A_1, A_2\}$  with elements uniformly distributed in  $[-1, 1]$  to compare the performance of graphs  $G_1, G_2$  and  $\bar{G}_2$ . If in each case we also consider the relabeled matrices (i.e.,  $\{A_2, A_1\}$ ) as our input, then, out of the total 200 instances, graph  $G_1$  produced strictly better bounds on the JSR 58 times, whereas graphs  $G_2$  and  $\bar{G}_2$  each produced the best bound of the three graphs only 23 times. (The numbers do not add up to 200 due to ties.) In addition to this superior performance, the bound  $\hat{\rho}_{V,G_1}(\{A_1, A_2\})$  obtained by analysis via the graph  $G_1$  is invariant under (i) permutation of the labels  $A_1$  and  $A_2$  (obvious), and (ii)

transposing of  $A_1$  and  $A_2$  (Corollary 5.3). These are desirable properties which fail to hold for  $G_2$  and  $\overline{G}_2$  or their duals. Motivated by these observations, we generalize  $G_1$  and its dual  $G'_1$  in the next section to the case of  $m$  matrices and  $m$  Lyapunov functions and establish that they have certain appealing properties. We will prove (cf. Theorem 5.4) that these graphs always perform better than a common Lyapunov function in 2 steps (i.e., the graph  $H_2$  in Figure 2.3), whereas, this is not the case for  $G_2$  and  $\overline{G}_2$  or their duals.

**5. Further analysis of a particular family of path-complete graphs.** The framework of path-complete graphs provides a multitude of semidefinite programming based techniques for the approximation of the JSR whose performance vary with computational cost. For instance, as we increase the number of nodes of the graph, or the degree of the polynomial Lyapunov functions assigned to the nodes, or the number of edges of the graph that instead of labels of length one have labels of higher length, we obtain better results but at a higher computational cost. Many of these approximation techniques are asymptotically tight, so in theory they can be used to achieve any desired accuracy of approximation. For example,

$$\hat{\rho}_{\mathcal{V}^{SOS,2d}}(\mathcal{A}) \rightarrow \rho(\mathcal{A}) \text{ as } 2d \rightarrow \infty,$$

where  $\mathcal{V}^{SOS,2d}$  denotes the class of sum of squares homogeneous polynomial Lyapunov functions of degree  $2d$ . (Recall our notation for bounds from Section 4.2.) It is also true that a common quadratic Lyapunov function for products of higher length achieves the true JSR asymptotically [19]; i.e.<sup>5</sup>,

$$\sqrt[t]{\hat{\rho}_{\mathcal{V}^2}(\mathcal{A}^t)} \rightarrow \rho(\mathcal{A}) \text{ as } t \rightarrow \infty.$$

Nevertheless, it is desirable for practical purposes to identify a class of path-complete graphs that provide a good tradeoff between quality of approximation and computational cost. Towards this objective, we propose the use of  $m$  quadratic Lyapunov functions assigned to the nodes of the De Bruijn graph<sup>6</sup> of order 1 on  $m$  symbols for the approximation of the JSR of a set of  $m$  matrices. This graph and its dual are particular path-complete graphs with  $m$  nodes and  $m^2$  edges and will be the subject of study in this section. If we denote the quadratic Lyapunov functions by  $x^T P_i x$ , then we are proposing the use of linear matrix inequalities

$$\begin{aligned} P_i &\succ 0 \quad \forall i = 1, \dots, m, \\ \gamma^2 A_i^T P_j A_i &\preceq P_i \quad \forall i, j = \{1, \dots, m\}^2 \end{aligned} \tag{5.1}$$

or the set of LMIs

$$\begin{aligned} P_i &\succ 0 \quad \forall i = 1, \dots, m, \\ \gamma^2 A_i^T P_i A_i &\preceq P_j \quad \forall i, j = \{1, \dots, m\}^2 \end{aligned} \tag{5.2}$$

for the approximation of the JSR of  $m$  matrices. Throughout this section, we denote the path-complete graphs associated with (5.1) and (5.2) with  $G_1$  and  $G'_1$  respectively.

<sup>5</sup>By  $\mathcal{V}^2$  we denote the class of quadratic homogeneous polynomials. We drop the superscript “SOS” because nonnegative quadratic polynomials are always sums of squares.

<sup>6</sup>The De Bruijn graph of order  $k$  on  $m$  symbols is a labeled directed graph with  $m^k$  nodes and  $m^{k+1}$  edges whose nodes are indexed by all possible words of length  $k$  from the alphabet  $\{1, \dots, m\}$ , and whose edges have labels of length one and are obtained by the following simple rule: There is an edge labeled with the letter  $j$  (or for our purposes the matrix  $A_j$ ) going from node  $i_1 i_2 \dots i_{k-1} i_k$  to node  $i_2 i_3 \dots i_k j$ ,  $\forall i_1 \dots i_k \in \{1, \dots, m\}^k$  and  $\forall j \in \{1, \dots, m\}$ .

(The De Bruijn graph of order 1, by standard convention, is actually the graph  $G'_1$ .) Observe that  $G_1$  and  $G'_1$  are indeed dual graphs as they can be obtained from each other by reversing the direction of the edges. For the case  $m = 2$ , our notation is consistent with the previous section and these graphs are illustrated in Figure 3.1. Also observe from Corollary 3.4 and Corollary 3.5 that the LMIs in (5.1) give rise to max-of-quadratics Lyapunov functions, whereas the LMIs in (5.2) lead to min-of-quadratics Lyapunov functions. We will prove in this section that the approximation bound obtained by these LMIs (i.e., the reciprocal of the largest  $\gamma$  for which the LMIs (5.1) or (5.2) hold) is always the same and lies within a multiplicative factor of  $\frac{1}{\sqrt[4]{n}}$  of the true JSR, where  $n$  is the dimension of the matrices. The relation between the bound obtained by a pair of dual path-complete graphs has a connection to transposition of the matrices in the set  $\mathcal{A}$ . We explain this next.

**5.1. Duality and invariance under transposition.** In [11], [12], it is shown that absolute asymptotic stability of the linear difference inclusion in (1.3) defined by the matrices  $\mathcal{A} = \{A_1, \dots, A_m\}$  is equivalent to absolute asymptotic stability of (1.3) for the transposed matrices  $\mathcal{A}^T := \{A_1^T, \dots, A_m^T\}$ . Note that this fact is immediately seen from the definition of the JSR in (1.1), since  $\rho(\mathcal{A}) = \rho(\mathcal{A}^T)$ . It is also well-known that

$$\hat{\rho}_{\mathcal{V}^2}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}^2}(\mathcal{A}^T).$$

Indeed, if  $x^T Px$  is a common quadratic Lyapunov function for the set  $\mathcal{A}$ , then it is easy to show that  $x^T P^{-1}x$  is a common quadratic Lyapunov function for the set  $\mathcal{A}^T$ . However, this nice property is not true for the bound obtained from some other techniques. For instance, the next example shows that

$$\hat{\rho}_{\mathcal{V}\text{SOS},4}(\mathcal{A}) \neq \hat{\rho}_{\mathcal{V}\text{SOS},4}(\mathcal{A}^T), \quad (5.3)$$

i.e., the upper bound obtained by searching for a common quartic SOS polynomial is not invariant under transposition.

EXAMPLE 5.1. Consider the set of matrices  $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ , with

$$A_1 = \begin{bmatrix} 10 & -6 & -1 \\ 8 & 1 & -16 \\ -8 & 0 & 17 \end{bmatrix}, A_2 = \begin{bmatrix} -5 & 9 & -14 \\ 1 & 10 & 5 \\ 3 & 2 & 16 \end{bmatrix}, A_3 = \begin{bmatrix} -14 & 1 & 0 \\ -15 & -8 & -12 \\ -1 & -6 & 7 \end{bmatrix}, A_4 = \begin{bmatrix} 1 & -8 & -2 \\ 1 & 16 & 3 \\ 16 & 11 & 14 \end{bmatrix}.$$

We have  $\hat{\rho}_{\mathcal{V}\text{SOS},4}(\mathcal{A}) = 21.411$ , but  $\hat{\rho}_{\mathcal{V}\text{SOS},4}(\mathcal{A}^T) = 21.214$  (up to three significant digits).

Similarly, the bound obtained by non-convex inequalities proposed in [11] is not invariant under transposing the matrices. For such methods, one would have to run the numerical optimization twice—once for the set  $\mathcal{A}$  and once for the set  $\mathcal{A}^T$ —and then pick the better bound of the two. We will show that by contrast, the bound obtained from the LMIs in (5.1) and (5.2) are invariant under transposing the matrices. Before we do that, let us prove a general result which states that for path-complete graphs with quadratic Lyapunov functions as nodes, transposing the matrices has the same effect as dualizing the graph.

**THEOREM 5.1.** Let  $G(N, E)$  be a path-complete graph, and let  $G'(N, E')$  be its dual graph. Then,

$$\hat{\rho}_{\mathcal{V}^2,G}(\mathcal{A}^T) = \hat{\rho}_{\mathcal{V}^2,G'}(\mathcal{A}). \quad (5.4)$$

*Proof.* For ease of notation, we prove the claim for the case where the edge labels of  $G(N, E)$  have length one. The proof of the general case is identical. Pick an arbitrary edge  $(i, j) \in E$  going from node  $i$  to node  $j$  and labeled with some matrix  $A_l \in \mathcal{A}$ . By the application of the Schur complement we have

$$A_l P_j A_l^T \preceq P_i \Leftrightarrow \begin{bmatrix} P_i & A_l \\ A_l^T & P_j^{-1} \end{bmatrix} \succeq 0 \Leftrightarrow A_l^T P_i^{-1} A_l \preceq P_j^{-1}.$$

But this already establishes the claim since we see that  $P_i$  and  $P_j$  satisfy the LMI associated with edge  $(i, j) \in E$  when the matrix  $A_l$  is transposed if and only if  $P_j^{-1}$  and  $P_i^{-1}$  satisfy the LMI associated with edge  $(j, i) \in E'$ .  $\square$

COROLLARY 5.2.  $\hat{\rho}_{\mathcal{V}^2, G}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}^2, G}(\mathcal{A}^T)$  if and only if  $\hat{\rho}_{\mathcal{V}^2, G}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}^2, G'}(\mathcal{A})$ .

*Proof.* This is an immediate consequence of the equality in (5.4).  $\square$

It is an interesting question for future research to characterize the path-complete graphs for which one has  $\hat{\rho}_{\mathcal{V}^2, G}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}^2, G}(\mathcal{A}^T)$ . For example, the above corollary shows that this is obviously the case for any path-complete graph that is self-dual. Let us show next that this is also the case for graphs  $G_1$  and  $G'_1$  despite the fact that they are not self-dual.

COROLLARY 5.3. For the path-complete graphs  $G_1$  and  $G'_1$  associated with the inequalities in (5.1) and (5.2), and for any class of continuous, homogeneous, and positive definite functions  $\mathcal{V}$ , we have

$$\hat{\rho}_{\mathcal{V}, G_1}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}, G'_1}(\mathcal{A}). \quad (5.5)$$

Moreover, if quadratic Lyapunov functions are assigned to the nodes of  $G_1$  and  $G'_1$ , then we have

$$\hat{\rho}_{\mathcal{V}^2, G_1}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}^2, G_1}(\mathcal{A}^T) = \hat{\rho}_{\mathcal{V}^2, G'_1}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}^2, G'_1}(\mathcal{A}^T). \quad (5.6)$$

*Proof.* The proof of (5.5) is established by observing that the GLFs associated with  $G_1$  and  $G'_1$  can be derived from one another via  $V'_i(A_i x) = V_i(x)$ . (Note that we are relying here on the assumption that the matrices  $A_i$  are invertible, which as we noted in Remark 2.1, is not a limiting assumption.) Since (5.5) in particular implies that  $\hat{\rho}_{\mathcal{V}^2, G_1}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}^2, G'_1}(\mathcal{A})$ , we get the rest of the equalities in (5.6) immediately from Corollary 5.2 and this finishes the proof. For concreteness, let us also prove the leftmost equality in (5.6) directly. Let  $P_i$ ,  $i = 1, \dots, m$ , satisfy the LMIs in (5.1) for the set of matrices  $\mathcal{A}$ . Then, the reader can check that

$$\tilde{P}_i = A_i P_i^{-1} A_i^T, \quad i = 1, \dots, m,$$

satisfy the LMIs in (5.1) for the set of matrices  $\mathcal{A}^T$ .  $\square$

**5.2. An approximation guarantee.** The next theorem gives a bound on the quality of approximation of the estimate resulting from the LMIs in (5.1) and (5.2). Since we have already shown that  $\hat{\rho}_{\mathcal{V}^2, G_1}(\mathcal{A}) = \hat{\rho}_{\mathcal{V}^2, G'_1}(\mathcal{A})$ , it is enough to prove this bound for the LMIs in (5.1).

THEOREM 5.4. Let  $\mathcal{A}$  be a set of  $m$  matrices in  $\mathbb{R}^{n \times n}$  with JSR  $\rho(\mathcal{A})$ . Let  $\hat{\rho}_{\mathcal{V}^2, G_1}(\mathcal{A})$  be the bound on the JSR obtained from the LMIs in (5.1). Then,

$$\frac{1}{\sqrt[4]{n}} \hat{\rho}_{\mathcal{V}^2, G_1}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}_{\mathcal{V}^2, G_1}(\mathcal{A}). \quad (5.7)$$

*Proof.* The right inequality is just a consequence of  $G_1$  being a path-complete graph (Theorem 2.4). To prove the left inequality, consider the set  $\mathcal{A}^2$  consisting of all  $m^2$  products of length two. In view of (1.6), a common quadratic Lyapunov function for this set satisfies the bound

$$\frac{1}{\sqrt{n}} \hat{\rho}_{\mathcal{V}^2}(\mathcal{A}^2) \leq \rho(\mathcal{A}^2).$$

It is easy to show that

$$\rho(\mathcal{A}^2) = \rho^2(\mathcal{A}).$$

See e.g. [19]. Therefore,

$$\frac{1}{\sqrt[4]{n}} \hat{\rho}_{\mathcal{V}^2}^{\frac{1}{2}}(\mathcal{A}^2) \leq \rho(\mathcal{A}). \quad (5.8)$$

Now suppose for some  $\gamma > 0$ ,  $x^T Q x$  is a common quadratic Lyapunov function for the matrices in  $\mathcal{A}_\gamma^2$ ; i.e., it satisfies

$$\begin{array}{ccc} Q & \succ & 0 \\ \gamma^4 (A_i A_j)^T Q A_i A_j & \preceq & Q \quad \forall i, j = \{1, \dots, m\}^2. \end{array}$$

Then, we leave it to the reader to check that

$$P_i = Q + A_i^T Q A_i, \quad i = 1, \dots, m$$

satisfy (5.1). Hence,

$$\hat{\rho}_{\mathcal{V}^2, G_1}(\mathcal{A}) \leq \hat{\rho}_{\mathcal{V}^2}^{\frac{1}{2}}(\mathcal{A}^2),$$

and in view of (5.8) the claim is established.  $\square$

Note that the bound in (5.7) is independent of the number of matrices. Moreover, we remark that this bound is tighter, in terms of its dependence on  $n$ , than the known bounds for  $\hat{\rho}_{\mathcal{V}SOS,2d}$  for any finite degree  $2d$  of the sum of squares polynomials. The reader can check that the bound in (1.7) goes asymptotically as  $\frac{1}{\sqrt{n}}$ . Numerical evidence suggests that the performance of both the bound obtained by sum of squares polynomials and the bound obtained by the LMIs in (5.1) and (5.2) is much better than the provable bounds in (1.7) and in Theorem 5.4. The problem of improving these bounds or establishing their tightness is open. It goes without saying that instead of quadratic functions, we can associate sum of squares polynomials to the nodes of  $G_1$  and obtain a more powerful technique for which we can also prove better bounds with the exact same arguments.

**5.3. Numerical examples.** In the proof of Theorem 5.4, we essentially showed that the bound obtained from LMIs in (5.1) is tighter than the bound obtained from a common quadratic applied to products of length two. Our first example shows that the LMIs in (5.1) can in fact do better than a common quadratic applied to products of *any* finite length.

EXAMPLE 5.2. Consider the set of matrices  $\mathcal{A} = \{A_1, A_2\}$ , with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

This is a benchmark set of matrices that has been studied in [4], [24], [3] because it gives the worst case approximation ratio of a common quadratic Lyapunov function. Indeed, it is easy to show that  $\rho(\mathcal{A}) = 1$ , but  $\hat{\rho}_{\mathcal{V}^2}(\mathcal{A}) = \sqrt{2}$ . Moreover, the bound obtained by a common quadratic function applied to the set  $\mathcal{A}^t$  is

$$\hat{\rho}_{\mathcal{V}^2}^{\frac{1}{t}}(\mathcal{A}^t) = 2^{\frac{1}{2t}},$$

which for no finite value of  $t$  is exact. On the other hand, we show that the LMIs in (5.1) give the exact bound; i.e.,  $\hat{\rho}_{\mathcal{V}^2, G_1}(\mathcal{A}) = 1$ . Due to the simple structure of  $A_1$  and  $A_2$ , we can even give an analytical expression for our Lyapunov functions. Given any  $\varepsilon > 0$ , the LMIs in (5.1) with  $\gamma = 1/(1 + \varepsilon)$  are feasible with

$$P_1 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad P_2 = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix},$$

for any  $b > 0$  and  $a > b/2\varepsilon$ .

EXAMPLE 5.3. Consider the set of randomly generated matrices  $\mathcal{A} = \{A_1, A_2, A_3\}$ , with

$$A_1 = \begin{bmatrix} 0 & -2 & 2 & 2 & 4 \\ 0 & 0 & -4 & -1 & -6 \\ 2 & 6 & 0 & -8 & 0 \\ -2 & -2 & -3 & 1 & -3 \\ -1 & -5 & 2 & 6 & -4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5 & -2 & -4 & 6 & -1 \\ 1 & 1 & 4 & 3 & -5 \\ -2 & 3 & -2 & 8 & -1 \\ 0 & 8 & -6 & 2 & 5 \\ -1 & -5 & 1 & 7 & -4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 3 & -8 & -3 & 2 & -4 \\ -2 & -2 & -9 & 4 & -1 \\ 2 & 2 & -5 & -8 & 6 \\ -4 & -1 & 4 & -3 & 0 \\ 0 & 5 & 0 & -3 & 5 \end{bmatrix}.$$

A lower bound on  $\rho(\mathcal{A})$  is  $\rho(A_1 A_2 A_3)^{1/3} = 11.8015$ . The upper approximations for  $\rho(\mathcal{A})$  that we computed for this example are as follows:

$$\begin{aligned} \hat{\rho}_{\mathcal{V}^2}(\mathcal{A}) &= 12.5683 \\ \hat{\rho}_{\mathcal{V}^2}^{\frac{1}{2}}(\mathcal{A}^2) &= 11.9575 \\ \hat{\rho}_{\mathcal{V}^2, G_1}(\mathcal{A}) &= 11.8097 \\ \hat{\rho}_{\mathcal{V}^{SOS}, 4}(\mathcal{A}) &= 11.8015. \end{aligned} \tag{5.9}$$

The bound  $\hat{\rho}_{\mathcal{V}^{SOS}, 4}$  matches the lower bound numerically and is most likely exact for this example. This bound is slightly better than  $\hat{\rho}_{\mathcal{V}^2, G_1}$ . However, a simple calculation shows that the semidefinite program resulting in  $\hat{\rho}_{\mathcal{V}^{SOS}, 4}$  has 25 more decision variables than the one for  $\hat{\rho}_{\mathcal{V}^2, G_1}$ . Also, the running time of the algorithm leading to  $\hat{\rho}_{\mathcal{V}^{SOS}, 4}$  is noticeably larger than the one leading to  $\hat{\rho}_{\mathcal{V}^2, G_1}$ . In general, when the dimension of the matrices is large, it can often be cost-effective to increase the number of the nodes of our path-complete graphs but keep the degree of the polynomial Lyapunov functions assigned to its nodes relatively low.

**6. Converse Lyapunov theorems and approximation with arbitrary accuracy.** It is well-known that existence of a Lyapunov function which is the pointwise maximum of quadratics is not only sufficient but also necessary for absolute asymptotic stability of (1.2) or (1.3); see e.g. [23]. This is perhaps an intuitive fact if we recall that switched systems of type (1.2) and (1.3) always admit a convex Lyapunov function. Indeed, if we take “enough” quadratics, the convex and compact unit sublevel set of a convex Lyapunov function can be approximated arbitrarily well with sublevel sets of max-of-quadratics Lyapunov functions, which are intersections of ellipsoids. This of course implies that the bound obtained from max-of-quadratics Lyapunov functions is asymptotically tight for the approximation of the JSR. However, this converse Lyapunov theorem does not answer two natural questions of importance in practice: (i) How many quadratic functions do we need to achieve a desired quality

of approximation? (ii) Can we search for these quadratic functions via semidefinite programming or do we need to resort to non-convex formulations? Our next theorem provides an answer to these questions.

**THEOREM 6.1.** *Let  $\mathcal{A}$  be a set of  $m$  matrices in  $\mathbb{R}^{n \times n}$ . Given any positive integer  $l$ , there exists an explicit path-complete graph  $G$  consisting of  $m^{l-1}$  nodes assigned to quadratic Lyapunov functions and  $m^l$  edges with labels of length one such that the linear matrix inequalities associated with  $G$  imply existence of a max-of-quadratics Lyapunov function and the resulting bound obtained from the LMIs satisfies*

$$\frac{1}{\sqrt[2l]{n}} \hat{\rho}_{\mathcal{V}^2, G}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}_{\mathcal{V}^2, G}(\mathcal{A}). \quad (6.1)$$

*Proof.* Let us denote the  $m^{l-1}$  quadratic Lyapunov functions by  $x^T P_{i_1 \dots i_{l-1}} x$ , where  $i_1 \dots i_{l-1} \in \{1, \dots, m\}^{l-1}$  is a multi-index used for ease of reference to our Lyapunov functions. We claim that we can let  $G$  be the graph dual to the De Bruijn graph of order  $l-1$  on  $m$  symbols. The LMIs associated to this graph are given by

$$\begin{aligned} P_{i_1 i_2 \dots i_{l-2} i_{l-1}} &\succ 0 & \forall i_1 \dots i_{l-1} \in \{1, \dots, m\}^{l-1} \\ A_j^T P_{i_1 i_2 \dots i_{l-2} i_{l-1}} A_j &\preceq P_{i_2 i_3 \dots i_{l-1}} \\ &\quad \forall i_1 \dots i_{l-1} \in \{1, \dots, m\}^{l-1}, \\ &\quad \forall j \in \{1, \dots, m\}. \end{aligned} \quad (6.2)$$

The fact that  $G$  is path-complete and that the LMIs imply existence of a max-of-quadratics Lyapunov function follows from Corollary 3.5. The proof that these LMIs satisfy the bound in (6.1) is a straightforward generalization of the proof of Theorem 5.4. By the same arguments we have

$$\frac{1}{\sqrt[2l]{n}} \hat{\rho}_{\mathcal{V}^2}^{\frac{1}{l}}(\mathcal{A}^l) \leq \rho(\mathcal{A}). \quad (6.3)$$

Suppose  $x^T Q x$  is a common quadratic Lyapunov function for the matrices in  $\mathcal{A}^l$ ; i.e., it satisfies

$$\begin{aligned} Q &\succ 0 \\ (A_{i_1} \dots A_{i_l})^T Q A_{i_1} \dots A_{i_l} &\preceq Q \quad \forall i_1 \dots i_l \in \{1, \dots, m\}^l. \end{aligned}$$

Then, it is easy to check that<sup>7</sup>

$$\begin{aligned} P_{i_1 i_2 \dots i_{l-2} i_{l-1}} &= Q + A_{i_{l-1}}^T Q A_{i_{l-1}} \\ &+ (A_{i_{l-2}} A_{i_{l-1}})^T Q (A_{i_{l-2}} A_{i_{l-1}}) + \dots \\ &+ (A_{i_1} A_{i_2} \dots A_{i_{l-2}} A_{i_{l-1}})^T Q (A_{i_1} A_{i_2} \dots A_{i_{l-2}} A_{i_{l-1}}), \\ i_1 \dots i_{l-1} &\in \{1, \dots, m\}^{l-1}, \end{aligned}$$

satisfy (6.2). Hence,

$$\hat{\rho}_{\mathcal{V}^2, G}(\mathcal{A}) \leq \hat{\rho}_{\mathcal{V}^2}^{\frac{1}{l}}(\mathcal{A}^l),$$

and in view of (6.3) the claim is established.  $\square$

<sup>7</sup>The construction of the Lyapunov function here is a special case of a general scheme for constructing Lyapunov functions that are monotonically decreasing from those that decrease only every few steps; see [1, p. 58].

**REMARK 6.1.** A converse Lyapunov theorem identical to Theorem 6.1 can be proven for the min-of-quadratics Lyapunov functions. The only difference is that the LMIs in (6.2) would get replaced by the ones corresponding to the dual graph of  $G$ .

Our last theorem establishes approximation bounds for a family of path-complete graphs with one single node but several edges labeled with words of different lengths. Examples of such path-complete graphs include graph  $H_3$  in Figure 2.3 and graph  $H_4$  in Figure 3.2.

**THEOREM 6.2.** Let  $\mathcal{A}$  be a set of matrices in  $\mathbb{R}^{n \times n}$ . Let  $\tilde{G}(\{1\}, E)$  be a path-complete graph, and  $l$  be the length of the shortest word in  $\tilde{\mathcal{A}} = \{L(e) : e \in E\}$ . Then  $\hat{\rho}_{\mathcal{V}^2, \tilde{G}}(\mathcal{A})$  provides an estimate of  $\rho(\mathcal{A})$  that satisfies

$$\frac{1}{\sqrt[2l]{n}} \hat{\rho}_{\mathcal{V}^2, \tilde{G}}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}_{\mathcal{V}^2, \tilde{G}}(\mathcal{A}).$$

*Proof.* The right inequality is obvious, we prove the left one. Since both  $\hat{\rho}_{\mathcal{V}^2, \tilde{G}}(\mathcal{A})$  and  $\rho$  are homogeneous in  $\mathcal{A}$ , we may assume, without loss of generality, that  $\hat{\rho}_{\mathcal{V}^2, \tilde{G}}(\mathcal{A}) = 1$ . Suppose for the sake of contradiction that

$$\rho(\mathcal{A}) < 1/\sqrt[2l]{n}. \quad (6.4)$$

We will show that this implies that  $\hat{\rho}_{\mathcal{V}^2, \tilde{G}}(\mathcal{A}) < 1$ . Towards this goal, let us first prove that  $\rho(\tilde{\mathcal{A}}) \leq \rho^l(\mathcal{A})$ . Indeed, if we had  $\rho(\tilde{\mathcal{A}}) > \rho^l(\mathcal{A})$ , then there would exist<sup>8</sup> an integer  $i$  and a product  $A_\sigma \in \tilde{\mathcal{A}}^i$  such that

$$\rho^{\frac{1}{i}}(A_\sigma) > \rho^l(\mathcal{A}). \quad (6.5)$$

Since we also have  $A_\sigma \in \mathcal{A}^j$  (for some  $j \geq il$ ), it follows that

$$\rho^{\frac{1}{j}}(A_\sigma) \leq \rho(\mathcal{A}). \quad (6.6)$$

The inequality in (6.5) together with  $\rho(\mathcal{A}) \leq 1$  gives

$$\rho^{\frac{1}{j}}(A_\sigma) > \rho^{\frac{il}{j}}(\mathcal{A}) \geq \rho(\mathcal{A}).$$

But this contradicts (6.6). Hence we have shown

$$\rho(\tilde{\mathcal{A}}) \leq \rho^l(\mathcal{A}).$$

Now, by our hypothesis (6.4) above, we have that  $\rho(\tilde{\mathcal{A}}) < 1/\sqrt{n}$ . Therefore, there exists  $\epsilon > 0$  such that  $\rho((1+\epsilon)\tilde{\mathcal{A}}) < 1/\sqrt{n}$ . It then follows from (1.6) that there exists a common quadratic Lyapunov function for  $(1+\epsilon)\tilde{\mathcal{A}}$ . Hence,  $\hat{\rho}_{\mathcal{V}^2}((1+\epsilon)\tilde{\mathcal{A}}) \leq 1$ , which immediately implies that  $\hat{\rho}_{\mathcal{V}^2, \tilde{G}}(\mathcal{A}) < 1$ , a contradiction.  $\square$

A noteworthy immediate corollary of Theorem 6.2 (obtained by setting  $\tilde{\mathcal{A}} = \bigcup_{t=r}^k \mathcal{A}^t$ ) is the following: If  $\rho(\mathcal{A}) < \frac{1}{\sqrt[2l]{n}}$ , then there exists a quadratic Lyapunov function that decreases simultaneously for all products of lengths  $r, r+1, \dots, r+k$ , for any desired value of  $k$ . Note that this fact is obvious for  $r=1$ , but nonobvious for  $r \geq 2$ .

<sup>8</sup>Here, we are appealing to the well-known fact about the JSR of a general set of matrices  $\mathcal{B}$ :  $\rho(\mathcal{B}) = \limsup_{k \rightarrow \infty} \max_{B \in \mathcal{B}^k} \rho^{\frac{1}{k}}(B)$ . See e.g. [19, Chap. 1].

**7. Conclusions and future directions.** We introduced the framework of path-complete graph Lyapunov functions for the formulation of semidefinite programming based algorithms for approximating the joint spectral radius (or equivalently establishing absolute asymptotic stability of an arbitrarily switched linear system). We defined the notion of a path-complete graph, which was inspired by concepts in automata theory. We showed that every path-complete graph gives rise to a technique for the approximation of the JSR. This provided a unifying framework that includes many of the previously proposed techniques and also introduces new ones. (In fact, all families of LMIs that we are aware of are particular cases of our method.) We shall also emphasize that although we focused on switched *linear* systems because of our interest in the JSR, the analysis technique of multiple Lyapunov functions on path-complete graphs is clearly valid for switched *nonlinear* systems as well.

We compared the quality of the bound obtained from certain classes of path-complete graphs, including all path-complete graphs with two nodes on an alphabet of two matrices, and also a certain family of dual path-complete graphs. We proposed a specific class of such graphs that appear to work particularly well in practice and proved that the bound obtained from these graphs is invariant under transposition of the matrices and is always within a multiplicative factor of  $1/\sqrt[4]{n}$  from the true JSR. Finally, we presented two converse Lyapunov theorems, one for the well-known methods of minimum and maximum-of-quadratics Lyapunov functions, and the other for a new class of methods that propose the use of a common quadratic Lyapunov function for a set of words of possibly different lengths.

We believe the methodology proposed in this chapter should straightforwardly extend to the case of *constrained switching* by requiring the graphs to have a path not for all the words, but only the words allowed by the constraints on the switching. A rigorous treatment of this idea is left for future work.

Another question for future research is to determine the complexity of checking path-completeness of a given graph  $G(N, E)$ . As we explained in Section 2, well-known algorithms in automata theory (see e.g. [14, Chap. 4]) can check for path-completeness by testing whether the associated finite automaton accepts all finite words. When the automata are deterministic (i.e., when all outgoing edges from every node have different labels), these algorithms are very efficient and have running time of only  $O(|N|^2)$ . However, the problem of deciding whether a non-deterministic finite automaton accepts all finite words is known to be PSPACE-complete [10, p. 265]. We are yet to investigate whether the same is true for automata arising from path-complete graphs which have a little more structure. Vincent Blondel showed that the problem is at least NP-hard (personal communication). Of course, the step of checking path-completeness of a graph is done offline and prior to the run of our algorithms for approximating the JSR. Therefore, while checking path-completeness is in general difficult, the approximation algorithms that we presented indeed run in polynomial time since they work with a fixed (a priori chosen) path-complete graph. Nevertheless, the question on complexity of checking path-completeness is interesting in many other settings, e.g., when deciding whether a given set of Lyapunov inequalities imply stability of an arbitrarily switched system.

Some other interesting questions that can be explored in the future are the following. What are some other classes of path-complete graphs that lead to new techniques for proving stability of switched systems? How can we compare the performance of different path-complete graphs in a systematic way? Given a set of matrices, a class of Lyapunov functions, and a fixed size for the graph, can we efficiently come up with

the least conservative topology of a path-complete graph? Within the framework that we proposed, do all the Lyapunov inequalities that prove stability come from path-complete graphs? What are the analogues of the results of this chapter for continuous time switched systems? To what extent do the results carry over to the synthesis (controller design) problem for switched systems? These questions and several others show potential for much follow-up work on path-complete graph Lyapunov functions.

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