SIMULATION OF NON-RECIRCULATING FLOWS OF DILUTE FIBER SUSPENSIONS

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Abstract

An elementary continuum model is used in the simulation of the flow of dilute fiber suspensions in a Newtonian fluid. A pseudo-Newton–Raphson technique is developed in which the constitutive equation is solved via a stable and accurate integration along streamlines, while the remaining equations of motion are solved by a conventional finite element method. The numerical technique is used to simulate both planar extrusion and falling-ball rheometry. In planar extrusion, simulations predict a strong dependence of extrudate swell on fiber concentration and aspect ratio. Simulations of falling-ball rheometry predict Newtonian behavior with an intrinsic viscosity that is insensitive to initial fiber orientation.

Keywords: falling-ball rheometry; fiber suspensions; finite element method; streamline integration

1. Introduction

Dilute suspensions of short fibers in Newtonian fluids are known to exhibit large deviations from Newtonian behavior. In flows through contractions, recirculation vortices increase in size drastically with the addition of a small amount of fibers [1]. In extrusion, extrudate swell decreases substan-
tially with the presence of fibers [2]. Recent falling-ball rheometry experiments [3–5] have shown that short-fiber suspensions can also exhibit Newtonian behavior in complex flows.

Continuum models currently used to describe dilute and semi-dilute suspensions are hyperbolic in nature and can predict sharp boundary layers and singularities even in viscometric flows. Standard finite element and finite difference techniques are known to be unstable in the solution of hyperbolic systems and tend to fail in the presence of boundary layers and singularities. Lipscomb and co-workers [1] avoided this problem by the use of a fiber-alignment approximation which is only valid in a limited class of flows. In this work, a pseudo-Newton–Raphson technique is developed in which the suspension model is solved with stable and accurate integration along streamlines, while the remaining equations of motion are discretized by a finite element method. The numerical technique is used to simulate planar extrusion and falling-ball rheometry.

2. Governing equations

In this work, an elementary continuum model is used to represent the flow behavior of dilute and semi-dilute short-fiber suspensions. Here, dilute, semi-dilute, and concentrated suspensions are defined as

dilute: \( \phi a_r^2 < 1 \),

semi-dilute: \( 1 < \phi a_r^2 < a_r \),

concentrated: \( 1 < \phi a_r \),

where \( \phi \) is the fiber volume fraction and \( a_r \) is the fiber aspect ratio. The continuum model was apparently first developed by Evans [6]; a complete discussion of the theory is given by Lipscomb and co-workers [1]. The model represents fibers as a population of inertialess ellipsoids of revolution within a Newtonian suspending fluid. Each ellipsoid possesses a single preferred direction, denoted by the unit vector \( n \), which is allowed to vary spatially and temporally. In order to represent the suspension as a continuum, an orientation distribution function \( \psi(x, n, t) \) is introduced, where the quantity \( \psi(x, n, t) \, dn \) is defined as the fraction of particles in a fluid element at the spatial location \( x \) which have an orientation vector between \( n \) and \( n + dn \) at time \( t \). Prediction of the distribution function over an entire flow domain is computationally feasible only in some simple flow geometries [1]. The orientation is characterized by the second order tensor \( N \) defined by

\[
N = \int n n \, \psi(x, n, t) \, dn,
\]

where the integration is over all possible orientations \( n \). Since the fibers are
rigid particles, and \( n \) has unit magnitude, the orientation tensor \( N \) is constrained by

\[
\text{trace } N = 1. \tag{3}
\]

The remaining equations describing the continuum model are given by the orientation equation (which follows from a closure approximation [1]),

\[
\frac{D}{Dt} N = W \cdot N - N \cdot W + \lambda (D \cdot N + N \cdot D - 2D : NN), \tag{4}
\]

and the constitutive equation for the total stress \( \sigma \),

\[
\sigma = -pI + 2\mu_n D + T_f, \tag{5}
\]

where

\[
T_f = \phi \{-p_0 I + 2\mu_1 D + \mu_2 D : NN + 2\mu_3 (D \cdot N + N \cdot D)\}. \tag{6}
\]

Here, \( T_f \) is the stress due to the presence of the fibers, \( \mu_n \) is the viscosity of the Newtonian suspending fluid, \( D \) is the rate-of-deformation tensor, \( W \) is the vorticity tensor, \( p \) and \( p_0 \) are isotropic pressures, and \( \lambda \) and \( \{ \mu_i \} \) are material constants. It is important to note that eqn. (4) constitutes a hyperbolic system in \( N \) whose characteristic curves in steady incompressible flow are streamlines. (Equations (4)–(6) are equivalent to an orientation-averaging of the transversely isotropic fluid (TIF) with a closure approximation [1,6]. The TIF model was first proposed by Ericksen [7,8] as an elementary theory for anisotropic fluids.)

The material parameter \( \lambda \) is given by

\[
\lambda = \frac{a_r^2 - 1}{a_r^2 + 1}. \tag{7}
\]

Parameters \( \{ \mu_i \} \) are viscosity coefficients which are uniquely determined by \( a_r \) and \( \mu_n \) as follows:

\[
\frac{\mu_1}{\mu_n} = \frac{4(a_r^2 - 1)^2}{a_r^2(3\Theta + 2a_r^2 - 5)}, \tag{8}
\]

\[
\frac{\mu_2}{\mu_n} = \frac{8(a_r^2 - 1)^2}{a_r^2(3\Theta + 2a_r^2 - 5)} + \frac{8(a_r^2 - 1)^2}{(a_r^2 + 1)[(1 - 3\Theta)a_r^2 + 2]} \]

\[
+ \frac{2(a_r^2 - 1)^2[(1 - 4a_r^2)\Theta + 2a_r^2 + 1]}{a_r^2(3\Theta + 2a_r^2 - 5)[(2a_r^2 + 1)\Theta - 3]}, \tag{9}
\]

\[
\frac{\mu_3}{\mu_n} = \frac{2(a_r^2 - 1)^2}{a_r^2(a_r^2 + 1)[(2a_r^2 + 1)\Theta - 3]} - \frac{4(a_r^2 - 1)^2}{a_r^2(3\Theta + 2a_r^2 - 5)}, \tag{10}
\]
where the function $\Theta$ is defined for $a_r > 1$ by

$$
\Theta = \frac{1}{2 a_r^2 \sqrt{a_r^2 - 1}} \ln \left( \frac{a_r + \sqrt{a_r^2 - 1}}{a_r - \sqrt{a_r^2 - 1}} \right).
$$

For high aspect ratios (i.e., $a_r \to \infty$) the viscosity coefficients (8)–(10) reduce to

$$
\frac{\mu_1}{\mu_n} = 2, \quad \frac{\mu_2}{\mu_n} = \frac{a_r^2}{\ln a_r}, \quad \frac{\mu_3}{\mu_n} = 0.
$$

Thus, for large aspect ratio fibers, the constitutive equation (6) reduces approximately to

$$
T_f = \phi \mu_2 D : NN.
$$

An analogous constitutive form was proposed by Evans [6] for semi-dilute suspensions (see also Dinh and Armstrong [9]). In this model, the stress due to the presence of the fibers is given by

$$
T_f = \phi \mu_f D : NN,
$$

where the viscosity coefficient $\mu_f$ is defined by

$$
\frac{\mu_f}{\mu_n} = \frac{\pi N_p L_p^3}{6 \ln \left( \frac{2H}{D_p} \right)}.
$$

Here, $N_p$ is the number of fibers per unit volume, $L_p$ and $D_p$ are the fiber length and diameter, respectively, and $H$ is the average distance between a fiber and its nearest neighbor. Thus, $H$ is not uniquely defined, and two limits are considered, corresponding to a random orientation and an orientation fully-aligned along a single axis [10,11], as follows:

random: $H = \frac{1}{N_p L_p^2}$, \hspace{1cm} (16)

aligned: $H = \frac{1}{\sqrt{N_p L_p}}$. \hspace{1cm} (17)

The Dinh–Armstrong theory [9], which is an integral form of eqn. (14) without a closure approximation, takes $H$ to be given by (16) for a random orientation, regardless of the true local orientation distribution.

The equations of the suspension model (4)–(6), together with the linear momentum balance

$$
\nabla \cdot \left( 2\mu_n D + T_f - pI \right) + f = 0,
$$

(18)
and the mass balance
\[ \nabla \cdot \mathbf{u} = 0, \] (19)
comprise the governing equations for steady, isothermal, creeping motion. Here, \( f \) is a body force. It becomes convenient to rearrange the momentum equation (18) into the following form
\[ \nabla \cdot (2\mu_0 \mathbf{D} + \mathcal{I} - \mathcal{P} \mathbf{I}) + f = 0, \] (20)
with the following definitions
\[ \mu_0 = \mu_n + \phi \mu_1, \quad \mathcal{P} \equiv p + \phi p_0, \quad \mathcal{I} \equiv T - \phi(-p_0 \mathbf{I} + 2\mu_1 \mathbf{D}). \] (21)

3. Solution of full set of equations of motion

In simple steady shear flows, eqn. (4) predicts a spatial periodicity in the fiber orientation which results from an end-to-end tumbling of fibers [12]; the period of tumbling is given by
\[ T_r = \frac{2\pi (a_r^2 + 1)}{a_r \dot{\gamma}}, \] (22)
where \( \dot{\gamma} \) is the shear rate; the length scale over which tumbling occurs is thus given by
\[ L_r = |\mathbf{u}| T_r. \] (23)

It is important to note that the length \( L_r \) is not constant over the shear field since the period of tumbling approaches zero near no-slip solid boundaries. On solid boundaries, however, fibers have a fixed orientation dictated by the orientation equation (4). Thus there exists a singularity: fibers rotate over length scales approaching zero near solid boundaries, but have fixed orientations on these boundaries [1]. (A similar singularity can exist near planes and axes of symmetry where the velocity gradient is zero.) Furthermore, it has been shown that conventional Galerkin finite element methods (GFEMs) are inappropriate methods to predict the arbitrarily sharp orientation gradients arising from this singular behavior [1]. Physically, the singular behavior is in part a consequence of neglecting interactions between fibers and solid boundaries; such interactions will be important over distances on the order of the fiber length.

In order to eliminate the singular behavior, Lipscomb and co-workers [1] employed the ‘full-alignment’ assumption. Fibers of large aspect ratio tend to align with the flow field when a certain stability criterion is satisfied, and this will occur in flows possessing some degree of elongation [1]. Fibers with an infinite aspect ratio, once fully aligned with the flow field, remain fully
aligned [1,6]. Thus, Lipscomb and co-workers replaced the orientation equation (4) with the full-alignment assumption given by

$$\mathcal{N} = \frac{uu}{u \cdot u}. \quad (24)$$

The resulting simplified equations of motion were then solved successfully for flow through a 4:1 contraction by a conventional GFEM, giving excellent agreement with flow visualization experiments. It should be noted, however, that the full-alignment assumption (24) is not valid in portions of the corner recirculation in contraction flow.

An alternative approach was taken by Papanastasiou and Alexandrov [13] in the simulation of the Dinh–Armstrong integral model [9]. They employed a fully coupled Newton–Raphson technique, where the constitutive equation was integrated along streamlines while the remaining equations of motion were discretized by a GFEM. This particular numerical method, however, can require large computational resources, making intensive spatial mesh refinement difficult [14].

Streamlines are the characteristic curves for the hyperbolic orientation equation (4) in two-dimensional steady-state flows. The substantial derivative in eqn. (4) can thus be replaced by a derivative along streamlines. Streamline integration (SI) is a method by which eqn. (4) may be numerically integrated along streamlines [14–16]. Integration starts at an upstream location where $\mathcal{N}$ is known, and is continued downstream until a desired point is reached. In this work, SI is performed as follows (complete details are contained in Rosenberg [17]): First, the streamfunction is computed on the basis of a known velocity field by solution of a Poisson equation. A discrete number of streamlines are then constructed from the computed streamfunction. An Adams–Bashford predictor with a trapezoidal corrector is used to integrate the orientation equation (4) numerically along the constructed streamlines. The trapezoidal rule is known to be second-order accurate, and A-stable (see for example Ref. 18). The integration step size along a streamline is adapted such that a measure of the local truncation error at each integration step is less than some predefined tolerance whenever possible.

Here, the velocity, pressure, and streamfunction are represented by the finite sums

$$u = \sum_j u_j \varphi_j, \quad P = \sum_k P_k \pi_k, \quad \Psi = \sum_j \Psi_j \varphi_j, \quad (25)$$

where $u_j$, $P_k$, and $\Psi_j$ are nodal values of velocity, pressure and streamfunction, respectively; $\pi_k$ and $\varphi_j$ represent bilinear and biquadratic finite element shape functions, respectively.
Following Tanner and Luo [15,16], an iterative method is developed in which the constitutive equation is solved by SI, while the other conservation equations are solved by a standard GFEM. The iterative scheme used here, henceforth referred to as PSLE (for 'preconditioned streamline elements'), is as follows.

1. Determine Newtonian kinematics via a GFEM ($\phi = 0$).
2. For each Gauss point, determine new values of $N$ via SI, based upon current kinematics.
3. Determine new kinematics based upon $T$ via a GFEM.
4. Calculate a streamfunction based upon current kinematics via a GFEM.
5. Adapt the mesh so that elements conform to streamlines.
6. Check for convergence. Return to step (2) if necessary.

Step (2) in the iterative scheme calls for the computation of $N$ at the Gauss points in order to compute integrals in a weak form of the equations of motion. (In general, the SI method itself cannot compute $N$ at the nodal points, since integration steps onto element-element boundaries would often be required. For the finite element methods used here, the velocity is $C^0$ across such boundaries. Therefore, the components of the velocity gradient are not uniquely defined on these boundaries.) Each Gauss point, however, can lie on different streamlines. The construction and integration along all required streamlines can be quite time consuming. Tanner and Luo [15,16] use biquadratic quadrilateral elements which conform to streamlines to avoid this difficulty. When a side of a quadratic quadrilateral becomes even slightly curved inwards, the mapping of the element to the parent element can become singular [19]. To guard against this, bilinear quadrilateral streamline elements are used in this work. The finite element mesh is formed by strips of bilinear isoparametric elements. Each strip is defined by two streamlines such that the vertices of each element in the strip lie on one of the two streamlines (see Appendix A for details). Spatial coordinates are represented by bilinear interpolations, whereas the streamfunction is represented by a biquadratic interpolation. Within each element, streamlines are approximated by lines of constant local coordinate. In general, streamline elements are not used in flows with recirculation.

It should be noted that lines of constant streamfunction do not necessarily correspond to lines of constant local coordinate for bilinear streamline elements (see Appendix A). Within each element, the computed streamfunction is potentially biquadratic, while the streamlines are approximated by straight lines. Only straight streamlines correspond to lines of constant local coordinate for such bilinear streamline elements. In regions where streamlines are curved, the use of relatively small elements is anticipated. Thus, the error associated with this approximation of streamlines is expected to remain small.
Step (3) calls for the prediction of kinematics via a GFEM. A different approach is taken here from that of Tanner and Luo [15,16]. The weak form used in the iterative scheme is preconditioned to enhance overall convergence properties. Preconditioning is accomplished as follows: Consider a case where the full-alignment assumption is adopted. Since extra stress is now explicit in velocity, it may be eliminated directly from the equations of motion. A standard GFEM is applied to the resulting equations of motion, yielding the following weak form

\[
\begin{align*}
\mathbf{r}_m &= \left\langle \nabla \mathbf{q}_m ; 2\mu_0 \mathbf{D} - \mathbf{P} \mathbf{I} + \mathbf{\mathcal{T}} \right\rangle - \left\langle \left\langle \mathbf{q}_m ; \mathbf{t} \right\rangle \right\rangle - \left\langle \mathbf{q}_m ; \mathbf{f} \right\rangle = 0, \\
\mathbf{r}_n &= \left\langle \pi_n ; \nabla \cdot \mathbf{u} \right\rangle = 0,
\end{align*}
\]

where \(\mathbf{r}_m\) and \(\mathbf{r}_n\) are a family of residuals. A Newton–Raphson technique is easily applied to the weak form (26), (27). The Newton–Raphson iteration is shown schematically as follows

\[
\begin{bmatrix}
\frac{\partial \mathbf{r}_m}{\partial \mathbf{u}_j} \\
\frac{\partial \mathbf{r}_n}{\partial \mathbf{u}_j}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_j^{(n+1)} - \mathbf{u}_j^{(n)} \\
\mathbf{P}_k^{(n+1)} - \mathbf{P}_k^{(n)}
\end{bmatrix}
= -\begin{bmatrix}
\mathbf{r}_m^{(n)} \\
\mathbf{r}_n^{(n)}
\end{bmatrix},
\]

where superscripts \((n)\) and \((n+1)\) denote iteration number. Preconditioning is achieved by using SI predicted values of \(\mathbf{\mathcal{T}}\) to calculate the residuals \(\mathbf{r}_m\) on the right side of eqn. (28), while retaining the full-alignment assumption in calculating the terms \(\partial \mathbf{r}_m / \partial \mathbf{u}_j\) in the Jacobian matrix. This preconditioning is now used in step (3) of the PSLE description.

PSLE enjoys several advantages over a Picard scheme. A dissipative viscosity, as used in Picard methods, is not necessary to stabilize PSLE iterations. In many cases, PSLE iterations were found to converge faster than Picard schemes. At low fiber concentrations, PSLE sometimes enjoyed nearly quadratic convergence, even in cases where full fiber alignment was a poor approximation. Furthermore, in some cases, PSLE obtained converged results at fiber concentrations slightly higher than were obtainable with a Picard scheme.

Convergence criteria for PSLE are rather simple. Maximum relative changes (MRCs) are introduced and defined as follows

\[
\begin{align*}
\max_g \left| \mathcal{g}^{n+1} - \mathcal{g}^n \right|, & \quad \max_j \left| \mathbf{u}_j^{n+1} - \mathbf{u}_j^n \right| \\
\max_g \left[ \max_{\mathcal{g}} \left| \mathcal{g}^{n+1} \right| , 1 \right], & \quad \max_j \left[ \max_{\mathbf{u}_j} \left| \mathbf{u}_j^{n+1} \right| , 1 \right], \\
\max_k \left| \mathcal{P}_k^{n+1} - \mathcal{P}_k^n \right|, & \quad \max_j \left| \Psi_j^{n+1} - \Psi_j^n \right| \\
\max_k \left[ \max_{\mathcal{P}_k} \left| \mathcal{P}_k^{n+1} \right| , 1 \right], & \quad \max_j \left[ \max_{\Psi_j} \left| \Psi_j^{n+1} \right| , 1 \right].
\end{align*}
\]
Here, subscript $g$ represents values at Gauss points, and superscripts $n$ and $n+1$ refer to iteration number. Different MRCs are assumed for each component of $T$ and $u$. The MRCs (32) are measures of the relative changes in the computed field variables from iteration to iteration. Decreasing or small MRCs do not guarantee convergence; MRCs can be arbitrarily small while the solution still changes slowly but constantly with iteration. For this reason, 2-norm residuals are introduced and defined by

$$
\sqrt{\sum_m (r_{1,m}^i)^2}, \quad \sqrt{\sum_n (r_{2,n}^i)^2},
$$

where $r_{1,m}^i$ are the spatial components of $r_m^i$ in eqn. (26), and $r_{2,n}^i$ is given by eqn. (27). Convergence for PSLE is defined as the situation where all MRCs remain below $5 \times 10^{-2}$ and the 2-norm residuals converge with iteration.

4. Results in planar extrudate swell

The PSLE method was used to simulate planar extrusion using the mesh shown in Fig. 1. At the domain inlet, fibers were specified to be axially aligned. The predicted extrudate swell is dramatically reduced by the addition of a relatively small amount of fibers (see Fig. 2). The same effect was noted by Papanastasiou and Alexandrov in simulations with the Dinh–Armstrong model [13]. (In Ref. 13 extrudate swell is plotted as a function of $N_p$, the number of fibers per unit volume, for various aspect ratios. It is not possible, however, to determine the volume fraction $\phi$ in terms of just $N_p$,
and \( a_r \), nor is the viscosity coefficient \( \mu_r \) (15) used in the Dinh–Armstrong model determined uniquely by just \( N_p \) and \( a_r \). Quantitative comparison with the results of Ref. 13 is, therefore, not possible.

Upstream of the exit, fibers exhibit the expected tumbling behavior in the viscometric flow field. The orientation field near to, and downstream of, the exit is essentially fully-aligned with the flow field, hence velocity fields predicted by PSLE are nearly identical to those predicted using the full-alignment assumption. This suggests that the flow characteristics in this problem are governed by the behavior of the suspension in the elongational flow at the exit.

In the extrusion problem, PSLE iterations did not converge much past the dilute concentration limit \( \phi a_r^2 = 1 \) for aspect ratios greater than 10. (For an aspect ratio of 10, converged results were obtained for \( \phi = 0.1 \).) Loss of convergence was marked by a small region encompassing the exit, where small perturbations in velocity produced large disturbances in the computed \( F \) field. When convergence occurred, it was typically linear but rapid, the 2-norm residuals (30) decreasing roughly one order of magnitude with each iteration. (Even though the decoupled stresses appear in a pseudo-Newton–Raphson technique, the prediction of the free surface location occurs in step (5) on a Picard-type adaptation of the mesh to the streamlines. Thus, linear convergence is not surprising.) Depending on the fiber aspect ratio, approximately one minute or less on a Cray X-MP (single processor) was required to obtain converged results in the concentration range of \( \phi a_r^2 \approx 1 \).

5. Results in falling-ball rheometry

The PSLE method was used to simulate falling-ball rheometry measurements, where the flow geometry consisted of a sphere falling with constant velocity down the center-line of a tube (see Fig. 3). Here, the ratio of sphere diameter to tube diameter was 0.2. Simulations were performed on the two meshes shown in Fig. 4. Since bilinear streamline elements were used in the PSLE method, the surface of the sphere was represented by a piecewise linear approximation. Calculations were performed in a reference frame fixed with the sphere without body forces; the sphere was fixed in space, while the cylinder and bulk fluid moved. The velocity field was assumed to be two-dimensional and axi-symmetric.

Three different conditions were chosen for the initial unperturbed fiber orientations. In case (A), fibers were initially in a three-dimensional random configuration. Case (B) corresponded to an initially two-dimensional random orientation in the \( r-z \) plane, with no component of orientation out of the \( r-z \) plane. In case (C), fibers were initially fully-aligned with the \( z \)-axis.
Initial Conditions on $N$:

A $\Rightarrow \bigcirc$ 3-D
B $\Rightarrow \bigcirc$ 2-D
C $\Rightarrow \bigvert$

Fig. 3. Falling-ball rheometry with different initial fiber orientations: case (A) is 3-D random; case (B) is 2-D random in $r-z$ plane, and case (C) is aligned in the $z$-axis.

With these initial conditions, simulations were performed for fiber aspect ratios ranging from 20 to 1000.

The orientation state as described by $N$ is represented graphically using ‘orientation ellipsoids.’ The axes of these ellipsoids adopt the direction and

mesh 1

mesh 2

Fig. 4. Meshes used in falling-ball simulations.
length of the eigenvectors and eigenvalues of \(N\), respectively. The major axis of the orientation ellipsoid is an indication of the direction of preferred fiber orientation. The degree of fiber alignment is indicated by the roundness of the ellipsoid. A circle or sphere is indicative of a random orientation, whereas a sharp line is indicative of full alignment.

Given computed kinematics and extra stress, drag on the sphere was determined by integrating the resulting contact force along the sphere surface. The drag on the falling sphere is simply

\[
F_{\text{drag}} = - \int_A \mathbf{t} \, \mathrm{d}A \bigg|_{\text{z-component}},
\]

where \(\mathbf{t}\) is the contact force and \(\mathrm{d}A\) is the differential surface of the sphere. The integral (31) cannot be evaluated in this form because the contact force \(\mathbf{t}\) was not directly computed at the sphere surface; the extra stress \(\mathcal{F}\) was computed at Gauss points and not along solid boundaries. Furthermore, the components of the velocity gradient are not defined on the sphere surface; velocity interpolations are \(C^0\) across element boundaries. Instead, the contact force was evaluated from the known total stress in the domain by using the divergence theorem

\[
\langle \nabla \varphi_j \rangle \cdot (2\mu_0 \mathbf{D} - \mathbf{p} \mathbf{I} + \mathcal{F}) - \langle \varphi_j \cdot \mathbf{f} \rangle = \langle \varphi_j ; \mathbf{t} \rangle.
\]

The drag was obtained by summing the index \(j\) over the nodes along the sphere surface, as follows

\[
\sum_{j \text{ on } A} \langle \varphi_j ; \mathbf{t} \rangle \bigg|_{\text{z-component}} = \int_A \mathbf{t} \, \mathrm{d}A \bigg|_{\text{z-component}}.
\]

(See Rosenberg [17] for details.) The calculated Newtonian drag is in excellent agreement with Stokes’ law [20] using the Faxén correction [4] to account for the effect of a finite sphere-to-tube diameter ratio. Discrepancy between the two is less than 0.8% and 0.4% using the coarse and refined meshes, respectively.

Within the dilute regime, the predicted velocity fields are nearly Newtonian; the maximum relative deviation from the Newtonian velocity is less than 5% for all simulations. The bulk fluid viscosity as measured by falling-ball rheometry can thus be expressed in terms of an effective Newtonian viscosity, \(\mu_e\). The effective viscosity was determined by the drag on the sphere as

\[
\mu^* = \frac{\mu_e}{\mu_n} = \frac{F_s}{F_n},
\]

where \(\mu^*\) is the relative viscosity, \(F_s\) is the drag on the sphere in the suspension, and \(F_n\) is the drag in the pure suspending fluid.
The suspension model predicts a linear dependence of $T_f$ on the fiber volume fraction, thus, simulation results are expected to follow the relationship

$$\mu^* - 1 = [\mu] \phi,$$

where $[\mu]$ is the intrinsic viscosity. Computed values of $[\mu]$ as a function of $a_r$ and $\phi$ are shown in Figs. 5–7. The predicted intrinsic viscosity does not seem to be a strong function of the initial fiber orientation for the cases considered. It should be noted, however, that convergence with mesh refinement has not been convincingly demonstrated for this problem. For an aspect ratio of 20, simulations predict intrinsic viscosities of roughly 12 and 10 for the coarse and refined meshes, respectively. Further mesh refinement is not possible due to the behavior of the fibers near the sphere surface; this point is discussed in the following.
Close inspection of the predicted orientation fields reveals a rich structure, with sharp boundary layers transverse and parallel to the flow direction. Figures 8 and 9 show the final fiber orientation after the sphere has passed for initial orientation cases (A) and (B). Final orientations in the $r-z$ plane starting with two- and three-dimensional random orientations are very similar. For fibers initially fully-aligned along the $z$-axis, case (C), the final fiber orientation remains essentially fully-aligned.

The behavior of fibers close to the sphere is practically invariant to changes in the initial fiber orientation. Approaching the sphere from the bottom, fibers align horizontally in the compressive flow (see Figs. 10 to 13). Traveling around the sphere, fibers tumble in the quasi-shear flow; the length scale of tumbling tends to zero as the sphere surface is approached.

Fig. 7. Intrinsic viscosity as a function of fiber volume fraction for aspect ratio of 20. Points shown are for both meshes with initial fiber orientations of (A), 3-D random; (B), 2-D random, and (C), aligned.

Fig. 8. Final fiber orientation in the $r-z$ plane as a function of radial distance from the tube center-line for an aspect ratio of 20 and a fiber volume fraction of $10^{-4}$. Case (A) corresponds to the final orientation given an initial 3-D random orientation and case (B) to that given an initial 2-D random orientation.
Moving away from the top of the sphere, fibers align axially in the elongational flow field. The full-alignment assumption is clearly incorrect over an important portion of the flow field. It is surprising to note, however, that the computed intrinsic viscosity using the full-alignment assumption is approximately the same as that computed using the complete suspension model.
Fig. 11. Fiber orientation along the streamline in Fig. 10 closest to the sphere. Solid and dashed curves correspond to $N_{rr}$ and $N_{r}$ in cylindrical coordinates, respectively.

Mesh refinement was, for the most part, limited by the ability to predict this fiber tumbling numerically close to the sphere surface. For a given mesh, convergence with iteration was lost when small perturbations in

Fig. 12. The $rr$-component of $N$ in spherical coordinates as computed along the streamlines in Fig. 10. The dashed, solid, and dotted curves correspond to streamlines progressively closer to the sphere.
Fig. 13. The \( r\theta \)-component of \( N \) in spherical coordinates as computed along the streamlines in Fig. 10. The dashed, solid, and dotted curves correspond to streamlines progressively closer to the sphere.

velocity effected large disturbances in the computed \( \mathcal{I} \) in the predicted boundary layers. At low fiber concentrations, convergence with iteration was nearly quadratic. Roughly 0.5 min on a Cray X-MP (single processor) was required to obtain converged results in the concentration range \( \phi a_r^2 \approx 1 \) using the coarse mesh. For the refined mesh, the same results required roughly 1–10 min, depending upon the fiber aspect ratio and the initial orientation.

Powell and co-workers [3,4,5] performed falling-ball rheometry experiments with suspensions of neutrally-buoyant cylindrical rods. The rod length in the experiments was comparable to the diameter of the sphere, while the theory assumes that the fibers have no length and are only characterized by an aspect ratio; thus, the theory used here is not rigorously applicable to their experiments. For an aspect ratio of 20, they measured an intrinsic viscosity of approximately 28 for fibers that were initially randomly ordered in space [4,5]. For fibers that were initially aligned with the tube axis, they measured an intrinsic viscosity of approximately 10 for the same aspect ratio [3]. Rigorous comparison between the behavior of cylinders and ellipsoids cannot be made, although there are equivalent aspect ratios for cylinders when using equations developed for ellipsoids [21–24]. Such corrections are slight, however, and cannot account for a three-fold difference in viscosity.

Milliken and co-workers [4] found that their experiments with initially randomly oriented rods compared well in the dilute regime with Brenner's
theory [25] for suspensions subject to strong Brownian motion. The work of Haber and Brenner [26] gives some justification for this comparison. Haber and Brenner showed that the shear viscosity of a suspension of randomly oriented particles is the same as that for a suspension of particles subject to strong Brownian motion if the particle is a body of revolution. The simulations reported here clearly show, however, that fibers do not remain randomly oriented in falling-ball rheometry.

For large aspect ratios ($a_r > 100$), the intrinsic viscosity appears to be linearly dependent on the viscosity coefficient $\mu_2$ (see Fig. 6). Thus in the dilute regime, the simulations show that

$$[\mu] \approx \frac{1}{20} \frac{\mu_2}{\mu_n} \approx \frac{1}{20} \frac{a_r^2}{\ln a_r}.$$  (36)

This form can be rationalized by approximate analyses given in Appendix B, but the numerical value of the coefficient is very sensitive to details of the orientation distribution. Due to the formal equivalence between dilute and semi-dilute systems represented by eqns. (13) and (14), expression (36) may be extrapolated into the semi-dilute regime by replacing $\mu_2$ with $\mu_f$, giving

$$[\mu] \approx \frac{1}{20} \frac{\mu_f}{\mu_n}.$$  (37)

Using the random and aligned values of the spacing parameter $H$ (eqns. (16) and (17)) produces an upper and lower estimate, respectively, for the intrinsic viscosity. The resulting bounds then enable the use of the calculations performed for the dilute regime theory to be extrapolated into the semi-dilute regime (see Fig. 14). In order to make the comparison, the effect of $\mu_1$ and $\mu_3$ on the intrinsic viscosity was ignored; $\mu_1$, $\mu_2$, and $\mu_3$ are

![Fig. 14. Intrinsic viscosity as a function of fiber volume fraction for an aspect ratio of 20. Points correspond to simulations. The solid curves correspond to the dilute theory extrapolated into the semi-dilute range. Vertical bars correspond to experiments of Milliken and co-workers [4,5] for an initially random orientation.](image-url)
approximately 2, 175, and $2 \times 10^{-2}$, respectively, at an aspect ratio of 20. The upper curve corresponds to an estimate of the results that would be obtained for the Dinh–Armstrong theory [9], which uses eqn. (16) to define the average fiber spacing. The spread between the bounds is too great, however, to allow any definitive conclusions to be drawn regarding agreement with experiments. The poor agreement with experiments in the dilute regime suggests that the theory can only be used qualitatively when the fiber length is comparable to the sphere diameter.

6. Conclusions

Simulations indicate that the flow characteristics of fiber suspensions in extrusion are governed by the behavior of the suspension model in the elongational flow field surrounding the exit. The behavior of the suspension model in elongation is described well by the full-alignment assumption. Similarly, in flow through a contraction, Lipscomb and co-workers [1] showed that the full-alignment assumption leads to good qualitative agreement with experiments. It is likely, then, that the character of contraction flows is also governed by the behavior of the suspension in the elongational flow surrounding the re-entrant corner.

In falling-ball rheometry, streamline integration is capable of predicting sharp boundary layers in fiber orientation. The iterative scheme tends to fail, however, when small disturbances in the velocity field produce large perturbations in the computed extra stress field in these boundary layers. The intrinsic viscosity predicted by falling-ball simulation appears to be the consequence of a nearly Newtonian flow in which fibers tumble over arbitrarily small length scales near the sphere surface. Further mesh refinement is severely restricted by these small distances over which fiber orientation changes. The absence of interactions between fibers and solid boundaries is undoubtedly the cause of this computational problem. For both physical and computational reasons, therefore, there is a need for suspension models to describe the nature of these interactions in more detail.

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References


Appendix A. Streamline elements

This appendix illustrates the major similarities and differences between biquadratic and bilinear streamline elements. Biquadratic streamline elements are formed by strips of isoparametric quadrilateral elements. Each strip is defined by three streamlines. Nodes of all elements within the strip lie on one of the three streamlines (see Fig. A.1).

The spatial coordinates and streamfunction are approximated by the finite sums

\[ x = \sum_j x_j \varphi_j, \quad y = \sum_j y_j \varphi_j, \]  \hspace{1cm} (A.1)

\[ \Psi = \sum_j \Psi_j \varphi_j. \]  \hspace{1cm} (A.2)
Here, $x_j$ and $y_j$ are nodal coordinates; $\Psi_j$ are nodal values of the stream-function; and $\varphi_j$ are biquadratic shape functions. For the element shown in Fig. A.1

$$\Psi_2 = \Psi_3 = \Psi_6 = \Psi_{(a)}, \quad \Psi_5 = \Psi_7 = \Psi_9 = \Psi_{(b)}, \quad \Psi_1 = \Psi_4 = \Psi_8 = \Psi_{(c)}. \quad (A.3)$$

Thus, the streamfunction within this element is given by the following quadratic equation

$$\Psi = \left(\frac{1}{2} \Psi_{(a)} - \Psi_{(b)} + \frac{1}{2} \Psi_{(c)}\right) \xi^2 + \frac{1}{2} \left(\Psi_{(c)} - \Psi_{(a)}\right) \xi + \Psi_{(b)}. \quad (A.4)$$

Within this element, therefore, any streamline is given by a line of constant $\xi$. It follows, then, that the three streamlines defining a strip of elements are given by

$$\chi = -1, \quad \chi = 0, \quad \chi = 1, \quad (A.5)$$

where $\chi$ refers to one of the two local coordinates ($\xi$ or $\eta$) of an element.

Bilinear streamline elements are used in the PSLE iterative method presented in this work. Again, these streamline elements are formed by strips of elements (see Fig. A.2). Each strip, now, is defined by two streamlines. All vertices of the elements within a strip lie on one of the two defining streamlines. The remaining nodes are placed as follows: nodes on the element boundaries are placed equidistantly between adjacent vertices while the interior node is placed in the center of the quadrilateral. Thus, the
locations of the non-vertex nodes are given by

\[ x_5 = \frac{1}{2}(x_1 + x_2), \quad y_3 = \frac{1}{2}(y_1 + y_2), \quad x_6 = \frac{1}{2}(x_2 + x_3), \]
\[ y_6 = \frac{1}{2}(y_2 + y_3), \quad x_7 = \frac{1}{2}(x_3 + x_4), \quad y_7 = \frac{1}{2}(y_3 + y_4), \]
\[ x_8 = \frac{1}{2}(x_1 + x_4), \quad y_8 = \frac{1}{2}(y_1 + y_4), \quad x_9 = \frac{1}{4}(x_1 + x_2 + y_3 + x_4), \]
\[ y_9 = \frac{1}{4}(y_1 + y_2 + y_3 + y_4). \]  

(A.6)

Substitution of nodal locations (A.6) into eqn. (A.1) results in a bilinear isoparametric element, given by

\[ x = \sum_k x_k \phi_k, \quad y = \sum_k y_k \phi_k, \quad \phi_k \]  

(A.7)

where \( x_k \) and \( y_k \) are vertex coordinates, and \( \phi_k \) are bilinear shape functions.

Within each element, streamlines are approximated by lines of constant \( \chi \). Consider, for instance, the bilinear element shown in Fig. A.2. The vertices of the element lie on the two defining streamlines. The other nodes of the element, however, potentially lie on different streamlines. Thus, for this element

\[ \Psi_{(\alpha)} = \Psi_2 = \Psi_3 \neq \Psi_6, \quad \Psi_5 \neq \Psi_7 \neq \Psi_9, \quad \Psi_{(\beta)} = \Psi_1 = \Psi_4 \neq \Psi_8. \]  

(A.8)

Hence, lines of constant streamfunction do not necessarily correspond to lines of constant \( \xi \) or \( \eta \). Within each element, the computed streamfunction
is potentially biquadratic, while the streamlines are approximated by straight lines. (Only straight streamlines correspond to lines of constant $\chi$ for such bilinear streamline elements.) The spatial discrepancy between this approximation of streamlines and lines of constant streamfunction is proportional to the curvature of the streamline and to the size of the element. The use of small elements is anticipated in regions where streamlines are curved.

Appendix B. Limiting-case predictions of falling-ball intrinsic viscosity

Overall, the falling-ball simulations suggest that the drag on the sphere in excess of that from the suspending fluid could be the result of fibers near the sphere which tumble over arbitrarily small length scales, all imbedded in a flow exhibiting Newtonian kinematics. In this appendix, analytical expressions for the intrinsic viscosity are developed using such an assumption. Consider the flow around a sphere in an infinite domain of suspension. Consider, furthermore, that the resulting velocity fields is strictly Newtonian. The Newtonian velocity and pressure field in the spherical coordinate system fixed with the sphere are given by [20]

\[ u_r = u_\infty \left( 1 - \frac{3}{2} \left( \frac{R}{r} \right) + \frac{1}{2} \left( \frac{R}{r} \right)^3 \right) \cos(\theta), \]
\[ u_\theta = -u_\infty \left( 1 - \frac{3}{4} \left( \frac{R}{r} \right) - \frac{1}{4} \left( \frac{R}{r} \right)^3 \right) \sin(\theta), \]
\[ p_n = -\frac{3R}{2r^2} \mu_n u_\infty \cos(\theta), \]

where $p_n$ is the Newtonian pressure; $r$ and $\theta$ are the radial coordinate and

![Fig. B.1. Spherical coordinate system used in falling-ball rheometry.](image)
azimuthal angle, respectively; \( u_\infty \) is fluid velocity far from the sphere, and \( R \) is the radius of the sphere (see Fig. B.1).

Close to the sphere surface, the length scale over which fibers tumble becomes infinitesimally small. In this region, the constitutive equation (6) can effectively be replaced by
\[
T_\ell \approx \phi \left( -p_0 I + 2\mu_1 D + \mu_2 D : \overline{N N} + 2\mu_3 (D \cdot \overline{N} + \overline{N} \cdot D) \right),
\]
with the following definitions
\[
\overline{N} \equiv \frac{1}{s_1 - s_0} \int_{s_0}^{s_1} N \, ds, \quad \overline{N N} \equiv \frac{1}{s_1 - s_0} \int_{s_0}^{s_1} NN \, ds,
\]
where \( s \) is a curvilinear distance along a streamline. Equation (B.5) represents an orientation state averaged over the distance traveled by the fiber. These average values are constants for fibers which tumble over infinitesimally small length scales.

The force exerted on the sphere by the pure Newtonian suspending fluid is given by Stokes' law [20]
\[
F_n = 6\pi R \mu_n u_\infty. \quad \text{(B.6)}
\]
Consider now the force exerted on the sphere by the suspension. This force, denoted \( F_s \), is given by
\[
F_s = \int_A \left( 2\mu_n D - pI + T_\ell \right) \cdot dA. \quad \text{(B.7)}
\]
The vertical component of the tangential force is a friction drag, given by
\[
F_{s, \text{friction}} = -\int_0^{2\pi} \int_0^\pi (2\mu_n D_{r\theta} + T_{r\theta}) \bigg|_{r=R} R^2 \sin^2(\theta) d\theta d\phi
\]
\[
= 4\pi R \mu_n u_\infty \left( \frac{\phi}{\mu_n} + \left( \mu_1 + \frac{\overline{N N}}{R^2} \mu_2 + \mu_3 \right) \right). \quad \text{(B.8)}
\]
The vertical component of the normal force is a form drag given by
\[
F_{s, \text{form}} = \int_0^{2\pi} \int_0^\pi (2\mu_n D_{rr} - p + T_{rr}) \bigg|_{r=R} R^2 \sin(\theta) \cos(\theta) \, d\theta \, d\Phi
\]
\[
-\int_0^{2\pi} \int_0^\pi \mathcal{P} \bigg|_{r=R} R^2 \sin(\theta) \cos(\theta) \, d\theta \, d\Phi, \quad \text{(B.9)}
\]
where the modified pressure \( \mathcal{P}(21) \) must be determined from \( u \) and \( T_\ell \). Rearrangement of the momentum equation (18) yields
\[
\mu_n \nabla^2 u - \nabla \mathcal{P} + \phi \nabla \cdot \tau = 0. \quad \text{(B.10)}
\]
Here, the tensor \( \tau \) is given by
\[
\tau \equiv \frac{1}{\phi} T_\ell + p_0 I. \quad \text{(B.11)}
\]
The Newtonian velocity and pressure field (B.1)–(B.3) satisfy the following relationship
\[ \mu_n \nabla^2 u - \nabla p_n = 0. \]  
(B.12)

Combination of eqn. (B.12) and (B.10) yields
\[ \nabla p' = \phi \nabla \cdot \tau, \]  
(B.13)

where the pressure \( p' \) is defined by
\[ p' \equiv \rho - p_n. \]  
(B.14)

The form drag (B.9) becomes
\[ F_{\text{form}} = 2\pi R \mu_n u_\infty - \int_0^{2\pi} \int_0^\pi p' \left|_{r=R} \right. R^2 \sin(\theta) \cos(\theta) \, d\theta \, d\Phi. \]  
(B.15)

The \( \theta \)-component of eqn. (B.13) is given by
\[ \frac{1}{\phi} \frac{\partial p'}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \tau_{r\theta} \right) + \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \tau_{\theta\theta} \right) + \tau_{r\theta} - \frac{\cos(\theta)}{\sin(\theta)} \tau_{\Phi\Phi}. \]  
(B.16)

\( \bar{N} \) is a constant in shear flow everywhere except on solid boundaries. Hence, it is taken that
\[ \frac{\partial}{\partial r} \bar{N} \bigg|_{r=R} = 0, \]  
(B.17)
in which case eqn. (B.16) can be simplified to
\[ \left. \frac{\partial p'}{\partial \theta} \right|_{r=R} = \frac{3\phi u_\infty}{2R} \left\{ \left( \mu_1 + \bar{N}_{r\theta} \right) \mu_2 + \mu_3 \right\} \sin(\theta) \]
\[ - \left[ \left( 3 \bar{N}_{r\theta} \bar{N}_{\theta\theta} - 2 \bar{N}_{rr} \bar{N}_{r\theta} \right) \mu_2 + 2 \bar{N}_{r\theta} \mu_3 \right] \cos(\theta) \}. \]  
(B.18)

Integration of eqn. (B.18) gives the pressure \( p' \) as
\[ p' \big|_{r=R} = p^* \frac{3\phi u_\infty}{2R} \left\{ \left( \mu_1 + \bar{N}_{r\theta} \right) \mu_2 + \mu_3 \right\} \cos(\theta) \]
\[ + \left( \left( 3 \bar{N}_{r\theta} \bar{N}_{\theta\theta} - 2 \bar{N}_{rr} \bar{N}_{r\theta} \right) \mu_2 + 2 \bar{N}_{r\theta} \mu_3 \right) \sin(\theta) \}, \]  
(B.19)

where \( p^* \) represents the value of \( p' \) at the point \( (r = R, \theta = \pi/2) \) on the sphere. Substitution of eqn. (B.19) into eqn. (B.15) gives the form drag as
\[ F_{\text{form}} = F_n \left\{ 1 + \frac{\phi}{\mu_n} \left( \mu_1 + \bar{N}_{r\theta} \mu_2 + \mu_3 \right) \right\}. \]  
(B.20)

Thus, the total drag exerted by the suspension is
\[ F_s = F_{\text{friction}} + F_{\text{form}} - F_n \left\{ 1 + \frac{\phi}{\mu_n} \left( \mu_1 + \bar{N}_{r\theta} \mu_2 + \mu_3 \right) \right\}. \]  
(B.21)
The intrinsic viscosity is determined from the various drags as

\[ [\mu] = \frac{F_s - F_n}{\phi F_n}. \]  

(B.22)

Thus for fibers which tumble over small length scales imbedded in a Newtonian velocity field, the falling-ball prediction of the intrinsic viscosity is

\[ [\mu] = \frac{1}{\mu_n} (\mu_1 + \frac{N_{r\theta}^2}{\mu_2 + \mu_3}). \]  

(B.23)

In order to determine an appropriate average value of \( N_{r\theta}^2 \), the local flow near the sphere surface is approximated as a simple shear flow. The orientation of a single fiber in simple shear is given by Okagawa [12] as

\[ \theta = \tan^{-1}(a_r \tan(\xi)) + \theta_0, \]  

(B.24)

where

\[ \xi = \frac{\dot{\gamma} s}{|\mathbf{u}|(a_r + a_r^{-1})}. \]  

(B.25)

Here, \( \theta \) is the angle formed by the unit vector \( \mathbf{n} \), \( \theta_0 \) is the initial fiber orientation, \( \dot{\gamma} \) is the shear rate, and \( s \) is the curvilinear distance traveled by the fiber. From eqn. (B.24), it follows that the component \( N_{r\theta} \) is given by

\[ N_{r\theta} \approx \sin(\theta) \cos(\theta) = \frac{a_r \tan(\xi)}{a_r^2 \tan^2(\xi) + 1}. \]  

(B.26)

The average value of \( N_{r\theta}^2 \) is determined as follows

\[ \overline{N_{r\theta}^2} = \frac{1}{s_1 - s_0} \int_{s_0}^{s_1} N_{r\theta}^2 \, ds = \frac{1}{\xi_1 - \xi_0} \int_{\xi_0}^{\xi_1} N_{r\theta}^2 \, d\xi. \]  

(B.27)

It has been assumed that the velocity and the shear rate are constant, at least locally, along a streamline. It is convenient to confine the limits of integration in eqn. (B.27) over a single period of \( \tan(\xi) \). Thus, the average value of \( N_{r\theta}^2 \) is given by

\[ \overline{N_{r\theta}^2} = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} N_{r\theta}^2 \, d\xi = \frac{a_r (a_r^2 - 2a_r + 1)}{2(a_r^4 - 2a_r^2 + 1)}. \]  

(B.28)

Combination of eqns. (B.23) and (B.28) gives

\[ [\mu] = \frac{\mu_1}{\mu_n} + \frac{a_r (a_r^2 - 2a_r + 1)}{2(a_r^4 - 2a_r^2 + 1)} \frac{\mu_2}{\mu_n} + \frac{\mu_3}{\mu_n}. \]  

(B.29)

As expected, this estimate predicts a linear dependence of the intrinsic viscosity on the viscosity coefficients \( \mu_1 \), \( \mu_2 \) and \( \mu_3 \); the drag on the sphere
depends upon $T_I$, which is linearly dependent on the viscosity coefficients. Equation (B.29), however, shows poor agreement with simulations, especially for large aspect ratios (see Fig. B.2). The poor agreement may be a consequence both of the approximations in the analysis and the possible lack of convergence of the simulations. The simulations were limited by the ability to predict fiber orientation close to the sphere surface numerically because of the problem of resolving the orientation gradients. Consequently, the simulations were unable to predict orientations in the region near the sphere, where the averaging of the orientation tensor $N$, as in eqn. (B.5), would be appropriate.

A similar analysis can be carried out where no assumptions need be made about the details of the fiber behavior near the sphere, but where the flow and orientation must be approximated far from the sphere. Consider the following application of the divergence theorem

$$\int_V \nabla \cdot \sigma \, dV = \oint_A t \, dA + \oint_{A_\infty} t \, dA_{\infty},$$

where $dA$ and $dA_\infty$ represent the differential areas of the sphere surface and an arbitrary surface surrounding the sphere, respectively; $dV$ represents the volume formed by surfaces $dA$ and $dA_\infty$. (The integrations on the right-hand side of eqn. (B.30) are taken in the counter-clockwise sense.) Since the divergence of the total stress $\sigma$ is zero at any point within the volume, eqn. (B.30) reduces to

$$\oint_A t \, dA = -\oint_{A_\infty} t \, dA_{\infty}.$$
The drag is given by

\[ F_{\text{drag}} = - \oint_A t \, dA \bigg|_{\text{z-component}} = \oint_{A_\infty} t \, dA_{\infty} \bigg|_{\text{z-component}} \]  \hspace{1cm} (B.32)

Therefore, the drag can be determined from the contact force on the arbitrary surface \( A_\infty \). It is convenient to define this arbitrary surface as a spherical surface of infinite radius. The velocity field is assumed to be Newtonian, as in the previous limiting case analysis; the fiber orientation at infinite distance from the falling-sphere is assumed to be the unperturbed initial fiber orientation. Under these assumptions, the form and friction drags, respectively, are given by

\[ F_{\text{form}} = \lim_{r \to \infty} \left( - \int_{0}^{2\pi} \int_{0}^{\pi} \sigma_{rr} |_{r} r^2 \sin^2(\theta) \, d\theta \, d\Phi \right) \]

\[ = F_{n \text{friction}} + \lim_{r \to \infty} \left( -2\pi r^2 \int_{0}^{\pi} T_{r\theta} |_{r} \sin^2(\theta) \, d\theta \right), \]  \hspace{1cm} (B.33)

\[ F_{\text{friction}} = \lim_{r \to \infty} \left( \int_{0}^{2\pi} \int_{0}^{\pi} \sigma_{rr} |_{r} r^2 \sin(\theta) \cos(\theta) \, d\theta \, d\Phi \right) \]

\[ = F_{n \text{form}} + \lim_{r \to \infty} \left( 2\pi r^2 \int_{0}^{\pi} (-p' + \phi \tau_{rr}) |_{r} \sin(\theta) \cos(\theta) \, d\theta \right), \]  \hspace{1cm} (B.34)

where the pressure \( p' \) is given by

\[ p' = \int \frac{\partial p'}{\partial \theta} \, d\theta + C(r). \]  \hspace{1cm} (B.35)

Here, \( C(r) \) is an arbitrary function of the radial coordinate, and the derivative \( \partial p'/\partial \theta \) is given by eqn. (B.16). Combination of eqns. (B.22), (B.33), and (B.34) yield the following expression for the intrinsic viscosity

\[ [\mu] = \frac{F_s - F_n}{\phi F_n} \]

\[ = \lim_{r \to \infty} \left( 2\pi r^2 \int_{0}^{\pi} \left( \tau_{rr} - \tan(\theta) \tau_{\theta} - \frac{p'}{\phi} \right) |_{r} \sin(\theta) \cos(\theta) \, d\theta \right). \]  \hspace{1cm} (B.36)

Given an initially three-dimensional random fiber orientation, case (A), the intrinsic viscosity (B.36) reduces to

\[ [\mu] = \frac{1}{\mu_n} \left( \mu_1 + \frac{2}{3} \mu_3 \right). \]  \hspace{1cm} (B.37)

Given an initial fiber orientation which is randomly oriented in the \( r \) \( z \)
Fig. B.3. Comparison between simulation results and the limiting case analysis for behavior at infinite distance from the sphere. Dark circles correspond to simulation results. Curves (A), (B), and (C) correspond to initial fiber orientations which are 3-D random, 2-D random, and fully aligned with the z-axis, respectively. The dashed line corresponds to an empirical fit of the simulation results.

plane with no orientation out of the $r-z$ plane, case (B), the intrinsic viscosity becomes

$$[\mu] = \frac{1}{\mu_n} \left( \mu_1 + \frac{1}{48} \mu_2 + \frac{5}{6} \mu_3 \right). \tag{B.38}$$

An initial fiber orientation that is fully-aligned with the z-axis, case (C), results in the following intrinsic viscosity

$$[\mu] = \frac{1}{\mu_n} \left( \mu_1 + \frac{2}{21} \mu_2 + \frac{13}{15} \mu_3 \right). \tag{B.39}$$

It is interesting to note that the intrinsic viscosity (B.37) predicted for case (A) has no dependence upon $\mu_2$, which is contrary to the simulations and the predictions made for cases (B) and (C). This lack of dependence upon $\mu_2$ is a consequence of both the Newtonian velocity field and the three-dimensional random fiber orientation. A perturbation in either the velocity field or the orientation field would result in a dependence upon $\mu_2$.

Predictions (B.38) and (B.39) are in good qualitative agreement with the simulation results, which can be fit empirically (Fig. B.3) by the expression

$$[\mu] = \frac{1}{\mu_n} \left( \mu_1 + 0.054 \mu_2 \right). \tag{B.40}$$

(The coefficient $\mu_3$ is relatively small at any aspect ratio and has essentially no influence on the intrinsic viscosity.) The solution is clearly sensitive to the deviations from the ideal cases considered here, however, and any agreement should be considered fortuitous. Indeed, the ordering of the magnitude of the coefficient multiplying $\mu_2$ in the three cases is counter-intuitive.