Non-linear temporal stability analysis of viscoelastic plane channel flows using a fully-spectral method

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Abstract

A non-linear analysis of the temporal evolution of finite, two-dimensional disturbances is conducted for plane Poiseuille and Couette flows of viscoelastic fluids. A fully-spectral method of solution is used with a stream-function formulation of the problem. The upper-convected Maxwell (UCM), Oldroyd-B and Giesekus models are considered. The bifurcation of solutions for increasing elasticity is investigated both in the high and low Reynolds number regimes. The transition mechanism is discussed in terms of both the transient linear growth of misfit disturbances due to non-normality, and their possible saturation into finite-amplitude periodic solutions due to non-linear effects.

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1. Introduction

The stability of fluid flows to infinitesimal and finite disturbances, using linear and non-linear approaches, respectively, and the details of the transition mechanism from laminar to turbulent regime, is still an active research topic even for Newtonian fluids. It has been known for long that viscoelasticity alters and enriches the transition picture at high Reynolds number. In this regime, linear stability analyses showed first a destabilising effect of viscoelasticity [1], followed at higher values of the elasticity number by a restabilising effect [2–4]. This has also been observed experimentally [5,6] in cylindrical pipe flows of polymer solutions. In the fully developed turbulent regime, however, there is experimental and numerical evidence of drag reduction phenomena observed in flows of dilute polymer solutions [7,8].

In the inertialess regime, recent experimental investigations with viscoelastic fluids indicate another loss of stability due to purely-elastic effects, and the possibility of “elastic turbulence” [9]. In curved
geometries, linear stability analyses have shown the development of purely-elastic instabilities in Taylor–Couette, Taylor–Dean, cone-and-plate and plate-and-plate flows [10]. In the case of plane geometries, however, available linear stability analyses did not show any loss of stability although the numerical treatment of the equations suffers from severe problems of artificial instabilities [11,12]. On the other hand, a related open question is the origin of instabilities known as melt fracture, which occur in polymer processing at low speeds when a melt or concentrated solution emerging from a slit shows surface distortions beyond a critical flow rate [1]. In order to explain these observations, a number of mechanisms have been investigated including instabilities in the inflow and/or outflow region, as well as non-monotonic rheology [13,14]. Regarding the development of hydrodynamic instabilities in the upstream channel flow, available linear stability results using constitutive equations [11,12] or kinetic theory models [15] with monotonic flow curves do not show transitions in the relevant parameter space. The above studies are limited to infinitesimal disturbances, and it could happen that the upstream flow be unstable to finite-amplitude disturbances due to non-linear effects.

In order to explain the onset of shear flow turbulence in Newtonian fluids, the mechanism of non-normality has been recently put forward [16,17]. Basically, disturbances that are misfit to the eigen-directions of the linearised stability operator can draw energy from the laminar shear flow and grow before they eventually die out, the base flow being linearly stable. If the intermediate growth is significant enough, however, the misfit disturbances will interact in a non-linear fashion. This leads to a feed-back loop which eventually drives the system to a non-linear instability. It should be noted that, in this case, the shear flow can become turbulent without being linearly unstable. Non-normality effects in viscoelastic fluids have been observed numerically for plane Couette flow of UCM and Oldroyd-B fluids [18].

In the present work, we aim to conduct a qualitative analysis of non-linear instabilities in viscoelastic plane channel flows at high and low Reynolds numbers using the Giesekus model and its particular cases (UCM, Oldroyd-B). The analysis is qualitative in the sense that it is based on a dynamical system of relatively low order (typically 1000) derived from the governing equations using a fully-spectral method. This analysis permits to follow the evolution of two-dimensional finite disturbances which is governed by non-normality and non-linearity effects. The paper is organised as follows. In Section 2, we outline the mathematical formulation of the problem. In Section 3, we illustrate the effects of non-normality as a transition mechanism on a viscoelastic model problem. In Section 4, we present the stream-function formulation of the non-linear stability problem, and we describe the fully-spectral method of solution in Section 5. The need for the inclusion of an artificial diffusivity term in the constitutive equation in the context of fully-spectral methods is also raised there. In Section 6, we discuss several tests regarding the numerical stability and convergence of the method in both the linear and non-linear regimes. The development of finite-amplitude waves upon increasing elasticity is discussed in Sections 7.1 and 7.2 for the inertial (high $Re$) and elastic (low $Re$) regimes, respectively. Concluding remarks are given in Section 8.

2. Governing equations

We consider the channel flow between two infinite parallel plates located at $y = \pm h$ where the $x$ and $y$ axes are in the streamwise and cross-stream directions, respectively. The Poiseuille flow is driven by a constant mass flux while the two plates move in opposite directions with the same speed to generate the Couette flow.
The conservation equations for isothermal, incompressible viscoelastic flow can be written in dimensionless form as,

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{\text{Re}} (-\nabla p + \beta \nabla^2 \mathbf{u} + (1 - \beta) \nabla \cdot \mathbf{T}),
\]

(1)

\[
\nabla \cdot \mathbf{u} = 0.
\]

(2)

Here, \( \mathbf{u} \) represents the velocity vector, \( \mathbf{T} \) is the viscoelastic extra-stress tensor, and \( p \) is the pressure. The dimensionless parameter \( \beta \) is the ratio of the solvent viscosity \( \mu_s \) to the total viscosity \( \mu = \mu_s + \mu_p \), with \( \mu_p \) being the zero-shear rate polymer viscosity. The channel half-width \( h \) is chosen as the characteristic length scale, and the centreline velocity (Poiseuille) or the plate speed (Couette) is selected as the characteristic velocity scale \( U \). The flow time scale is thus \( h/U \). The scales for the viscoelastic stress and pressure are \( \mu_p U/h \) and \( \mu U/h \), respectively. The Reynolds number is defined as \( \text{Re} = \rho U h / \mu \) where \( \rho \) is the fluid density. In dimensionless form, the Giesekus model reads,

\[
\mathbf{T} + \text{We} \left[ \frac{\partial \mathbf{T}}{\partial t} + (\nabla \mathbf{u})^T \cdot \mathbf{T} - \mathbf{T} \cdot \nabla \mathbf{u} + a \mathbf{T} \cdot \mathbf{T} \right] = \nabla \mathbf{u} + (\nabla \mathbf{u})^T,
\]

(3)

where \( a \) is the non-dimensional mobility factor. The Weissenberg number is defined as \( \text{We} = \lambda U / h \), where \( \lambda \) is the zero-shear rate relaxation time of the fluid. When the viscosity ratio \( \beta \) and the mobility factor \( a \) vanish, we recover the upper-convected Maxwell model. For a non-zero viscosity ratio and a vanishing mobility factor, we have the Oldroyd-B model.

In the present work, we consider spatially periodic solutions of Eqs. (1)–(3) in a computational domain spreading over a dimensionless periodicity length \( L_x \) in the streamwise direction. We thus apply periodic boundary conditions in this direction for all dependent variables, and no-slip boundary conditions at the channel walls for the velocity components.

3. Non-normality

It has recently been suggested that a purely linear mechanism can play an important role in the transition process from laminar to turbulent regime in Newtonian shear flows [16, 17]. The transient growth (before final decay) of infinitesimal perturbations has generally been attributed to the existence of degenerate or nearly degenerate eigenvalues of the linearised stability problem derived from the Navier–Stokes equations. It has been argued, however, that the non-normality associated to the non-commutability with its adjoint of the linearised differential operator, can be systematically responsible for the transient linear growth of disturbances [16, 17]. Misfit disturbances with respect to the eigendirections of the linear stability operator can then grow, and if their growth is sufficient enough, they can interact through non-linearity and cause a transition to a new flow state.

In order to show non-normality and its effects on a viscoelastic model problem, we consider the two-dimensional Couette flow of an upper-convected Maxwell or Oldroyd-B fluid. We linearise the system of equations consisting of the conservation Eqs. (1) and (2) together with the corresponding constitutive equation (3) for the viscoelastic stresses. The flow variables are decomposed into their base and perturbation parts as,

\[
z(x, t) = z^0(x, t) + z^p(x, t), \quad \text{where} \quad z = (\mathbf{u}, p, \mathbf{T}).
\]

(4)
In Couette flow, the base solution is given by,
\[ u_0(x, t) = \begin{bmatrix} y \end{bmatrix}, \quad T_0(x, t) = \begin{bmatrix} 2 \text{Re} & 1 \\ 0 & 0 \end{bmatrix}. \] (5)

Inserting the decomposition (4) in the system of Eqs. (1)–(3) and neglecting the quadratic non-linearities in the disturbance variables, we then obtain the linearised evolution equations for \( z^p \). The unknowns are sought in the usual form as,
\[ z^p(x, t) = \exp(\mathbf{c}t) \phi_c(x), \] (6)
where \( \mathbf{c}, \phi_c \) are, respectively, the eigenvalues and eigenfunctions of the linear stability operator \( L \). The \( L \) operator has two contributions arising from the original non-linear terms. The first term is of the form \( (z_0 \cdot \nabla) z^p \), and it describes the advection of the disturbance with the base flow. If one assumes \( \phi_c \propto \exp(\mathbf{i}k \cdot \mathbf{x}) \), where \( k \) represents the disturbance wavenumber, the contribution of this term is of the form \( \mathbf{i}k \text{Re} \) for the momentum equations and \( \mathbf{i}k \text{We} \) for the constitutive equations. Since it has no real part, it does not determine the growth or decay of the disturbances, hence it is globally energy conserving.

The second term is of the form \( (z^p \cdot \nabla) z_0 \), and it represents the advection and deformation of the base flow field by the disturbance. It is proportional to \( \text{Re} \) or \( \text{We} \), and being real it contributes directly to growth or decay of disturbances; it is thus not energy conserving. In our case, the linear operator \( A \) corresponding to this second term is as follows,
\[ A z^p = \begin{bmatrix} 0 & \text{Re} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\text{We} & 0 \\ 0 & 0 & 0 & 0 & -\text{We} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} z^p. \] (7)

(Here, we have omitted the pressure terms and the continuity equation supposing a divergence free velocity field.) As it can be observed, the matrix \( A \) is not symmetric, reflecting the fact that the base solution is translationally invariant in the streamwise direction and varies in the cross-stream direction. As \( A \) is a part of the linear stability operator \( L \), both operators share the property,
\[ AA^T \neq A^T A \quad \text{and} \quad LL^T \neq L^T L. \] (8)

Since, \( A \) and \( L \) are not commutative with their adjoint \( A^T \) and \( L^T \), their eigenfunctions are not mutually orthogonal. We conclude that the linear operator \( L \) describing the time evolution of infinitesimal disturbances of plane Couette flow of an Oldroyd-B fluid is non-normal. As a result, although all the eigenvalues of \( L \) have negative real parts so that any infinitesimal disturbance will finally lose its energy and decay, transient growth of the disturbance is possible.

We can demonstrate the effect of non-normality using the following model problem. We consider a dynamical system for the unknown vector field \( \mathbf{v} \) such that the linear dynamics of its infinitesimal perturbations \( \mathbf{v}^p \) satisfy,
\[
\frac{d\mathbf{w}^p}{dt} = \hat{A}\mathbf{w}^p = \begin{bmatrix}
\lambda_1 & Re & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & -2We & 0 \\
0 & 0 & 0 & \lambda_4 & -We \\
0 & 0 & 0 & 0 & \lambda_5
\end{bmatrix} \mathbf{w}^p,
\]  
(9)

where the \(\lambda_i\)'s are the eigenvalues of the linear operator \(\hat{A}\). In fact, \(\hat{A}\) is part of the linear operator \(L\) for the Couette flow of an Oldroyd-B fluid. We assume that all \(\lambda_i\)'s have negative real parts, so that the perturbation \(\mathbf{w}^p\) eventually decays. The solution of the system (9) reads,

\[
v_{\mathbf{p}}^1(t) = v_{\mathbf{p}}^1(0) + Re(\lambda_2 - \lambda_1) t \exp(\lambda_2 t - 1) v_{\mathbf{p}}^2(0) \exp(\lambda_1 t), \\
v_{\mathbf{p}}^2(t) = v_{\mathbf{p}}^2(0) \exp(\lambda_2 t), \\
v_{\mathbf{p}}^3(t) = v_{\mathbf{p}}^3(0) + 2 We (\lambda_5 - \lambda_3)(\lambda_4 - \lambda_5) (1 - \exp(\lambda_5 t - \lambda_4) t) v_{\mathbf{p}}^5(0) \exp(\lambda_3 t), \\
v_{\mathbf{p}}^4(t) = v_{\mathbf{p}}^4(0) + We (\lambda_5 - \lambda_4) (1 - \exp(\lambda_5 t - \lambda_4) t) v_{\mathbf{p}}^5(0) \exp(\lambda_4 t), \\
v_{\mathbf{p}}^5(t) = v_{\mathbf{p}}^5(0) \exp(\lambda_5 t).
\]  
(10)

Here, the \(v_{\mathbf{p}}^i(0)\) are the components of the initial disturbance at \(t = 0\). It is clear from Eq. (10) that, in general, the disturbance \(\mathbf{w}^p\) will have the possibility to grow at small time before the eventual decay. In fact, only the disturbances that are initially aligned with an eigenvector of the operator \(\hat{A}\) are guaranteed not to show a transient growth. Following [17], we call them “fit disturbances”. For example, an initial disturbance equal at time \(t = 0\) to the eigenvector \([0, 0, -2 We^2(\lambda_5 - \lambda_4)(\lambda_5 - \lambda_3), We(\lambda_5 - \lambda_4), 1]^T\) of \(\hat{A}\) decays monotonically as follows:

\[
v_{\mathbf{p}}^1(t) = 0, \quad v_{\mathbf{p}}^2(t) = 0, \quad v_{\mathbf{p}}^3(t) = -2 We^2 (\lambda_5 - \lambda_3)(\lambda_5 - \lambda_4) t \exp(\lambda_3 t), \\
v_{\mathbf{p}}^4(t) = -We (\lambda_5 - \lambda_4) t \exp(\lambda_4 t), \quad v_{\mathbf{p}}^5(t) = \exp(\lambda_5 t).
\]  
(11)

Disturbances that are initially misfit will show a transient growth governed by \(Re\) or \(We\).

4. Formulation of the non-linear stability problem

We consider two-dimensional flows and adopt a stream-function formulation of the problem. The components of the velocity vector are related to the stream-function \(\Psi\) as follows:
\[ u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x} \]  

(12)

The continuity Eq. (2) is thus automatically satisfied. Then, taking the curl of Eq. (1) and eliminating the pressure term, we obtain the following system,

\[ \frac{\partial}{\partial t} \left( \nabla^2 \Psi \right) + \frac{\partial \Psi}{\partial y} \frac{\partial}{\partial x} \left( \nabla^2 \Psi \right) - \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial y} \left( \nabla^2 \Psi \right) - \frac{\beta}{6} \left( \nabla^4 \Psi \right) - \left( 1 - \beta \right) \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial T_{xx}}{\partial t} - \frac{\partial T_{yy}}{\partial t} \right) + \left( \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} \right) \Psi = 0, \]  

(13)

\[ T_{xx} + \text{We} \left( \frac{\partial T_{xx}}{\partial t} + \frac{\partial \Psi}{\partial y} \frac{\partial T_{xx}}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial T_{xx}}{\partial y} - 2 \frac{\partial^2 \Psi}{\partial x \partial y} T_{xx} - 2 \frac{\partial^2 \Psi}{\partial x^2} T_{xx} + a \left( T_{xx}^2 + T_{xx} \right) \right) = \frac{\partial^2 \Psi}{\partial x \partial y}, \]  

(14)

\[ T_{xy} + \text{We} \left( \frac{\partial T_{xy}}{\partial t} + \frac{\partial \Psi}{\partial y} \frac{\partial T_{xy}}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial T_{xy}}{\partial y} - \frac{\partial^2 \Psi}{\partial y^2} T_{xy} + \frac{\partial^2 \Psi}{\partial x^2} T_{xy} + a \left( T_{xy}^2 + T_{xy} \right) \right) \]  
\[ = \left( \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} \right) \Psi, \]  

(15)

\[ T_{yy} + \text{We} \left( \frac{\partial T_{yy}}{\partial t} + \frac{\partial \Psi}{\partial y} \frac{\partial T_{yy}}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial T_{yy}}{\partial y} + 2 \frac{\partial^2 \Psi}{\partial x \partial y} T_{yy} + 2 \frac{\partial^2 \Psi}{\partial x^2} T_{yy} + a \left( T_{yy}^2 + T_{yy} \right) \right) = -2 \frac{\partial^2 \Psi}{\partial x \partial y}. \]  

(16)

In order to enforce the constant-flux formulation (Poiseuille flow) or the moving plate conditions (Couette flow), we decompose the stream-function into two parts: the first part corresponding to the steady-state flux \((y - y^3/3)\) and \((y^2/2)\) in Poiseuille and Couette flow, respectively, and the second part corresponding to the perturbation, whose flux vanishes. For the velocity components, we apply periodic boundary conditions in the streamwise direction and no-slip boundary conditions at the walls. In the above formulation, this corresponds to the perturbation stream-function and its \(x\) and \(y\) derivatives vanishing at the walls.

The choice of periodic boundary conditions in the streamwise direction corresponds to the analysis of the temporal evolution of disturbances rather than their spatial evolution which would necessitate inflow–outflow boundary conditions in this direction. For the viscoelastic stress components, due to the hyperbolic nature of the constitutive equations, we apply the periodic boundary conditions in the streamwise direction and we impose no condition in the cross-stream direction.

5. Fully-spectral numerical method

In view of the flow geometry and the nature of the boundary conditions, it is quite natural to solve the governing equations by means of a fully-spectral method. This approach is well suited to a stream-function formulation, since the stream-function is more regular than the primitive velocity variables [19]. In view of its high accuracy relative to standard finite difference or finite element methods, the fully-spectral method
will allow us to obtain a discrete dynamical system of relatively small order that remains reasonably faithful to the original continuum problem.

The stream-function and the viscoelastic stress are spectrally decomposed into Fourier and Chebyshev modes, respectively, in the streamwise and cross-stream directions:

\[
\Psi = \sum_{n=1}^{M} \sum_{k=-N}^{N} a_{kn}(t) \exp\left(\frac{12\pi k x}{L_x}\right) \Phi_m(y),
\]

\[
T_{ij} = \sum_{n=1}^{M} \sum_{k=-N}^{N} (b_{ij})_{km}(t) \exp\left(\frac{12\pi k x}{L_x}\right) T_m(y).
\]

Here, \(a_{kn}\) and \((b_{ij})_{km}\) are unknown, time-dependent complex coefficients which satisfy the reality condition,

\[
a_{km} = \overline{a_{(-k)m}}, \quad (b_{ij})_{km} = (b_{ij})_{(-k)m}.
\]

The function \(T_m(y)\) is the Chebyshev polynomial of order \(m\). The basis function \(\Phi_m(y)\) is defined as a linear combination of the Chebyshev polynomials of order \((m+2)\), \(m\) and \((m-2)\):

\[
\Phi_m(y) = \frac{T_{m+2}(y)}{m(m+1)} - 2 \frac{T_{m}(y)}{m^2-1} + \frac{T_{m-2}(y)}{m(m-1)}.
\]

This particular choice is such that \(\Phi'_m(\pm1) = \Phi_m(\pm1) = 0\), which implies that the no-slip boundary condition is exactly satisfied. Similarly, the periodicity boundary conditions are automatically specified with the use of Fourier modes in the streamwise direction.

Evolution equations for the expansion coefficients are derived using the Galerkin method. Substitution of Eqs. (17) and (18) into the governing Eqs. (13)–(16) yields residuals which are made orthogonal to the basis functions. In contrast, pseudo-spectral methods set the weighted residuals to zero at collocation points only, leading to aliasing errors which must be accounted for using appropriate procedures [20].

An important aspect of the method is the full satisfaction of the boundary conditions by the above linear combination of Chebyshev polynomials. This is aimed to avoid the fast propagation of boundary errors into the whole domain of integration which is characteristic of hyperbolic problems [20,21].

After performing the spatial integrations analytically, a dynamical system is obtained,

\[
A \dot{c} + Bc = f(c),
\]

where \(c\) is the vector made of the time-dependent coefficients \(a_{kn}(t)\) and \((b_{ij})_{km}(t)\) in Eqs. (17) and (18). The constant matrices \(A\) and \(B\) result from the discretisation of linear differential operators while \(f\) is a non-linear vector function arising from the quadratic couplings between spatial modes. The time integration of the above dynamical system is performed using a second-order, semi-implicit time integration scheme used also in Newtonian simulations [22,23]. Linear terms are integrated using the implicit Crank–Nicolson scheme and non-linear terms are integrated using the explicit Adams–Bashforth method. The CPU time of a typical simulation over a dimensionless time period of order 100 is between 5 and 25 h on a single DEC Alpha 667 MHz processor, depending on the number of spectral modes.

The hyperbolic nature of the viscoelastic constitutive equation is known to cause severe numerical problems as the Weissenberg number increases in both the high and low Reynolds number regimes.
The accumulation of numerical instabilities induces the artificial loss of positive definiteness of the configuration tensor \( \tilde{\mathbf{c}} = \mathbf{W} \cdot \mathbf{T} + \mathbf{I} \). Hence a Hadamard type of instability develops which leads to the blow-up of the numerical solutions. In the literature, various numerical schemes have been proposed to remedy this situation. In the context of pseudo-spectral methods, an artificial stress diffusion term, proportional to the diffusivity constant \( \kappa \), is added to the constitutive equation [24]. The constitutive equation thus becomes,

\[
\mathbf{T} + \mathbf{W} \left( \frac{\partial \mathbf{T}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{T} - (\nabla \mathbf{u})^T \cdot \mathbf{T} - \mathbf{T} \cdot \nabla \mathbf{u} + \alpha \mathbf{T} \cdot \mathbf{T} + \kappa \nabla^2 \mathbf{T} \right) = \nabla \mathbf{u} + (\nabla \mathbf{u})^T,
\]

(22)

where \( \kappa \) is the non-dimensional stress diffusivity, defined as \( \kappa = D_s / U_h \), with \( D_s \) being the dimensional stress diffusivity. The product \( \kappa \mathbf{W} \) should be small enough for the artificial stress diffusion not to affect the flow dynamics qualitatively. The introduction of an elliptic term in the constitutive equation requires the specification of wall boundary conditions for the viscoelastic stresses. These boundary conditions are defined at each value of time as the wall viscoelastic stresses are computed through Eq. (22) with \( \kappa = 0 \).

In the present work, the need for an added stress diffusion term mainly arises in the elastic regime (low Reynolds and high Weissenberg numbers). We shall also consider the effect of stress diffusion in the inertial regime at high Reynolds number.

The implementation of the artificial diffusivity in the fully-spectral context follows the pseudo-spectral approach developed in [24]. At each time step, the intermediate viscoelastic stresses \( \tilde{T}^{n+1} \) are calculated without the inclusion of stress diffusive term, as described above. The updated viscoelastic stress \( T^{n+1} \) that takes into account stress diffusion is then computed by solving the Helmholtz equation,

\[
T^{n+1} - \frac{\kappa \Delta t}{4} \nabla^2 T^{n+1} = \frac{\kappa \Delta t}{4} \nabla^2 \tilde{T}^{n+1} + \tilde{T}^{n+1} \text{ with } T^{n+1} = \tilde{T}^{n+1} \text{ at } y = \pm 1.
\]

(23)

In order to impose the necessary boundary conditions on \( T^{n+1} \) and to solve Eq. (23) in the fully-spectral context, we introduce a modified spectral decomposition for the unknowns \( T^{n+1} \) as,

\[
[T^{n+1}]_y = [\tilde{T}^{n+1}]_y + \sum_{m=2}^{M} \sum_{k=-N}^{N} (o_{m})_{lm} (t) \exp \left( \frac{2\pi ikx}{L_x} \right) S_m(y),
\]

(24)

where \( (o_{m})_{lm} (t) \) are unknown time-dependent coefficients and \( S_m(y) = T_{m,0}(y) - 2T_{m,1}(y) + T_{m,-1}(y) \) is another linear combination of the Chebyshev polynomials chosen to satisfy the wall boundary conditions \( (S_m(\pm 1) = 0) \).

6. Numerical tests

6.1. Linear stability tests

It is useful to test the fully-spectral method in the case of infinitesimal disturbances to the base channel flow. This amounts to apply the method to the linear stability analysis described in Section 3, yielding a discrete eigenvalue problem whose solution can be compared with published results. In addition, these linear results will provide initial conditions for the non-linear, time-dependent simulations reported in
Table 1
Least stable eigenvalues for $Re = 10000$, $\alpha = 1$, $M = 50$, UCM Couette flow

<table>
<thead>
<tr>
<th>$We$</th>
<th>Eigenvalues given in [12]</th>
<th>This work</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>$-0.05205 \pm 0.81213i$</td>
<td>$-0.05205 \pm 0.81213i$</td>
</tr>
<tr>
<td>0.1</td>
<td>$-0.05167 \pm 0.81164i$</td>
<td>$-0.05167 \pm 0.81164i$</td>
</tr>
<tr>
<td>0.5</td>
<td>$-0.04984 \pm 0.80960i$</td>
<td>$-0.04984 \pm 0.80959i$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.04137 \pm 0.80409i$</td>
<td>$-0.04139 \pm 0.80415i$</td>
</tr>
</tbody>
</table>

Section 7. For these tests, we select the upper-convected Maxwell model for its notoriously difficult numerical behaviour.

First, we consider Couette flow at a high Reynolds number ($Re = 10000$), and focus on a unit perturbation wavenumber ($\alpha = 2\pi/L_x = 1$). The least stable eigenvalues are given in Table 1 as a function of the Weissenberg number. They are compared to those obtained by Renardy and Renardy [12]. The real part of the eigenvalues is the predicted growth/decay rate of the disturbance. Agreement between our results and those of [12] is excellent indeed.

The same problem is known [12] to become numerically much more difficult at low Reynolds number and large disturbance wavenumber, when $We$ is increased. In Table 2, we list some of the eigenvalues of lowest imaginary parts for the case $Re = 0.25$, $We = 1$, and $\alpha = 15$. Here again, agreement with the results reported in [12] is very good. It should be noted that the real part of these eigenvalues is nearly $-1/2We$. In [12], the authors observed the deterioration of the numerical accuracy of their (Chebyshev-tau) method for eigenvalues with imaginary parts beyond $\pm 120$. This can be seen in Table 2, where their last eigenvalue ($-0.50579 \pm 117.354i$) has a real part that deviates from $-1/2We$. The result obtained with the fully-spectral method (with the same number $M = 80$ of Chebyshev modes) is much more accurate. In this problem, both methods predict eigenvalues with positive real parts. These are clearly of a spurious nature, however, as they do not converge when the number of modes increases. The last test problem is the linear stability analysis of Poiseuille flow of the UCM model at high Reynolds number. Lee and Finlayson [11] have reported several sets of critical values for the parameters ($Re$, $We$, $\alpha$) where an instability sets in, i.e. the most dangerous eigenvalue has a real part which becomes positive. Examples are listed in Table 3. We observe that the real part is nearly zero, which confirms the criticality of the parameter values reported in [11].

Table 2
Eigenvalues with lowest imaginary parts, $Re = 0.25$, $We = 1$, $\alpha = 15$, $M = 80$, UCM Couette flow

<table>
<thead>
<tr>
<th>Eigenvalues given in [12]</th>
<th>This work</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.50596 \pm 33.020i$</td>
<td>$-0.50664 \pm 33.020i$</td>
</tr>
<tr>
<td>$-0.50263 \pm 54.631i$</td>
<td>$-0.50262 \pm 54.631i$</td>
</tr>
<tr>
<td>$-0.50595 \pm 72.494i$</td>
<td>$-0.50593 \pm 72.494i$</td>
</tr>
<tr>
<td>$-0.50579 \pm 117.37i$</td>
<td>$-0.50075 \pm 117.354i$</td>
</tr>
<tr>
<td>$-0.51427 \pm 120.35i$</td>
<td>$-0.50048 \pm 120.305i$</td>
</tr>
</tbody>
</table>
Table 3
Critical eigenvalues for the UCM Poiseuille flow ($M = 50$)

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$Re = 5537$, $We = 0.1$, $\alpha = 1.03$</td>
<td>$5.73 \times 10^{-6} - 0.27526i$</td>
</tr>
<tr>
<td>$Re = 3650$, $We = 1$, $\alpha = 1.15$</td>
<td>$-1.29 \times 10^{-5} - 0.34837i$</td>
</tr>
<tr>
<td>$Re = 2528$, $We = 2$, $\alpha = 1.29$</td>
<td>$-7.965 \times 10^{-6} - 0.43594i$</td>
</tr>
</tbody>
</table>

6.2. Non-linear stability tests

We now consider the solution of the full non-linear equations (13)–(16) by means of the fully-spectral method. As initial conditions, we specify an arbitrary finite perturbation of Poiseuille or Couette flow. The flow regime is ($Re = 10$, $We = 0.01$) for which no instability is expected. The initial perturbation is of order $10^{-3}$, and is the same for all numerical experiments. Our experience reveals that a perturbation of order $10^{-3}$ must be considered as finite indeed. Results are given for the UCM fluid, and a wavenumber $\alpha = 1$. In Fig. 1, we show the computed vorticity near the wall as a function of time for the perturbed Poiseuille flow. We have used four combinations of number of spectral modes ($M$, $N$). For a small number of modes ($M = 5$, $N = 2$), the solution settles to a periodic behaviour which would indicate the occurrence of a non-linear flow instability. When the number of modes increases, however, we find that the initial finite perturbation dies out as expected. This numerical experiment shows that it can be misleading to use a low-order dynamical system (i.e. the one produced with $M = 5$, $N = 2$) in order to study the qualitative non-linear behaviour of viscoelastic flow. Convergence of the numerical results with increasing number of modes is clearly seen in Fig. 1 (as well as in Fig. 2 which is limited to the early time interval $[0, 1]$). Indeed, results for $M = 9$ and 11 can hardly be distinguished.

Corresponding results for the finitely-perturbed Couette flow are shown in Fig. 3. In this case, all combinations of number of modes predicted the eventual non-linear decay of the disturbance. Here again, numerical convergence is clearly observable in the early-time plot of Fig. 4.

Fig. 1. Time evolution of near-wall perturbation vorticity at $x = \pi/2$, $y = -0.99$ for UCM plane Poiseuille flow ($Re = 10$, $We = 0.01$, $\beta = 0$, $\alpha = 1$, $\Delta t = 0.005$).
Fig. 2. Results of Fig. 1 restricted to the time interval [0, 1].

Fig. 3. Time evolution of near-wall perturbation vorticity at $x = \pi/2, y = -0.99$ for UCM plane Couette flow ($Re = 10, \text{We} = 0.01, \beta = 0; u = 1; \Delta t = 0.005$).

Fig. 4. Results of Fig. 3 restricted to the time interval [0, 1].

7.1. Inertial regime

7.1.1. Poiseuille flow

In Poiseuille flow of Newtonian fluids at high Re, it is well known that two-dimensional, finite perturbations to the base laminar flow cause a transition to a travelling wave. This occurs at a critical Reynolds number much lower than the critical number for linear instability [25]. The transition corresponds to a Hopf bifurcation with a periodic attractor. In the present section, we investigate the effect of viscoelasticity on the formation of finite-amplitude waves at high Reynolds number.

In the Newtonian case and for a wavenumber \( k = 1.3231 \), finite-amplitude waves develop at \( Re \approx 2800 \) while linear stability sets in at \( Re \approx 5800 \) [22,25]. In order to study the effect of elasticity in this reference problem, we form the initial conditions from the eigenfunctions of the Newtonian linear stability problem corresponding to the least stable eigenvalue. In the viscoelastic case, this particular choice introduces naturally misfit disturbances which then have the possibility to grow at high Re and/or We through the non-normality mechanism (Section 3). The magnitude of the initial disturbance is of order \( 10^{-2} \) to \( 10^{-3} \) (smaller values would decay as predicted by linear stability).

Results for the Oldroyd-B fluid at \( Re = 2400 \) are shown in Fig. 5, in terms of the temporal evolution of near-wall vorticity for increasing values of the elasticity number \( E \). The latter is defined as the ratio of the Weissenberg to the Reynolds number:

\[
E = \frac{\lambda}{\rho h^2/\mu}.
\]  

(25)

In the Newtonian case (\( E = 0 \)), the disturbance decays in an oscillatory fashion as expected. Increasing elasticity slightly (e.g. \( E = 10^{-4} \)) destabilises the flow in the sense that the wave amplitude increases relative to the Newtonian case, but it does also eventually decay. At higher elasticity, we find an interval \([E_1, E_2]\) where a periodic wave of finite-amplitude is predicted. Here, we have \( E_1 \approx 3 \times 10^{-4} \) and

![Fig. 5. Time evolution of near-wall perturbation vorticity at \( x = \pi/2, y = -0.99 \) in plane Poiseuille flow of an Oldroyd-B fluid for different elasticity numbers (\( Re = 2400, \beta = 0.5, a = 1.3231, M = 25, N = 4, \Delta t = 0.005 \)).](image-url)
Fig. 6. Critical Reynolds number for the formation of two-dimensional finite-amplitude waves in plane Poiseuille flow of an Oldroyd-B fluid as a function of the elasticity number ($E = 1.3231$). Results for different viscosity ratio: $\beta = 0.9$ (+), $\beta = 0.5$ ($\times$), $\beta = 0.1$ ($\circ$), $\beta = 0.01$ ($\triangle$).

$E_1 \simeq 6 \times 10^{-4}$. In this interval, the wave amplitude first increases with increasing $E$ until a critical value $E^* \simeq 5 \times 10^{-4}$ is reached. For $E$ in $[E^*, E_2]$, the wave amplitude remains constant in time but decreases with increasing $E$. This phenomenon we shall refer to as restabilisation. Finally, for $E$ larger than $E_2$ (i.e. $E \geq 6 \times 10^{-4}$), the disturbance decays again in an oscillatory manner. We note that these results are in qualitative agreement with the unpublished finite element UCM calculations described by Draad in his Ph.D. thesis [6].

The above findings are illustrated in Fig. 6 in terms of the critical Reynolds number for the formation of two-dimensional finite-amplitude waves versus the elasticity number. Four curves are reported, corresponding each to a particular value of the viscosity ratio $\beta$. They all have a similar shape, with a minimum located at a value $E^*$ independent of $\beta$.

It should be noted that a recent linear stability analysis (i.e. for infinitesimal disturbances) of the same flow problem [4] predicted a similar behaviour of the critical Reynolds number for linear instability with increasing elasticity. The difference, however, is that the minimum shifts towards increasing elasticity as the viscosity ratio decreases. Also, as found in the Newtonian case, the critical Reynolds number for linear instability is much higher (by about a factor of 1.5–2) than the critical Reynolds number for the formation of non-linear wave solutions.

Similar results have been obtained with the Giesekus model ($a \neq 0$). There is, however, an important qualitative difference relative to the Oldroyd-B results: the critical elasticity number $E^*$ for restabilisation does not exist beyond a critical value $a_c \simeq 0.0425$ of the mobility factor. Thus, for $a > a_c$, the amplitude of the periodic wave keeps on increasing very slowly with increasing $E$. For $a \leq a_c$, we find that the critical elasticity number $E^*$ for restabilisation decreases monotonically as a function of $a$. This is shown in Fig. 7. Finally, we observed that increasing the mobility factor $a$ introduces some stabilisation in the sense that, all other parameters being kept fixed, it slightly reduces the wave amplitude.

It is useful to relate the above findings to the direct numerical simulation of drag reduction by Beris and his co-workers [7,8]. Using the FENE-P constitutive equation, they found indeed that drag reduction only occurs for sufficiently large values of the finite extensibility parameter $L$. The latter controls the
Fig. 7. Critical elasticity number $E^*$ for restabilisation as a function of mobility factor in Poiseuille flow of the Giesekus fluid ($Re = 2400, \beta = 0.9, \alpha = 1.3231, M = 31, N = 4, \Delta t = 0.005$).

extensional behaviour of the fluid, and, by analogy, it is related to the mobility factor of the Giesekus model through $a \sim 1/L^2$. Thus, drag reduction is only predicted for sufficiently small values of $a$. This is in line with our observation that restabilisation in the non-linear transition regime can only occur for $a < 0.0425$.

7.1.2. Couette flow

In Couette flow of Newtonian fluids at high Reynolds number, two-dimensional infinitesimal and finite disturbances always decay. We have repeated for Couette flow the simulations described above using similar values for all the parameters involved (flow, rheology and numerical method). In all cases, we found that two-dimensional finite-amplitude initial disturbances always decay in an oscillatory fashion, as they do in Newtonian fluids. Thus, we conjecture from these numerical results that there is no finite-amplitude two-dimensional periodic solutions in Couette flow of Oldroyd-B and Giesekus fluids.

7.1.3. Numerical considerations

The numerical results discussed in this section have been checked for convergence in the sense that all our qualitative observations (e.g. existence of periodic solutions and occurrence of restabilisation) remain unaffected upon increasing the number of spectral modes. They have all been obtained without added numerical stress diffusion ($\kappa = 0$) in the constitutive equation. In the parameter range of the presented results, we never encountered problems of artificial loss of evolution of the governing equations induced by excessive numerical errors. When the Weissenberg number (rather than the elasticity number) is increased beyond that range (e.g. $We \geq 2$ for $\beta \leq 0.01$), however, we observed the numerical blow-up of solutions as a function of time, while more stable solutions would be expected due to restabilisation phenomenon for Poiseuille flow. Increasing the number of spectral modes remedied this situation in most cases, providing the expected restabilisation. At higher $We$ ($We \geq 3$ for $\beta < 0.01$), it was found impossible to get rid of the numerical blow-up, whether by increasing the number of modes or decreasing the time step. In these cases, use of a non-zero artificial stress diffusion was found useful. For example, for the case ($Re = 1500, We = 3, \beta = 0.01, \alpha = 1.3231$), using a spectral decomposition of $M = 31$, $N = 4$, $\Delta t = 0.005$.
$N = 4$ and a time step $\Delta t = 0.005$, the expected oscillatory decay of the finite disturbance was obtained for $\kappa = 10^{-3}$. In this case, results obtained with $\kappa \geq 10^{-2}$ are strongly affected by the added diffusion.

7.2. Elastic regime

7.2.1. Poiseuille flow

We now consider the non-linear stability of creeping plane Poiseuille flow of a Giesekus fluid at $Re = 0.1$. For all simulations, we have used the same initial disturbance of order $10^{-3}$ (smaller disturbances decay since the flow is linearly stable). This particular choice introduces naturally misfit disturbances to the flow which can grow through the non-normality mechanism discussed in Section 3.

A first result is that two-dimensional finite-amplitude disturbances appear to decay in all our simulations when the viscosity ratio $\beta \geq 10^{-2}$. At smaller values of $\beta$, which are relevant to melts or concentrated solutions, finite-amplitude waves can develop. In the sequel, we focus on results obtained for $\beta = 10^{-3}$.

Results for a perturbation wavenumber $\alpha = 1$ and the Oldroyd-B fluid are shown in Figs. 8–13. There we show the evolution of the near-wall perturbation vorticity and normal stress $T_{xx}$ for increasing values of $We$. Finite-amplitude quasi-periodic waves are predicted beyond a critical Weissenberg number $We_c = 0.1$ (namely for $E_c \geq 1$). It should be noted that the frequencies of oscillation are much larger than in the inertial regime (see, e.g. Fig. 5). Upon increasing $We$, the amplitude of the vorticity wave increases, and the form of the wave is strongly affected. The normal stress $T_{xx}$ oscillates around its base flow value ($8Wey^2 + 16Wey^2\kappa$), with a wave amplitude almost independent of $We$.

The results of Figs. 8–13 have been obtained with a small stress diffusivity ($\kappa = 0.005$) in order to prevent the long time blow-up of the numerical solution. Using $\kappa = 0$, we could indeed predict the development of the quasi-periodic waves but these could not be sustained numerically for long.

Quasi-periodic waves were also found with the Giesekus model. Increasing the mobility factor $a$ in the range $[0, 0.5]$ leads to some decrease of the wave amplitudes, as in the inertial regime. Results are shown in Figs. 14 and 15 for $a = 0.1$. For perturbation wavenumbers $\alpha < 1$, we obtained the eventual decay of

![Fig. 8. Time evolution of near-wall perturbation vorticity at $x = \pi/2, y = -0.99$ in plane Poiseuille flow of an Oldroyd-B fluid ($Re = 0.1, We = 0.5, \beta = 0.001, \alpha = 1, \kappa = 0.005, M = 51, N = 8, \Delta t = 0.001$).](image)
Fig. 9. Time evolution of near-wall normal stress $T_{xx}$ at $x = \pi/2$, $y = -0.99$ in plane Poiseuille flow of an Oldroyd-B fluid ($Re = 0.1$, $We = 0.5$, $\beta = 0.001$, $\alpha = 1$, $\kappa = 0.005$, $M = 51$, $N = 8$, $\Delta t = 0.001$).

Fig. 10. Time evolution of near-wall perturbation vorticity at $x = \pi/2$, $y = -0.99$ in plane Poiseuille flow of an Oldroyd-B fluid ($Re = 0.1$, $We = 1$, $\beta = 0.001$, $\alpha = 1$, $\kappa = 0.005$, $M = 51$, $N = 8$, $\Delta t = 0.001$).

Fig. 11. Time evolution of near-wall normal stress $T_{xx}$ at $x = \pi/2$, $y = -0.99$ in plane Poiseuille flow of an Oldroyd-B fluid ($Re = 0.1$, $We = 1$, $\beta = 0.001$, $\alpha = 1$, $\kappa = 0.005$, $M = 51$, $N = 8$, $\Delta t = 0.001$).
finite disturbances in the whole range parameters covered above. The effect of increasing \( \alpha \) beyond 1 is illustrated in Figs. 16–19 for the Oldroyd-B fluid at \( \text{We} = 1 \). Finite-amplitude waves are again obtained.

In order to investigate the uniqueness of the predicted finite-amplitude waves, we have run simulations with initial perturbations of increasing magnitude \( (10^{-2}, 10^{-1}, 10) \). We find that this increase does not affect the frequencies and amplitude of the waves, although they develop at earlier times.

7.2.2. Couette flow

We have performed for Couette flow the simulations described above using similar values for all the relevant parameters. As in the inertial regime, we found that two-dimensional finite-amplitude disturbances always decay in an oscillatory fashion.
Fig. 14. Time evolution of near-wall perturbation vorticity at \( x = \pi/2, y = -0.99 \) in plane Poiseuille flow of a Giesekus fluid \((Re = 0.1, We = 1, \beta = 0.001, \alpha = 0.1, \kappa = 0.005, M = 51, N = 8, \Delta t = 0.001)\).

Fig. 15. Time evolution of near-wall normal stress \( T_{xx} \) at \( x = \pi/2, y = -0.99 \) in plane Poiseuille flow of a Giesekus fluid \((Re = 0.1, We = 1, \beta = 0.001, \alpha = 0.1, \kappa = 0.005, M = 51, N = 8, \Delta t = 0.001)\).

Fig. 16. Time evolution of near-wall perturbation vorticity at \( x = \pi/2, y = -0.99 \) in plane Poiseuille flow of an Oldroyd-B fluid \((Re = 0.1, We = 1, \beta = 0.001, \alpha = 2, \kappa = 0.005, M = 51, N = 8, \Delta t = 0.001)\).
Fig. 17. Time evolution of near-wall perturbation vorticity at $x = \pi/2, y = -0.99$ in plane Poiseuille flow of an Oldroyd-B fluid ($Re = 0.1, We = 1, \beta = 0.001, \alpha = 4, x_s = 0.005, M = 51, N = 8, \Delta x = 0.001$).

Fig. 18. Time evolution of near-wall perturbation vorticity at $x = \pi/2, y = -0.99$ in plane Poiseuille flow of an Oldroyd-B fluid ($Re = 0.1, We = 1, \beta = 0.001, \alpha = 5, x_s = 0.005, M = 51, N = 8, \Delta x = 0.001$).

Fig. 19. Time evolution of near-wall perturbation vorticity at $x = \pi/2, y = -0.99$ in plane Poiseuille flow of an Oldroyd-B fluid ($Re = 0.1, We = 1, \beta = 0.001, \alpha = 6, x_s = 0.005, M = 51, N = 8, \Delta x = 0.001$).
8. Concluding remarks

In the present paper, we have investigated the non-linear stability of viscoelastic plane channel flows to finite-amplitude two-dimensional disturbances, using the Giesekus model and its particular cases.

We first noted the non-normality property of the linearised problem, and demonstrated its effect on a model problem. It was found that, even under conditions of linear stability, a transient growth of misfit disturbances occurs under the control of the Reynolds and Weissenberg numbers.

We then considered the evolution of two-dimensional, finite-amplitude disturbances in the inertial (high $Re$) and elastic (low $Re$) regimes. For this purpose, we constructed from the field equations a dynamical system of order $\sim 1000$ using a fully-spectral Galerkin method. In the inertial regime, finite-amplitude periodic waves develop beyond a critical Reynolds number. Increasing the elasticity number has a destabilising effect at first, followed by a restabilisation. For the Giesekus model, restabilisation does not occur beyond a critical value of the mobility factor. In the elastic regime, finite-amplitude quasi-periodic solutions are obtained in Poiseuille flow of Oldroyd-B and Giesekus fluids, for values of the viscosity ratio below $10^{-2}$.

Finally, for the case of Couette flow, we consistently observed the oscillatory decay of finite disturbances in the entire parameter range considered, both in the inertial and elastic regimes.

In the Newtonian case, it is known that the two-dimensional finite-amplitude waves which develop at high $Re$ in plane Poiseuille flow become unstable under three-dimensional infinitesimal perturbations through the secondary instability mechanism [23]. Also, Newtonian Couette flow is non-linearly stable in two dimensions, but becomes non-linearly unstable in three dimensions at $Re \approx 1000$ [22,23]. The present work is limited to two-dimensional perturbations of viscoelastic channel flows. Clearly, consideration of three-dimensional effects is likely to reveal new features.

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References