A new upper bound for Policy Iteration

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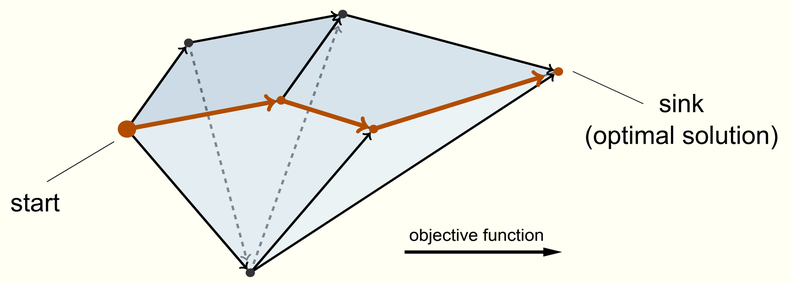
Linear Programming

Markov Decision Processes

Simplex algorithm

Follow the "best" outgoing edge around you and iterate until you reach the sink

Worst case : exponential complexity [Klee & Minty '72] [Friedmann '11]



Policy Iteration

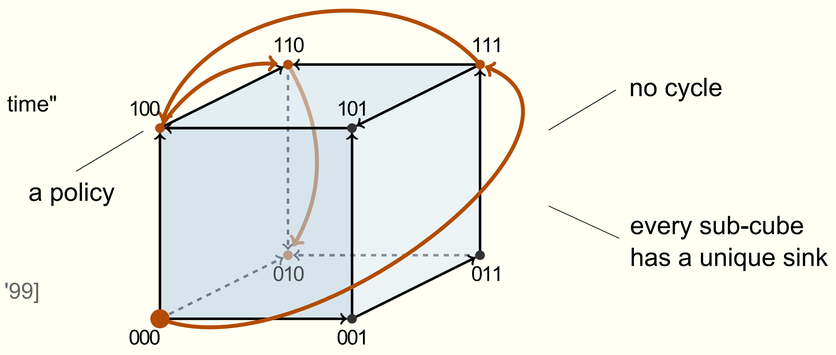
Follow all outgoing (improving) edges around you "at the same time" and iterate until you reach the sink

Worst case : exponential complexity [Fearnley '10]

Best lower bound : $\Omega\left(\sqrt[7]{2}^{n}\right)$ Iterations [Fearnley '10]

Best upper bound : $6 \cdot \frac{2^n}{n}$ Iterations [Mansour & Singh '99]

Can we get better bounds ?



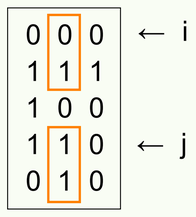
Relaxation 1 : Order-Regular matrices

Translate the cube structure into a matrix condition

$\Pi \in \{0,1\}^{m \times n}$ is Order-Regular

iff $\forall\ 1 \leq i < j < m,\ \exists\ 1 \leq k \leq n$

s.t.: $\Pi_{i,k} \neq \Pi_{i+1,k} = \Pi_{j,k} = \Pi_{j+1,k}$



Goal : Bound the number of rows (number of iterations or explored policies) of any Order-Regular matrix with n columns (dimensions or states)

Experimentally :

n	1	2	3	4	5	6	7
Maximum m	2	3	5	8	13	21	?

Conjecture [Hansen & Zwick '12]

Maximum $m = F_{n+2} = O(\phi^n)$ (Fibonacci numbers ; golden ratio)

Result 1 : for $n = 7$, we showed that $m \leq 33$ (using a "smart" exhaustive search ; deviates from the Fibonacci sequence)

The conjecture does not hold !

Using this relaxation, the best bound we can get for PI so far is $2^{n-1} + 1$...

Relaxation 2 : Pseudo-PI-sequences

Impose only two conditions on the sequence of policies explored by PI

Let $T_1 \prec ... \prec T_{2^n}$ be a total ordering of all distinct $T_i \in \{0,1\}^n$ (each T_i corresponds to the set of outgoing edges of a distinct policy of the cube)

Let $I \subseteq \{1, ..., 2^n\}$ (corresponds to the explored policies)

The set $\{i \in I\}$ is a Pseudo-PI-sequence iff I and the T_i's ensure :

The non-inclusion condition : $\forall\, i < j\ :\ \mathrm{supp}\,(T_i) \not\subseteq \mathrm{supp}\,(T_j)$

The jumping condition : $\forall\, i, j \in I, i < j\ :\ i + |\mathrm{supp}\,(T_i)| \leq j$

Goal : Bound the number of elements in any Pseudo-PI-sequence

Result 2 : the size of any Pseudo-PI-sequence is bounded above by $2 \cdot \frac{2^n}{n} + o\left(\frac{2^n}{n}\right)$

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n	1	2	3	4	5	6	7
Maximum m	2	3	5	8	13	21	33
$2 \cdot \frac{2^n}{n}$	4	4	5.3	8	12.8	21.3	36.6

Not such a bad fit for the real empirical bound

Result 3 : we can build a Pseudo-PI-sequence of size $2 \cdot \frac{2^n}{n} + o\left(\frac{2^n}{n}\right)$

The upper bound is optimal for the relaxation

Our construction of an extremal Pseudo-PI-sequence for n = 4 :

T_1 1111 $\prec$ T_2 1110 $\prec$ T_3 1101 $\prec$ T_4 1011 $\prec$ T_5 0111 $\prec$ T_6 1100 $\prec$ T_7 1010 $\prec$ T_8 1001 $\prec$ T_9 0110 $\prec$ T_{10} 0101 $\prec$ T_{11} 0011 $\prec$ T_{12} 1000 $\prec$ T_{13} 0100 $\prec$ T_{14} 0010 $\prec$ T_{15} 0001 $\prec$ T_{16} 0000

4 jumps ; 3 jumps ; 2 jumps ; 2 jumps ; 1 jump ; 1 jump ; 1 jump ; 1 jump

ordered by decreasing cardinality → non-inclusion condition OK

So $I = \{1, 5, 8, 10, 12, 13, 14, 15, 16\}$ and $|I| = 9 \sim 2 \cdot \frac{2^n}{n} + o\left(\frac{2^n}{n}\right)$ (chosen greedily such that the jumping condition is ensured)

More info on http:/perso.uclouvain.be/romain.hollanders