On the complexity of optimizing PageRank

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Joint work with Raphaël Jungers and Jean-Charles Delvenne

Université catholique de Louvain

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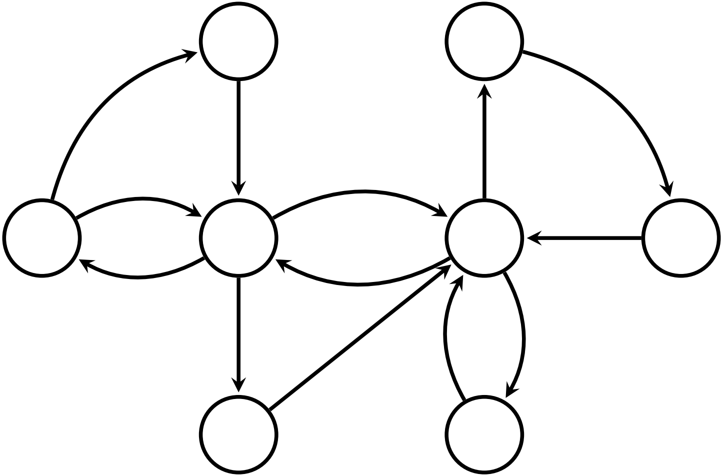
PageRank is the average time-portion spent in a node

during an infinite random walk



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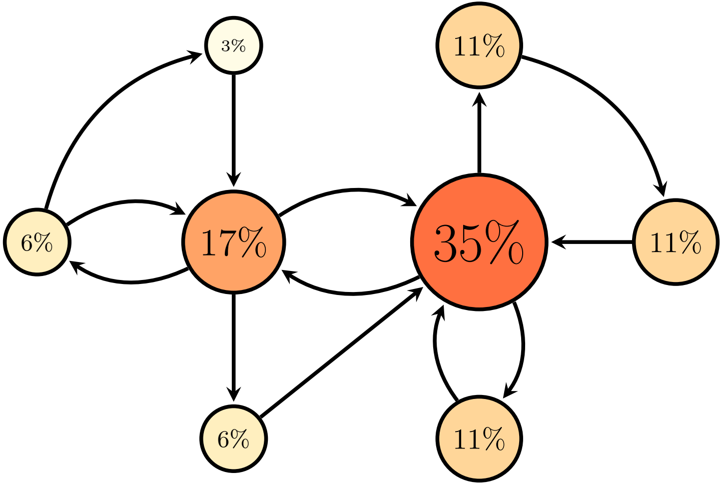

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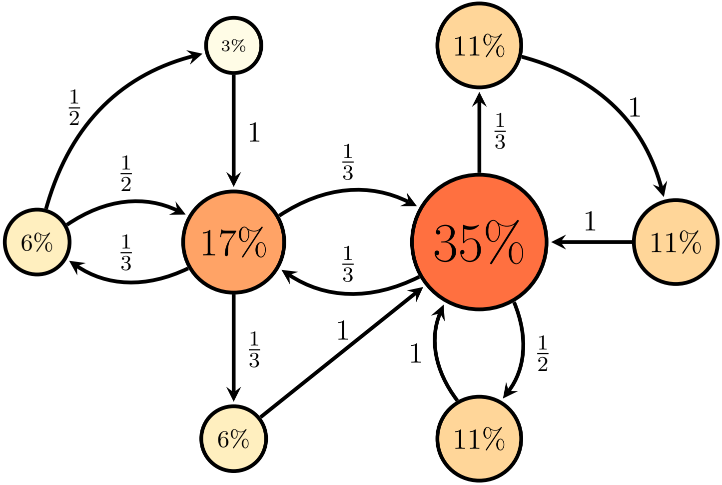
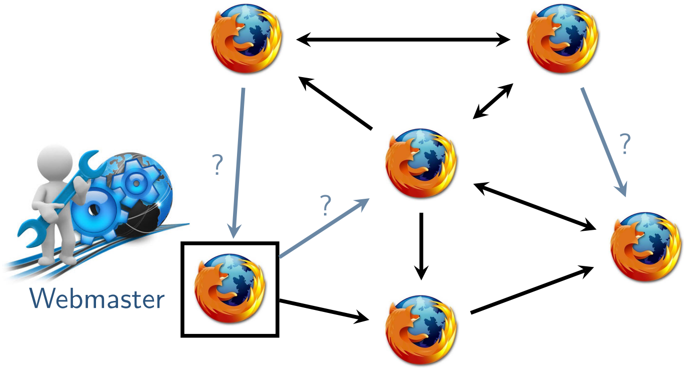
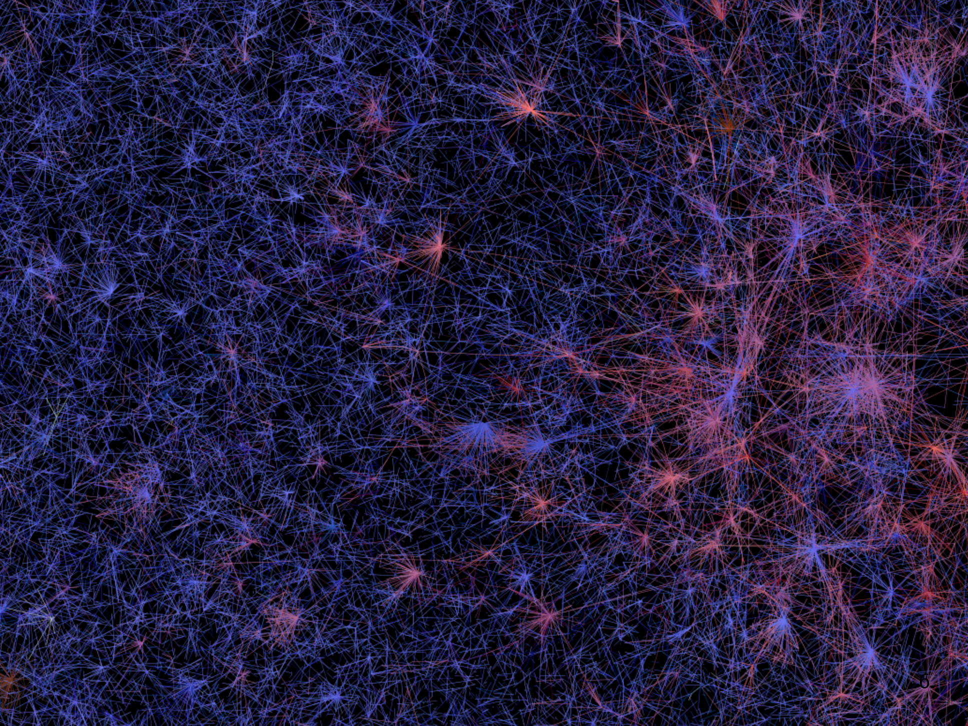
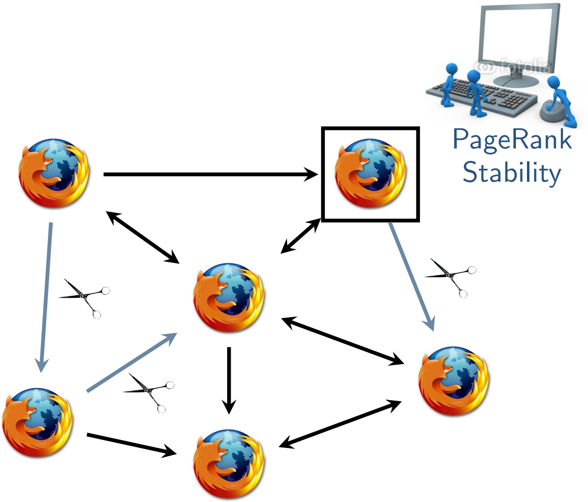


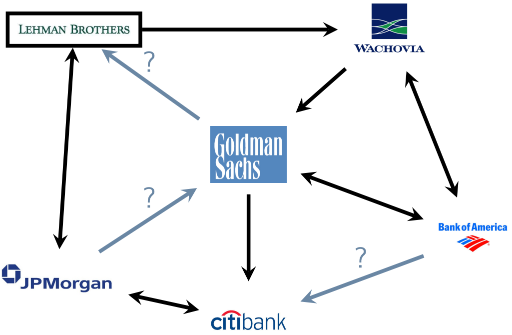


$$\min_{\pi \in \Pi} x_1 \quad \text{such that} \quad x = P^{\pi} x + 1$$




PageRank
Stability

Centrality


LEHMAN BROTHERS
WACHOVIA
Goldman Sachs
Bank of America
JPMorgan
citibank
?
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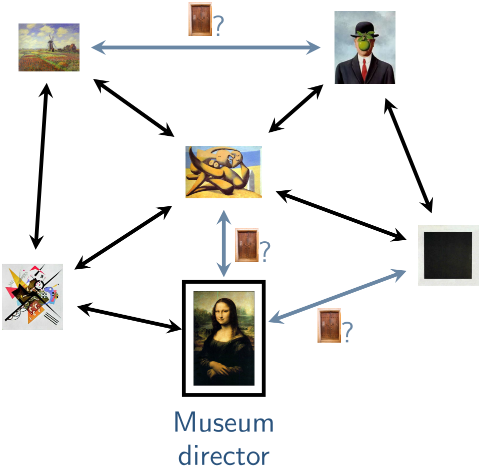

?
?
?
Museum
director

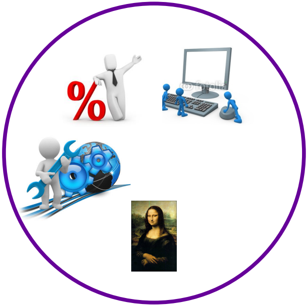
Centrality

PageRank stability

PageRank Optimization (PRO)

Webmaster

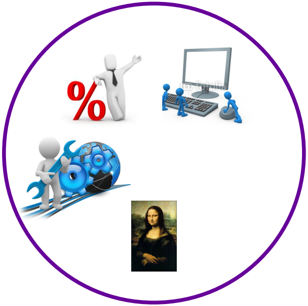
Museum Director



PRO

PageRank

Optimization

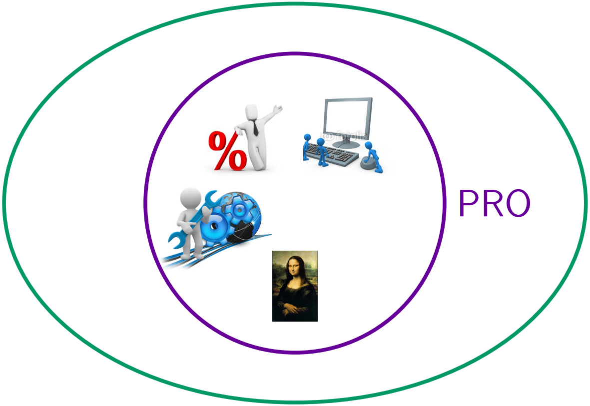


PRO

PageRank

Optimization

1. A set of nodes
2. Sets of fixed and free edges
3. Uniform transition probabilities
4. Unit transition costs
5. Minimize the expected first return time

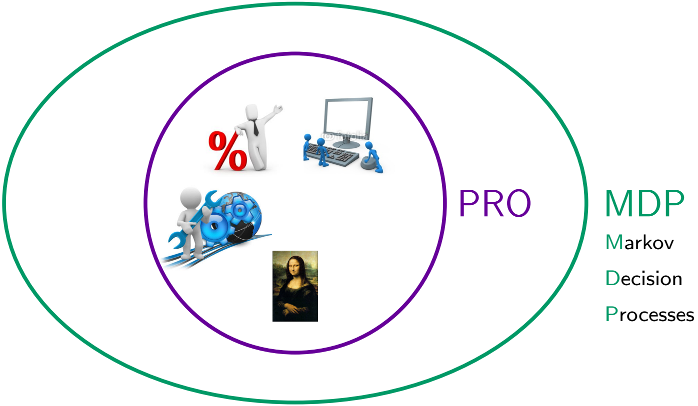


MDP

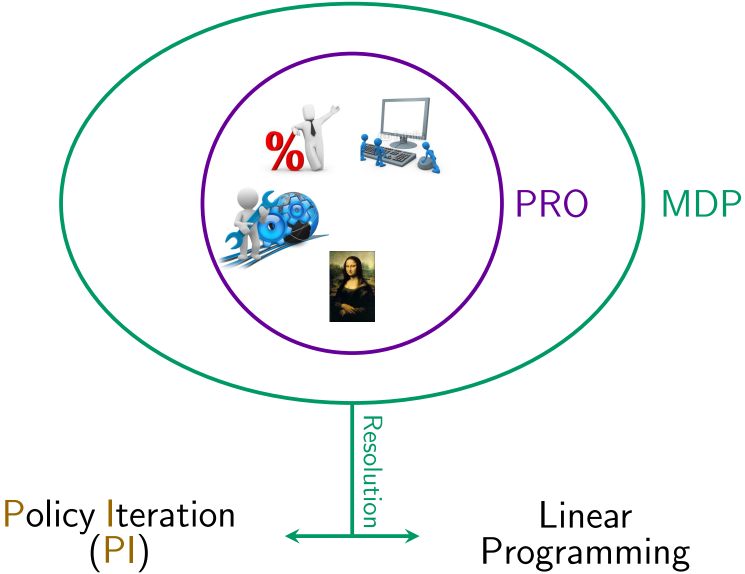
Markov

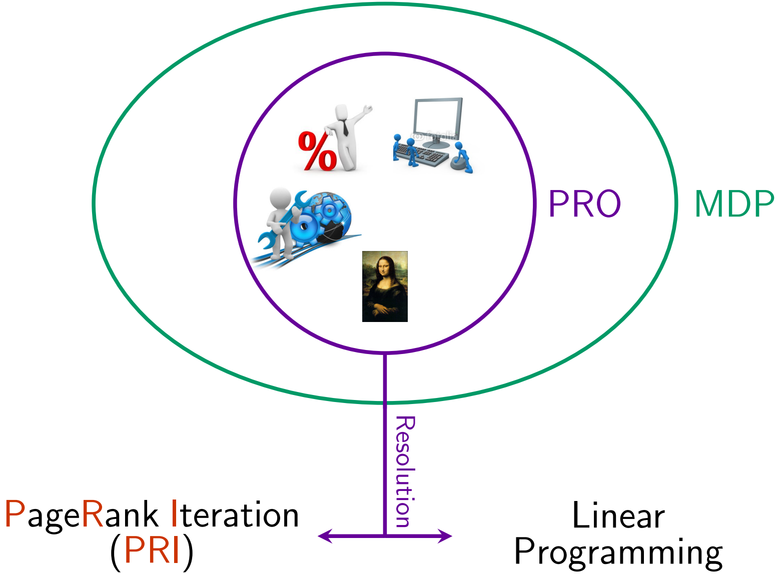
Decision

Processes



1. A set of states
2. A set of actions
3. Any transition probabilities
4. Any transition costs
5. Minimize the expected total-cost


PRO
MDP
Resolution
Policy Iteration
(PI)
Linear
Programming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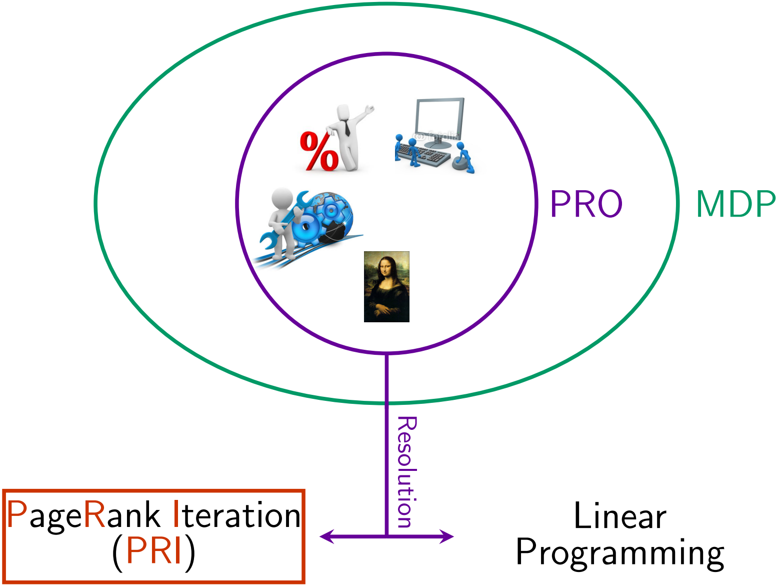
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PageRank Iteration : how it works

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while $\pi_k \neq \pi_{k-1}$ do

1. Evaluate $\pi_\mathbf{k}$: compute the first return times x^{π_k} of the nodes:

$$x^{\pi_k} = P^{\pi_k} x^{\pi_k} + 1$$

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1. Evaluate $\pi_{\mathbf{k}}$: compute the first return times x^{π_k} of the nodes:

$$x^{\pi_k} = P^{\pi_k} x^{\pi_k} + 1$$

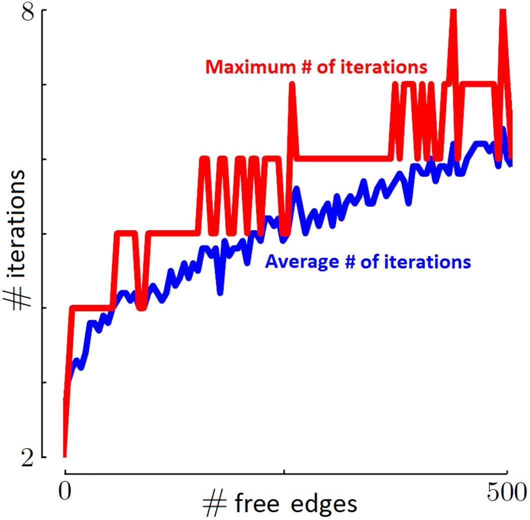
2. Improve $\pi_{\mathbf{k}}$: greedily switch all free edges that enhance the first return times such that:

$$P^{\pi_{k+1}} x^{\pi_k} \leq P^{\pi_k} x^{\pi_k}$$

$k \to k+1$

end

In practice, PRI converges in a linear number of iterations



In theory, PRI might need an exponential number of iterations to converge

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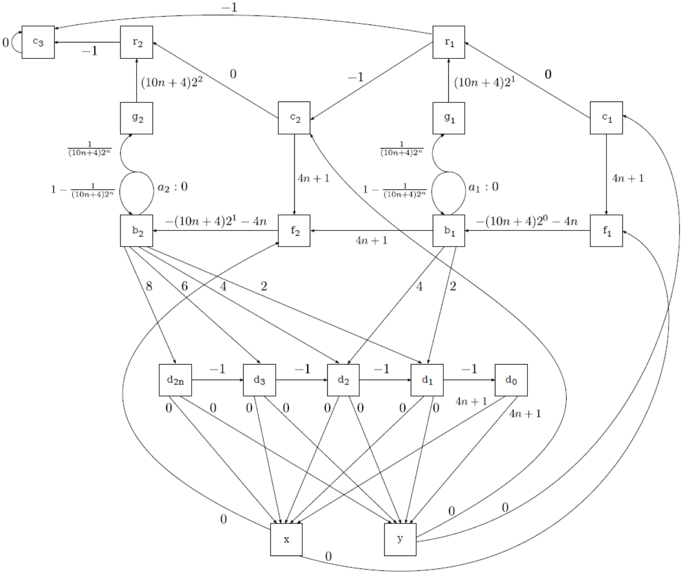
- Do they apply to PageRank Optimization?

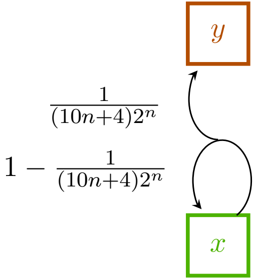
$\Rightarrow$ The answer is *no, but almost*...

Fearnley's example.



Transform Fearnley's example into a PRO problem.

Small transition probabilities?



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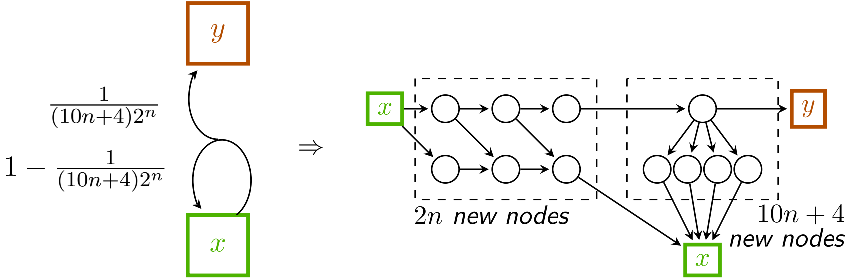

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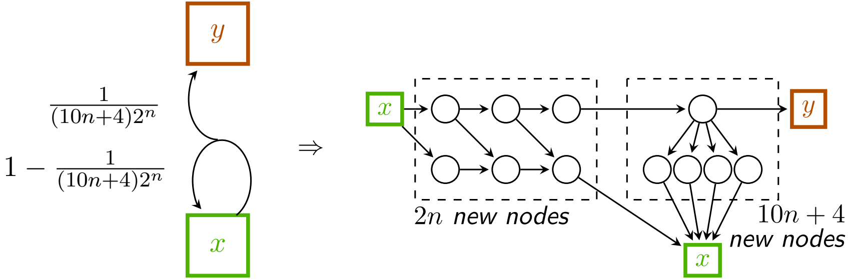

An exponentially small transition probability can be replaced by a polynomial sized structure with uniform transition probabilities.

Transform Fearnley's example into a PRO problem.

High costs?

$$x \xrightarrow{(10n+4)2^n} y$$

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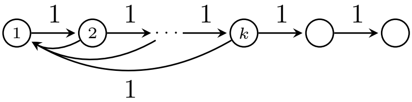
High costs?

$$\boxed{x} \xrightarrow{(10n+4)2^n} \boxed{y}$$

$G_k \equiv$



$$\Rightarrow \begin{cases} \textit{cost} = 2^k + 2^{k-1} \\ \#\ \textit{nodes} = k+2 \end{cases}$$

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High costs?

$$\boxed{x} \xrightarrow{(10n + 4)2^n} \boxed{y}$$

$$G_k \equiv \; 1 \xrightarrow{1} 2 \xrightarrow{1} \cdots \xrightarrow{1} k \xrightarrow{1} \bigcirc \xrightarrow{1} \bigcirc \qquad \Rightarrow \begin{cases} \textit{cost} = 2^k + 2^{k-1} \\ \#\ \textit{nodes} = k + 2 \end{cases}$$

$$H_k \equiv \; \boxed{G_{k-1}} \xrightarrow{1} \boxed{G_{k-3}} \xrightarrow{1} \cdots \xrightarrow{1} \boxed{G_1} \xrightarrow{1} \bigcirc \qquad \Rightarrow \begin{cases} \textit{cost} = 2^k \\ \#\ \textit{nodes} = \frac{3}{4}k^2 + k + 1 \end{cases}$$

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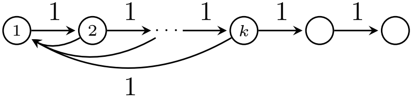
High costs?

$$x \xrightarrow{(10n+4)2^n} y \quad \Rightarrow \quad x \xrightarrow{1} H_n \xrightarrow{1} H_n \xrightarrow{1} \cdots \xrightarrow{1} H_n \xrightarrow{1} y$$

repeated $(10n+4)$ *times*

$G_k \equiv$



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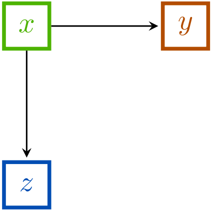
Transform Fearnley's example into a PRO problem.

From actions to free edges?



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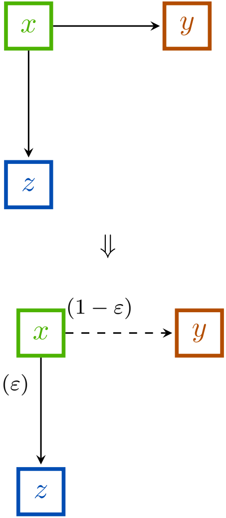

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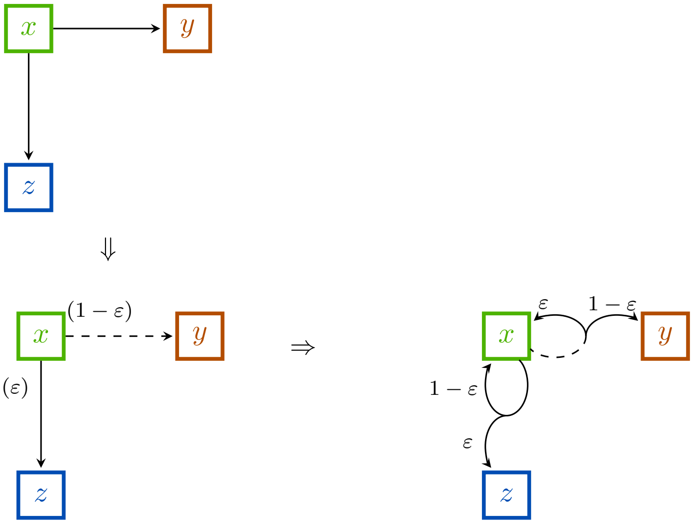

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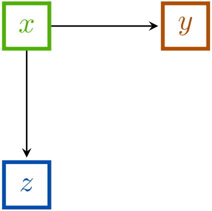
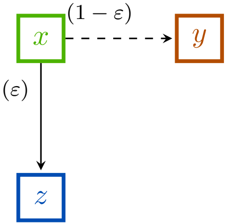
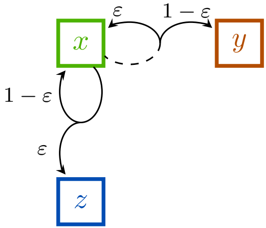


$\Downarrow$



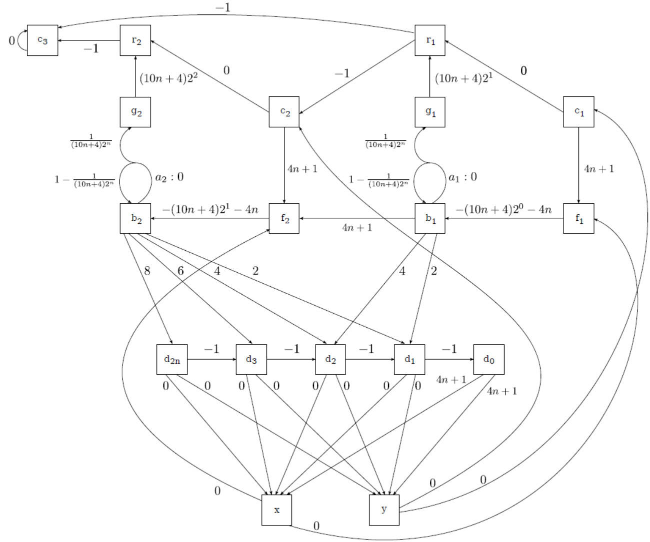
$\Rightarrow$

It works for some $\varepsilon < \frac{1}{2^{O(n^2)}}$



Transform Fearnley's example into a PRO problem.

What about zero and negative costs?



Our main result.

Theorem

If $+1$ *and* -1 *costs are allowed, then there exists an infinite family of PageRank Optimization problems on which the number of iterations that PI takes is lower bounded by an exponential function of the size of the problem.*

PRI runs in polynomial time in some particular cases.

1. If zapping is included in the problem,
   then PRI runs in weakly polynomial time.
   Google's case!

2. If every free edge leaves either the target node or some other node,
   then PRI runs in strongly polynomial time.
   The case of a webmaster with one friend...

Take home messages.

- PageRank Optimization modelizes most problems that consist in **optimizing centrality**.
- PageRank Optimization is **polynomially equivalent to MDPs with only positive costs**, provided some regularity assumptions.
- PageRank Iteration is efficient in practice for solving **large instances**.
- Optimizing PageRank is essentially useful when one seeks to improve the **ranking** and not the absolute value of its PageRank.

Thanks for your attention!