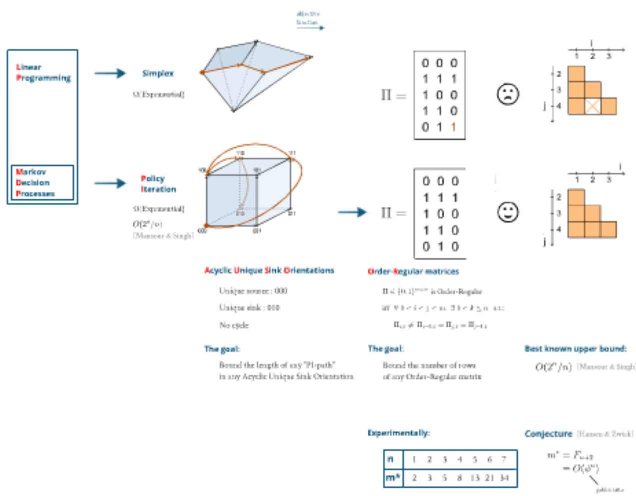
A combinatorial open problem

for the complexity of Policy Iteration

Romain Hollanders, Raphaël M. Jungers, Jean-Charles Delvenne

UCLouvain

Benelux Meeting 2013



The state of the art and some inspiring ideas

Upper bounds on m* $\quad m^* \leq\ O(2^n/n) \leq\ O(2^n)$

Conjecture on m* $\quad m^* \sim\ \Theta(\phi^n) \quad = O(1.618^n)$

Lower bounds on m* $\quad m^* \geq\ \Omega\left(\sqrt{2}^n\right) = \Omega(1.4142^n)$

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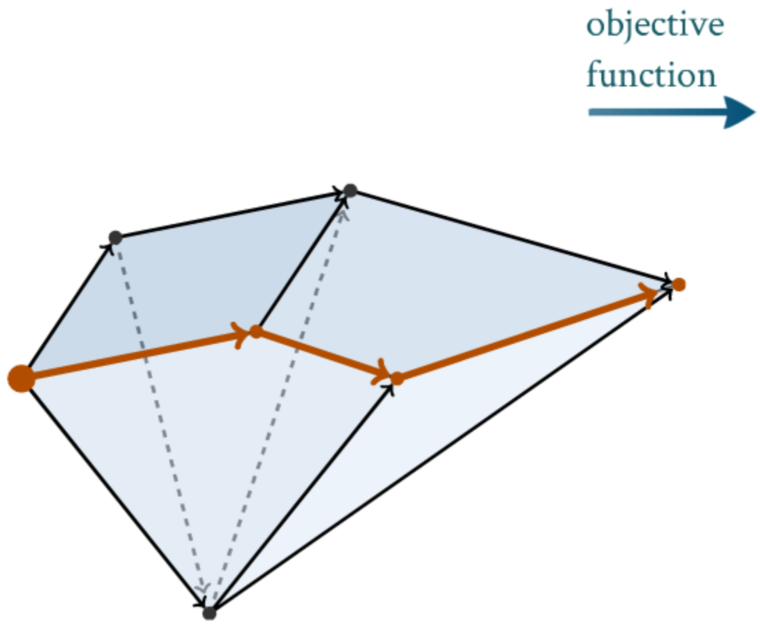
Linear Programming

Markov Decision Processes

Simplex

$\Omega(\text{Exponential})$

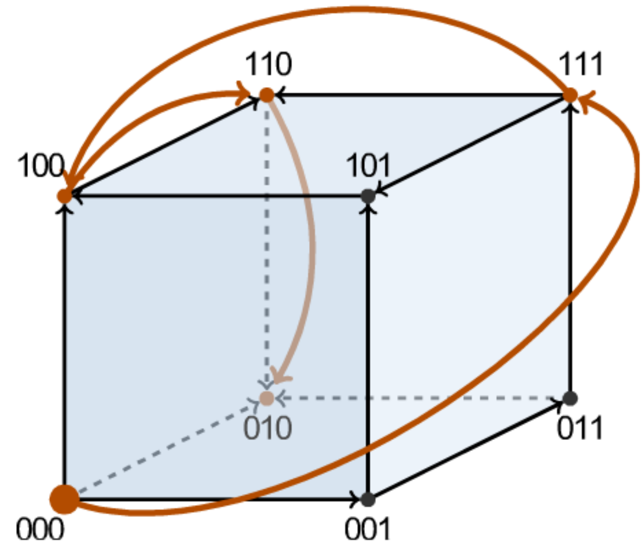


Policy Iteration

$\Omega(\text{Exponential})$

$O(2^n/n)$

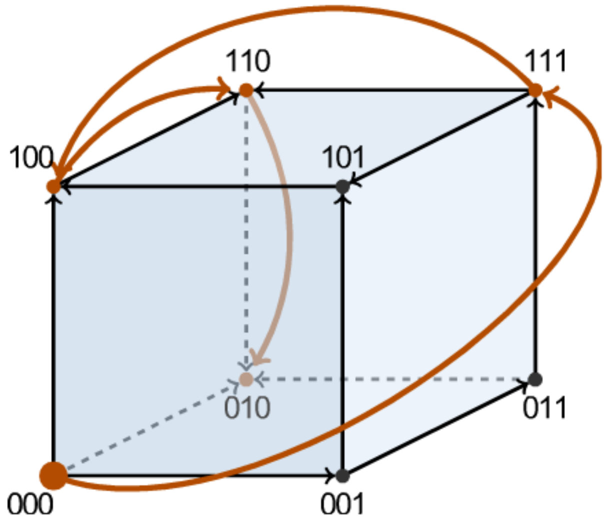
[Mansour & Singh]



$\begin{array}{|ccc|} 0 & 1 & 1 \\ \hline \end{array}$

al)

Singh]



$$\Pi = \begin{array}{|ccc|} \hline 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline \end{array}$$

Acyclic Unique Sink Orientations

Unique source : 000

Unique sink : 010

No cycle

Order-Regular matrices

$\Pi \in \{0,1\}^{m \times n}$ is Order-Regular

iff $\forall\ 0 < i < j < m,\ \exists\ 0 < k \leq n$ s.t.:

$$\Pi_{i,k} \neq \Pi_{i+1,k} = \Pi_{j,k} = \Pi_{j+1,k}$$

The goal:

Bound the length of any "PI-path"
in any Acyclic Unique Sink Orientation

The goal:

Bound the number of rows
of any Order-Regular matrix

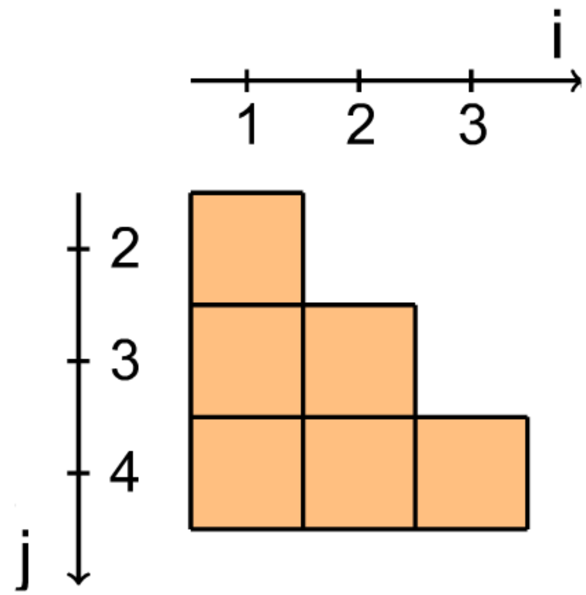
Best

O

0 1 1

$$\Pi = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

☺



Order-Regular matrices

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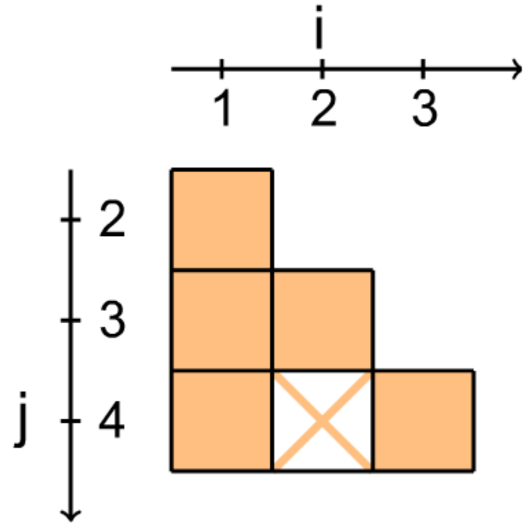
$$\Pi_{i,k} \neq \Pi_{i+1,k} = \Pi_{j,k} = \Pi_{j+1,k}$$

The goal:

Best known upper bound:

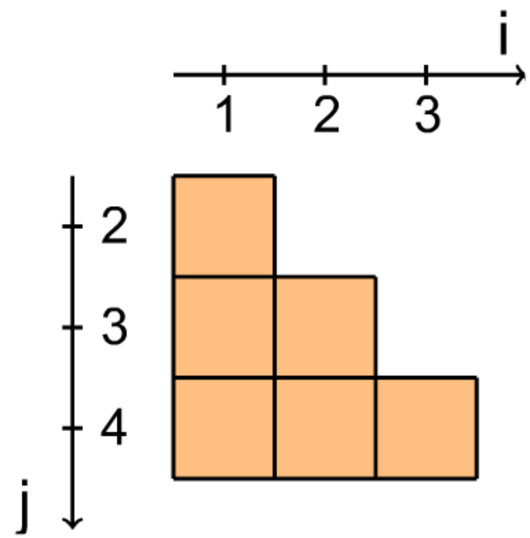
$$\Pi = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

☹



$$\Pi = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

☺



$$\Pi_{i,k} \neq \Pi_{i+1,k} = \Pi_{j,k} = \Pi_{j+1,k}$$

The goal:

Bound the number of rows
of any Order-Regular matrix

Best known upper bound:

$O(2^n/n)$ [Mansour & Singh]

Experimentally:

n	1	2	3	4	5	6	7
m*	2	3	5	8	13	21	34

Conjecture [Hansen & Zwick]

$$m^* = F_{n+2}$$
$$= O(\phi^n)$$

golden ratio

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Upper bounds on m*

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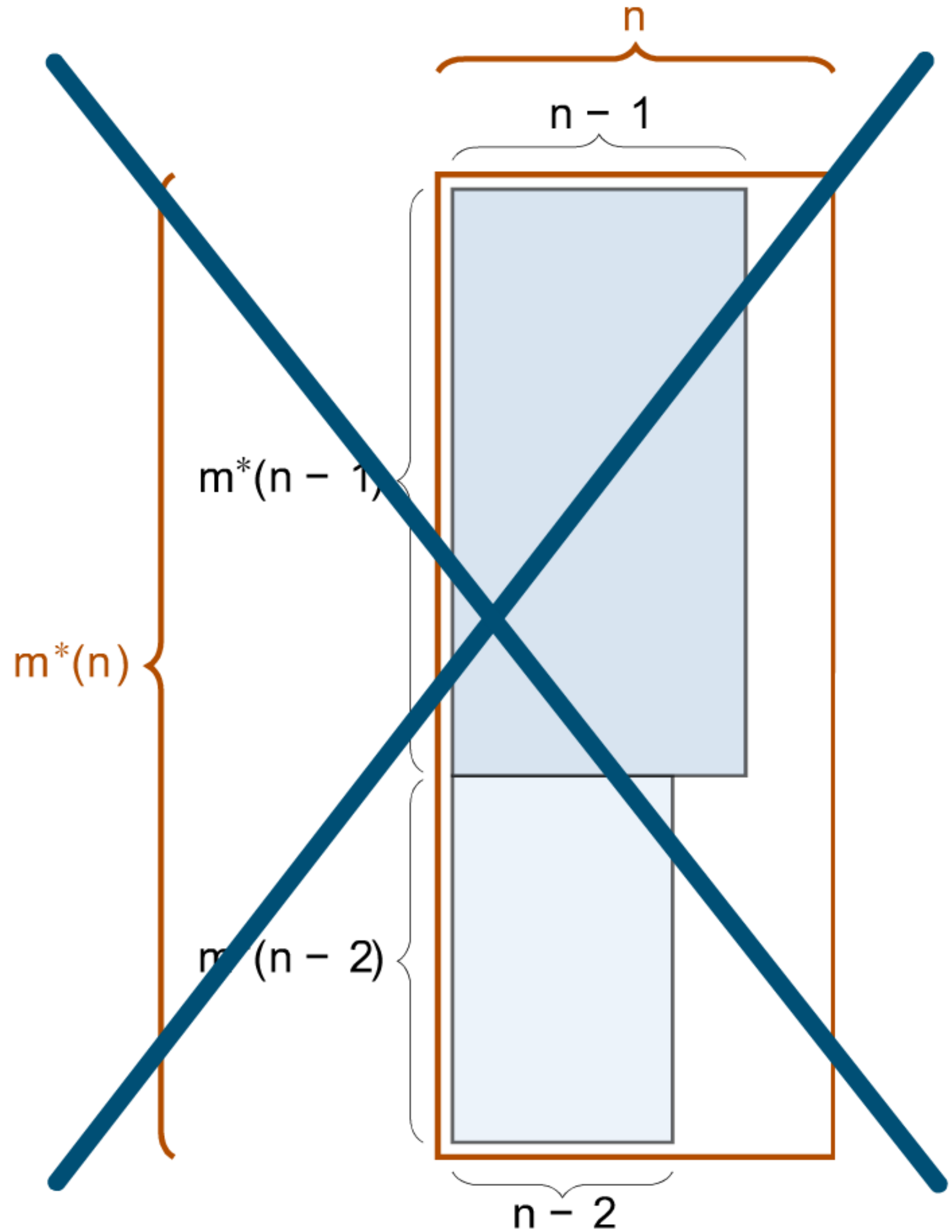
Conjecture on m*

$$m^* \sim O(\phi^n) \quad = O(1.618^n)$$

Lower bounds on m*

$$m^* \geq \Omega\left(\sqrt{2}^n\right) = \Omega(1.4142^n)$$

A natural idea



fails...

The state of the art and some inspiring ideas

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$$m^* \sim O(\phi^n) \quad = O(1.618^n)$$

Lower bounds on m*

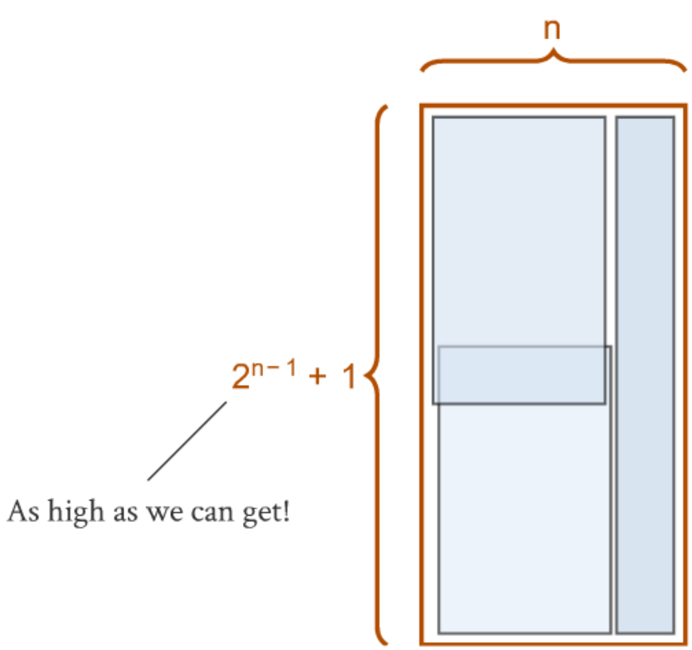
$$m^* \geq \Omega\left(\sqrt{2}^n\right) = \Omega(1.4142^n)$$

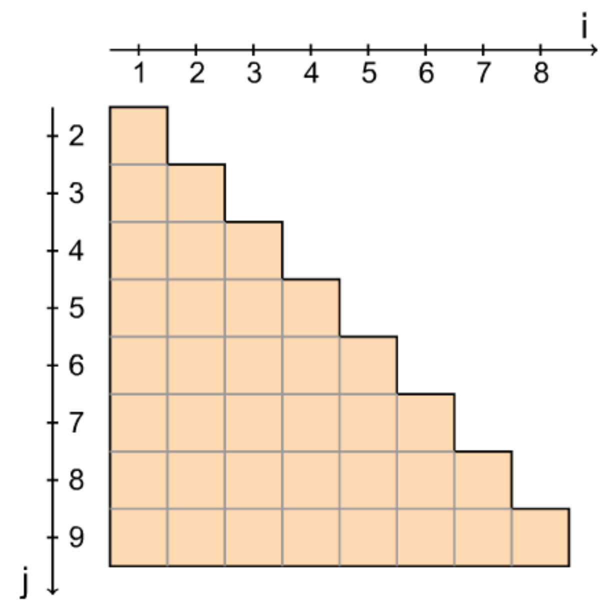
A relaxation

$\Pi \in \{0,1\}^{m \times n}$ is quasi-Order-Regular

iff $\forall\ 0 < i < j < m,\ \exists\ 0 < k \leq n$ s.t.:

$$\Pi_{i,k} \neq \Pi_{i+1,k} = \Pi_{j,k}$$ ~~$= \Pi_{j+1,k}$~~





We can build quasi-Order-Regular matrices with $2^{n-1} + 1$ rows!

The state of the art and some inspiring ideas

Upper bounds on m* $m^* \leq O(2^n/n) \leq O(2^n)$

Conjecture on m* $m^* \sim O(\phi^n) = O(1.618^n)$

Lower bounds on m* $m^* \geq \Omega(\sqrt{2}^n) = \Omega(1.4142^n)$