On the complexity of Policy Iteration for PageRank Optimization

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Joint work with Raphaël Jungers and Jean-Charles Delvenne

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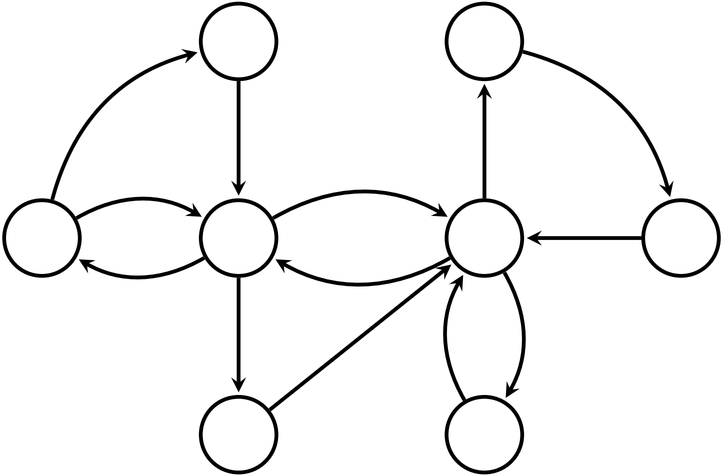
PageRank is the average time-portion spent in a node

during an infinite random walk



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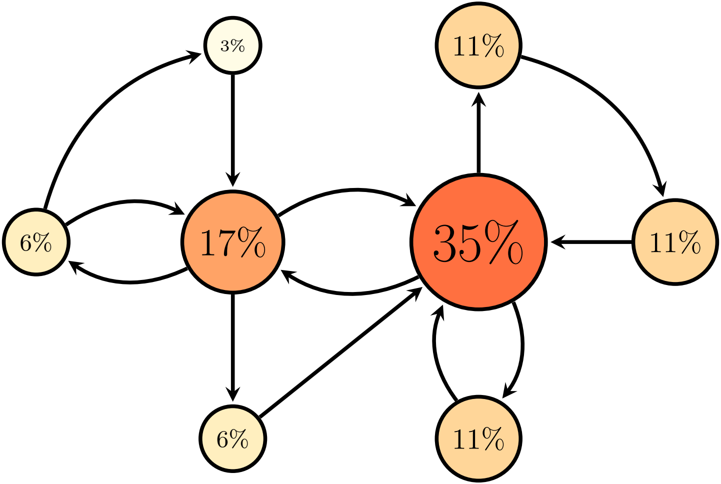

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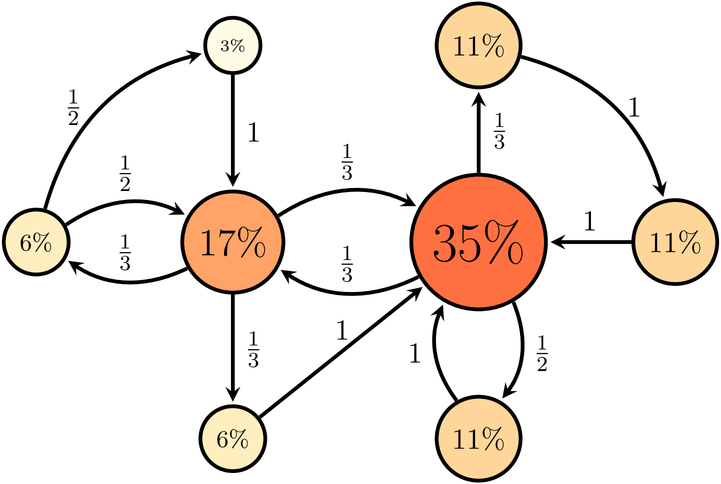
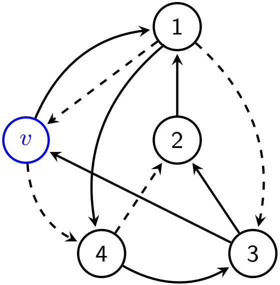


PageRank Optimization by edge selection

A not so trivial task...

PageRank Optimization

by edge selection



Several algorithms have been proposed

But which one should we use?

1. Original approach : polynomial time
   But only approximates the optimal solution
   (Ishii & Tempo, 2008)

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→ **Our main focus**

Outline

1. The Max-PageRank Problem
   Which problem do we want to solve?

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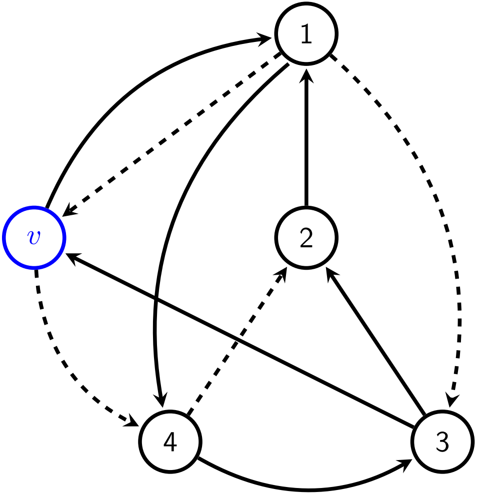
   What did we find about the algorithm?

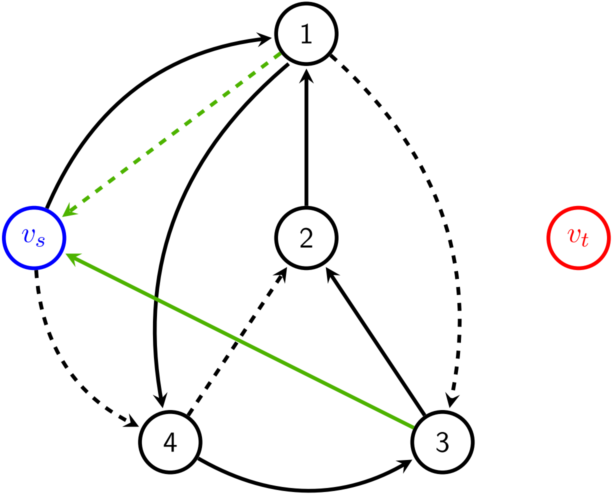
Which fragile edge should we activate?

To maximize the PageRank of v or minimize its first hitting time



Which fragile edge should we activate?

We formulate the problem as a Stochastic Shortest Path problem



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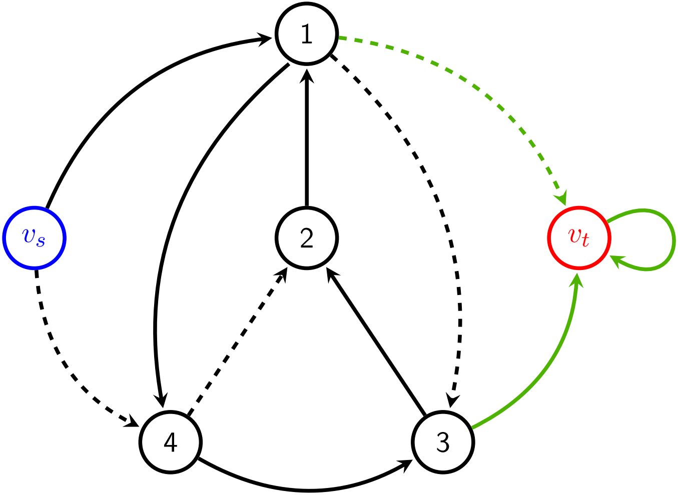
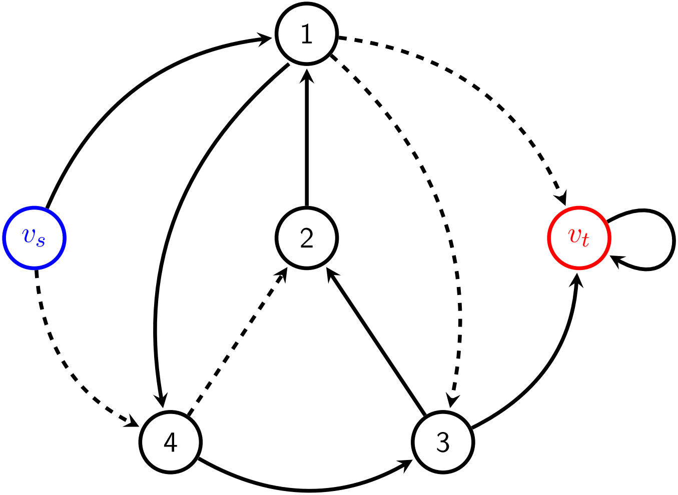


Which fragile edge should we activate?

To maximize the PageRank of v or minimize the distance from v_s to v_t



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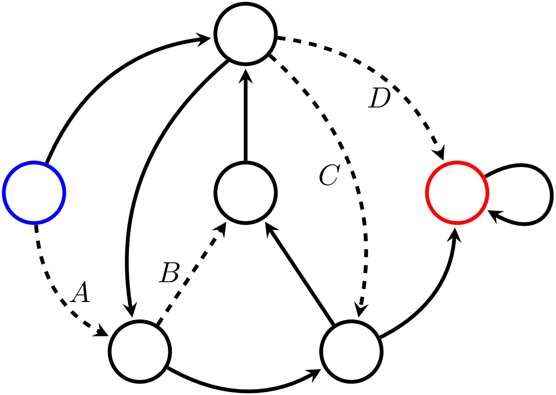
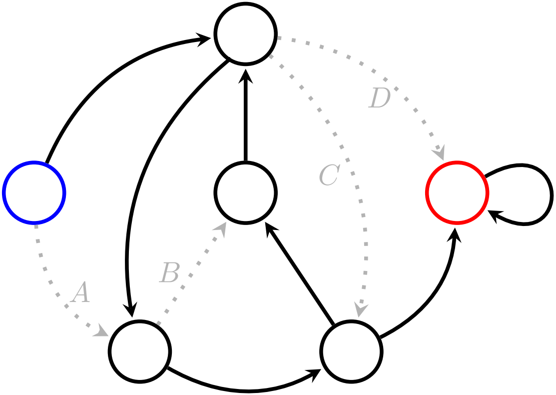
The PageRank Iteration algorithm

At each step, we switch all fragile edges that greedily improve the first return time of v



Iteration 1 : Evaluation step

Initial policy : all fragile edges are OFF



$S_1 = \{ \qquad \}$

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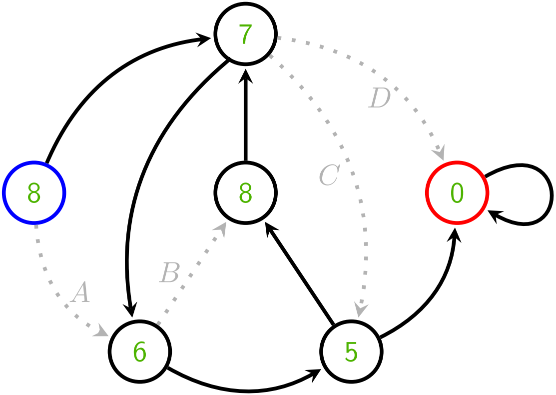

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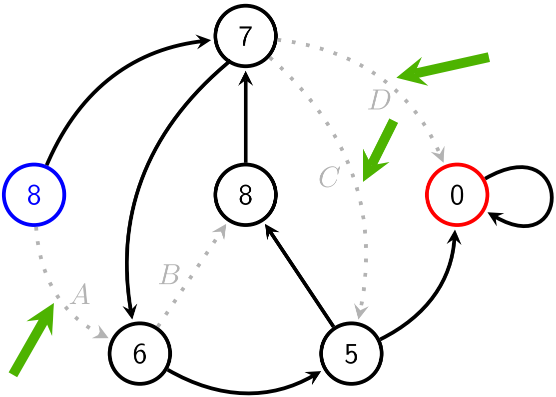
Iteration 1 : Improvement step

It is good to step from a distance d from v_t to a distance $< d-1$



$S_1 = \{ \quad \}$

$T_1 = \{ A \quad C \; D \}$

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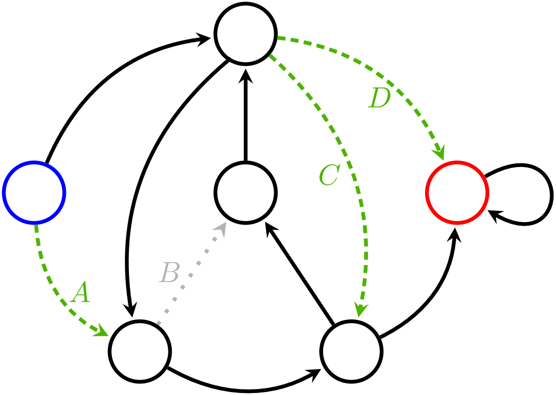
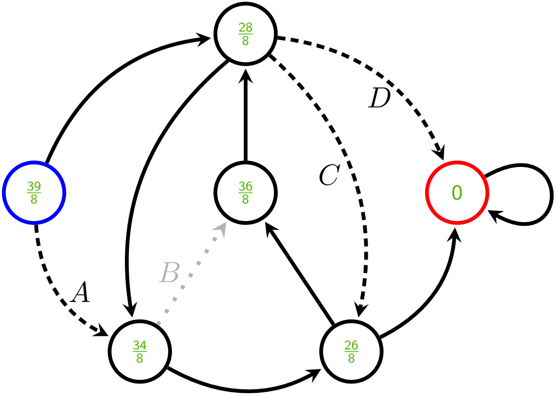


$$S_1 = \{ \quad \}$$
$$S_2 = \{ A \quad C \ D \}$$

$$T_1 = \{ A \quad C \ D \}$$

Iteration 2 : Evaluation step



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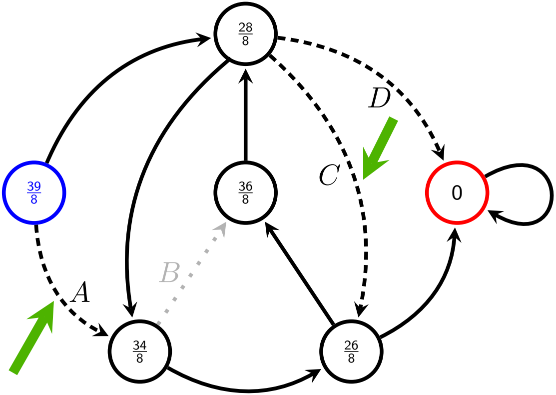

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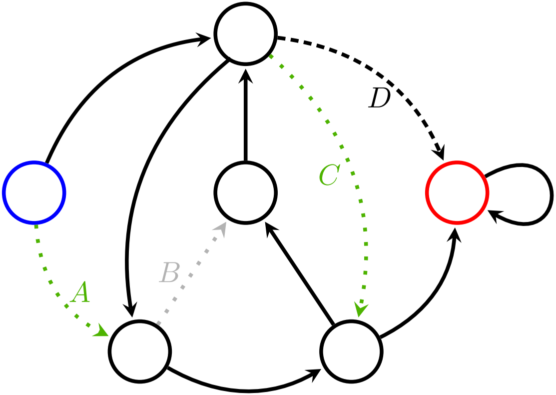

$S_1 = \{ \quad \}$

$S_2 = \{ A \quad C \ D \}$

$S_3 = \{ \quad D \}$

$T_1 = \{ A \quad C \ D \}$

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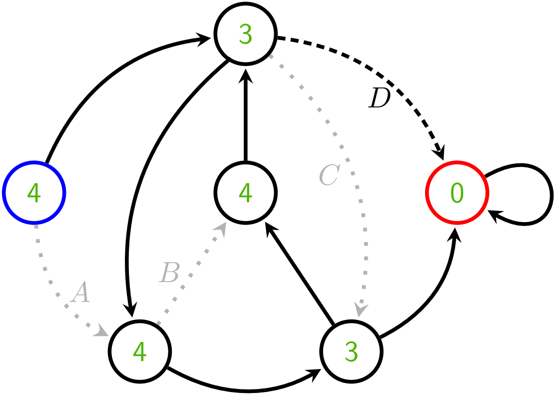
Iteration 3 : Evaluation step



$$S_1 = \{ \quad \}$$
$$S_2 = \{ A \quad C \; D \}$$
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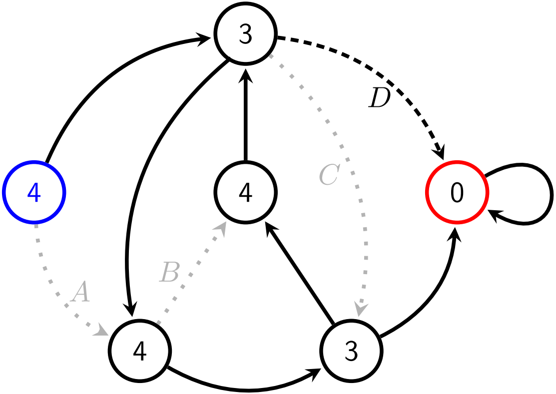
Iteration 3 : Improvement step

No improvements available



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The solution is optimal! : $K = 3$

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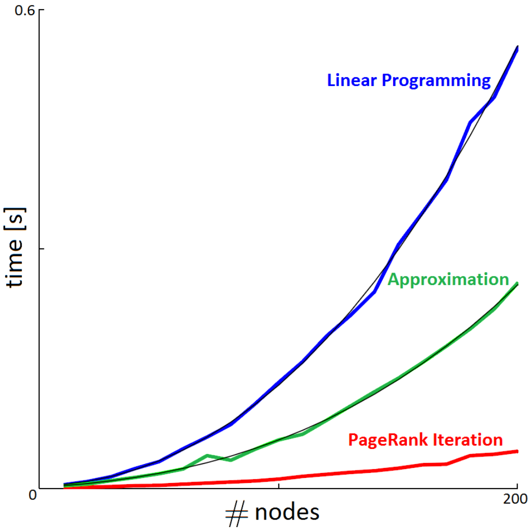
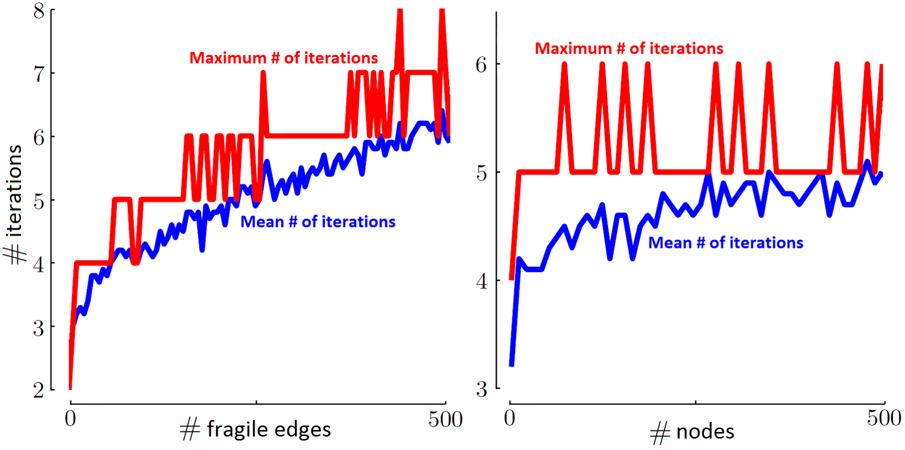
In practice, PRI is by far the best method

In terms of execution time



The number of iterations of PRI seems to grow at most linearly with respect to the problem size



But what is the worst case complexity of PRI?

How many iterations?

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Can we do better ? : Maybe!

Our tools

We define :

1. The configuration set S_k ($\sim$ the policy at iteration k)
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   S_k contains all activated fragile edges from iteration k
2. The improvement set T_k
   Switching elements of T_k will improve S_k

In the improvement step of PRI, we update S_k as follows :

$$S_{k+1} = S_k \oplus T_k$$

Strong properties hint toward a linear bound on the complexity of PRI

But something is still missing...

The following properties hold for the improvement sets :

1. $\nexists\ i < j$ such that $T_i \subseteq T_j$
2. $\nexists\ i < j$ such that $T_i \oplus \cdots \oplus T_{j-1} \subseteq T_j$
3. $\nexists\ i < j$ such that $T_i \subseteq T_{i+1} \oplus \cdots \oplus T_j$

Properties derived from Mansour & Singh (1999)

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  Or even logarithmically?
- In theory, K could be exponential
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We have new properties that can be used to reduce the gap and eventually solve our webmaster's problem efficiently.

Thanks for your attention!