**Introduction to Goodwillie-Weiss manifold calculus.**

**Short summary:** The historical root of Goodwillie-Weiss manifold calculus can be found in the famous “eversion of the sphere”, in other words in the proof in the late 1950’s by Steve Smale that a sphere in the usual 3-dimensional Euclidean space can be turned inside out through a continuous family of immersions. Indeed this proof leads to a description of the homotopy type of the space of immersions of the 2-dimensional sphere into the 3-dimensional euclidean space, \( \text{Imm}(S^2, \mathbb{R}^3) \), in terms of homotopical universal construction; the key fact here is that the homotopy functor \( \text{Imm}(\bullet, \mathbb{R}^3) \) defined on the category of open subsets of \( S^2 \) is a linear functor, that is it converts homotopy pushouts into homotopy pullbacks. This approach was generalized by Goodwillie, Klein, and Weiss in the 1990’s to the study of spaces of smooth embedding of a smooth manifold \( M \) into another manifold \( W \), \( \text{Emb}(M, W) \). In that case the functor \( \text{Emb}(\bullet, W) \) is not linear anymore but it can be arbitrarily well approximated by polynomial functor, leading to an approximation of functors akin to approximations of real functions by Taylor polynomials. The theory is also very much related to the little \( n \)-disks operad. In particular the rational formality of this operad, established by Kontsevich, has applications to the computation of the rational homology of the space of embeddings of \( M \) into \( \mathbb{R}^n \).

**Outline of the lectures**

- Spaces of immersions \( \text{Imm}(M, W) \) and embeddings \( \text{Emb}(M, W) \) between smooth manifolds \( M \) and \( W \).
- Quick review on homotopy limits and colimits.
- Linear, polynomial and analytic functor. A (contravariant) functor is called linear it it sends homotopy pushouts to homotopy pullbacks. A functor is polynomial of degree \( \leq k \) if it has an analogous property where the square diagrams (of the homotopy pushout/pullback) are replaced by \( k+1 \)-dimensional cubical diagrams. A functor is analytic if it is well approximated by polynomial functors.
- The Smale-Hirsh theorem: the functor \( \text{Imm}(\cdot, W) : O(M) \to \text{Top} \), \( U \mapsto \text{Imm}(U, W) \), where \( O(M) \) is the category of open subsets in \( M \), is a linear functor.
- The Goodwillie-Klein-Weiss theorem: the functor \( \text{Emb}(\cdot, W) \) is analytic. Description of its approximations as homotopy limits of configuration spaces.
- The operad \( E_n \) of little \( n \)-disks and its relation with Goodwillie-Weiss calculus applied to the study of \( \text{Emb}(M, \mathbb{R}^n) \).
- Kontsevich’s formality of the little disks operad and its application to the computation of the rational homology of spaces of embeddings.
References


