Categorical-algebraic methods in group cohomology

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Group cohomology via categorical algebra

In my work I mainly develop and apply categorical algebra
in its interactions with homology theory.

- My concrete aim: to understand (co)homology of groups.
- Several aspects:
  - general categorical versions of known results;
  - problems leading to further development of categorical algebra;
  - categorical methods leading to new results for groups.

Today, I would like to

- explain how the concept of a *higher central extension* unifies the interpretations of homology and cohomology;
- give an overview of some categorical-algebraic methods used for this aim.

This is joint work with many people, done over the last 15 years.

Homology vs. cohomology via higher central extensions

Several streams of development are relevant to us:

- categorical Galois theory + semi-abelian categories $\rightsquigarrow$ *higher central extensions* $\rightsquigarrow$ interpretation of homology objects via Hopf formulae

  [Janelidze, 1991] [Everaert, Gran & VdL, 2008] [Duckerts-Antoine, 2013]

- in an abelian context: Yoneda's interpretation of $H^{n+1}(X, A)$ through equivalence classes of exact sequences of length $n + 1$

  [Yoneda, 1960]

- in Barr-exact categories: cohomology classifies higher torsors

  [Barr & Beck, 1969] [Duskin, 1975] [Glenn, 1982]

- "directions approach to cohomology"

  [Bourn & Rodelo, 2007] [Rodelo, 2009]

What are the connections between these developments?

Overview, $n = 1$

	Homology $H_2(X)$	Cohomology $H^2(X,(A,\xi))$	
		trivial action ξ	arbitrary action ξ
Gp	$\frac{R \wedge [F,F]}{[R,F]}$	$CentrExt^1(X,A)$	$OpExt^1(X,A,\xi)$
abelian categories	0	$Ext^1(X,A)$	
Barr-exact categories		$Tors^1[X,(A,\xi)]$	
semi-abelian categories	$\frac{R \wedge [F,F]}{[R,F]}$	$CentrExt^1(X,A)$	$OpExt^1(X,A,\xi)$

Low-dimensional cohomology of groups, I

An **extension** from A to X is a short exact sequence

$$0 \longrightarrow A \rightarrowtail E \xrightarrow{f} X \longrightarrow 0.$$

It is **central** if and only if $[A, E] = 0$: all $eae^{-1}a^{-1}$ vanish, $a \in A$, $e \in E$.
Then, in particular, A is an abelian group.

Theorem [Eckmann 1945-46; Eilenberg & Mac Lane, 1947]

For any abelian group A we have $H^2(X, A) \cong \mathit{CentrExt}^1(X, A)$,
the group of equivalence classes of central extensions from A to X.

- $H^2(-, A)$ is the first derived functor of $\mathit{Hom}(-, A)\colon \mathit{Gp}^{\mathrm{op}} \to \mathit{Ab}$.
- By the Short Five Lemma, *equivalence class* = *isomorphism class*:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \rightarrowtail & E & \xrightarrow{f} & X & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle e} & & \| & & \\
0 & \longrightarrow & A & \rightarrowtail & E' & \xrightarrow[f']{} & X & \longrightarrow & 0
\end{array}$$

The theorem remains true [Gran & VdL, 2008] in any semi-abelian category [Janelidze, Márki & Tholen, 2002] with enough projectives;
centrality may be defined via commutator theory or via categorical Galois theory.

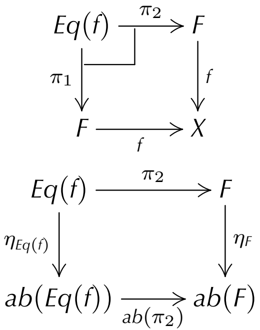
Low-dimensional homology of groups

Theorem (Hopf formula for $H_2(X)$, [Hopf, 1942])

Consider a **projective presentation** $X \cong F/R$ of X: an extension $0 \to R \to F \to X \to 0$ where F is projective. Then the *second integral homology group* $H_2(X)$ is $\frac{R \wedge [F,F]}{[R,F]}$.

Basic analysis

- H_2 is a derived functor of the reflector $ab\colon Gp \to Ab\colon X \mapsto \frac{X}{[X,X]}$.
- The commutator $[R,F]$ occurs in/is determined by the reflector $ab_1\colon Ext(Gp) \to CExt(Gp)\colon (f\colon F \to X) \mapsto (ab_1(f)\colon \frac{F}{[R,F]} \to X)$.
- Through categorical Galois theory [Janelidze & Kelly, 1994], the second adjunction may be obtained from the first.
- In fact, f is central iff the bottom right square is a pullback.



All ingredients of the formula may be obtained from the reflector *ab*.

The theorem remains true [Everaert & VdL, 2004] for reflectors of **semi-abelian** varieties of algebras to their subvarieties: $[X,X]$ is commutator (Gp vs. Ab), Lie bracket ($Lie_{\mathbb{K}}$ vs. $Vect_{\mathbb{K}}$), product XX (Alg_R vs. Mod_R), or …

What is a semi-abelian category?

A category is **Barr-exact** [Barr, 1971] when

1. finite limits and coequalisers of kernel pairs exist;
2. regular epimorphisms are pullback-stable;
3. every internal equivalence relation is a kernel pair.

All varieties of algebras and all elementary toposes are such.

- An **abelian category** is a Barr-exact category which is also **additive**: it has finitary biproducts and is enriched over *Ab*.
  [Buchsbaum, 1955; Grothendieck, 1957; Yoneda, 1960; Freyd, 1964]

Examples: Mod_R, sheaves of abelian groups.

- A Barr-exact category is **semi-abelian** when it is pointed, has binary coproducts and is **protomodular**: the *Split Short Five Lemma* holds [Bourn, 1991].

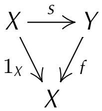
This definition [Janelidze, Márki & Tholen, 2002] unifies "old" approaches towards an axiomatisation of categories "close to *Gp*" such as [Higgins, 1956] and [Huq, 1968] with "new" categorical algebra—the concepts of Barr-exactness and **Bourn-protomodularity**.

Examples: Gp, $Lie_{\mathbb{K}}$, $Alg_{\mathbb{K}}$, $XMod$, $Loop$, $HopfAlg_{\mathbb{K},coc}$, $C^*\text{-}Alg$, Set_*^{op}, varieties of Ω-groups.

More on protomodularity

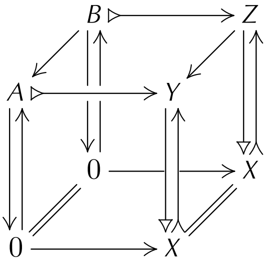
Protomodular categories [Bourn, 1991] arose out of the idea that in algebra, *categories of points* may be more fundamental than slice categories.

A **point** (f, s) **over** X is a split epimorphism $f\colon Y \to X$ with a chosen splitting $s\colon X \to Y$.



$Pt_X(\mathscr{X}) = (1_X \downarrow (\mathscr{X} \downarrow X))$ is the **category of points** over X in $\mathscr{X}$.

The **Split Short Five Lemma** is precisely the condition that the pullback functor $Pt_X(\mathscr{X}) \to Pt_0(\mathscr{X}) \cong \mathscr{X}$ reflects isomorphisms.



Points are actions.

If $\mathscr{X}$ is semi-abelian, then this change-of-base functor is monadic [Bourn & Janelidze, 1998]; the algebras for the monad are called **internal actions**, and correspond to split extensions: if X acts on A via ξ, we obtain

$$0 \longrightarrow A \rightarrowtail A \rtimes_\xi X \underset{f_\xi}{\overset{s_\xi}{\leftrightarrows}} X \longrightarrow 0.$$

Overview, $n = 1$

	Homology $H_2(X)$	Cohomology $H^2(X, (A, \xi))$	
		trivial action ξ	arbitrary action ξ
Gp	$\frac{R \wedge [F, F]}{[R, F]}$	$CentrExt^1(X, A)$	$OpExt^1(X, A, \xi)$
abelian categories	0	$Ext^1(X, A)$	
Barr-exact categories		$Tors^1[X, (A, \xi)]$	
semi-abelian categories	$\frac{R \wedge [F, F]}{[R, F]}$	$CentrExt^1(X, A)$	$OpExt^1(X, A, \xi)$

- $0 \to R \to F \to X \to 0$ is a projective presentation.
- A priori, H_2 is a derived functor of $ab \colon \mathscr{X} \to Ab(\mathscr{X})$.
- The Hopf formula is valid for any reflector $I \colon \mathscr{X} \to \mathscr{Y}$ from a semi-abelian category $\mathscr{X}$ to a Birkhoff subcategory $\mathscr{Y}$; then the commutators are relative with respect to I. Also in the abelian case, this gives something non-trivial.

Low-dimensional cohomology of groups, II

Consider an extension $0 \longrightarrow A \rhd\!\!\longrightarrow E \overset{f}{\longrightarrow}\!\!\rhd X \longrightarrow 0$.

- Any action $\xi\colon X \to Aut(A)$ of X on A pulls back along f to an action $f^*(\xi)\colon E \to Aut(A)\colon e \mapsto \xi(f(e))$ of E on A.
- If A is abelian, then there is a unique action ξ of X on A such that $f^*(\xi)$ is the conjugation action of E on A: put $\xi(x)(a) = eae^{-1}$ for $e \in E$ with $f(e) = x$.
- This action ξ is called the **direction** of the given extension. It determines a left $\mathbb{Z}(X)$-module structure on A.

Theorem (Cohomology with non-trivial coefficients)
$H^2(X, (A, \xi)) \cong OpExt^1(X, A, \xi)$, the group of equivalence classes of extensions from A to X with direction (A, ξ).

This agrees with the above: an extension with abelian kernel is central iff its direction is trivial.

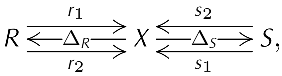
$$
\begin{aligned}
f \text{ is central} &\Leftrightarrow \forall_{a\in A}\forall_{e\in E} \quad a = eae^{-1}\\
&\Leftrightarrow \forall_{a\in A}\forall_{e\in E} \quad a = \xi(f(e))(a)\\
&\Leftrightarrow \forall_{x\in X} \quad 1_A = \xi(x)
\end{aligned}
$$

How to extend this to semi-abelian categories?

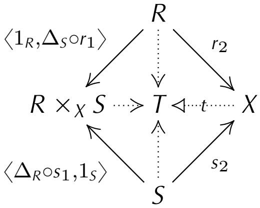
Three commutators

Smith-Pedicchio

For equivalence relations R, S on X

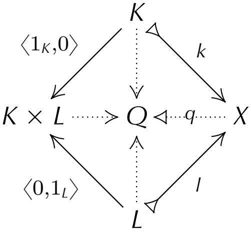


the **Smith-Pedicchio commutator** $[R, S]^{\mathrm{S}}$ is the kernel pair of t:

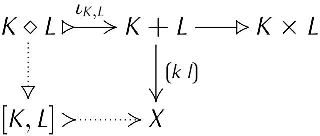


Huq & Higgins

For $K, L \lhd X$, the **Huq commutator** $[K, L]^{\mathrm{Q}}$ is the kernel of q:



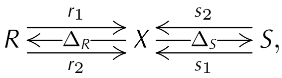
The **Higgins commutator** $[K, L] \leqslant X$ is the image of $(k\ l) \circ \iota_{K,L}$:



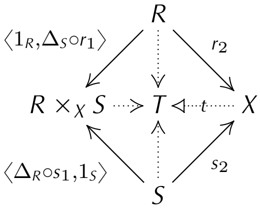
Pregroupoids

Smith-Pedicchio

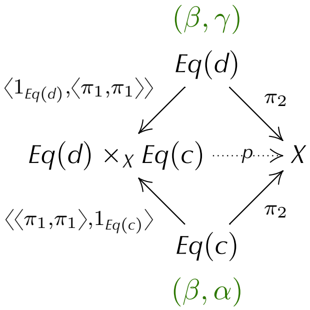
For equivalence relations R, S on X

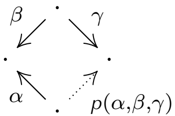


the **Smith-Pedicchio commutator** $[R, S]^S$ is the kernel pair of t:



A span $D \xleftarrow{d} X \xrightarrow{c} C$ is a **pregroupoid** iff $[Eq(d), Eq(c)]^S = \Delta_X$. [Kock, 1989]





$$\begin{cases} p(\alpha, \beta, \beta) = \alpha \\ p(\beta, \beta, \gamma) = \gamma \end{cases}$$

The *Smith is Huq* condition

Several categorical-algebraic conditions have been considered which "make a semi-abelian category behave more like *Gp* does".

One (weak and well-studied) such is the ***Smith is Huq* condition (SH)**, which holds when two equivalence relations R and S on an object X commute iff their normalisations $K, L \lhd X$ do.

$$K \xrightarrow{r_2 \circ \ker(r_1)} X \xleftarrow{s_2 \circ \ker(s_1)} L \quad \text{normalisations of} \quad R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \underset{s_1}{\overset{s_2}{\leftleftarrows}} S$$

- One implication is automatic [Bourn & Gran, 2002].
- All *Orzech categories of interest* [Orzech, 1972] satisfy (SH). *Loop* does not.
- By [Martins-Ferreira & VdL, 2012] and [Hartl & VdL, 2013], under (SH) the description of internal crossed modules of [Janelidze, 2003] simplifies. This is, essentially, because then, a span $D \xleftarrow{d} X \xrightarrow{c} C$ is a pregroupoid iff $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0$, so a reflexive graph $G_1 \underset{c}{\overset{d}{\rightrightarrows}} G_0$ is an internal groupoid iff $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0$.

This is important when defining abelian extensions.

The semi-abelian case: abelian extensions, I

Let $\mathscr{X}$ be a semi-abelian category. An **abelian extension** in $\mathscr{X}$ is a short exact sequence

$$0 \longrightarrow A \overset{a}{\rightarrowtail} E \overset{f}{\twoheadrightarrow} X \longrightarrow 0$$

where f is an **abelian object** in $(\mathscr{X} \downarrow X)$: this means that, equivalently,

1. the span (f, f) is a pregroupoid;
2. the commutator $[Eq(f), Eq(f)]^{S}$ is trivial;
3. $\langle 1_E, 1_E \rangle \colon E \to Eq(f)$ is a normal monomorphism $f \to f\pi_1$ in $(\mathscr{X} \downarrow X)$;
4. $\langle a, a \rangle \colon A \to Eq(f)$ is a normal monomorphism in $\mathscr{X}$.

Example: a **split extension** (a point (f, s) with $a = \ker(f)$) is abelian iff it is a **Beck module** [Beck, 1967]: an abelian group object in $(\mathscr{X} \downarrow X)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \overset{\langle a,a \rangle}{\rightarrowtail} & Eq(f) & \overset{1_A \rtimes f}{\twoheadrightarrow} & A \rtimes_\xi X & \longrightarrow & 0 \\
 & & \| & & \pi_1 \downarrow \uparrow \langle 1_E, 1_E \rangle & & f_\xi \downarrow \uparrow s_\xi & & \\
0 & \longrightarrow & A & \underset{a}{\rightarrowtail} & E & \underset{f}{\twoheadrightarrow} & X & \longrightarrow & 0
\end{array}$$

Given an abelian extension, we may take cokernels as in the diagram on the left to find its **direction**: the X-module (A, ξ).

The pullback $f^*(\xi)$ of ξ along f is the conjugation action of E on A.

The semi-abelian case: abelian extensions, II

- There are examples (e.g. in *Loop*) where A is abelian but f is not.
- The condition (SH) implies that all extensions with abelian kernel are abelian, because $[A, A]^{\mathrm{Q}} = 0$ implies that $[Eq(f), Eq(f)]^{\mathrm{S}}$ is trivial.

  In particular then, any internal action on an abelian group object is a Beck module. (Actions are non-abelian modules.)

Theorem (Cohomology with non-trivial coefficients)
$H^2(X, (A, \xi)) \cong OpExt^1(X, A, \xi)$, the group of equivalence classes of extensions from A to X with direction (A, ξ).
Under (SH), cohomology classifies all extensions with abelian kernel.

- By [Bourn & Janelidze, 2004], abelian extensions are *torsors*, which by [Duskin, 1975] [Glenn, 1982] are classified by means of comonadic cohomology [Barr & Beck, 1969].
- $H^2(-, (A, \xi))$ is a derived functor of $Hom(-,\ A \rtimes_\xi X \to X) \colon (\mathscr{X} \downarrow X)^{\mathrm{op}} \to Ab$.
  We assume that $\mathscr{X}$ carries a comonad $\mathbb{G}$ whose projectives are the regular projectives.

Overview, $n = 1$

	Homology $H_2(X)$	Cohomology $H^2(X, (A, \xi))$	
		trivial action ξ	arbitrary action ξ
Gp	$\frac{R \wedge [F, F]}{[R, F]}$	$CentrExt^1(X, A)$	$OpExt^1(X, A, \xi)$
abelian categories	0	$Ext^1(X, A)$	
Barr-exact categories		$Tors^1[X, (A, \xi)]$	
semi-abelian categories	$\frac{R \wedge [F, F]}{[R, F]}$	$CentrExt^1(X, A)$	$OpExt^1(X, A, \xi)$

Overview, arbitrary degrees ($n \geqslant 1$)

	Homology $H_{n+1}(X)$	Cohomology $H^{n+1}(X,(A,\xi))$	
		trivial action ξ	arbitrary action ξ
Gp	$\dfrac{\bigwedge_{i\in n} K_i \wedge [F_n, F_n]}{\bigvee_{I\subseteq n}[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\setminus I} K_i]}$	$CentrExt^n(X,A)$	$OpExt^n(X,A,\xi)$
abelian categories	0	$Ext^n(X,A)$	
Barr-exact categories		$Tors^n[X,(A,\xi)]$	
semi-abelian categories	$\dfrac{\bigwedge_{i\in n} K_i \wedge [F_n, F_n]}{L_n[F]}$	$CentrExt^n(X,A)$	$OpExt^n(X,A,\xi)$

Yoneda's extensions

Let X and A be objects in an abelian category $\mathscr{A}$.
A **Yoneda 1-extension** from A to X is a short exact sequence

$$0 \longrightarrow A \rightarrowtail E^1 \xrightarrow{f^1} X \longrightarrow 0.$$

Consider $n \geqslant 2$. A **Yoneda n-extension** from A to X is an exact sequence

$$0 \longrightarrow A \rightarrowtail E^n \xrightarrow{f^n} E^{n-1} \longrightarrow \cdots \xrightarrow{f^1} X \longrightarrow 0.$$

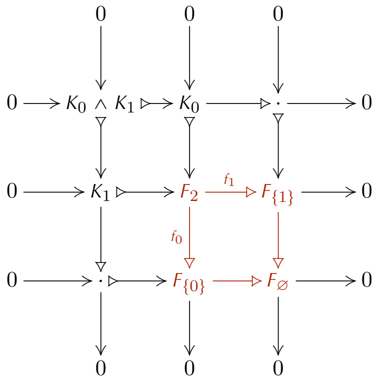
Taking commutative ladders between those as morphisms gives a category $EXT^n(X, A)$.
Its set/abelian group of connected components is denoted $Ext^n(X, A)$.

Theorem [Yoneda, 1960]
If $\mathscr{A}$ has enough projectives, then for $n \geqslant 1$ we have $H^{n+1}(X, A) \cong Ext^n(X, A)$.

- The cohomology on the left is a derived functor of $Hom(-, A)\colon \mathscr{A}^{\mathrm{op}} \to Ab$.

How to extend this to semi-abelian categories?

Non-abelian higher extensions: 3^n-diagrams



A **double extension** is a 3×3 diagram.
Its rows and columns are short exact sequences.

The red square $F \in Arr^2(\mathscr{X})$ which determines it is a **regular pushout**: its arrows and the comparison $F_2 \to F_{\{0\}} \times_{F_\varnothing} F_{\{1\}}$ are regular epimorphisms.

F is usually considered as a functor $\mathscr{P}(2)^{\mathrm{op}} \to \mathscr{X}$.

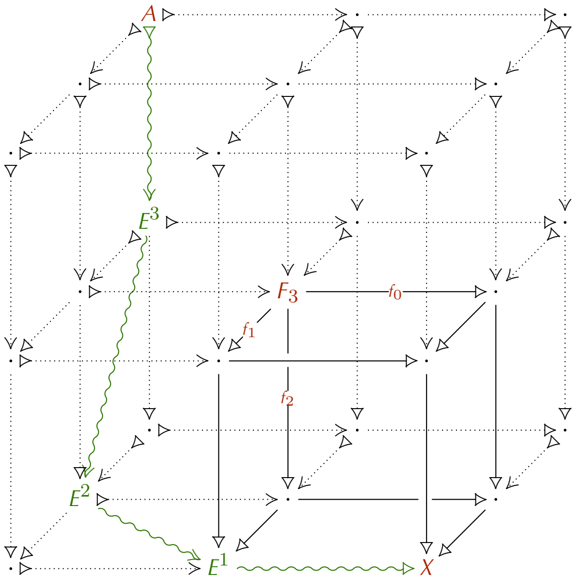
An *n*-**fold extension** is a 3^n-diagram.

It is determined by an n-fold arrow $F \in Arr^n(\mathscr{X})$, an n-cube viewed as a functor $\mathscr{P}(n)^{\mathrm{op}} \to \mathscr{X}$.

Example: the n-truncation of any aspherical augmented simplicial object (in particular, any simplicial **resolution**) determines an $(n+1)$-fold extension (**presentation**).
In fact, the extension property characterises being aspherical [Everaert, Goedecke & VdL, 2012].

In the abelian case, Yoneda *n*-extensions are equivalent to *n*-fold extensions (by Dold-Kan).

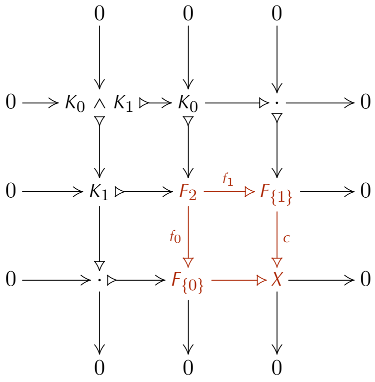
Abelian case: 3-fold extension vs. Yoneda 3-extension



Overview, arbitrary degrees ($n \geqslant 1$)

	Homology $H_{n+1}(X)$	Cohomology $H^{n+1}(X,(A,\xi))$	
		trivial action ξ	arbitrary action ξ
Gp	$\dfrac{\bigwedge_{i\in n} K_i \wedge [F_n, F_n]}{\bigvee_{I\subseteq n}[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\setminus I} K_i]}$	$CentrExt^n(X,A)$	$OpExt^n(X,A,\xi)$
abelian categories	0	$Ext^n(X,A)$	
Barr-exact categories		$Tors^n[X,(A,\xi)]$	
semi-abelian categories	$\dfrac{\bigwedge_{i\in n} K_i \wedge [F_n, F_n]}{L_n[F]}$	$CentrExt^n(X,A)$	$OpExt^n(X,A,\xi)$

What is a double *central* extension?



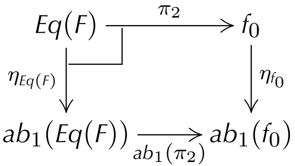
This question was answered in [Janelidze, 1991].

Theorem

Given a double extension of groups as on the left, $F \in Arr^2(Gp)$, viewed as an arrow $f_0 \to c$, is central with respect to the adjunction

$$Ext(Gp) \underset{\supset}{\overset{ab_1}{\rightleftarrows}} CExt(Gp) \quad (\bot)$$

iff the square on the right is a pullback



if and only if $[K_0, K_1] = 0 = [K_0 \wedge K_1, F_2]$.

- $[K_0 \wedge K_1, F_2] = 0$ means that the comparison $F_2 \to F_{\{0\}} \times_X F_{\{1\}}$ is a central extension.
- $[K_0, K_1] = 0$ iff the span (f_0, f_1) is a pregroupoid in $(Gp \downarrow X)$, since (SH) holds in Gp.
- Valid in (SH) semi-abelian categories. [Everaert, Gran & VdL, 2008] [Rodelo & VdL, 2010]

Repeating this construction gives a definition of *n-fold central extensions* for all *n*.

The higher homology objects

Categorical Galois theory says when an $(n+1)$-extension F is **central**: this happens if, considered as an arrow between n-fold extensions $F\colon D \to C$, it is central with respect to the adjunction

$$\mathit{Ext}^n(\mathscr{X}) \underset{\supset}{\overset{ab_n}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathit{CExt}^n(\mathscr{X}).$$

$$\begin{array}{ccc} Eq(F) & \xrightarrow{\ \pi_2\ } & D \\ {\scriptstyle \eta_{Eq(F)}}\downarrow & \lrcorner & \downarrow{\scriptstyle \eta_D} \\ ab_n(Eq(F)) & \xrightarrow[ab_n(\pi_2)]{} & ab_n(D) \end{array}$$

Theorem [Everaert, Gran & VdL, 2008]

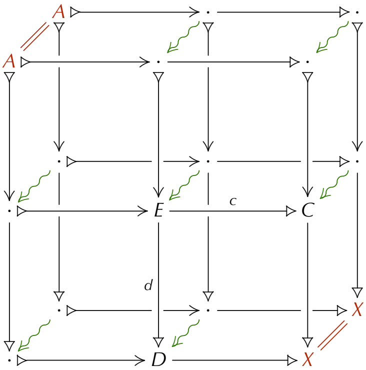
The derived functors of $ab\colon \mathscr{X} \to Ab(\mathscr{X})$ are $H_{n+1}(X, ab) \cong \dfrac{\bigwedge_{i\in n} K_i \wedge [F_n, F_n]}{L_n[F]}$.

- F is an *n-fold projective presentation*; its "initial maps" $f_i\colon F_n \to F_{n\setminus\{i\}}$ have kernel K_i.
- The object $L_n[F]$ is what must be divided out of F_n to make F central.
- By [Rodelo & VdL, 2012], under (SH), the object $L_n[F]$ is a join $\bigvee_{I\subseteq n}\Big[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\setminus I} K_i\Big]$ as in [Brown & Ellis, 1988] [Donadze, Inassaridze & Porter, 2005].
- In fact, the Hopf formula is valid for any Birkhoff reflector $I\colon \mathscr{X} \to \mathscr{Y}$.
- Alternatively, $H_{n+1}(X, I) \cong \lim(\mathit{CExt}^n_{I,X}(\mathscr{X}) \to \mathscr{Y}\colon F \mapsto \bigwedge_{i\in n} K_i)$.
  [Goedecke & VdL, 2009]

Overview, arbitrary degrees ($n \geqslant 1$)

	Homology $H_{n+1}(X)$	Cohomology $H^{n+1}(X,(A,\xi))$ trivial action ξ	arbitrary action ξ
Gp	$\dfrac{\bigwedge_{i\in n} K_i \wedge [F_n, F_n]}{\bigvee_{I\subseteq n}[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\setminus I} K_i]}$	$CentrExt^n(X,A)$	$OpExt^n(X,A,\xi)$
abelian categories	0	$Ext^n(X,A)$	
Barr-exact categories		$Tors^n[X,(A,\xi)]$	
semi-abelian categories	$\dfrac{\bigwedge_{i\in n} K_i \wedge [F_n, F_n]}{L_n[F]}$	$CentrExt^n(X,A)$	$OpExt^n(X,A,\xi)$

Cohomology classifies higher central extensions



End of 2008, with Diana Rodelo we proved that cohomology in the sense of [Bourn & Rodelo, 2007] [Rodelo, 2009] classifies double central extensions.

Defining a category with maps as on the left, its set/abelian group of connected components $CentrExt^2(X, A)$ is isomorphic to $H^3_{\mathrm{BR}}(X, A)$.

Indeed any pregroupoid over X is connected to a groupoid over X with the same direction A: pull back $\langle d, c \rangle$ along $d \times_X c \colon E \times_X E \to D \times_X C$.

We failed to prove $H^{n+1}_{\mathrm{BR}}(X, A) \cong CentrExt^n(X, A)$.

Instead, we used Duskin and Glenn's interpretation of comonadic cohomology [Barr & Beck, 1969] in terms of ***higher torsors*** [Duskin, 1975] [Glenn, 1982] to show for $n \geqslant 2$

Theorem [Rodelo & VdL, 2016]

$H^{n+1}(X, A) \cong CentrExt^n(X, A)$ if X is an object, and A an abelian object, in any semi-abelian variety **that satisfies** (SH).

Higher torsors: Duskin and Glenn's interpretation of cohomology

Theorem [Duskin, 1975] [Glenn, 1982]

Let $\mathscr{X}$ be Barr-exact and $\mathbb{G}$ a comonad on $\mathscr{X}$ where ($\mathbb{G}$-projectives = regular projectives). For any X in $\mathscr{X}$ and any X-module (A,ξ), the cotriple cohomology $H^{n+1}_{\mathbb{G}}(X,(A,\xi))$ is

$$H^n Hom_{(\mathscr{X}\downarrow X)}\big(\mathbb{G}(1_X),\ A \rtimes_\xi X \rightleftarrows X\big)$$
$$\cong \pi_0 Tors^n(X,(A,\xi))$$
$$=: Tors^n[X,(A,\xi)]$$

- $Tors^n(X,(A,\xi))$ denotes the category of *torsors* over $\mathbb{K}((A,\xi),n)$ in $(\mathscr{X}\downarrow X)$.
- $\mathbb{K}((A,\xi),n)$ is determined by
$$(A,\xi)^{n+1}\rtimes X \underset{\pi_0\rtimes 1_X}{\overset{\partial_{n+1}\rtimes 1_X,\ \pi_n\rtimes 1_X}{\rightrightarrows\!\vdots}} (A,\xi)\rtimes X \underset{f_\xi}{\overset{f_\xi}{\rightrightarrows\!\vdots}} X \ \overset{}{\underset{}{\equiv\!\vdots\!\equiv}}\ X \quad \cdots \quad X = X$$
where $\partial_{n+1} = (-1)^n \sum_{i=0}^{n} (-1)^i \pi_i$.
- An augmented simplicial morphism $\mathbb{t}\colon \mathbb{T} \to \mathbb{K}((A,\xi),n)$ is called a **torsor** when
  - (T1) $\mathbb{t}$ is a fibration which is exact from degree n on;
  - (T2) $\mathbb{T} \cong \mathrm{Cosk}_{n-1}(\mathbb{T})$;
  - (T3) $\mathbb{T}$ is aspherical.

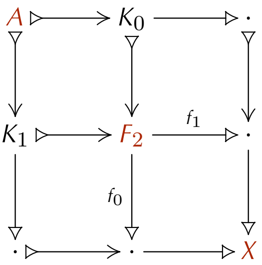
If (A,ξ) is a trivial X-module in a semi-abelian category with (SH), then (1) any torsor, viewed as an n-extension, is central; and (2) every class in $CentrExt^n(X,A)$ contains a torsor.

Overview, arbitrary degrees ($n \geqslant 1$)

	Homology $H_{n+1}(X)$	Cohomology $H^{n+1}(X,(A,\xi))$	
		trivial action ξ	arbitrary action ξ
Gp	$\dfrac{\bigwedge_{i\in n} K_i \wedge [F_n, F_n]}{\bigvee_{I\subseteq n}[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\setminus I} K_i]}$	$CentrExt^n(X,A)$	$OpExt^n(X,A,\xi)$
abelian categories	0	$Ext^n(X,A)$	
Barr-exact categories		$Tors^n[X,(A,\xi)]$	
semi-abelian categories	$\dfrac{\bigwedge_{i\in n} K_i \wedge [F_n, F_n]}{L_n[F]}$	$CentrExt^n(X,A)$	$OpExt^n(X,A,\xi)$

Non-trivial coefficients

[Peschke, Simeu & VdL, work-in-progress]



If (A, ξ) is a trivial X-module, then an n-extension from A to X is connected to a torsor over $\mathbb{K}((A, \xi), n)$ iff it is central.

- When $n = 2$ this means that $[K_0, K_1] = 0 = [A, F_2]$.

The case of non-trivial coefficients is much harder, because here the proof techniques by induction of **categorical Galois theory** are no longer available.

Question: When is an n-extension connected to an (A, ξ)-torsor?
Answer: When it is an n-pregroupoid with direction (A, ξ).

An n-extension is in a class in $OpExt^n(X, A, \xi)$ iff it satisfies the following two conditions:

n-pregroupoid condition

An n-fold analogue $[Eq(f_0), \ldots, Eq(f_{n-1})]^S$ of the Smith commutator of the $Eq(f_i)$ is trivial $\rightsquigarrow$ *higher-order Mal'tsev operation*

Is it $\bigvee_{\varnothing \neq I \subsetneq n} [\bigwedge_{i \in I} K_i, \bigwedge_{i \in n \setminus I} K_i] = 0$?

direction is (A, ξ)

The pullback $(F_n \to X)^*(\xi)$ of ξ is the conjugation action of F_n on A.

Under (SH), any n-extension from A to X has a direction which is an X-module (A, ξ).

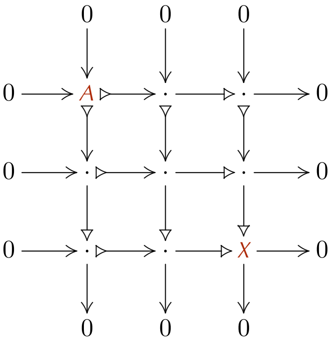
Some final remarks

- For a complete picture of cohomology with non-trivial coefficients, mainly certain aspects of commutator theory need to be further developed: in particular, higher Smith commutators, and their decomposition into (potentially non-binary) Higgins commutators.

  It seems here something stronger than (SH) may be needed.
- Results in group theory/non-abelian algebra may only extend to the semi-abelian context when certain additional conditions are satisfied.

  We made heavy use of the condition (SH), but a whole hierarchy of categorical-algebraic conditions has been introduced and studied over the last few years: some examples are (local) algebraic cartesian closedness, action representability, action accessibility, algebraic coherence, strong protomodularity, normality of Higgins commutators.
- These categorical conditions may help us understand algebra from a new perspective. For instance, they might lead to a categorical characterisation of Gp, $\mathsf{Lie}_{\mathbb{K}}$, etc.

Coda



Higher central extensions play "dual" roles in the interpretation of homology and cohomology (with trivial coefficients):

Homology $H_{n+1}(X)$: take the limit over the diagram of all n-fold central extensions over X of the functor which forgets to A.

Cohomology $H^{n+1}(X, A)$: take connected components of the category with maps of n-fold central extensions that keep A and X fixed.

The relationship between homology and cohomology *of groups* (with trivial coefficients) may be simplified by viewing it yet another way:

Theorem [Peschke & VdL, 2016]

If X is a group and $n \geqslant 1$, then $H_{n+1}(X) \cong Hom(H^{n+1}(X, -), 1_{Ab})$.

- This may also be shown via a *non-additive derived Yoneda lemma*.

Thank you!