CLOSED SETS OF REAL ROOTS IN KAC–MOODY ROOT SYSTEMS

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ABSTRACT. In this note, we provide a complete description of the closed sets of real roots in a Kac–Moody root system.

1. Introduction

Let $A = (a_{ij})_{i,j\in I}$ be a generalised Cartan matrix, and let $\mathfrak{g}(A)$ be the associated Kac–Moody algebra (see [Kac90]). Like (finite-dimensional) semisimple complex Lie algebras, $\mathfrak{g}(A)$ possesses a *root space decomposition*

$$\mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

with respect to the adjoint action of a *Cartan subalgebra* $\mathfrak{h}$, with associated set of *roots* $\Delta \subseteq \mathfrak{h}^*$ contained in the $\mathbf{Z}$-span of the set $\Pi = \{\alpha_i \mid i \in I\}$ of *simple roots*, as well as a *Weyl group* $\mathcal{W} \subseteq \mathrm{GL}(\mathfrak{h}^*)$ stabilising Δ. However, as soon as A is not a Cartan matrix (i.e. as soon as $\mathfrak{g}(A)$ is infinite-dimensional), the set $\Delta^{re} := \mathcal{W}.\Pi$ of *real roots* is properly contained in Δ. In some sense, the real roots of Δ are those that behave as the roots of a semisimple Lie algebra; in particular, $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta^{re}$.

A subset $\Psi \subseteq \Delta$ is *closed* if $\alpha + \beta \in \Psi$ whenever $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Delta$. Note that, denoting by $\mathfrak{g}_\Psi$ the subspace $\mathfrak{g}_\Psi := \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha$ of $\mathfrak{g}(A)$, a subset $\Psi \subseteq \Delta^{re}$ is closed if and only if $\mathfrak{h} \oplus \mathfrak{g}_\Psi$ is a subalgebra of $\mathfrak{g}(A)$; in particular, if $\Psi \subseteq \Delta^{re}$ is closed, the subalgebra generated by $\mathfrak{g}_\Psi$ is contained in $\mathfrak{h} \oplus \mathfrak{g}_\Psi$. Closed sets of real roots in Kac–Moody root systems thus arise naturally, and the purpose of this note is to provide a complete description of these sets. Our main theorem is as follows. For each $\alpha \in \Delta^{re}$, let $\alpha^\vee \in \mathfrak{h}$ be the *coroot* of α, i.e. the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ with $\langle \alpha, \alpha^\vee \rangle = 2$.

Main Theorem. *Let $\Psi \subseteq \Delta^{re}$ be a closed set of real roots and let $\mathfrak{g}$ be the subalgebra of $\mathfrak{g}(A)$ generated by $\mathfrak{g}_\Psi$. Set $\Psi_s := \{\alpha \in \Psi \mid -\alpha \in \Psi\}$ and $\Psi_n := \Psi \setminus \Psi_s$. Set also $\mathfrak{h}_s := \sum_{\gamma\in\Psi_s} \mathrm{C}\gamma^\vee$, $\mathfrak{g}_s := \mathfrak{h}_s \oplus \mathfrak{g}_{\Psi_s}$ and $\mathfrak{g}_n := \mathfrak{g}_{\Psi_n}$. Then*

(1) $\mathfrak{g}_s$ is a subalgebra and $\mathfrak{g}_n$ is an ideal of $\mathfrak{g}$. In particular, $\mathfrak{g} = \mathfrak{h}_s \oplus \mathfrak{g}_\Psi = \mathfrak{g}_s \ltimes \mathfrak{g}_n$.

(2) $\mathfrak{g}_n$ is nilpotent; it is the largest nilpotent ideal of $\mathfrak{g}$.

(3) $\mathfrak{g}_s$ is a semisimple finite-dimensional Lie algebra with Cartan subalgebra $\mathfrak{h}_s$ and set of roots Ψ_s.

Note that the possible closed root subsystems Ψ_s in the statement of the Main Theorem were explicitely determined in [KV18] when A is an affine GCM.

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2. Preliminaries

2.1. **Kac–Moody algebras.** The general reference for this section is [Kac90, Chapters 1–5].

Let $A = (a_{ij})_{i,j\in I}$ be a **generalised Cartan matrix** with indexing set I, and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realisation of A in the sense of [Kac90, §1.1], with set of **simple roots** $\Pi = \{\alpha_i \mid i \in I\}$ and set of **simple coroots** $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$. Let $\mathfrak{g}(A)$ be the corresponding **Kac–Moody algebra** (see [Kac90, §1.2-1.3]). Then $\mathfrak{g}(A)$ admits a root space decomposition

$$\mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

with respect to the adjoint action of the **Cartan subalgebra** $\mathfrak{h}$, with associated set of **roots** $\Delta \subseteq \mathfrak{h}^*$. Set $Q := \bigoplus_{i\in I} \mathbf{Z}\alpha_i \subseteq \mathfrak{h}^*$ and $Q_+ := \bigoplus_{i\in I} \mathbf{N}\alpha_i \subseteq Q$. Then $\Delta \subseteq Q_+ \cup -Q_+$, and we let $\Delta_+ := \Delta \cap Q_+$ (resp. $\Delta_- := \Delta \setminus \Delta_+ = -\Delta_+$) denote the set of **positive** (resp. **negative**) **roots**. The **height** of a root $\alpha = \sum_{i\in I} n_i\alpha_i \in Q$ is the integer $\mathrm{ht}(\alpha) := \sum_{i\in I} n_i$. Thus a root is positive if and only if it has positive height. We also introduce a partial order $\leq$ on Q defined by

$$\alpha \leq \beta \iff \beta - \alpha \in Q_+.$$

The **Weyl group** of $\mathfrak{g}(A)$ is the subgroup $\mathcal{W}$ of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the **fundamental reflections**

$$r_i = r_{\alpha_i} \colon \mathfrak{h}^* \to \mathfrak{h}^* : \alpha \mapsto \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i$$

for $i \in I$. Alternatively, $\mathcal{W}$ can be identified with the subgroup of $\mathrm{GL}(\mathfrak{h})$ generated by the reflections $r_i \colon \mathfrak{h} \to \mathfrak{h} : h \mapsto h - \langle \alpha_i, h \rangle \alpha_i^\vee$. Then $\mathcal{W}$ stabilises Δ, and we let $\Delta^{re} := \mathcal{W}.\Pi$ (resp. $\Delta_+^{re} := \Delta^{re} \cap \Delta_+$) denote the set of (resp. **positive**) **real roots**. For each $\alpha \in \Delta^{re}$, say $\alpha = w\alpha_i$ for some $w \in \mathcal{W}$ and $i \in I$, the element $\alpha^\vee := w\alpha_i^\vee$ only depends on α, and is called the **coroot** of α. One can then also define the **reflection** $r_\alpha \in \mathcal{W}$ associated to α as

$$r_\alpha = w r_i w^{-1} \colon \mathfrak{h}^* \to \mathfrak{h}^* : \beta \mapsto \beta - \langle \beta, \alpha^\vee \rangle \alpha.$$

Let $\Psi \subseteq \Delta$ be a subset of roots. We call Ψ **closed** if $\alpha + \beta \in \Psi$ whenever $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Delta$. The **closure** $\overline{\Psi}$ of Ψ is the smallest closed subset of Δ containing Ψ. A subset $\Psi' \subseteq \Psi$ is an **ideal** in Ψ if $\alpha + \beta \in \Psi'$ whenever $\alpha \in \Psi$, $\beta \in \Psi'$ and $\alpha + \beta \in \Delta$. The set Ψ is **prenilpotent** if there exist some $w, w' \in \mathcal{W}$ such that $w\Psi \subseteq \Delta_+$ and $w'\Psi \subseteq \Delta_-$; in that case, Ψ is finite and contained in Δ^{re}. If Ψ is both prenilpotent and closed, it is called **nilpotent**. We further call Ψ **pro-nilpotent** if it is a directed union of nilpotent subsets.

The above terminology is motivated by its Lie algebra counterpart: consider the subspace $\mathfrak{g}_\Psi := \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha$ for each $\Psi \subseteq \Delta$. If Ψ is closed, then $\mathfrak{h} \oplus \mathfrak{g}_\Psi$ is a subalgebra. If, moreover, $\Psi \cap -\Psi = \varnothing$ and Ψ' is an ideal in Ψ, then $\mathfrak{g}_{\Psi'}$ is an ideal in $\mathfrak{g}_\Psi$. If Ψ is nilpotent, then $\mathfrak{g}_\Psi$ is a nilpotent subalgebra. This remains valid for pro-nilpotent sets of roots.

Lemma 1. *Let $\Psi \subseteq \Delta$ be a pro-nilpotent set of roots. Then $\mathfrak{g}_\Psi$ is a nilpotent subalgebra.*

Proof. By definition, $\mathfrak{g}_\Psi$ is a directed union of nilpotent subalgebras $\mathfrak{g}_{\Psi_n}$ associated to nilpotent sets of roots $\Psi_n \subseteq \Psi$. Since there is a uniform bound on the nilpotency class of these subalgebras by [Cap07, Theorem 1.1], the claim follows. □

2.2. **Davis complexes.** The general reference for this section is [AB08] (see also [Nos11]).

The pair $(\mathcal{W}, S := \{r_i \mid i \in I\})$ is a Coxeter system, and we let $\Sigma = \Sigma(\mathcal{W}, S)$ denote the corresponding **Davis complex**. Thus Σ is a CAT(0) cell complex whose underlying 1-skeleton $\Sigma^{(1)}$ is the Cayley graph of $(\mathcal{W}, S)$. Moreover, the $\mathcal{W}$-action on Σ is by cellular isometries, is proper, and induces on $\Sigma^{(1)}$ the canonical $\mathcal{W}$-action on its Cayley graph. The vertices of $\Sigma^{(1)}$ (which we identify with the elements of $\mathcal{W}$) are called the **chambers** of Σ, and their set is denoted $\mathrm{Ch}(\Sigma)$. The vertex $1_{\mathcal{W}}$ is called the **fundamental chamber** of Σ, and is denoted C_0. Two chambers of Σ are **adjacent** if they are adjacent in the Cayley graph of $(\mathcal{W}, S)$. A **gallery** is a sequence $\Gamma = (x_0, x_1, \dots, x_d)$ of chambers such that x_{i-1} and x_i are distinct adjacent chambers for each $i = 1, \dots, d$. The integer $d \in \mathbf{N}$ is the **length** of Γ, and Γ is called **minimal** if it is a gallery of minimal length between x_0 and x_d. In that case, d is called the **chamber distance** between x_0 and x_d, denoted $\mathrm{d}_{\mathrm{Ch}}(x_0, x_d)$. If X is a nonempty subset of $\mathrm{Ch}(\Sigma)$ and $x \in \mathrm{Ch}(\Sigma)$, we also set $\mathrm{d}_{\mathrm{Ch}}(x, X) := \min_{x' \in X} \mathrm{d}_{\mathrm{Ch}}(x, x')$.

Given two distinct adjacent chambers $x, x' \in \mathrm{Ch}(\Sigma)$, any chamber of Σ is either closer to x or to x' (for the chamber distance). This yields a partition of $\mathrm{Ch}(\Sigma)$ into two subsets, which are the underlying chamber sets of two closed convex subcomplexes of Σ, called **half-spaces**. If $x = C_0$ and $x' = sC_0$ for some $s \in S$, we let $H(\alpha_s)$ denote the corresponding half-space containing C_0 and $H(-\alpha_s)$ the other half-space. If $x = wC_0$ and $x' = wsC_0$ ($w \in \mathcal{W}$ and $s \in S$), the corresponding half-spaces $H(\pm\alpha) = wH(\pm\alpha_s)$ depend only on $\alpha = w\alpha_s$. The map

$$\Delta^{re} \to \{\text{half-spaces of } \Sigma\} : \alpha \mapsto H(\alpha)$$

is a $\mathcal{W}$-equivariant bijection mapping Δ^{re}_+ onto the set of half-spaces of Σ containing C_0. The **wall** associated to $\alpha \in \Delta^{re}$ is the closed convex subset $\partial\alpha := H(\alpha) \cap H(-\alpha)$ of Σ, and coincides with the fixed-point set of the reflection $r_\alpha \in \mathcal{W}$. A wall $\partial\alpha$ **separates** two chambers $x, x' \in \mathrm{Ch}(\Sigma)$ if $x \in H(\epsilon\alpha)$ and $x' \in H(-\epsilon\alpha)$ for some $\epsilon \in \{\pm 1\}$.

We conclude this preliminary section with a short dictionary between roots and half-spaces. Call two roots $\alpha, \beta \in \Delta^{re}$ **nested** if either $H(\alpha) \subseteq H(\beta)$ or $H(\beta) \subseteq H(\alpha)$.

Lemma 2. *Let $\alpha, \beta \in \Delta^{re}$ with $\alpha \neq \pm\beta$, and set $m := \langle \alpha, \beta^\vee \rangle$ and $n := \langle \beta, \alpha^\vee \rangle$.*

(1) $\{\alpha, \beta\}$ is not prenilpotent $\iff$ $\{\alpha, -\beta\}$ is nested $\iff$ $m, n < 0$ and $mn \geq 4$ $\iff$ $\overline{\{\alpha, \beta\}} \not\subseteq \Delta^{re}$.

(2) $\langle r_\alpha, r_\beta \rangle \subseteq \mathcal{W}$ is finite $\iff$ $\partial\alpha \cap \partial\beta \neq \varnothing$ $\iff$ $\{\pm\alpha, \pm\beta\}$ does not contain any nested pair $\iff$ $\overline{\{\pm\alpha, \pm\beta\}}$ is a root system of type $A_1 \times A_1$, A_2, B_2 or G_2.

(3) A subset $\Psi \subseteq \Delta^{re}$ of roots is prenilpotent if and only if the subcomplexes $\bigcap_{\alpha \in \Psi} H(\alpha)$ and $\bigcap_{\alpha \in \Psi} H(-\alpha)$ of Σ both contain at least one chamber.

Proof. (1) The first equivalence is [AB08, Lemma 8.42(3)], and the second and third equivalences follow from [Mar13, Proposition 4.31(3)] (see also [Mar18, Exercise 7.42]).

(2) For the first equivalence, note that $\langle r_\alpha, r_\beta\rangle$ fixes $\partial\alpha\cap\partial\beta$. Hence if $\partial\alpha\cap\partial\beta\neq\varnothing$, then $\langle r_\alpha, r_\beta\rangle$ is finite (because the $\mathcal{W}$-action on Σ is proper, hence has finite point stabilisers). Conversely, if $\partial\alpha\cap\partial\beta=\varnothing$, then r_α, r_β generate an infinite dihedral group.

For the second equivalence, note that since half-spaces are convex, they are in particular arc-connected, and hence if $\partial\alpha\cap\partial\beta=\varnothing$ then either $\{\alpha,\beta\}$ or $\{\alpha,-\beta\}$ is nested. Conversely, if $\{\pm\alpha,\pm\beta\}$ contains a nested pair, then $\langle r_\alpha, r_\beta\rangle\subseteq\mathcal{W}$ is an infinite dihedral group.

Finally, for the third equivalence, note that $\langle r_\alpha, r_\beta\rangle$ stabilises $R:=\overline{\{\pm\alpha,\pm\beta\}}$, and that $\langle r_\alpha, r_\beta\rangle$ is finite if and only if R is finite. In this case, R is a (reduced) root system of rank 2 in the sense of [Bou68, VI, §1.1], and hence of one of the types $A_1\times A_1$, A_2, B_2 or G_2 by [Bou68, VI, §4.2 Théorème 3], as desired.

(3) Ψ is prenilpotent if and only if there exist $v,w\in\mathcal{W}$ such that $v\Psi\subseteq\Delta^{re}_+$ (i.e. $v^{-1}C_0\in\bigcap_{\alpha\in\Psi}H(\alpha)$) and $w\Psi\subseteq\Delta^{re}_-$ (i.e. $w^{-1}C_0\in\bigcap_{\alpha\in\Psi}H(-\alpha)$). □

3. Closed and prenilpotent sets of roots

We shall need the following lemma, which is a slight variation of [Cap06, Lemma 12].

Lemma 3. *Let $x\in\mathrm{Ch}(\Sigma)$ be a chamber and $\alpha\in\Delta^{re}$ be a root such that $x\notin H(\alpha)$. Let $y\in\mathrm{Ch}(\Sigma)$ be a chamber contained in $H(\alpha)$ and at minimal distance from x, and let $\beta\in\Delta^{re}$ be a root such that $H(\beta)$ contains x but not y, and that $H(-\beta)$ contains a chamber adjacent to x. Assume that $\beta\neq-\alpha$. Then:*

(i) $\langle\alpha,\beta^\vee\rangle<0$.
(ii) $x\notin H(r_\beta(\alpha))$.

Proof. For the first assertion, we invoke [Cap06, Lemma 12] that we apply to the roots $\phi:=\alpha$ and $\psi:=\beta$. We deduce that $r_\alpha(\beta)\neq\beta$ and that $H(r_\alpha(\beta))\supseteq H(\alpha)\cap H(\beta)$. If $\langle r_\alpha, r_\beta\rangle$ is infinite, then $\partial\alpha\cap\partial\beta=\varnothing$, and since $\{\alpha,\beta\}$ is not nested by construction, the pair $\{\alpha,-\beta\}$ is nested and $\langle\alpha,\beta^\vee\rangle<0$ (see Lemma 2(1)). If $\langle r_\alpha, r_\beta\rangle$ is finite, on the other hand, the root system generated by α and β is an irreducible root system of rank 2 (see Lemma 2(2)). A quick case-by-case inspection of the rank 2 root systems of type A_2, B_2 and G_2 (see [Hum78, §9.3]) reveals that the condition $H(r_\alpha(\beta))\supseteq H(\alpha)\cap H(\beta)$ implies that the angle between the walls of α and β (in the Euclidean plane spanned by α,β and with scalar product $(\cdot|\cdot)$ such that $\langle\alpha,\beta^\vee\rangle=2(\alpha|\beta)/(\alpha|\alpha)$, see [Hum78, §9.1]) is acute. Thus the angle between the corresponding roots is obtuse, hence $\langle\alpha,\beta^\vee\rangle<0$. This proves (i).

For the second assertion, let $x'\in\mathrm{Ch}(\Sigma)$ be the unique chamber adjacent to x and contained in $H(-\beta)$. If $x\in H(r_\beta(\alpha))$, then $x'=r_\beta(x)\in r_\beta(H(r_\beta(\alpha)))=H(\alpha)$. Since $x\notin H(\alpha)$ by hypothesis, it follows that the wall $\partial\alpha$ separates x from x'. Since x and x' are adjacent, we deduce that $\beta=-\alpha$, contradicting the hypotheses. This proves (ii). □

Given a set of roots $\Phi\subseteq\Delta$ and a subset $\Psi\subseteq\Phi$, we set

$$\Phi_{\geq\Psi}:=\{\phi\in\Phi\mid\phi\geq\psi\text{ for some }\psi\in\Psi\}.$$

Lemma 4. *Let $\Phi \subseteq \Delta^{re}$ be a closed set of roots and let $\Psi \subseteq \Phi$ be a non-empty finite subset. Assume that $\Phi \cap -\Phi = \varnothing$. Then $\bigcap_{\phi \in \Phi_{\geq \Psi}} H(\phi)$ contains a chamber. In particular, $\bigcap_{\psi \in \Psi} H(\psi)$ contains a chamber.*

Proof. For simplicity, we identify each half-space $H(\phi)$ ($\phi \in \Delta^{re}$) with its underlying set of chambers; we thus have to show that $\bigcap_{\phi \in \Phi_{\geq \Psi}} H(\phi) \neq \varnothing$. For each set of roots $B \subseteq \Delta$, we set $B_\varepsilon = B \cap \Delta_\varepsilon$ for $\varepsilon \in \{+, -\}$.

Set $A = \Phi_{\geq \Psi}$. Observe that the set A_- is finite, since Ψ is finite by hypothesis. We shall proceed by induction on $|A_-|$.

In the base case $|A_-| = 0$, we have $A \subseteq \Delta_+^{re}$, and each half-space $H(\phi)$ with $\phi \in A$ contains the fundamental chamber C_0. Thus $\bigcap_{\phi \in A} H(\phi)$ is nonempty in this case.

We assume henceforth that A_- is non-empty. We set $\mathcal{X} = \bigcap_{\phi \in \Phi_+} H(\phi)$. In the special case where Φ_+ is empty, we adopt the convention that $\mathcal{X} = \mathrm{Ch}(\Sigma)$. In all cases, we observe that $\mathcal{X}$ is non-empty since it contains C_0. We now distinguish two cases.

Assume first that there exists $\psi \in A_-$ such that $\mathcal{X}$ is not entirely contained in the half-space $H(-\psi)$. We may then choose a chamber $C \in \mathcal{X} \cap H(\psi)$. Let $w \in W$ be such that $wC = C_0$. For each $\phi \in \Phi_+$, we have $C \in \mathcal{X} \subseteq H(\phi)$, hence $C_0 \in H(w\phi)$. Therefore $w\Phi_+ \subseteq \Delta_+$, and in particular $wA_+ \subseteq \Delta_+$. Moreover $wA \cap \Delta_+$ contains $w\psi$, since $C \in H(\psi)$. It follows that $|wA \cap \Delta_-| = |wA_- \cap \Delta_-| < |wA_-| = |A_-|$. Set

$$B = wA = w(\Phi_{\geq \Psi}) = w(\Phi)_{\geq w(\Psi)}.$$

We have just seen that $|B_-| < |A_-|$. Hence the induction hypothesis ensures that $\bigcap_{\phi \in B} H(\phi) \neq \varnothing$. Therefore, $\bigcap_{\phi \in A} H(\phi) = w^{-1}\Big(\bigcap_{\phi \in B} H(\phi)\Big)$ is also non-empty, and we are done in this case.

Assume finally that for all $\psi \in A_-$, we have $\mathcal{X} \subseteq H(-\psi)$. Choose $x \in \mathcal{X}$ and $\alpha \in A_-$ such that $\mathrm{d}_{\mathrm{Ch}}(x, H(\alpha))$ is minimal. Choose also $y \in H(\alpha)$ such that $\mathrm{d}_{\mathrm{Ch}}(x, y) = \mathrm{d}_{\mathrm{Ch}}(x, H(\alpha))$. Let $x = x_0, x_1, \dots, x_m = y$ be a minimal gallery from x to y, and let $\beta \in \Delta^{re}$ be the unique root such that $H(\beta)$ contains x_0 but not x_1.

We have $\beta \in \Phi_+$, since otherwise we would have $x_1 \in H(\phi)$ for all $\phi \in \Phi_+$, so that $x_1 \in \mathcal{X}$, contradicting the minimality condition in the definition of x. In particular, we have $\alpha \neq -\beta$ since $\Phi \cap -\Phi = \varnothing$.

Notice that x is a chamber contained in $H(\beta)$ but not in $H(\alpha)$, whereas y is a chamber contained in $H(\alpha)$ but not $H(\beta)$. We now invoke Lemma 3. It follows that $\langle \alpha, \beta^\vee \rangle < 0$ and that $x \notin H(r_\beta(\alpha))$. Since β is a positive root, it follows that $r_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta \geq \alpha \in A = \Phi_{\geq \Psi}$. Moreover $r_\beta(\alpha) \in \overline{\{\alpha, \beta\}} \subseteq \Phi$. Hence $r_\beta(\alpha) \in A$. In particular, $r_\beta(\alpha) \in \Delta_-$, for otherwise $x \in \mathcal{X} \subseteq H(r_\beta(\alpha))$, a contradiction. We conclude that $r_\beta(\alpha) \in A_-$. Now we observe that the gallery $x = x_0 = r_\beta(x_1), r_\beta(x_2), \dots, r_\beta(x_m) = r_\beta(y)$ is of length $m - 1$ and joins x to a chamber in $H(r_\beta(\alpha))$. Since $r_\beta(\alpha)$ belongs to A_-, this contradicts the minimality condition in the definition of α. Thus this final case does not occur, and the proof is complete. $\square$

4. Nilpotent sets

Lemma 5. *A subset $\Phi \subseteq \Delta^{re}$ is nilpotent if and only if it satisfies the following three conditions:*

(i) Φ *is closed.*
(ii) Φ *is finite.*
(iii) $\Phi \cap -\Phi = \varnothing$.

Proof. Assume that $\Phi \subseteq \Delta^{re}$ satisfies the three conditions (i)–(iii). Applying Lemma 4 with $\Phi = \Psi$, we deduce that $\bigcap_{\phi\in\Phi} H(\phi)$ contains a chamber. Similarly, Lemma 4 applied to $-\Phi$ implies that $\bigcap_{\phi\in\Phi} H(-\phi)$ also contains a chamber. Hence Lemma 2(3) implies that Φ is nilpotent, as desired.

The converse assertion is clear by the definition of a nilpotent set of roots. $\square$

5. On the closure of a finite set

Lemma 6. *Let $\Phi \subseteq \Delta^{re}$ be a closed set of roots such that $\Phi \cap -\Phi = \varnothing$. For any finite subset $\Psi \subseteq \Phi$, the closure $\overline{\Psi}$ is finite.*

Proof. Applying Lemma 4 to Ψ and $-\Psi$, we deduce that the intersections $\bigcap_{\psi\in\Psi} H(\psi)$ and $\bigcap_{\psi\in\Psi} H(-\psi)$ both contain a chamber. Hence Ψ is prenilpotent by Lemma 2(3). The closure of any prenilpotent set of real roots is nilpotent, hence finite by Lemma 5. $\square$

6. Pro-nilpotent sets

Proposition 7. *Let $\Phi \subseteq \Delta^{re}$ be a closed set of real roots such that $\Phi \cap -\Phi = \varnothing$. Then Φ is pro-nilpotent.*

Proof. Let n be a positive integer and let $\Phi_{\leq n}$ denote the subset of those $\phi \in \Phi$ with $|\operatorname{ht}(\phi)| \leq n$. Thus $\Phi_{\leq n}$ is finite for all n, and the sets $\Phi_{\leq n}$ are linearly ordered by inclusion. By Lemma 6, the closure $\overline{\Phi_{\leq n}}$ is finite, hence nilpotent by Lemma 5. Thus Φ is the union of an ascending chain of nilpotent subsets. $\square$

7. Levi decomposition

Lemma 8. *Let $\Psi \subseteq \Delta^{re}$ be a closed set of real roots. Set $\Psi_s := \{\alpha \in \Psi \mid -\alpha \in \Psi\}$ and $\Psi_n := \Psi \setminus \Psi_s$. Then Ψ_s is closed and Ψ_n is an ideal in Ψ.*

Proof. If $\alpha, \beta \in \Psi_s$ and $\alpha + \beta \in \Delta$, then $-\alpha, -\beta \in \Psi_s$ and hence $-(\alpha+\beta) \in \Psi$, that is, $\alpha + \beta \in \Psi_s$. This shows that Ψ_s is closed.

If now $\alpha \in \Psi$ and $\beta \in \Psi_n$ are such that $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Psi_n$. Otherwise, $-(\alpha+\beta) \in \Psi$, and hence $-\beta = -(\alpha+\beta) + \alpha \in \Psi$, contradicting the fact that $\beta \in \Psi_n$. This shows that Ψ_n is an ideal in Ψ. $\square$

Lemma 9. *Let $\Psi \subseteq \Delta^{re}$ be a closed set of real roots such that $\Psi = -\Psi$. Then Ψ is the root system of a finite-dimensional semisimple complex Lie algebra*

Proof. Let H be the reflection subgroup of $\mathcal{W}$ generated by $R_\Psi := \{r_\alpha \mid \alpha \in \Psi\}$. Then H is a Coxeter group in its own right (see [Deo89]). Moreover, the set M_Ψ of walls of $\Sigma(\mathcal{W}, S)$ corresponding to reflections in R_Ψ is stabilised by H. Indeed, if $\alpha, \beta \in \Psi$, we have to check that $r_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha \in \Psi$. But as $\pm\alpha \in \Psi$, this follows from the fact that Ψ is closed. In particular, H is finite, for otherwise it would contain two reflections $r_\alpha, r_\beta \in R_\Psi$ generating an infinite dihedral group by [Hée93, Prop. 8.1, p. 309], and hence Ψ would contain a non-prenilpotent pair of roots by Lemma 2(1,2), contradicting the fact that $\Psi \subseteq \Delta^{re}$ (see Lemma 2(1)). This shows that Ψ is a finite (reduced) root system in the sense of [Bou68, VI, §1.1], hence the root system of a finite-dimensional semisimple complex Lie algebra (see [Bou68, VI, §4.2 Théorème 3]), as desired. □

Theorem 10. *Let $\Psi \subseteq \Delta^{re}$ be a closed set of real roots and let $\mathfrak{g}$ be the subalgebra of $\mathfrak{g}(A)$ generated by $\mathfrak{g}_\Psi$. Set $\Psi_s := \{\alpha \in \Psi \mid -\alpha \in \Psi\}$ and $\Psi_n := \Psi \setminus \Psi_s$. Set also $\mathfrak{h}_s := \sum_{\gamma \in \Psi_s} \mathbb{C}\gamma^\vee$, $\mathfrak{g}_s := \mathfrak{h}_s \oplus \mathfrak{g}_{\Psi_s}$ and $\mathfrak{g}_n := \mathfrak{g}_{\Psi_n}$. Then*

(1) $\mathfrak{g}_s$ is a subalgebra and $\mathfrak{g}_n$ is an ideal of $\mathfrak{g}$. In particular, $\mathfrak{g} = \mathfrak{g}_s \ltimes \mathfrak{g}_n$.
(2) $\mathfrak{g}_n$ is the unique maximal nilpotent ideal of $\mathfrak{g}$.
(3) $\mathfrak{g}_s$ is a semisimple finite-dimensional Lie algebra with Cartan subalgebra $\mathfrak{h}_s$ and set of roots Ψ_s.

Proof. (1) follows from Lemma 8 and (3) from Lemma 9. For (2), note that Ψ_n is a pronilpotent set of roots by Proposition 7, and hence $\mathfrak{g}_n$ is nilpotent by Lemma 1. Finally, since $\mathfrak{g}_s$ is semisimple, the image of any nilpotent ideal $\mathfrak{i}$ of $\mathfrak{g}$ under the quotient map $\mathfrak{g} \to \mathfrak{g}_s$ must be zero, that is, $\mathfrak{i} \subseteq \mathfrak{g}_n$. Therefore, $\mathfrak{g}_n$ is the unique maximal nilpotent ideal of $\mathfrak{g}$, as desired. □

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