

OPEN SUBGROUPS OF LOCALLY COMPACT KAC–MOODY GROUPS

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ABSTRACT. Let G be a complete Kac–Moody group over a finite field. It is known that G possesses a BN-pair structure, all of whose parabolic subgroups are open in G . We show that, conversely, every open subgroup of G is contained with finite index in some parabolic subgroup; moreover there are only finitely many such parabolic subgroups. The proof uses some new results on parabolic closures in Coxeter groups. In particular, we give conditions ensuring that the parabolic closure of the product of two elements in a Coxeter group contains the respective parabolic closures of those elements.

1. INTRODUCTION

This paper is devoted to the study of open subgroups of complete Kac–Moody groups over finite fields. The interest in the structure of those groups is motivated by the fact that they constitute a prominent family of locally compact groups which are simultaneously *topologically simple* and *non-linear over any field* (see [Rém04] and [CR09]). They show some resemblance with the simple linear locally compact groups arising from semi-simple algebraic groups over local fields of positive characteristic.

The first question on open subgroups of a given locally compact group G one might ask is: How many such subgroups are there? Let us introduce some terminology providing possible answers to this question. We say that G **has few open subgroups** if every proper open subgroup of G is compact. We say that G is **Noetherian** if G satisfies an ascending chain condition on open subgroups. Equivalently G is Noetherian if and only if every open subgroup of G is compactly generated (see Lemma 3.22 below). Clearly, if G has few open subgroups, then it is Noetherian. Basic examples of locally compact groups that are Noetherian — and in fact, even have few open subgroups — are connected groups and compact groups. Noetherianity can thus be viewed as a finiteness condition which generalizes simultaneously the notion of connectedness and of compactness. It is highlighted in [CM11], where it is notably shown that a Noetherian group admits a subnormal series with every subquotient compact, or abelian, or simple. An example of a non-Noetherian group is given by the additive group \mathbf{Q}_p of the p -adics. Other examples, including simple ones, can be constructed as groups acting on trees.

According to a theorem of G. Prasad [Pra82] (which he attributes to Tits), simple locally compact groups arising from algebraic groups over local fields have few open subgroups. Locally compact Kac–Moody groups are however known to have a broader variety of open subgroups in general. Indeed, Kac–Moody groups

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are equipped with a BN -pair all of whose parabolic subgroups are open. In particular, if the Dynkin diagram of a Kac–Moody group admits proper subdiagrams that are not of spherical type, then the corresponding Kac–Moody groups have proper open subgroups that are not compact.

Our main result is that parabolic subgroups in Kac–Moody groups are essentially the only source of open subgroups.

Theorem A. *Every open subgroup of a complete Kac–Moody group G over a finite field has finite index in some parabolic subgroup.*

Moreover, given an open subgroup O , there are only finitely many distinct parabolic subgroups of G containing O as a finite index subgroup.

A more precise statement of this theorem will be given later, see Theorem 3.3. As a consequence, we deduce the following.

Corollary B. *Complete Kac–Moody groups over finite fields are Noetherian.*

In fact, Theorem A allows us to characterize those locally compact Kac–Moody groups having few open subgroups, as follows.

Corollary C. *Let G be a complete Kac–Moody group of irreducible type over a finite field. Then G has few open subgroups if and only if the Weyl group of G is of affine type, or of compact hyperbolic type.*

Notice that the list of all compact hyperbolic types of Weyl groups is finite and contains diagrams of rank at most 5 (see *e.g.* Exercise V.4.15 on p. 133 in [Bou68]). The groups in Corollary C include in particular all complete Kac–Moody groups of rank two.

Another application of Theorem A is that it shows how the BN -pair structure is encoded in the topological group structure of a Kac–Moody group. Here is a precise formulation of this.

Corollary D. *Let G be a complete Kac–Moody group over a finite field and $P < G$ be an open subgroup. If P is maximal in its commensurability class, then P is a parabolic subgroup of G .*

Our proof of Theorem A relies on some new results on parabolic closures in Coxeter groups, which we now proceed to describe. Let thus (W, S) be a Coxeter system with S finite. Recall that any intersection of parabolic subgroups in W is itself a parabolic subgroup. Following D. Krammer [Kra09], it thus makes sense to define the **parabolic closure** of a subset of W as the intersection of all parabolic subgroups containing it. The parabolic closure of a set $E \subseteq W$ is denoted by $\text{Pc}(E)$.

Theorem E. *Let $w \in W$ be an element of infinite order and let λ be a translation axis for w in the Davis complex. Assume that the parabolic closure $\text{Pc}(w)$ is of irreducible type.*

Then there is a constant C such that for any two parallel walls m, m' transverse to λ , if $d(m, m') > C$, then $\text{Pc}(w) = \text{Pc}(r_m, r_{m'})$.

In particular, we get the following.

Corollary F. *Any irreducible non-spherical parabolic subgroup of a Coxeter group is the parabolic closure of a pair of reflections.*

Our main result on Coxeter groups concerns the parabolic closure of the product of two elements.

Theorem G. *There is a finite index normal subgroup $W_0 < W$ enjoying the following property.*

For all $g, h \in W_0$, there exists a constant $K = K(g, h) \in \mathbf{N}$ such that for all $m, n \in \mathbf{Z}$ with $\min\{|m|, |n|, |m/n| + |n/m|\} \geq K$, we have $\text{Pc}(g^m h^n) \supseteq \text{Pc}(g) \cup \text{Pc}(h)$.

The following corollary is an essential ingredient in the proof of Theorem A.

Corollary H. *Let H be a subgroup of W . Then there exists $h \in H$ such that the parabolic closure of h has finite index in the parabolic closure of H .*

2. WALLS AND PARABOLIC CLOSURES IN COXETER GROUPS

Throughout this section, we let (W, S) be a Coxeter system with W finitely generated (equivalently S is finite). Let Σ be the associated Coxeter complex, and let $|\Sigma|$ denote its standard geometric realization. Also, let X be the Davis realization of Σ . Thus X is a CAT(0) subcomplex of the barycentric subdivision of $|\Sigma|$.

Let $\Phi = \Phi(\Sigma)$ denote the set of half-spaces of Σ . A half-space $\alpha \in \Phi$ will also be called a **root**. Given a root $\alpha \in \Phi$, we write $r_\alpha = r_{\partial\alpha}$ for the unique reflection of W fixing the wall $\partial\alpha$ of α pointwise.

We say that two walls m, m' of X are **parallel** if either they coincide or they are disjoint. We say that the walls m, m' are **perpendicular** if they are distinct and if the reflections r_m and $r_{m'}$ commute.

Finally, for a subset $J \subseteq S$, we set $J^\perp := \{s \in S \setminus J \mid sj = js \ \forall j \in J\}$.

In this paper, we call a subset $J \subseteq S$ **essential** if each irreducible component of J is non-spherical.

2.1. The normalizer of a parabolic subgroup.

Lemma 2.1. *Let $L \subseteq S$ be essential. Then $\mathcal{N}_W(W_L) = W_L \times \mathcal{Z}_W(W_L)$ and is again parabolic. Moreover, $\mathcal{Z}_W(W_L) = W_{L^\perp}$.*

Proof. See [Deo82, Proposition 5.5] and [Kra09, Chapter 3]. \square

2.2. Preliminaries on parabolic closures. A subgroup of W of the form W_J for some $J \subset S$ is called a **standard parabolic subgroup**. Any of its conjugates is called a **parabolic subgroup** of W . Since any intersection of parabolic subgroups is itself a parabolic subgroup (see [Tit74]), it makes sense to define the **parabolic closure** $\text{Pc}(E)$ of a subset $E \subset W$ as the smallest parabolic subgroup of W containing E . For $w \in W$, we will also write $\text{Pc}(w)$ instead of $\text{Pc}(\{w\})$.

Lemma 2.2. *Let G be a reflection subgroup of W , namely a subgroup of W generated by a set T of reflections. We have the following:*

- (i) *There is a set of reflections $R \subset G$, each conjugate to some element of T , such that (G, R) is a Coxeter system.*
- (ii) *If T has no nontrivial partition $T = T_1 \cup T_2$ such that $[T_1, T_2] = 1$, then (G, R) is irreducible.*
- (iii) *If (G, R) is irreducible (resp. spherical, affine of rank ≥ 3), then so is $\text{Pc}(G)$.*

- (iv) If G' is a reflection subgroup of irreducible type which centralizes G and if G is of irreducible non-spherical type, then either $\mathrm{Pc}(G \cup G') \cong \mathrm{Pc}(G) \times \mathrm{Pc}(G')$ or $\mathrm{Pc}(G) = \mathrm{Pc}(G')$ is of irreducible affine type.

Proof. For (i) and (iii), see [Cap09, Lemma 2.1]. Assertion (ii) is easy to verify. For (iv), see [Cap09, Lemma 2.3]. \square

Lemma 2.3. Let $\alpha_0 \subsetneq \alpha_1 \subsetneq \cdots \subsetneq \alpha_k$ be a nested sequence of half-spaces such that $A = \langle r_{\alpha_i} \mid i = 0, \dots, k \rangle$ is infinite dihedral. If $k \geq 7$, then for any wall m which meets every $\partial\alpha_i$, either r_m centralizes $\mathrm{Pc}(A)$, or $\langle A \cup \{r_m\} \rangle$ is a Euclidean triangle group.

Proof. This follows from [Cap06, Lemma 11] together with Lemma 2.2(iv). \square

2.3. Parabolic closures and finite index subgroups.

Lemma 2.4. Let $H_1 < H_2$ be subgroups of W . If H_1 is of finite index in H_2 , then $\mathrm{Pc}(H_1)$ is of finite index in $\mathrm{Pc}(H_2)$.

Proof. For $i = 1, 2$, set $P_i := \mathrm{Pc}(H_i)$. Since the kernel N of the action of H_2 on the coset space H_2/H_1 is a finite index normal subgroup of H_2 that is contained in H_1 , so that in particular $\mathrm{Pc}(N) \subseteq \mathrm{Pc}(H_1)$, we may assume without loss of generality that H_1 is normal in H_2 . But then H_2 normalizes P_1 . Up to conjugating by an element of W , we may also assume that P_1 is standard, namely $P_1 = W_I$ for some $I \subseteq S$. Finally, it is sufficient to prove the lemma when I is essential, which we assume henceforth. Lemma 2.1 then implies that $P_2 < W_I \times W_{I^\perp}$. We thus have an action of H_2 on the residue $W_I \times W_{I^\perp}$, and since H_1 stabilizes W_I and has finite index in H_2 , the induced action of H_2 on W_{I^\perp} possesses finite orbits. By the Bruhat–Tits fixed point theorem (see for example [AB08, Th.11.23]), it follows that H_2 fixes a point in the Davis realization of W_{I^\perp} , that is, it stabilizes a spherical residue of W_{I^\perp} . This shows $[P_2 : W_I] < \infty$. \square

2.4. Parabolic closures and essential roots. Our next goal is to present a description of the parabolic closure $\mathrm{Pc}(w)$ of an element $w \in W$, which is essentially due to D. Krammer [Kra09].

Let $w \in W$. A root $\alpha \in \Phi$ is called **w -essential** if either $w^n\alpha \subsetneq \alpha$ or $w^{-n}\alpha \subsetneq \alpha$ for some $n > 0$. A wall is called w -essential if it bounds a w -essential root. We denote by

$$\mathrm{Ess}(w)$$

the set of w -essential walls. Clearly $\mathrm{Ess}(w)$ is empty if w is of finite order. If w is of infinite order, then it acts on X as a hyperbolic isometry and thus possesses some translation axis. We say that a wall is **transverse** to such an axis if it intersects this axis in a single point. We recall that the intersection of a wall and any geodesic segment which is not completely contained in that wall is either empty or consists of a single point (see [NV02, Lemma 3.4]). Given $x, y \in X$, we say that a wall m **separates** x from y if the intersection $[x, y] \cap m$ consists of a single point.

Lemma 2.5. Let $w \in W$ be of infinite order and let λ be a translation axis for w in X . Then $\mathrm{Ess}(w)$ coincides with those walls which are transverse to λ .

The proof requires a subsidiary fact. Recall that Selberg’s lemma ensures that any finitely generated linear group over \mathbf{C} admits a finite index torsion-free subgroup. This is thus the case for Coxeter groups. The following lemma provides important combinatorial properties of those torsion-free subgroups of Coxeter groups. Throughout the rest of this section, we let $W_0 < W$ be a torsion-free finite index normal subgroup.

Lemma 2.6. *For all $w \in W_0$ and $\alpha \in \Phi$, either $w\alpha = \alpha$ or $w.\partial\alpha \cap \partial\alpha = \emptyset$.*

Proof. See Lemma 1 in [DJ99]. □

Proof of Lemma 2.5. It is clear that if $\alpha \in \Phi$ is w -essential, then $\partial\alpha$ is transverse to any w -axis. To see the converse, let $n > 0$ be such that $w^n \in W_0$. Since λ is also a w^n -axis, we deduce from Lemma 2.6 that for all roots α such that $\partial\alpha$ is transverse to λ , we have either $w^n\alpha \subsetneq \alpha$ or $\alpha \subsetneq w^n\alpha$. The result follows. □

We also set

$$\text{Pc}^\infty(w) = \langle r_\alpha \mid \alpha \text{ is a } w\text{-essential root} \rangle.$$

Notice that every nontrivial element of W_0 is hyperbolic. Moreover, in view of Lemma 2.6, we deduce that if $w \in W_0$, then a root α is w -essential if and only if $w\alpha \subsetneq \alpha$ or $w^{-1}\alpha \subsetneq \alpha$.

Lemma 2.7. *Let $w \in W$ be of infinite order, let λ be a translation axis for w in X and let $x \in \lambda$.*

Then we have the following.

- (i) $\text{Pc}^\infty(w) = \langle r_\alpha \mid \partial\alpha \text{ is a wall transverse to } \lambda \rangle$
 $= \langle r_\alpha \mid \partial\alpha \text{ is a wall transverse to } \lambda \text{ and separates } x \text{ from } wx \rangle.$
- (ii) $\text{Pc}^\infty(w)$ coincides with the essential component of $\text{Pc}(w)$, i.e. the product of its non-spherical components. In particular $\text{Pc}(w) = \text{Pc}^\infty(w)$ if and only if $\text{Pc}(w)$ is of essential type.
- (iii) If $w \in W_0$, then $\text{Pc}(w) = \text{Pc}^\infty(w)$.

Proof. The first equality in Assertion (i) follows from Lemma 2.5. To check the second, it suffices to remark that if $\partial\alpha$ is any wall transverse to λ , then there exists a power w^k of w such that $w^k\partial\alpha$ separates x from wx .

Assertion (ii) follows from Corollary 5.8.7 in [Kra09] (notice that what we call *essential* roots here are called *odd* roots in *loc. cit.*). Assertion (iii) follows from Lemma 2.6 and Theorem 5.8.3 from [Kra09]. □

2.5. The Grid Lemma. The following lemma is an unpublished observation due to the first author and Piotr Przytycki.

Lemma 2.8 (Caprace–Przytycki). *There exists a constant N , depending only on (W, S) , such that the following property holds. Let $\alpha_0 \subsetneq \alpha_1 \subsetneq \dots \alpha_k$ and $\beta_0 \subsetneq \beta_1 \subsetneq \dots \subsetneq \beta_l$ be two nested families of half-spaces of X such that $\min\{k, l\} > 2N$. Set $A = \langle r_{\alpha_i} \mid i = 0, \dots, k \rangle$, $A' = \langle r_{\alpha_i} \mid i = N, N + 1, \dots, k - N \rangle$, $B = \langle r_{\beta_j} \mid j = 0, \dots, l \rangle$ and $B' = \langle r_{\beta_j} \mid j = N, N + 1, \dots, k - N \rangle$. If $\partial\alpha_i$ meets $\partial\beta_j$ for all i, j , then either of the following assertions holds:*

- (i) *The groups A and B are both infinite dihedral, their union generates a Euclidean triangle group and the parabolic closure $\text{Pc}(A \cup B)$ coincides with $\text{Pc}(A)$ and $\text{Pc}(B)$ and is of irreducible affine type.*

- (ii) *The parabolic closures $\text{Pc}(A)$, $\text{Pc}(A')$, $\text{Pc}(B)$ and $\text{Pc}(B')$ are all of irreducible type; furthermore we have*

$$\text{Pc}(A' \cup B) \cong \text{Pc}(A') \times \text{Pc}(B) \quad \text{and} \quad \text{Pc}(A \cup B') \cong \text{Pc}(A) \times \text{Pc}(B').$$

We shall use the following related result.

Lemma 2.9. *There exists a constant L , depending only on (W, S) , such that the following property holds. Let $\alpha_0 \subsetneq \alpha_1 \subsetneq \dots \subsetneq \alpha_k$ be a nested sequence of half-spaces and m, m' be walls such that $\emptyset \neq m \cap m' \subset \partial\alpha_0$, and that both m and m' meets $\partial\alpha_i$ for each i . If $k \geq L$, then $\langle r_m, r_{m'}, r_{\alpha_i} \mid i = 0, \dots, k \rangle$ is a Euclidean triangle group and $\langle r_{\alpha_i} \mid i = 0, \dots, k \rangle$ is infinite dihedral.*

Proof. See [Cap06, Th. 8]. □

Proof of Lemma 2.8. We let $N = \max\{8, L\}$ where L is the constant appearing in Lemma 2.9.

Assume first that for some $i \in \{0, 1, \dots, k\}$ and some $j \in \{N, N+1, \dots, l-N\}$, the reflections r_{α_i} and r_{β_j} do not centralize one another. Let $\phi = r_{\alpha_i}(\beta_j)$; thus $\phi \notin \{\pm\alpha_i, \pm\beta_j\}$. Let $x_0 \in \partial\alpha_0 \cap \partial\beta_j$ and $x_k \in \partial\alpha_k \cap \partial\beta_j$. Then the geodesic segment $[x_0, x_k]$ lies entirely in $\partial\beta_j$ and crosses $\partial\alpha_i$. Since $\partial\alpha_i \cap \partial\beta_j$ is contained in $\partial\phi$, it follows that $[x_0, x_k]$ meets $\partial\phi$. This shows that the wall $\partial\phi$ separates x_0 from x_k .

Let now $p_0 \in \partial\alpha_0 \cap \partial\beta_0$ and $p_k \in \partial\alpha_k \cap \partial\beta_0$. Then the piecewise geodesic path $[x_0, p_0] \cup [p_0, p_k] \cup [p_k, x_k]$ is a continuous path joining x_0 to x_k . This path must therefore cross $\partial\phi$. Thus $\partial\phi$ meets either $\partial\alpha_0$ or $\partial\beta_0$ or $\partial\alpha_k$. We now deal with the case where $\partial\phi$ meets $\partial\alpha_0$. The other two cases may be treated with analogous arguments; the straightforward adaption will be omitted here.

Then $\partial\phi$ meets $\partial\alpha_m$ for each $m = 0, 1, \dots, i$. Therefore Lemma 2.9 may be applied, thereby showing that $A_i = \langle r_{\alpha_m} \mid m = 0, \dots, i \rangle$ is infinite dihedral and that the subgroup $T = \langle r_{\alpha_m}, r_{\beta_j} \mid m = 0, \dots, i \rangle$ is a Euclidean triangle group. Furthermore Lemma 2.2(iii) shows that $\text{Pc}(T)$ is of irreducible affine type. Since $\text{Pc}(A_i)$ is infinite (because A_i is infinite) and contained in $\text{Pc}(T)$ (because A_i is contained in T), it follows that $\text{Pc}(A_i) = \text{Pc}(T)$ since any proper parabolic subgroup of $\text{Pc}(T)$ is finite. We set $P := \text{Pc}(A_i) = \text{Pc}(T)$.

Let now $n \in \{0, 1, \dots, l\}$ with $n \neq j$. Then r_{β_n} does not centralize r_{β_i} ; in particular it does not centralize T . On the other hand the wall $\partial\beta_n$ meets $\partial\alpha_m$ for all $m = 0, \dots, i$, which implies by Lemma 2.3 that $\langle A_i \cup \{r_{\beta_n}\} \rangle$ is a Euclidean triangle group. Therefore $r_{\beta_n} \in P$ by Lemma 2.2(iii).

We have already seen that P is of irreducible affine type. We have just shown that B is contained in $\text{Pc}(A_i) = P$; in particular this shows that B is infinite dihedral since the walls $\partial\beta_0, \dots, \partial\beta_l$ are pairwise parallel. Moreover, the group $\langle B \cup \{r_{\alpha_i}\} \rangle$ must be a Euclidean triangle group since it is a subgroup of P . In particular we have $\text{Pc}(B) = P$ by Lemma 2.2(iii). Since every $\partial\alpha_m$ meets every $\partial\beta_j$, the same arguments as before now show that $r_{\alpha_m} \in \text{Pc}(B) = P$ for all $m = i+1, \dots, k$. Finally we conclude that $\text{Pc}(A) = \text{Pc}(B) = P$ in this case.

Notice that, in view of the symmetry between the α 's and the β 's, the previous arguments yield the same conclusion if one assumed instead that for some $i \in \{N, N+1, \dots, k-N\}$ and some $j \in \{0, 1, \dots, l\}$, the reflections r_{α_i} and r_{β_j} do not centralize one another.

Assume now that for all $i \in \{0, \dots, k\}$ and all $j \in \{N, N+1, \dots, l-N\}$, the reflections r_{α_i} and r_{β_j} commute and that, furthermore, for all $i \in \{N, N+1, \dots, k-N\}$ and all $j \in \{0, 1, \dots, l\}$, the reflections r_{α_i} and r_{β_j} commute. By Lemma 2.2(ii) the parabolic closures $\text{Pc}(A)$, $\text{Pc}(A')$, $\text{Pc}(B)$ and $\text{Pc}(B')$ are of irreducible type. By assumption A' centralizes B . By Lemma 2.2(iv), either $\text{Pc}(A') = \text{Pc}(B)$ is of affine type or else $\text{Pc}(A' \cup B) \cong \text{Pc}(A') \times \text{Pc}(B)$. In the former case, we may argue as before to conclude again that $\text{Pc}(A) = \text{Pc}(B)$ is of affine type and we are in case (i) of the alternative. Otherwise, we have $\text{Pc}(A' \cup B) \cong \text{Pc}(A') \times \text{Pc}(B)$ and by similar arguments we deduce that $\text{Pc}(A \cup B') \cong \text{Pc}(A) \times \text{Pc}(B')$. \square

2.6. Orbits of essential roots: affine versus non-affine. Using the Grid Lemma, we can now establish a basic description of the w -orbit of a w -essential wall for some fixed $w \in W$. As before, we let $W_0 < W$ be a torsion-free finite index normal subgroup. Recall from Lemma 2.5 that for all $n > 0$ we have $\text{Ess}(w) = \text{Ess}(w^n)$ and, moreover, the set $\text{Ess}(w)$ has finitely many orbits under the action of $\langle w \rangle$ (and hence also under $\langle w^n \rangle$).

Proposition 2.10. *Let $w \in W$ be of infinite order, let $k > 0$ be such that $w^k \in W_0$ and let $\text{Ess}(w) = \text{Ess}(w^k) = M_1 \cup \dots \cup M_t$ be the partition of $\text{Ess}(w)$ into $\langle w^k \rangle$ -orbits. For each $i \in \{1, \dots, t\}$, let also $P_i = \text{Pc}(\{r_m \mid m \in M_i\})$.*

Then for all $i \in \{1, \dots, t\}$, the group P_i is an irreducible direct component of $\text{Pc}(w)$. In particular, for all $j \neq i$, we have either $P_i = P_j$ or $\text{Pc}(P_i \cup P_j) \cong P_i \times P_j$. More precisely, one of the following assertions holds.

- (i) $P_i = P_j$ and each $m \in M_i$ meets finitely many walls in M_j .
- (ii) $P_i = P_j$ is irreducible affine.
- (iii) $\text{Pc}(P_i \cup P_j) \cong P_i \times P_j$.

Proof. Let $i \in \{1, \dots, t\}$. Since M_i is $\langle w^k \rangle$ -invariant, it follows that P_i is normalized by w^k . As $\langle r_m \mid m \in M_i \rangle$ is an irreducible reflection group by Lemma 2.2(ii), P_i is of irreducible non-spherical type by Lemma 2.2(iii). It then follows from Lemma 2.1 that $\mathcal{N}(P_i) = P_i \times \mathcal{Z}(P_i)$ is itself a parabolic subgroup. In particular it contains $\text{Pc}(w^k)$. Since on the other hand we have $P_i \leq \text{Pc}(w^k)$ by Lemma 2.7, we infer that P_i is a direct component of $\text{Pc}(w^k)$. Since $\text{Pc}(w^k) = \text{Pc}^\infty(w)$ is the essential component of $\text{Pc}(w)$ by Lemma 2.7, we deduce that P_i is a direct component of $\text{Pc}(w)$ as desired.

Let now $j \neq i$. Since we already know that P_i and P_j are irreducible direct components of $\text{Pc}(w)$, it follows that either $P_i = P_j$ or (iii) holds. So assume that $P_i = P_j$ and that there exists a wall $m \in M_i$ meeting infinitely many walls in M_j . We have to show that (ii) holds.

Let λ be a w -axis. By Lemma 2.5, all walls in $M_i \cup M_j$ are transverse to λ . Moreover, by Lemma 2.6 the elements of M_i (resp. M_j) are pairwise parallel. Therefore, we deduce that infinitely many walls in M_i meet infinitely many walls in M_j . Since M_i and M_j are both $\langle w^k \rangle$ -invariant, it follows that all walls in M_i meet all walls in M_j . Thus $M_i \cup M_j$ forms a grid and the desired conclusion follows from Lemma 2.8. \square

We shall now deduce a rather subtle, but nevertheless important, difference between the affine and non-affine cases concerning the $\langle w \rangle$ -orbit of a w -essential root α .

Let us start by considering a specific example, namely the Coxeter group $W = \langle r_a, r_b, r_c \rangle$ of type \tilde{A}_2 , acting on the Euclidean plane. One verifies easily that W contains a nonzero translation t which preserves the r_a -invariant wall m_a . Let $w = tr_a$. Then w is of infinite order so that $\text{Pc}(w) = W$. Moreover the walls m_b and m_c , respectively fixed by r_b and r_c , are both w -essential by Lemma 2.5. Now we observe that, for each even integer n the walls m_b and $w^n m_b$ are parallel, while for each odd integer the walls m_b and $w^n m_b$ have a non-empty intersection.

The following result (in the special case $m = m'$) shows that the situation we have just described cannot occur in the non-affine case.

Proposition 2.11. *Let $w \in W$, m be a w -essential wall and P be the irreducible component of $\text{Pc}(w)$ that contains r_m .*

If P is not of affine type, then for each w -essential wall m' such that $r_{m'} \in P$, there exists an $l_0 \in \mathbf{N}$ such that for all $l \in \mathbf{Z}$ with $|l| \geq l_0$, the wall m' lies between $w^{-l}m$ and $w^l m$.

Proof. First notice that if m is a w -essential wall, then the reflection r_m belongs to $\text{Pc}(w)$ by Lemma 2.7, so that P is well defined. Moreover, we have $r_{w^l m} = w^l r_m w^{-l} \in P$ for all $l \in \mathbf{Z}$.

Let $k > 0$ be such that $w^k \in W_0$ and let $\text{Ess}(w) = \text{Ess}(w^k) = M_1 \cup \dots \cup M_t$ be the partition of $\text{Ess}(w)$ into $\langle w^k \rangle$ -orbits. Upon reordering the M_i , we may assume that $m' \in M_1$. Let also $I \subseteq \{1, \dots, t\}$ be the set of those i such that $w^l m \in M_i$ for some l . In other words the $\langle w \rangle$ -orbit of m coincides with $\bigcup_{i \in I} M_i$.

For all j , set $P_j = \text{Pc}(\{r_\mu \mid \mu \in M_j\})$. By Proposition 2.10, each P_j is an irreducible direct component of $\text{Pc}(w)$. By hypothesis, this implies that $P = P_1 = P_i$ for all $i \in I$.

Suppose now that for infinitely many values of l , the wall $w^l m$ has a non-empty intersection with m' . We have to deduce that P is of affine type.

Recall from Lemma 2.6 that the elements of M_j are pairwise parallel for all j . Therefore, our assumption implies that for some $i \in I$, the wall m' meets infinitely many walls in M_i . By Proposition 2.10, this implies that either $P = P_1 = P_i$ is of affine type, or $\text{Pc}(P_1 \cup P_i) \cong P_1 \times P_i$. The second case is impossible since $P_1 = P_i$. \square

2.7. On parabolic closures of a pair of reflections. The following consequence of Proposition 2.10 was stated as Theorem E in the introduction.

Corollary 2.12. *For each $w \in W$ with infinite irreducible parabolic closure $\text{Pc}(w)$, there is a constant C such that the following holds. For all $m, m' \in \text{Ess}(w)$ with $d(m, m') > C$, we have $\text{Pc}(w) = \text{Pc}(r_m, r_{m'})$.*

We shall use the following.

Lemma 2.13. *Let $\alpha, \beta, \gamma \in \Phi$ such that $\alpha \subsetneq \beta \subsetneq \gamma$. Then $r_\beta \in \text{Pc}(\{r_\alpha, r_\gamma\})$.*

Proof. See [Cap06, Lemma 17]. \square

Proof of Corollary 2.12. Retain the notation of Proposition 2.10. Since $P = \text{Pc}(w)$ is irreducible, we have $P = P_i$ for all $i \in \{1, \dots, t\}$ by Proposition 2.10. Recall that M_i is the $\langle w^k \rangle$ -orbit of some w -essential wall m . For all $n \in \mathbf{Z}$, we set $m_n = w^{kn} m$. By Lemma 2.6 the elements of M_i are pairwise parallel and hence for all $i < j < n$, it follows that m_j separates m_i from m_n . For all $n \geq 0$ let now $Q_n = \text{Pc}(\{r_{m_n}, r_{m_{-n}}\})$. By Lemma 2.13 we have $Q_n \leq Q_{n+1} \leq P$ for all $n \geq 0$. In

particular $\bigcup_{n \geq 0} Q_n$ is a parabolic subgroup, which must thus coincide with P . It follows that $Q_n = P$ for some n . Since this argument holds for all $i \in \{1, \dots, t\}$, the desired result follows. \square

Corollary 2.14. *Any irreducible non-spherical parabolic subgroup P is the parabolic closure of a pair of reflections.*

Proof. Let $w \in P$ such that $P = \text{Pc}(w)$. Such an w always exists by [CF10, Cor.4.3]. (Note that this can also be deduced from Corollary 2.17 below together with [AB08, Prop.2.43].) The conclusion now follows from Corollary 2.12. \square

2.8. The parabolic closure of a product of two elements in a Coxeter group. We are now able to present the main result of this section, which was stated as Theorem G in the introduction.

Before we state it, we prove one more technical lemma about $\text{CAT}(0)$ spaces. Recall that W acts on the $\text{CAT}(0)$ space X . For a hyperbolic $w \in W$, let $|w|$ denote its translation length and set $\text{Min}(w) = \{x \in X \mid d(x, wx) = |w|\}$.

Lemma 2.15. *Let $w \in W$ be hyperbolic and suppose it decomposes as a product $w = w_1 w_2 \dots w_t$ of pairwise commuting hyperbolic elements of W . Let m be a w -essential wall. Then m is also w_i -essential for some $i \in \{1, \dots, t\}$.*

Proof. Write $w_0 := w$. Then, since the w_i are pairwise commuting for $i = 0, \dots, t$, each w_i stabilizes $\text{Min}(w_j)$ for all j . Thus $M := \bigcap_{j=1}^t \text{Min}(w_j)$ and $\text{Min}(w)$ are both non-empty by $\text{CAT}(0)$ -convexity, and are stabilized by each w_i , $i = 0, \dots, t$. Therefore, if $x \in M \cap \text{Min}(g)$, there is a piecewise geodesic path $x, w_1 x, w_1 w_2 x, \dots, w_1 \dots w_t x = wx$ inside $M \cap \text{Min}(g)$, where each geodesic segment is part of a w_i -axis for some $i \in \{1, \dots, t\}$. Since any wall intersecting the geodesic segment $[x, wx]$ must intersect one of those axis, the conclusion follows from Lemma 2.5. \square

Theorem 2.16. *For all $g, h \in W_0$, there exists a constant $K = K(g, h) \in \mathbf{N}$ such that for all $m, n \in \mathbf{Z}$ with $\min\{|m|, |n|, |m/n| + |n/m|\} \geq K$, we have $\text{Pc}(g) \cup \text{Pc}(h) \subseteq \text{Pc}(g^m h^n)$.*

Proof. Fix $g, h \in W_0$. Let $\text{Ess}(g) = M_1 \cup \dots \cup M_k$ (resp. $\text{Ess}(h) = N_1 \cup \dots \cup N_l$) be the partition of $\text{Ess}(g)$ into $\langle g \rangle$ -orbits (resp. $\text{Ess}(h)$ into $\langle h \rangle$ -orbits). For all $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$, set $P_i = \text{Pc}(\{r_m \mid m \in M_i\})$ and $Q_j = \text{Pc}(\{r_m \mid m \in N_j\})$.

By Lemma 2.7, we have $\text{Pc}(g) = \langle \{r_m \mid m \in M_i, i = 1, \dots, k\} \rangle$, and Proposition 2.10 ensures that P_i is an irreducible direct component of $\text{Pc}(g)$ for all i . Thus there is a subset $I \subseteq \{1, \dots, k\}$ such that $\text{Pc}(g) = \prod_{i \in I} P_i$. Similarly, there is a subset $J \subseteq \{1, \dots, l\}$ such that $\text{Pc}(h) = \prod_{j \in J} Q_j$.

For all $i \in I$ and $j \in J$, we finally let g_i and h_j denote the respective projections of g and h onto P_i and Q_j , so that $P_i = \text{Pc}(g_i)$ and $Q_j = \text{Pc}(h_j)$.

We define a collection $E(g, h)$ of subsets of W as follows: a set $Z \subseteq W$ belongs to $E(g, h)$ if and only if there exists a constant $K = K(g, h, Z) \in \mathbf{N}$ such that for all $m, n \in \mathbf{Z}$ with $\min\{|m|, |n|, |m/n| + |n/m|\} \geq K$ we have $Z \subseteq \text{Pc}(g^m h^n)$.

Our goal is to prove that $\text{Pc}(g)$ and $\text{Pc}(h)$ both belong to $E(g, h)$. To this end, it suffices to show that P_i and Q_j belong to $E(g, h)$ for all $i \in I$ and $j \in J$. This will be achieved in Claim 6 below.

Claim 1. $M_s \subseteq \text{Ess}(g_i)$ for all $s \in \{1, \dots, k\}$ and $i \in I$ such that $P_s = P_i$. Similarly, $N_s \subseteq \text{Ess}(h_j)$ for all $s \in \{1, \dots, l\}$ and $j \in J$ such that $Q_s = Q_j$.

Indeed, let $m \in M_s$ for some $s \in \{1, \dots, k\}$. Then $r_m \in P_s = P_i$. Moreover, as m is g -essential, it must be $g_{i'}$ -essential for some $i' \in I$ by Lemma 2.15. But then $r_m \in \text{Pc}(g_{i'}) = P_{i'}$ and so $i' = i$. The proof of the second statement is similar.

Claim 2. If $i \in I$ is such that $[P_i, Q_j] = 1$ for all $j \in J$, then P_i belongs to $E(g, h)$.

Similarly, if $j \in J$ is such that $[P_i, Q_j] = 1$ for all $i \in I$, then Q_j belongs to $E(g, h)$.

Indeed, suppose $[P_i, Q_j] = 1$ for some $i \in I$ and for all $j \in J$. Then P_i commutes with $\text{Pc}(h)$. Thus h fixes every wall of M_i . In particular, any wall $\mu \in M_i$ is $g^m h^n$ -essential for all $m, n \in \mathbf{Z}^*$ since $g^m h^n = g_i^m w$ for some $w \in W$ fixing μ and commuting with g_i . Therefore $P_i \subseteq \text{Pc}(g^m h^n)$ for all $m, n \in \mathbf{Z}^*$ and so P_i belongs to $E(g, h)$. The second statement is proven in the same way.

Claim 3. Let $i \in I$ and $j \in J$ be such that $P_i = Q_j$. Then, for all $m, n \in \mathbf{Z}$, every $g_i^m h_j^n$ -essential root is also $g^m h^n$ -essential.

Indeed, take $\alpha \in \Phi$ and $k > 0$ such that $(g_i^m h_j^n)^k \alpha \subsetneq \alpha$. Notice that $\text{Pc}(g_i^m h_j^n) \subseteq P_i = Q_j$, and hence $r_\alpha \in P_i = Q_j$ by Lemma 2.7. Moreover, setting $g' := \prod_{t \neq i} g_t^m$ and $h' := \prod_{t \neq j} h_t^n$, we have $g' \alpha = \alpha = h' \alpha$ since g' and h' centralize $P_i = Q_j$. Therefore $(g^m h^n)^k \alpha = (g_i^m h_j^n)^k (g' h')^k \alpha = (g_i^m h_j^n)^k \alpha \subsetneq \alpha$ so that α is also $g^m h^n$ -essential.

Claim 4. Let $i \in I$ and $j \in J$ be such that $P_i = Q_j$. If P_i is of affine type, then $P_i = Q_j$ belongs to $E(g, h)$.

Since $P_i = Q_j$ is of irreducible affine type, we have $\text{Pc}(w) = P_i$ for all $w \in P_i$ of infinite order. Thus, in order to prove the claim, it suffices to show that there exists some constant K such that $g_i^m h_j^n$ is of infinite order for all $m, n \in \mathbf{Z}$ with $\min\{|m|, |n|, |m/n| + |n/m|\} \geq K$. Indeed, we will then get that $\text{Pc}(g_i^m h_j^n) = P_i$ is of essential type and so $P_i = \text{Pc}(g_i^m h_j^n) \leq \text{Pc}(g^m h^n)$ by Claim 3 and Lemma 2.7(ii).

Recalling that P_i is of affine type, we can argue in the geometric realization of a Coxeter complex of affine type, which is a Euclidean space. We deduce that if g_i and h_j have non-parallel translation axes, then $g_i^m h_j^n$ is of infinite order for all nonzero m, n . On the other hand, if g_i and h_j have some parallel translation axes, we consider a Euclidean hyperplane H orthogonal to these and let ℓ_i and ℓ_j denote the respective translation lengths of g_i and h_j . Then, upon replacing g_i by its inverse (which does not affect the conclusion since $E(g, h) = E(g^{-1}, h)$), we have $d(g_i^m h_j^n H, H) = |m\ell_i - n\ell_j|$. Since $g_i^m h_j^n$ is of infinite order as soon as this distance is nonzero, the claim now follows by setting $K = \ell_i/\ell_j + \ell_j/\ell_i + 1$.

Claim 5. Let $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$ be such that $M_i \cap N_j$ is infinite. Then $P_i = Q_j$ and these belong to $E(g, h)$.

Indeed, remember that the walls in M_i are pairwise parallel by Lemma 2.6. Since $M_i \cap N_j \subseteq \text{Ess}(g_{i'}) \cap \text{Ess}(h_{j'})$ for some $i' \in I$ such that $P_i = P_{i'}$ and some $j' \in J$ such that $Q_j = Q_{j'}$ by Claim 1, Corollary 2.12 then yields $P_i = Q_j$.

Let now C denote the minimal distance between two parallel walls in X and set $K := \frac{|g|+|h|}{C} + 1$. Let $m, n \in \mathbf{Z}$ be such that $\min\{|m|, |n|, |m/n| + |n/m|\} \geq K$. We

now show that $P_i \leq \text{Pc}(g^m h^n)$. By Lemma 2.7 and Corollary 2.12, it is sufficient to check that infinitely many walls in $M_i \cap N_j$ are $g^m h^n$ -essential.

Note first that for any wall $\mu \in M_i \cap N_j$, we have $g^{\epsilon m} \mu \in M_i$ and $h^{\epsilon n} \mu \in N_j$ for $\epsilon \in \{+, -\}$. Thus, since $M_i \cap N_j$ is infinite, there exist infinitely many such $\mu \in M_i \cap N_j$ with the property that $g^{\epsilon m} \mu$ lies between μ and some $\mu_\epsilon \in M_i \cap N_j$ and $h^{\epsilon n} \mu$ lies between μ and some $\mu'_\epsilon \in M_i \cap N_j$ for $\epsilon \in \{+, -\}$. We now show that any such μ is $g^m h^n$ -essential, as desired. Consider thus such a μ .

Let D be a g -axis and D' be an h -axis. Since $M_i \cap N_j \subseteq \text{Ess}(g) \cap \text{Ess}(h)$, Lemma 2.5 implies that each of the walls μ , μ_ϵ and μ'_ϵ for $\epsilon \in \{+, -\}$ is transverse to both D and D' . In particular, the choice of μ implies that $g^{\epsilon m} \mu$ and $h^{\epsilon n} \mu$ for $\epsilon \in \{+, -\}$ are also transverse to both D and D' .

Let $\alpha \in \Phi$ be such that $\partial\alpha = \mu$ and $g^m \alpha \subsetneq \alpha$. If $h^n \alpha \subsetneq \alpha$ then clearly $g^m h^n \alpha \subsetneq \alpha$, as desired. Suppose now that $h^n \alpha \supsetneq \alpha$.

Note that the walls in $\langle g \rangle \mu \cup \langle h \rangle \mu$ are pairwise parallel since this is the case for the walls in $W_0 \cdot \mu$ by Lemma 2.6 and since $g, h \in W_0$.

Assume now that $|n| > |m|$, the other case being similar. In particular, $|n/m| > |g|/C$. Then $d(\mu, g^{-m} \mu) \leq |m| \cdot |g| < |n| \cdot C \leq d(\mu, h^n \mu)$ and so the wall $g^{-m} \mu$ lies between μ and $h^n \mu$. Thus $\alpha \subsetneq g^{-m} \alpha \subsetneq h^n \alpha$ and so $g^m h^n \alpha \supsetneq \alpha$, as desired.

Claim 6. *For all $i \in I$ and $j \in J$, the sets P_i and Q_j both belong to $E(g, h)$.*

We only deal with P_i ; the argument for Q_j is similar.

Let D denote a g -axis, and D' an h -axis in X . By Claim 5 we may assume that $M_i \cap \text{Ess}(h)$ is finite. Moreover, by Claim 3 we may assume there exists a $j \in J$ such that $[P_i, Q_j] \neq 1$.

If $N_j \cap \text{Ess}(g)$ is infinite, then $N_j \cap M_{i'}$ is infinite for some $i' \in \{1, \dots, k\}$ and thus Claim 5 yields that $Q_j = P_{i'} \in E(g, h)$. In particular, $[Q_j, P_s] = 1$ as soon as $P_s \neq P_{i'}$. This implies $P_i = P_{i'} \in E(g, h)$, as desired. We now assume that $N_j \cap \text{Ess}(g)$ is finite.

Thus by Lemma 2.5, only finitely many walls in M_i intersect D' and only finitely walls in N_j intersect D .

Take $m_1 \in M_i$ and $m_2 \in N_j$. By Claim 1 and Corollary 2.12, there exists some $k_0 \in \mathbf{N}$ such that if one sets $M := \{g^{sk_0} m_1 \mid s \in \mathbf{Z}\} \subseteq M_i$ and $N := \{h^{tk_0} m_2 \mid t \in \mathbf{Z}\} \subseteq N_j$, then any two reflections associated to distinct walls of M (respectively, N) generate P_i (respectively, Q_j) as parabolic subgroups. Also, we may assume that no wall in M intersects D' and that no wall in N intersects D .

If every wall of M intersects every wall of N , then since $[P_i, Q_j] \neq 1$, Lemma 2.8 yields that $P_i = Q_j$ is of affine type and Claim 4 allows us to conclude. Up to making a different choice for m_1 and m_2 inside M and N respectively, we may thus assume that m_1 is parallel to m_2 . For the same reason, we may also choose $m'_1 \in M$ and $m'_2 \in N$ such that D' lies between m_1 and m'_1 , D lies between m_2 and m'_2 , and such that $m_1 \cap m'_2 = m_2 \cap m'_1 = m'_1 \cap m'_2 = \emptyset$.

Let now $s_0, t_0 \in \mathbf{Z}$ be such that $g^{s_0 k_0} m_1 = m'_1$ and $h^{t_0 k_0} m_2 = m'_2$. Up to interchanging m_1 and m'_1 (respectively, m_2 and m'_2), we may assume that $s_0 > 0$ and $t_0 > 0$.

Let $\alpha, \beta \in \Phi$ be such that $\partial\alpha = m_1$, $\partial\beta = m_2$ and such that D' is contained in $\alpha \cap -g^{s_0 k_0} \alpha$ and D is contained in $\beta \cap -h^{t_0 k_0} \beta$. For each $s, t \in \mathbf{Z}$, set $\alpha_s := g^{sk_0} \alpha$ and $\beta_t := h^{tk_0} \beta$ (see Figure 1). Since for two roots $\gamma, \delta \in \Phi$ with $\partial\gamma$ parallel to $\partial\delta$, one of the possibilities $\gamma \subseteq \delta$ or $\gamma \subseteq -\delta$ or $-\gamma \subseteq \delta$ or $-\gamma \subseteq -\delta$ must hold, this

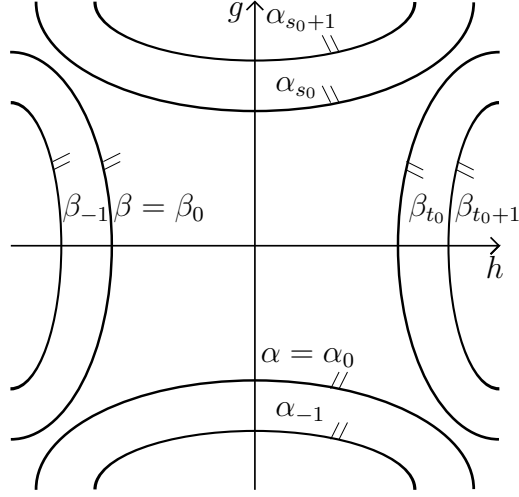


FIGURE 1. Claim 6.

implies that

$$\alpha_{s_0} \subseteq -\beta_{t_0}, \quad -\alpha \subseteq \beta \quad \text{and} \quad \beta_{t_0} \subseteq \alpha.$$

Set $K := (s_0 + t_0 + 1)k_0$ and let $m, n \in \mathbf{Z}$ be such that $|m|, |n| > K$. We now prove that $P_i \leq \text{Pc}(g^m h^n)$. By Lemma 2.7, it is sufficient to show that either α_{-1} and α or α_{s_0} and α_{s_0+1} are $g^m h^n$ -essential. We distinguish several cases depending on the respective signs of m, n .

- If $m, n > 0$, then

$$g^m h^n \alpha_{s_0+1} \subseteq g^m h^n \alpha_{s_0} \subseteq g^m h^n \beta \subsetneq g^m \beta_{t_0} \subseteq g^m \alpha \subsetneq \alpha_{s_0+1} \subseteq \alpha_{s_0}$$

so that α_{s_0} and α_{s_0+1} are $g^m h^n$ -essential.

- If $m, n < 0$, then

$$g^m h^n \alpha_{-1} \supseteq g^m h^n \alpha \supseteq g^m h^n \beta_{t_0} \supsetneq g^m \beta \supseteq g^m \alpha_{s_0} \supsetneq \alpha_{-1} \supseteq \alpha$$

so that α_{-1} and α are $g^m h^n$ -essential.

- If $m > 0$ and $n < 0$, then

$$g^m h^n \alpha_{s_0+1} \subseteq g^m h^n \alpha_{s_0} \subseteq g^m h^n (-\beta_{t_0}) \subsetneq g^m (-\beta) \subseteq g^m \alpha \subsetneq \alpha_{s_0+1} \subseteq \alpha_{s_0}$$

so that α_{s_0} and α_{s_0+1} are $g^m h^n$ -essential.

- If $m < 0$ and $n > 0$, then

$$g^m h^n \alpha_{-1} \supseteq g^m h^n \alpha \supseteq g^m h^n (-\beta) \supsetneq g^m (-\beta_{t_0}) \supseteq g^m \alpha_{s_0} \supsetneq \alpha_{-1} \supseteq \alpha$$

so that α_{-1} and α are $g^m h^n$ -essential.

This concludes the proof of the theorem. \square

The following corollary will be of fundamental importance in the rest of the paper. It was stated as Corollary H in the introduction.

Corollary 2.17. *Let H be a subgroup of W . Then there exists $h \in H \cap W_0$ such that $[\text{Pc}(H) : \text{Pc}(h)] < \infty$.*

Proof. Take $h \in H \cap W_0$ such that $\text{Pc}(h)$ is maximal. Then $\text{Pc}(h) = \text{Pc}(H \cap W_0)$, for otherwise there would exist $g \in H \cap W_0$ such that $\text{Pc}(g) \not\subseteq \text{Pc}(h)$, and hence Theorem 2.16 would yield integers m, n such that $\text{Pc}(h) \subsetneq \text{Pc}(g^m h^n)$, contradicting the choice of h . The result now follows from Lemma 2.4 since $[H : H \cap W_0] < \infty$. \square

Remark 2.18. Note that the conclusion of Corollary 2.17 cannot be improved: indeed, one cannot expect that there is some $h \in H$ such that $\text{Pc}(H) = \text{Pc}(h)$ in general. Consider for example the Coxeter group $W = \langle s \rangle \times \langle t \rangle \times \langle u \rangle$, which is a direct product of three copies of $\mathbf{Z}/2\mathbf{Z}$. Then the parabolic closure of the subgroup $H = \langle st, tu \rangle$ of W is the whole of W , but there is no $h \in H$ such that $\text{Pc}(h) = W$.

2.9. On walls at bounded distance from a residue. We finish this section with a couple of observations on Coxeter groups which we shall need in our study of open subgroups of Kac–Moody groups.

Given a subset $J \subseteq S$, we set $\Phi_J = \{\alpha \in \Phi \mid \exists v \in W_J, s \in J : \alpha = v\alpha_s\}$, where α_s denotes the positive root associated with the reflection s .

Lemma 2.19. *Let $L \subseteq S$ be essential. Then for each root $\alpha \in \Phi_L$, there exists $w \in W_L$ such that $w.\alpha \not\subseteq \alpha$. In particular α is w -essential.*

Proof. Let $\alpha \in \Phi_L$. By [Hée93, Prop. 8.1, p. 309], there exists a root $\beta \in \Phi_L$ such that $\alpha \cap \beta = \emptyset$. We can then take $w = r_\alpha r_\beta$ or its inverse. \square

Lemma 2.20. *Let $L \subseteq S$ be essential, and let R be the standard L -residue of the Coxeter complex Σ of W .*

Then for each wall m of Σ , the following assertions are equivalent:

- (i) m is perpendicular to every wall of R ,
- (ii) $[r_m, W_L] = 1$,
- (iii) There exists $n > 0$ such that R is contained in an n -neighbourhood of m .

Proof. We first show that (iii) \Rightarrow (ii). By Lemma 2.19, if m' is a wall of R (that is, a wall intersecting R), then there exists $w \in W_L$ such that one of the two half-spaces associated to m' is w -essential. It follows that m and m' cannot be parallel since R is at a bounded distance from m . Hence m is transversal to every wall of R , and does not intersect R . Back to an arbitrary wall m' of R , consider a wall m'' of R that is parallel to m' and such that the reflection group generated by the two reflections $r_{m'}$ and $r_{m''}$ is infinite dihedral. Such a wall m'' exists by Lemma 2.19. Then r_m centralizes these reflections by Lemma 2.3 and [CR09, Lem.12]. As m' was arbitrary, this means that r_m centralizes W_L .

The equivalence of (i) and (ii) is trivial.

Finally, to show (i) \Rightarrow (iii), notice that if C is a chamber of R and t a reflection associated to a wall of R , then the distance from C to m equals the distance from $t \cdot C$ to m . Indeed, if α is the root associated to m not containing R and D is the projection of C onto α , then $t \cdot D$ is the projection of $t \cdot C$ onto α . As W_L is transitive on R , (iii) follows. \square

3. OPEN AND PARABOLIC SUBGROUPS OF KAC–MOODY GROUPS

Basics on Kac–Moody groups and their completions can be found in [Rém02], [CR09] and references therein. We focus here on the case of a finite ground field.

Let $\mathcal{G} = \mathcal{G}(\mathbf{F}_q)$ be a (minimal) Kac–Moody group over a finite field \mathbf{F}_q of order q . The group \mathcal{G} is endowed with a root group datum $\{U_\alpha \mid \alpha \in \Phi = \Phi(\Sigma(W, S))\}$ for some Coxeter system (W, S) , which yields a twin BN-pair $(\mathcal{B}_+, \mathcal{B}_-, \mathcal{N})$ with associated twin building (Δ_+, Δ_-) . Let C_0 be the fundamental chamber of Δ_+ ,

namely the chamber such that $\mathcal{B}_+ = \text{Stab}_G(C_0)$, and let $A_0 \subset \Delta_+$ be the fundamental apartment, so that $\mathcal{N} = \text{Stab}_G(A_0)$ and $\mathcal{H} := \mathcal{B}_+ \cap \mathcal{N} = \text{Fix}_G(A_0)$. We identify Φ with the set of half-spaces of A_0 .

We next let G be the completion of \mathcal{G} with respect to the positive building topology. Thus the finitely generated group \mathcal{G} embeds densely in the topological group G , which is locally compact, totally disconnected and acts properly and continuously on $\Delta := \Delta_+$ by automorphisms. A completed Kac–Moody group over a finite field shall be called a **locally compact Kac–Moody group**. Let $B = \overline{\mathcal{B}_+}$ be the closure of \mathcal{B}_+ in G , let $N = \text{Stab}_G(A_0)$ and $H = B \cap N = \text{Fix}_G(A_0)$. [We warn the reader that \mathcal{N} and \mathcal{H} are discrete, whence closed in G while N and H are non-discrete closed subgroups.] The pair (B, N) is a BN-pair of type (W, S) for G ; in particular we have $N/H \cong W$. Moreover, the group B is a compact open subgroup, and every standard parabolic subgroup $P_J = BW_JB$ for some $J \subseteq S$ is thus open in G . Important to our later purposes is the fact that the group G acts transitively on the *complete* apartment system of Δ . In particular B acts transitively on the apartments containing C_0 .

For a root $\alpha \in \Phi$, we denote as before the unique reflection of W fixing the wall $\partial\alpha$ pointwise by r_α . In addition, we choose some element $n_\alpha \in N \cap \langle U_\alpha \cup U_{-\alpha} \rangle$ which maps onto r_α under the quotient map $N \rightarrow N/H \cong W$.

Before we state a more precise version of Theorem A, we will need some additional results on the BN-pair structure of G . This is the object of the following paragraph.

3.1. On Levi decompositions in complete Kac–Moody groups. Given $J \subseteq S$, we denote by $\mathcal{P}_J = \mathcal{B}_+ W_J \mathcal{B}_+$ (resp. $P_J = BW_JB$) the standard parabolic subgroup of \mathcal{G} (resp. G) of type J and by $R_J(C_0)$ the J -residue of Δ containing the chamber C_0 . Thus $\mathcal{P}_J = \text{Stab}_G(R_J(C_0))$, $P_J = \text{Stab}_G(R_J(C_0))$ and \mathcal{P}_J is dense in P_J .

We further set $\Phi_J = \{\alpha \in \Phi \mid \exists v \in W_J, s \in J : \alpha = v\alpha_s\}$ and

$$\mathcal{L}_J^+ = \langle U_\alpha \mid \alpha \in \Phi_J \rangle.$$

Finally, we set $\mathcal{L}_J = \mathcal{H} \cdot \mathcal{L}_J^+$ and denote by \mathcal{U}_J the normal closure of $\langle U_\alpha \mid \alpha \in \Phi, \alpha \supset R_J(C_0) \cap A_0 \rangle$ in \mathcal{B}_+ . Following [Rém02, 6.2.2], there is a semidirect decomposition

$$\mathcal{P}_J = \mathcal{L}_J \rtimes \mathcal{U}_J.$$

The group \mathcal{U}_J is called the **unipotent radical** of the parabolic subgroup \mathcal{P}_J , and \mathcal{L}_J is called the **Levi factor**.

We next define

$$L_J^+ = \overline{\mathcal{L}_J^+}, \quad L_J = \overline{\mathcal{L}_J} \quad \text{and} \quad U_J = \overline{\mathcal{U}_J}.$$

Thus U_J and L_J are closed subgroups of P_J , respectively called the **unipotent radical** and the **Levi factor**.

Lemma 3.1. *We have the following:*

- (i) U_J is a compact normal subgroup of P_J , and we have $P_J = L_J \cdot U_J$.
- (ii) L_J^+ is normal in L_J and we have $L_J = \mathcal{H} \cdot L_J^+$.

Proof. Since \mathcal{U}_J is normal in \mathcal{P}_J , which is dense in P_J , it is clear that U_J is normal in P_J . Moreover U_J is compact (since it is contained in B) and the product $L_J \cdot U_J$ is thus closed in P_J . Assertion (i) follows since $L_J \cdot U_J$ contains \mathcal{P}_J .

For assertion (ii), we remark that \mathcal{H} normalizes \mathcal{L}_J^+ and hence also L_J^+ . Moreover, since \mathcal{H} is finite, hence compact, the product $\mathcal{H} \cdot L_J^+$ is closed. Since $\mathcal{H} \cdot \mathcal{L}_J^+$ is dense in L_J , the conclusion follows. \square

Remark that the decomposition $P_J = L_J \cdot U_J$ is even semidirect when J is spherical, see [RR06, section 1.C.]. It is probably also the case in general, but this will not be needed here.

Lemma 3.2. *Let $J \subseteq S$. Then every open subgroup O of P_J that contains the product $L_J^+ \cdot U_{J \cup J^\perp}$ has finite index in P_J .*

Proof. Set $K := J^\perp$ and $U := U_{J \cup J^\perp}$. Note that $U \triangleleft P_{J \cup K} = L_{J \cup K} \cdot U$. Moreover, L_J^+ is normal in $L_{J \cup K}$. Indeed, as $[U_\alpha, U_\beta] = 1$ for all $\alpha \in \Phi_J$ and $\beta \in \Phi_K$, the subgroups \mathcal{L}_J^+ and \mathcal{L}_K^+ centralize each other. Since in addition \mathcal{H} normalizes each root group, we get a decomposition $\mathcal{L}_{J \cup K} = \mathcal{H} \cdot \mathcal{L}_J^+ \cdot \mathcal{L}_K^+$. In particular, $\mathcal{L}_{J \cup K}$ normalizes \mathcal{L}_J^+ , whence also L_J^+ . As the normalizer of a closed subgroup is closed, this implies that $L_{J \cup K}$ normalizes L_J^+ , as desired.

Let $\pi_1: P_{J \cup K} \rightarrow P_{J \cup K}/U$ denote the natural projection. Then $\pi_1(L_J^+)$ is normal in $P_{J \cup K}/U$, since it is the image of L_J^+ under the composition map

$$L_{J \cup K} \rightarrow \frac{L_{J \cup K}}{L_{J \cup K} \cap U} \xrightarrow{\cong} \frac{P_{J \cup K}}{U} : l \mapsto l(L_{J \cup K} \cap U) \mapsto lU.$$

Let $\pi: P_{J \cup K} \rightarrow \pi_1(P_{J \cup K})/\pi_1(L_J^+)$ denote the composition of π_1 with the canonical projection onto $\pi_1(P_{J \cup K})/\pi_1(L_J^+)$. Note that π is an open continuous group homomorphism. Then $\pi(P_J) = \pi_1(L_J^+ \cdot U_J \cdot \mathcal{H})/\pi_1(L_J^+)$ is compact. Indeed, it is homeomorphic to the quotient of the compact group $\pi_1(U_J \cdot \mathcal{H})$ by the normal subgroup $\pi_1(L_J^+ \cap U_J \cdot \mathcal{H})$ under the map

$$\frac{\pi_1(L_J^+ \cdot U_J \cdot \mathcal{H})}{\pi_1(L_J^+)} \xrightarrow{\cong} \frac{\pi_1(U_J \cdot \mathcal{H})}{\pi_1(L_J^+ \cap U_J \cdot \mathcal{H})} : \pi_1(l \cdot u)\pi_1(L_J^+) \mapsto \pi_1(u)\pi_1(L_J^+ \cap U_J \cdot \mathcal{H}).$$

In particular, since $\pi(O)$ is open in $\pi(P_J)$, it has finite index in $\pi(P_J)$. But then since $O = \pi^{-1}(\pi(O))$ by hypothesis, O has finite index in $\pi^{-1}(\pi(P_J)) = P_J$, as desired. \square

3.2. A refined version of Theorem A. We will prove the following statement, having Theorem A as an immediate corollary.

Theorem 3.3. *Let O be an open subgroup of G . Let $J \subseteq S$ be the type of a residue which is stabilized by some finite index subgroup of O and minimal with respect to this property.*

Then there exist a spherical subset $J' \subseteq J^\perp$ and an element $g \in G$ such that

$$L_J^+ \cdot U_{J \cup J^\perp} < gOg^{-1} < P_{J \cup J'}.$$

In particular, gOg^{-1} has finite index in $P_{J \cup J'}$.

Moreover, any subgroup of G containing gOg^{-1} as a finite index subgroup is contained in $P_{J \cup J''}$ for some spherical subset $J'' \subseteq J^\perp$. In particular, only finitely many distinct parabolic subgroups contain O as a finite index subgroup.

3.3. Proof of Theorem 3.3: outline and first observations. This section and the next ones are devoted to the proof of Theorem 3.3 itself.

Let thus O be an open subgroup of G . We define the subset J of S as in the statement of the theorem, namely, J is minimal amongst the subsets L of S for which there exists a $g \in G$ such that $O \cap g^{-1}P_Lg$ has finite index in O . For such a $g \in G$, we set $O_1 = gOg^{-1} \cap P_J$. Thus O_1 stabilizes $R_J(C_0)$ and is an open subgroup of G contained in gOg^{-1} with finite index.

We first observe that the desired statement is essentially empty when O is compact. Indeed, in that case the Bruhat–Tits fixed point theorem ensures that O stabilizes a spherical residue of G , and hence Theorem 3.3 stands proven with $J = \emptyset$. It thus remains to prove the theorem when O , and hence also O_1 , is non-compact, which we assume henceforth.

Recall from the previous section that we call a subset $J \subseteq S$ **essential** if all its irreducible components are non-spherical. We begin with the following simple observation.

Lemma 3.4. *J is essential.*

Proof. Let $J_1 \subseteq J$ denote the union of the non-spherical irreducible components of J . As P_{J_1} has finite index in P_J , the subgroup $O_1 \cap P_{J_1}$ is open of finite index in O_1 and stabilizes $R_{J_1}(C_0)$. The definition of J then yields $J_1 = J$. \square

Let us now describe the outline of the proof. Our first task will be to show that O_1 contains L_J^+ . We will see that this is equivalent to prove that O_1 acts transitively on the standard J -residue $R_J(C_0)$, or else that the stabilizer in O_1 of any apartment A containing C_0 is transitive on $R_J(C_0) \cap A$. Since each group $\text{Stab}_{O_1}(A)/\text{Fix}_{O_1}(A)$ can be identified with a subgroup of the Coxeter group W acting on A , we will be in a position to apply the results on Coxeter groups from the previous section. This will allow us to show that each $\text{Stab}_{O_1}(A)/\text{Fix}_{O_1}(A)$ contains a finite index parabolic subgroup of type $I_A \subseteq J$, and hence acts transitively on the corresponding residue.

We thus begin by defining some “maximal” subset I of J such that $\text{Stab}_{O_1}(A_1)$ acts transitively on $R_I(C_0) \cap A_1$ for a suitably chosen apartment A_1 containing C_0 . We then establish that I contains all the types I_A when A varies over all apartments containing C_0 . This eventually allows us to prove that in fact $I = J$, so that $\text{Stab}_{O_1}(A_1)$ is transitive on $R_J(C_0) \cap A_1$, or else that O_1 contains L_J^+ , as desired.

We next show that O_1 contains the unipotent radical $U_{J \cup J^\perp}$. Finally, we make use of the transitivity of O_1 on $R_J(C_0)$ to prove that O is contained in the desired parabolic subgroup.

3.4. Proof of Theorem 3.3: O_1 contains L_J^+ . We first need to introduce some additional notation which we will retain until the end of the proof.

Let $\mathcal{A}_{\geq C_0}$ denote the set of apartments of Δ containing C_0 . For $A \in \mathcal{A}_{\geq C_0}$, set $N_A := \text{Stab}_{O_1}(A)$ and $\bar{N}_A = N_A/\text{Fix}_{O_1}(A)$, which one identifies with a subgroup of W . Finally, for $h \in N_A$, denote by \bar{h} its image in $\bar{N}_A \leq W$. Here is the main tool developed in the previous section.

Lemma 3.5. *For all $A \in \mathcal{A}_{\geq C_0}$, there exists $h \in N_A$ such that*

$$\text{Pc}(\bar{h}) = \langle r_\alpha \mid \alpha \bar{h}\text{-essential root of } \Phi \rangle$$

and is of finite index in $\text{Pc}(\bar{N}_A)$.

Proof. This is an immediate consequence of Corollary 2.17 and Lemma 2.7. \square

Lemma 3.6. *Let $(g_n)_{n \in \mathbf{N}}$ be an infinite sequence of elements of O_1 . Then there exist an apartment $A \in \mathcal{A}_{\geq C_0}$, a subsequence $(g_{\psi(n)})_{n \in \mathbf{N}}$ and elements $z_n \in O_1$, $n \in \mathbf{N}$, such that for all $n \in \mathbf{N}$ we have*

- (1) $h_n := z_0^{-1}z_n \in N_A$,
- (2) $d(C_0, z_n R) = d(C_0, g_{\psi(n)} R)$ for every residue R containing C_0 and
- (3) $|\text{d}(C_0, h_n C_0) - \text{d}(C_0, g_{\psi(n)} C_0)| < \text{d}(C_0, z_0 C_0)$.

Proof. As O_1 is open, it contains a finite index subgroup $K := \text{Fix}_G(B(C_0, r))$ of B for some $r \in \mathbf{N}$. Since B is transitive on the set $\mathcal{A}_{\geq C_0}$, we deduce that K has only finitely many orbits in $\mathcal{A}_{\geq C_0}$, say $\mathcal{A}_1, \dots, \mathcal{A}_k$. So, up to choosing a subsequence, we may assume that all chambers $g_n C_0$ belong to the same K -orbit \mathcal{A}_{i_0} of apartments. Hence there exist elements $x_n \in K \subset O_1$ and an apartment $A' \in \mathcal{A}_{i_0}$ containing C_0 such that $g'_n := x_n g_n \in O_1$, $g'_n C_0 \in A'$ and $\text{d}(C_0, g'_n C_0) = \text{d}(C_0, g_n C_0)$. For each n , we now choose an element of G stabilizing A' and mapping C_0 to $g'_n C_0$. Thus such an element is in the same right coset modulo B as g'_n . In particular, up to choosing a subsequence, we may assume it has the form $g'_n y_n b \in \text{Stab}_G(A')$ for some $y_n \in K$ and some $b \in B$ independent of n . Denote by $\{\psi(n) \mid n \in \mathbf{N}\}$ the resulting indexing set for the subsequence. Then setting $A := bA' \in \mathcal{A}_{\geq C_0}$, the sequence $z_n := g'_{\psi(n)} y_{\psi(n)} \in O_1$ is such that $h_n := z_0^{-1}z_n \in b \text{Stab}_G(A') b^{-1} \cap O_1 = \text{Stab}_{O_1}(A) = N_A$ and

$$|\text{d}(C_0, h_n C_0) - \text{d}(C_0, g_{\psi(n)} C_0)| = |\text{d}(z_0 C_0, z_n C_0) - \text{d}(C_0, z_n C_0)| < \text{d}(C_0, z_0 C_0).$$

\square

Lemma 3.7. *There exists an apartment $A \in \mathcal{A}_{\geq C_0}$ such that the orbit $N_A \cdot C_0$ is unbounded. In particular, the parabolic closure in W of \bar{N}_A is non-spherical.*

Proof. Since O_1 is non-compact, the orbit $O_1 \cdot C_0$ is unbounded in Δ . For $n \in \mathbf{N}$, choose $g_n \in O_1$ such that $\text{d}(C_0, g_n C_0) \geq n$. Then by Lemma 3.6, there exist an apartment $A \in \mathcal{A}_{\geq C_0}$ and elements $h_n \in N_A$ for n in some unbounded subset of \mathbf{N} such that $\text{d}(C_0, h_n C_0)$ is arbitrarily large when n varies. This proves the lemma. \square

Let $A_1 \in \mathcal{A}_{\geq C_0}$ be an apartment such that the type of the product of the non-spherical irreducible components of $\text{Pc}(\bar{N}_{A_1})$ is nonempty and maximal for this property. Such an apartment exists by Lemma 3.7. Now choose $h_{A_1} \in N_{A_1}$ as in Lemma 3.5, so that in particular $[\text{Pc}(\bar{N}_{A_1}) : \text{Pc}(\bar{h}_{A_1})] < \infty$. Up to conjugating O_1 by an element of P_J , we may then assume without loss of generality that $\text{Pc}(\bar{h}_{A_1})$ is standard, non-spherical, and has essential type I . Moreover, it is maximal in the following sense: if $A \in \mathcal{A}_{\geq C_0}$ is such that $\text{Pc}(\bar{N}_A)$ contains a parabolic subgroup of essential type I_A with $I_A \supseteq I$, then $I = I_A$.

Now that I is defined, we need some tool to show that O_1 contains sufficiently many root groups U_α . This will ensure that O_1 is “transitive enough” in two ways: first on residues in the building by showing it contains subgroups of the form \mathcal{L}_T^+ , and second on residues in apartments by establishing the presence in O_1 of enough

$n_\alpha \in \langle U_\alpha \cup U_{-\alpha} \rangle$, since these lift reflections r_α in stabilizers of apartments. This tool is provided by the so-called (FPRS) property from [CR09, 2.1], which we now state. Note for this that as O_1 is open, it contains the fixator in G of a ball of Δ : we fix $r \in \mathbf{N}$ such that $O_1 \supset K_r := \text{Fix}_G(B(C_0, r))$.

Lemma 3.8. *There exists a constant $N = N(W, S, r) \in \mathbf{N}$ such that for every root $\alpha \in \Phi$ with $d(C_0, \alpha) > N$, the root group $U_{-\alpha}$ is contained in $\text{Fix}_G(B(C_0, r)) = K_r$.*

Proof. See [CR09, Prop. 4]. \square

We also record a version of this result in a slightly more general setting.

Lemma 3.9. *Let $g \in G$ and let $A \in \mathcal{A}_{\geq C_0}$ containing the chamber $D := gC_0$. Also, let $b \in B$ such that $A = bA_0$, and let $\alpha = b\alpha_0$ be a root of A , with $\alpha_0 \in \Phi$. Then there exists $N = N(W, S, r) \in \mathbf{N}$ such that if $d(D, -\alpha) > N$ then $bU_{\alpha_0}b^{-1} \subseteq gK_rg^{-1}$.*

Proof. Take for $N = N(W, S, r)$ the constant of Lemma 3.8 and suppose that $d(D, -\alpha) > N$. Let $h \in \text{Stab}_G(A_0)$ be such that $hC_0 = b^{-1}D$. Then

$$N < d(D, -\alpha) = d(bhC_0, -b\alpha_0) = d(hC_0, -\alpha_0) = d(C_0, -h^{-1}\alpha),$$

and so Lemma 3.8 implies $h^{-1}U_{\alpha_0}h = U_{h^{-1}\alpha_0} \subseteq K_r$. Let $b_1 \in B$ such that $bh = gb_1$. Then

$$bU_{\alpha_0}b^{-1} \subseteq bhK_rh^{-1}b^{-1} = gb_1K_rb_1^{-1}g^{-1} = gK_rg^{-1}.$$

\square

This will prove especially useful in the following form, when we will use the description of the parabolic closure of some $w \in W$ in terms of w -essential roots as in Lemma 3.5.

Lemma 3.10. *Let $A \in \mathcal{A}_{\geq C_0}$ and $b \in B$ such that $A = bA_0$. Also, let $\alpha = b\alpha_0$ ($\alpha_0 \in \Phi$) be a w -essential root of A for some $w \in \text{Stab}_G(A)/\text{Fix}_G(A)$, and let $g \in \text{Stab}_G(A)$ be a representative of w . Then there exists $n \in \mathbf{Z}$ such that for $\epsilon \in \{+, -\}$ we have $U_{\epsilon\alpha_0} \subseteq b^{-1}g^{\epsilon n}K_rg^{-\epsilon n}$.*

Proof. Choose $n \in \mathbf{Z}$ such that $d(g^{\epsilon n}C_0, -\epsilon\alpha) > N$ for $\epsilon \in \{+, -\}$, where $N = N(W, S, r)$ is the constant appearing in the statement of Lemma 3.8. Thus, for $\epsilon \in \{+, -\}$ we have $d(b^{-1}g^{\epsilon n}C_0, -\epsilon\alpha_0) > N$, and so $d(C_0, -\epsilon(b^{-1}g^{-\epsilon n}b)\alpha_0) > N$. Lemma 3.8 then yields

$$(b^{-1}g^{-\epsilon n}b)U_{\epsilon\alpha_0}(b^{-1}g^{-\epsilon n}b)^{-1} = U_{\epsilon(b^{-1}g^{-\epsilon n}b)\alpha_0} \subseteq K_r,$$

and so

$$U_{\epsilon\alpha_0} \subseteq (b^{-1}g^{\epsilon n}b)K_r(b^{-1}g^{-\epsilon n}b) = (b^{-1}g^{\epsilon n})K_r(g^{-\epsilon n}b).$$

\square

We are now ready to prove how the different transitivity properties of O_1 are related.

Lemma 3.11. *Let $T \subseteq S$ be essential, and let $A \in \mathcal{A}_{\geq C_0}$. Then the following are equivalent:*

- (1) O_1 contains \mathcal{L}_T^+ ;
- (2) O_1 is transitive on $R_T(C_0)$;
- (3) N_A is transitive on $R_T(C_0) \cap A$;
- (4) \bar{N}_A contains the standard parabolic subgroup W_T of W .

Proof. The equivalence (3) \Leftrightarrow (4), as well as the implications (1) \Rightarrow (2), (3) are trivial.

To see that (4) \Rightarrow (2), note that if $b \in B$ maps A_0 onto A , then for each $\alpha_0 \in \Phi_T$, we have $bU_{\pm\alpha_0}b^{-1} \subseteq O_1$, and so $O_1 \supseteq b\mathcal{L}_T^+b^{-1}$ is transitive on $R_T(C_0)$. Indeed, let $\alpha_0 \in \Phi_T$ and consider the corresponding root $\alpha := b\alpha_0 \in \Phi_T(A)$ of A . By Lemma 2.19, there exists $w \in W_T \subseteq \overline{N}_A$ such that α is w -essential. Then if $g \in O_1$ is a representative for w , Lemma 3.10 yields an $n \in \mathbf{Z}$ such that for $\epsilon \in \{+, -\}$ we have $U_{\epsilon\alpha} \subseteq b^{-1}g^{\epsilon n}K_r g^{-\epsilon n}b \subseteq b^{-1}O_1b$.

Finally, we show (2) \Rightarrow (1). Again, it is sufficient to check that if $\alpha \in \Phi_T$, then O_1 contains $U_{\epsilon\alpha}$ for $\epsilon \in \{+, -\}$. By Lemma 2.19, there exists $g \in \text{Stab}_G(A_0)$ stabilizing $R_T(C_0) \cap A_0$ such that α is \bar{g} -essential, where \bar{g} denotes the image of g in the quotient group $\text{Stab}_G(A_0)/\text{Fix}_G(A_0)$. Then, by Lemma 3.10, one can find an $n \in \mathbf{Z}$ such that $U_{\epsilon\alpha} \subseteq g^{\epsilon n}K_r g^{-\epsilon n}$ for $\epsilon \in \{+, -\}$. Now, since O_1 is transitive on $R_T(C_0)$, there exist $h_\epsilon \in O_1$ such that $h_\epsilon C_0 = g^{\epsilon n}C_0$, and so we find $b_\epsilon \in B$ such that $g^{\epsilon n} = h_\epsilon b_\epsilon$. Therefore

$$U_{\epsilon\alpha} \subseteq h_\epsilon b_\epsilon K_r b_\epsilon^{-1} h_\epsilon^{-1} = h_\epsilon K_r h_\epsilon^{-1} \subseteq O_1.$$

□

Now, to ensure that O_1 indeed satisfies one of those properties for some “maximal T ”, we use Lemma 3.5 to show that stabilizers in O_1 of apartments contain finite index parabolic subgroups.

Lemma 3.12. *Let $A \in \mathcal{A}_{\geq C_0}$. Then there exists $I_A \subseteq S$ such that \overline{N}_A contains a parabolic subgroup P_{I_A} of W of type I_A as a finite index subgroup.*

Proof. Choose $h \in N_A$ as in Lemma 3.5, so that in particular $\text{Pc}(\bar{h})$ is generated by the reflections r_α with α an \bar{h} -essential root of A . Let $\alpha = b\alpha_0$ be such a root ($\alpha_0 \in \Phi$), where $b \in B$ maps A_0 onto A . By Lemma 3.10, we then find $K \in \mathbf{Z}$ such that for $\epsilon \in \{+, -\}$,

$$U_{\epsilon\alpha} \subseteq (b^{-1}h^{\epsilon K})K_r(h^{-\epsilon K}b) \subseteq b^{-1}O_1b.$$

In particular, $n_{\alpha_0} \in \langle U_{\alpha_0} \cup U_{-\alpha_0} \rangle \subseteq b^{-1}O_1b$. As r_{α_0} is the image in W of n_{α_0} and since $r_\alpha = br_{\alpha_0}b^{-1}$, we finally obtain $\text{Pc}(\bar{h}) \subseteq \overline{N}_A$. Then $P_{I_A} := \text{Pc}(\bar{h})$ is the desired parabolic subgroup, of type I_A . □

For each $A \in \mathcal{A}_{\geq C_0}$, we fix such an $I_A \subseteq S$ which, without loss of generality, we assume essential. We also consider the corresponding parabolic P_{I_A} contained in \overline{N}_A . Note then that $P_{I_{A_1}}$ has finite index in $\text{Pc}(\overline{N}_{A_1})$ by Lemma 2.4, and so $I = I_{A_1}$.

Lemma 3.13. *O_1 contains L_I^+ .*

Proof. As noted above, we have $I = I_{A_1}$ and $P_I = W_I$. Since O_1 is closed in G , Lemma 3.11 allows us to conclude. □

We now have to show that I is “big enough”, that is, $I = J$. For this, we first need to know that I is “uniformly” maximal amongst all apartments containing C_0 .

Lemma 3.14. *Let $A \in \mathcal{A}_{\geq C_0}$. Then $I_A \subseteq I$.*

Proof. Set $R_1 := R_I(C_0) \cap A$ and let R_2 be an I_A -residue in A on which N_A acts transitively and that is at minimal distance from R_1 amongst such residues. Note that N_A is transitive on R_1 as well by Lemma 3.11.

If $R_1 \cap R_2$ is nonempty, then N_A is also transitive on the standard $I \cup I_A$ -residue of A and so \overline{N}_A contains $W_{I \cup I_A}$. By maximality of I and since $I \cup I_A$ is again essential, this implies $I_A \subseteq I$, as desired.

We henceforth assume that $R_1 \cap R_2 = \emptyset$. Let $b \in B$ such that $bA_0 = A$. Consider a root $\alpha = b\alpha_0$ of A , $\alpha_0 \in \Phi$, whose wall $\partial\alpha$ separates R_1 from R_2 .

If both R_1 and R_2 are at unbounded distance from $\partial\alpha$, then the transitivity of N_A on R_1 and R_2 together with Lemma 3.9 yield $bU_{\pm\alpha_0}b^{-1} \subseteq K_r \subseteq O_1$. Since $r_{\alpha_0} \in \langle U_{\alpha_0} \cup U_{-\alpha_0} \rangle$, we thus have $r_\alpha := br_{\alpha_0}b^{-1} \in O_1$ and so $r_\alpha \in \overline{N}_A$. But then $\overline{N}_A = r_\alpha \overline{N}_A r_\alpha^{-1}$ is also transitive on the I_A -residue $r_\alpha R_2$ which is closer to R_1 , a contradiction.

If R_2 is at bounded distance from $\partial\alpha$ then by Lemma 2.20, r_α centralizes the stabilizer P in W of R_2 , that is, $P = r_\alpha P r_\alpha^{-1}$. Note that \overline{N}_A contains P since it is transitive on R_2 . Thus N_A is transitive on the I_A -residue $r_\alpha R_2$, which is closer to R_1 , again a contradiction.

Thus we are left with the case where R_1 is contained in a tubular neighbourhood of every wall $\partial\alpha$ separating R_1 from R_2 . But in that case, Lemma 2.20 again yields that W_I is centralized by every reflection r_α associated to such walls. Choose chambers C_i in R_i , $i = 1, 2$, such that $d(C_1, C_2) = d(R_1, R_2)$, and let $\partial\alpha_1, \dots, \partial\alpha_k$ be the walls separating C_1 from C_2 , crossed in that order by a minimal gallery from C_1 to C_2 . Then each α_i , $1 \leq i \leq k$, separates R_1 from R_2 and so $w := r_{\alpha_k} \dots r_{\alpha_1}$ centralizes W_I and maps C_1 to C_2 . So $W_I = wW_I w^{-1} \subseteq \overline{N}_A$ is transitive on wR_1 and R_2 , and hence also on $R_{I \cup I_A}(C_2) \cap A$. Therefore \overline{N}_A contains a parabolic subgroup of essential type $I \cup I_A$, so that $I \supseteq I_A$ by maximality of I , as desired. \square

Lemma 3.15. *Let $A \in \mathcal{A}_{\geq C_0}$. Then \overline{N}_A contains W_I as a subgroup of finite index.*

Proof. We know by Lemmas 3.11 and 3.13 that \overline{N}_A contains W_I . Also, by Lemma 3.12, \overline{N}_A contains a finite index parabolic subgroup $P_{I_A} = wW_{I_A}w^{-1}$ of type I_A , for some $w \in W$. Since $I_A \subseteq I$ by Lemma 3.14, we get $W_{I_A} \subseteq \overline{N}_A$ and so the parabolic subgroup $P := W_{I_A} \cap wW_{I_A}w^{-1}$ has finite index in W_{I_A} . As I_A is essential, [AB08, Prop.2.43] then yields $P = W_{I_A}$ and so $W_{I_A} \subseteq wW_{I_A}w^{-1}$. Finally, since the chain $W_{I_A} \subseteq wW_{I_A}w^{-1} \subseteq w^2W_{I_A}w^{-2} \subseteq \dots$ stabilizes, we find that $W_{I_A} = P_{I_A}$ has finite index in \overline{N}_A . The result follows. \square

We are now ready to make the announced connection between I and J .

Lemma 3.16. $I = J$.

Proof. Let \mathcal{R} denote the set of I -residues of Δ containing a chamber of $O_1 \cdot C_0$, and set $R := R_I(C_0)$. We first show that the distance from C_0 to the residues of \mathcal{R} is bounded, and hence that \mathcal{R} is finite.

Indeed, suppose for a contradiction that there exists a sequence of elements $g_n \in O_1$ such that $d(C_0, g_n R) \geq n$ for all $n \in \mathbf{N}$. Then, up to choosing a subsequence and relabeling, Lemma 3.6 yields an apartment $A \in \mathcal{A}_{\geq C_0}$ and a sequence $(z_n)_{n \geq n_0}$ of elements of O_1 such that $h_n := z_n^{-1} z_{n_0} \in N_A$ and $d(C_0, z_n R) = d(C_0, g_n R)$. Moreover by Lemma 3.15, we have a finite coset decomposition of the form $\overline{N}_A = \coprod_{j=1}^t v_j W_I$. Denote by $\pi: N_A \rightarrow \overline{N}_A$ the natural projection. Again

up to choosing a subsequence and relabeling, we may assume that $\pi(h_n) = v_{j_0} u_n$ for all $n \geq n_1$ (for some fixed $n_1 \in \mathbf{N}$), where each $u_n \in W_I$ and where j_0 is independant of n . Then the elements $w_n := \pi(h_{n_1}^{-1} h_n) = \pi(z_{n_1}^{-1} z_n)$ belong to W_I . Thus the chambers $z_{n_1} C_0$ and $z_n C_0$ belong to the same I -residue since z_{n_1} maps an I -gallery between C_0 and $w_n C_0$ to an I -gallery between $z_{n_1} C_0$ and $z_n C_0$. Therefore

$$d(C_0, g_n R) = d(C_0, z_n R) \leq d(C_0, z_{n_1} C_0)$$

and so $d(C_0, g_n R)$ is bounded, a contradiction.

So \mathcal{R} is finite and is stabilized by O_1 . Hence the kernel O' of the induced action of O_1 on \mathcal{R} is a finite index subgroup of O_1 stabilizing an I -residue. Up to conjugating by an element of O_1 , we thus have $O' < P_I$ and $[O_1 : O'] < \infty$. Then $O'' := O_1 \cap P_I$ is open and contains O' , and has therefore finite index in O_1 . The definition of J finally implies that $I = J$. \square

In particular, Lemmas 3.13 and 3.16 yield the following.

Corollary 3.17. *O_1 contains L_J^+ .*

3.5. Proof of Theorem 3.3: O_1 contains the unipotent radical $U_{J \cup J^\perp}$. To show that O_1 contains the desired unipotent radical, we again make use of the (FPRS) property.

Lemma 3.18. *O_1 contains the unipotent radical $U_{J \cup J^\perp}$.*

Proof. By definition of $U_{J \cup J^\perp}$, we just have to check that for every $b \in B$ and every $\alpha \in \Phi$ containing $R_{J \cup J^\perp}(C_0) \cap A_0$, we have $bU_\alpha b^{-1} \in O_1$. Fix such b and α . In particular, α contains $R := R_J(C_0) \cap A_0$. We claim that R is at unbounded distance from the wall $\partial\alpha$ associated to α . Indeed, if it were not, then as J is essential by Lemma 3.4, the reflection r_α would centralize W_J by Lemma 2.20, and hence would belong to W_{J^\perp} by Lemma 2.1, contradicting $\alpha \supset R_{J^\perp}(C) \cap A_0$.

Set now $A = bA_0$. Then $\alpha' = b\alpha$ is a root of A containing $R' := R_J(C_0) \cap A$. Moreover, R' is at unbounded distance from $-\alpha'$. Since O_1 is transitive on $R_J(C_0)$ by Corollary 3.17, there exists $g \in O_1$ such that $D := gC_0 \in R_J(C_0) \cap A$ and $d(D, -\alpha') > N$, where N is provided by Lemma 3.9. This lemma then implies that $bU_\alpha b^{-1} \subseteq gK_r g^{-1} \subseteq O_1$, as desired. \square

3.6. Proof of Theorem 3.3: endgame. We can now prove that gOg^{-1} is contained in a parabolic subgroup that has P_J as a finite index subgroup.

Lemma 3.19. *Every subgroup H of G containing O_1 as a subgroup of finite index is contained in some standard parabolic $P_{J \cup J'}$ of type $J \cup J'$, with J' spherical and $J' \subseteq J^\perp$.*

Proof. Recall that O_1 stabilizes the J -residue $R := R_J(C_0)$ and acts transitively on its chambers by Corollary 3.17. Let \mathcal{R} be the (finite) set of J -residues of Δ containing a chamber in the orbit $H \cdot C_0$.

We first claim that for any $R' \in \mathcal{R}$ there is a constant M such that R is contained in an M -neighbourhood of R' (and since \mathcal{R} is finite we may then as well assume that this constant M is independant of R'). Indeed, because \mathcal{R} is finite, there is a finite index subgroup H' of H which stabilizes R' . In particular $d(D, R') = d(H' \cdot D, R')$ for any chamber D of R . Moreover, the chambers of R are contained in finitely H' -orbits since H acts transitively on R . The claim follows.

Let now $J' \subseteq S \setminus J$ be minimal such that $\mathbf{R} := R_{J \cup J'}(C_0)$ contains the reunion of the residues of \mathcal{R} . In other words, $H < P_{J \cup J'}$ with J' minimal for this property.

We next show that $J' \subseteq J^\perp$. For this, it is sufficient to see that H stabilizes $R_{J \cup J^\perp}(C_0)$.

Note that, given $R' \in \mathcal{R}$, if A is an apartment containing some chamber C'_0 of R' , then every chamber D in $R \cap A$ is at distance at most M from $R' \cap A$. Indeed, if $\rho = \rho_{A, C'_0}$ is the retraction of Δ onto A centered at C'_0 , then for every $D' \in R'$ such that $d(D, D') \leq M$, the chamber $\rho(D')$ belongs to $R' \cap A$ and is at distance at most M from $D = \rho(D)$ since ρ is distance decreasing (see [Dav98, Lemma 11.2]).

Let now $g \in H$ and set $R' := gR \in \mathcal{R}$. Let Γ be a minimal gallery from C_0 to its combinatorial projection onto R' , which we denote by C'_0 . Let A be an apartment containing Γ . Finally, let $w \in W = \text{Stab}_G(A)/\text{Fix}_G(A)$ such that $wC_0 = C'_0$. We want to show that Γ is a J^\perp -gallery, that is, $w \in W_{J^\perp}$.

To this end, we first observe that, since Γ joins C_0 to its projection onto R' , it does not cross any wall of $R' \cap A$. We claim that Γ does not cross any wall of $R \cap A$ either. Indeed, assume on the contrary that Γ crosses some wall m of $R \cap A$. Then by Lemma 2.19 we would find a wall $m' \neq m$ intersecting $R \cap A$ and parallel to m , and therefore also chambers of $R \cap A$ at unbounded distance from $R' \cap A$, a contradiction.

Thus every wall crossed by Γ separates $R \cap A$ from $R' \cap A$. In particular, $R \cap A$ is contained in an M -neighbourhood of any such wall m since it is contained in an M -neighbourhood of $R' \cap A$ and since every minimal gallery between a chamber in $R \cap A$ and a chamber in $R' \cap A$ crosses m . Then, by Lemmas 2.1 and 2.20, the reflection associated to m belongs to W_{J^\perp} . Therefore w is a product of reflections that belong to W_{J^\perp} , as desired.

Finally, we show that J' is spherical. As \mathbf{R} splits into a product of buildings $\mathbf{R} = R_J \times R_{J'}$, where $R_J := R_J(C_0)$ and $R_{J'} := R_{J'}(C_0)$, we get a homomorphism $H \rightarrow \text{Aut}(R_J) \times \text{Aut}(R_{J'})$. As O_1 stabilizes R_J and has finite index in H , the image of H in $\text{Aut}(R_{J'})$ has finite orbits in $R_{J'}$. In particular, by the Bruhat–Tits fixed point theorem, H fixes a point in the Davis realization of $R_{J'}$, and thus stabilizes a spherical residue of $R_{J'}$. But this residue must be the whole of $R_{J'}$ by minimality of J' . This concludes the proof of the lemma. \square

Proof of Theorem 3.3. The first statement summarizes Corollary 3.17 and Lemmas 3.18 and 3.19, since some conjugate gOg^{-1} of O contains O_1 as a finite index subgroup. The second statement then follows from Lemma 3.2 applied to the open subgroup O_1 of P_J . Finally, the two last statements are a consequence of Lemma 3.19. Indeed, any subgroup H containing gOg^{-1} with finite index also contains O_1 with finite index. Then H is a subgroup of some standard parabolic $P_{J \cup J'}$ for some spherical subset $J' \subset J^\perp$. Moreover, since the index of O_1 in $P_{J \cup J'}$ is finite, and since there are only finitely many spherical subsets of J^\perp , it follows that there are only finitely many possibilities for H . \square

Remark 3.20. Let O be a subgroup of G , and let $J \subseteq S$ be as in the statement of Theorem 3.3. Assume that J^\perp is spherical. Then $L_J^+ \cdot U_{J \cup J^\perp}$ has finite index in $P_{J \cup J^\perp}$ and is thus open since it is closed. Thus, in that case, O is open if and only if $L_J^+ \cdot U_{J \cup J^\perp} < gOg^{-1} < P_{J \cup J^\perp}$ for some $g \in G$.

Corollary 3.21. *Let O be an open subgroup of G and let $J \subseteq S$ be minimal such that O virtually stabilizes a J -residue. If $J^\perp = \emptyset$, then there exists some $g \in G$ such that $L_J^+ \cdot U_J < gOg^{-1} < P_J = \mathcal{H} \cdot L_J^+ \cdot U_J$.*

Proof. This readily follows from Theorem 3.3. \square

To prove Corollary B, we use the following general fact, which is well known in the discrete case.

Lemma 3.22. *Let G be a locally compact group. Then G is Noetherian if and only if every open subgroup is compactly generated.*

Proof. Assume that G is Noetherian and let $O < G$ be open. Let $U_1 < O$ be the subgroup generated by some compact identity neighbourhood V in O . If $U_1 \neq O$, there is some $g_1 \in O \setminus U_1$ and we let $U_2 = \langle U_1 \cup \{g_1\} \rangle$. Proceeding inductively we obtain an ascending chain of open subgroups $U_1 < U_2 < \dots < O$, and the ascending chain condition ensures that $O = U_n$ for some n . In other words O is generated by the compact set $V \cup \{g_1, \dots, g_n\}$.

Assume conversely that every open subgroup is compactly generated, and let $U_1 < U_2 < \dots$ be an ascending chain of open subgroups. Then $U = \bigcup_n U_n$ is an open subgroup. Let C be a compact generating set for U . By compactness, the inclusion $C \subset \bigcup_n U_n$ implies that C is contained in U_n for some n since every U_j is open. Thus $U = \langle C \rangle < U_n$, whence $U = U_n$ and G is Noetherian. \square

Proof of Corollary B. By Theorem A, every open subgroup of a complete Kac–Moody group G over a finite field is contained as a finite index subgroup in some parabolic subgroup. Notice that parabolic subgroups are compactly generated by the Svarc–Milnor Lemma since they act properly and cocompactly on the residue of which they are the stabilizer. Since a cocompact subgroup of a group acting cocompactly on a space also acts cocompactly on that space, it follows for the same reason that all open subgroups of G are compactly generated; hence G is Noetherian by Lemma 3.22. \square

Proof of Corollary C. Immediate from Theorem A since Coxeter groups of affine and compact hyperbolic type are precisely those Coxeter groups all of whose proper parabolic subgroups are finite. \square

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