POSITIVE ENERGY REPRESENTATIONS
FOR LOCALLY FINITE SPLIT LIE ALGEBRAS

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ABSTRACT. Let $\mathfrak{g}$ be a locally finite split simple complex Lie algebra of type A_J, B_J, C_J or D_J and $\mathfrak{h} \subseteq \mathfrak{g}$ be a splitting Cartan subalgebra. Fix $D \in \operatorname{der}(\mathfrak{g})$ with $\mathfrak{h} \subseteq \ker D$ (a diagonal derivation). Then every unitary highest weight representation $(\rho_\lambda, V^\lambda)$ of $\mathfrak{g}$ extends to a representation $\tilde{\rho}_\lambda$ of the semidirect product $\mathfrak{g} \rtimes \mathbb{C}D$ and we say that $\tilde{\rho}_\lambda$ is a positive energy representation if the spectrum of $-i\tilde{\rho}_\lambda(D)$ is bounded from below. In the present note we characterise all pairs (λ, D) with λ bounded for which this is the case.

If $U_1(\mathcal{H})$ is the unitary group of Schatten class 1 on an infinite dimensional real, complex or quaternionic Hilbert space and λ is bounded, then we accordingly obtain a characterisation of those highest weight representations π_λ satisfying the positive energy condition with respect to the continuous $\mathbb{R}$-action induced by D. In this context the representation π_λ is norm continuous and our results imply the remarkable result that, for positive energy representations, adding a suitable inner derivation to D, we can achieve that the minimal eigenvalue of $\tilde{\rho}_\lambda(D)$ is 0 (minimal energy condition). The corresponding pairs (λ, D) satisfying the minimal energy condition are rather easy to describe explicitly.

1. INTRODUCTION

Locally finite split Lie algebras are natural infinite-dimensional generalisations of finite-dimensional Lie algebras. More precisely, a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ of characteristic zero is called *split* if it has a root decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ with respect to some maximal abelian subalgebra $\mathfrak{h}$. It is moreover *locally finite* if every finite subset of $\mathfrak{g}$ generates a finite-dimensional subalgebra. Such a Lie algebra $\mathfrak{g}$ possesses a generalised Levi decomposition (see [Stu99]), whose Levi factor is an $\mathfrak{h}$-invariant semisimple Lie algebra. In turn, the infinite-dimensional locally finite split simple Lie algebras have been classified (see [NS01]), and can be realised as subalgebras of the algebra $\mathfrak{gl}(J, \mathbb{K})$ of $J \times J$ matrices with only finitely many nonzero entries, for some infinite set J. They fall into four distinct families of isomorphism classes, parametrised by the *locally finite root systems* of type A_J, B_J, C_J and D_J (see [LN04]).

On the other hand, a locally finite split simple Lie algebra $\mathfrak{g}$ is the directed union of its finite-dimensional simple subalgebras ([Stu99, Section V]). Unitary highest weight representations for such Lie algebras over $\mathbb{K} = \mathbb{C}$ were studied in [Nee98]. By [Stu99, Section VIII], $\mathfrak{g}$ carries an antilinear involutive antiautomorphism $X \mapsto X^*$ such that $\mathfrak{g}_\mathbb{R} = \{X \in \mathfrak{g} \mid X^* = -X\}$ is a compact real form, that is, $\mathfrak{g}_\mathbb{R}$ is a union of finite dimensional compact Lie algebras. A $\mathfrak{g}$-module V is then called *unitary* if it carries a contravariant positive definite hermitian form $\langle \cdot, \cdot \rangle$, in the sense that $\langle X.v, w \rangle = \langle v, X^*.w \rangle$ for all $v, w \in V$, $X \in \mathfrak{g}$. A $\mathfrak{g}$-module $V = V^\lambda$ is called a *highest weight module* with highest weight $\lambda \in \mathfrak{h}^*$ (with respect to some positive system $\Delta_+ \subseteq \Delta$ of roots) if it is generated by some primitive element, that is, by some $\mathfrak{h}$-weight vector $v \in V$ with weight λ such that $\mathfrak{g}_\alpha.v = \{0\}$ for all $\alpha \in \Delta_+$. Unitary highest weight modules V^λ for $\mathfrak{g}$ were classified in [Nee98], and correspond to dominant integral weights λ. Moreover, if $W \leq \operatorname{GL}(\mathfrak{h}^*)$ denotes the Weyl group of $\mathfrak{g}$, the set $\mathcal{P}_\lambda$ of $\mathfrak{h}$-weights on V^λ is given by $\mathcal{P}_\lambda = \operatorname{conv}(W.\lambda) \cap (\lambda + \mathbb{Z}[\Delta])$ ([Nee98, Theorem I.11]).

In this paper, we characterise the unitary highest weight representations of $\mathfrak{g}$ satisfying the following "positive energy condition". As a subalgebra of some $\mathfrak{gl}(J, \mathbb{C})$, the Lie algebra $\mathfrak{g}$ inherits a hermitian scalar product given by $\langle X, Y \rangle = \operatorname{tr}(XY^*)$. A derivation $D \in \operatorname{der}(\mathfrak{g})$ of $\mathfrak{g}$ is *skew-hermitian*

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if $\langle DX, Y \rangle = -\langle X, DY \rangle$ for all $X, Y \in \mathfrak{g}$. A skew-hermitian derivation D of $\mathfrak{g}$ is called *graded* if it annihilates $\mathfrak{h}$, in which case D preserves the root spaces of $\mathfrak{g}$. Such a derivation is then described by a character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$ such that $D(x_\alpha) = i\chi(\alpha)x_\alpha$ for all $x_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta$. Any unitary highest weight representation $\rho_\lambda \colon \mathfrak{g} \to \mathrm{End}(V^\lambda)$ of $\mathfrak{g}$ may be extended to a representation $\widetilde{\rho}_\lambda \colon \mathfrak{g} \rtimes \mathbb{C}D \to \mathrm{End}(V^\lambda)$ by setting $\widetilde{\rho}_\lambda(D)v_\mu = i\chi(\mu - \lambda)v_\mu$ for any $\mu \in \mathcal{P}_\lambda$ and any weight vector $v_\mu \in V^\lambda$ of weight μ. The representation $\widetilde{\rho}_\lambda$ is called a *positive energy representation* if the spectrum of $H := -i\widetilde{\rho}_\lambda(D)$ is bounded from below. We call H a *Hamiltonian* of the representation ρ_λ.

In view of the above description of $\mathcal{P}_\lambda$, the positive energy condition can then be rewritten as

$$\inf \chi(W.\lambda - \lambda) > -\infty.$$

Let J be an infinite set such that Δ is of one of the types A_J, B_J, C_J or D_J. Then there exists some basis $(e_j)_{j \in J}$ of $\mathfrak{h}$ (respectively a one-dimensional extension of $\mathfrak{h}$) indexed by J such that the set of *coroots* $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$ (see [Stu99, Section I]) is contained in $\mathrm{span}_\mathbb{Z}(e_j)_{j \in J}$, while Δ is contained in the $\mathbb{Z}$-span of the linearly independent system $(\epsilon_j)_{j \in J} \subseteq \mathfrak{h}^*$ defined by $\epsilon_j(e_k) = \delta_{jk}$ for all $j, k \in J$ (see [NS01, Section I and Theorem IV.6]).

We recall that the weight $\lambda \in \mathfrak{h}^*$ is called *integral* if $\lambda(\alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in \Delta$. This implies in particular that λ is *discrete*, in the sense that $\{\lambda_j := \lambda(e_j) \mid j \in J\}$ is a discrete subset of $\mathbb{R}$. We moreover call λ *bounded* if $\sup_{j \in J} |\lambda_j| < \infty$. Finally, a character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$ is said to be *summable* if it is the restriction of a homomorphism $\widetilde{\chi} \colon \mathrm{span}_\mathbb{Z}(\epsilon_j)_{j \in J} \to \mathbb{R}$ satisfying

$$\sum_{j \in J} |\widetilde{\chi}(\epsilon_j)| < \infty.$$

We can now state the main result of this paper, which provides a characterisation of the positive energy highest weight representations of $\mathfrak{g}$ with bounded highest weight.

Theorem A. *Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite split simple Lie algebra with root system Δ and Weyl group $W \leq \mathrm{GL}(\mathfrak{h}^*)$. Let $\lambda \in \mathfrak{h}^*$ be discrete and bounded. Then for a character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$, the following are equivalent:*

(1) $\inf \chi(W.\lambda - \lambda) > -\infty$.
(2) $\chi = \chi_{\min} + \chi_{\mathrm{sum}}$ *for some "minimal energy" character $\chi_{\min}$, satisfying $\inf \chi_{\min}(W.\lambda - \lambda) = 0$, and some summable character χ_{sum}.*

We also provide a description of minimal energy characters (see Proposition 5.3 and Remark 5.9). In [HN12, Corollary 3.2] a similar description of the minimal energy characters was obtained by Coxeter geometry. A priori, these can be described much more easily than the "positive energy" characters and the main point of the theorem is that it reduces the latter problem to the former. Note that, as can be seen from Proposition 5.3, the cone of minimal energy characters contains many summable characters, so that the decomposition $\chi = \chi_{\min} + \chi_{\mathrm{sum}}$ is non-unique. The proof of Theorem A is given in Section 6 below.

The assumption that the highest weight λ be bounded in Theorem A is motivated by the study of positive energy (projective) unitary representations of the corresponding Lie groups, which we now briefly review. We then state a corollary of Theorem A in this context.

The restriction of the scalar product $\langle \cdot, \cdot \rangle$ on the locally finite split simple Lie algebra $\mathfrak{g}$ to its real form $\mathfrak{g}_\mathbb{R}$ defines an *invariant* scalar product on $\mathfrak{g}_\mathbb{R}$, in the sense that $\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle$ for all $X, Y, Z \in \mathfrak{g}_\mathbb{R}$. The Hilbert space completion of $\mathfrak{g}_\mathbb{R}$ is then a so-called *Hilbert–Lie algebra*, that is, a real Lie algebra and a real Hilbert space with compatibility of the two structures given by the invariance of the scalar product. For instance, if $\mathfrak{g} = \mathfrak{sl}(J, \mathbb{C})$ is the subalgebra of $\mathfrak{gl}(J, \mathbb{C})$ of traceless matrices, the corresponding completion is the space $\mathfrak{u}_2(\mathcal{H})$ of skew-symmetric Hilbert-Schmidt operators on the complex Hilbert space $\mathcal{H} = \ell^2(J, \mathbb{C})$.

By a theorem of Schue[1] ([Sch61]), Hilbert–Lie algebras decompose into an orthogonal direct sum of simple ideals (and center). Moreover, each simple infinite-dimensional Hilbert–Lie algebra is isomorphic

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[1]In [Sch61], the complexifications of Hilbert–Lie algebras are called L^*-algebras.

to $\mathfrak{u}_2(\mathcal{H})$, for some infinite-dimensional real, complex or quaternionic Hilbert space $\mathcal{H}$. These are, in turn, classified by the locally finite root systems (see [Nee12, Examples C.4,5,6]).

Going back to our example $\mathfrak{g} = \mathfrak{sl}(J, \mathbb{C})$, we let $\rho_\lambda \colon \mathfrak{g} \to \operatorname{End}(V^\lambda)$ be, as before, a unitary highest weight representation of $\mathfrak{g}$ with highest weight λ. Then, under the assumption that λ is bounded, the restriction of ρ_λ to $\mathfrak{g}_\mathbb{R}$ extends to a continuous unitary representation $\rho_\lambda \colon \mathfrak{u}_1(\mathcal{H}) \to \mathfrak{u}(\mathcal{H}_\lambda)$ with $\|\rho_\lambda\| \leq \sup_{j \in J} |\lambda_j|$, where $\mathfrak{u}_1(\mathcal{H}) \subseteq \mathfrak{u}_2(\mathcal{H})$ denotes the Banach space of skew-hermitian trace-class operators on $\mathcal{H}$ and $\mathcal{H}_\lambda$ is the Hilbert space completion of V^λ (see [Nee98, Proposition III.7]). Moreover, if $\lambda_j \in \mathbb{Z}$ for all $j \in J$, then ρ_λ exponentiates to a holomorphic representation

$$\widehat{\rho}_\lambda \colon U_1(\mathcal{H}) \to U(\mathcal{H}_\lambda)$$

from $U_1(\mathcal{H}) = \operatorname{GL}(\mathcal{H}) \cap (\mathbf{1} + \mathfrak{u}_1(\mathcal{H}))$ to the unitary group $U(\mathcal{H}_\lambda)$.

Assume now that the Lie group $G = U_1(\mathcal{H})$ is endowed with a continuous $\mathbb{R}$-action, given by a homomorphism $\alpha \colon \mathbb{R} \to \operatorname{Aut}(G) : t \mapsto \alpha_t$. If $U \colon \mathbb{R} \to U(\mathcal{H}_\lambda) : t \mapsto U_t$ is a unitary representation of $\mathbb{R}$ on $\mathcal{H}_\lambda$, then the map

$$\pi_\lambda \colon G \rtimes_\alpha \mathbb{R} \to U(\mathcal{H}_\lambda) : (g, t) \mapsto \rho_\lambda(g) U_t$$

is called a *covariant unitary representation* of $(G, \mathbb{R}, \alpha)$ if it defines a unitary representation on $\mathcal{H}_\lambda$ of the semi-direct product $G \rtimes_\alpha \mathbb{R}$. This representation is said to be of *positive energy* if the spectrum of the corresponding Hamiltonian $H := -i \frac{d}{dt}|_{t=0} U_t$ is bounded below.

In our setting, one can show[2] that α must be of the form

(1.1) $$\alpha_t(g) = e^{itA} g e^{-itA}$$

for some self-adjoint operator $A \in B(\mathcal{H})$. A sufficient condition for ρ_λ to extend to a covariant representation of $(G, \mathbb{R}, \alpha)$ is the diagonalisability of A: in this case, choosing the orthonormal basis $(e_j)_{j \in J}$ of $\mathcal{H}$ so that $A e_j = d_j e_j$ for all $j \in J$, for some $d_j \in \mathbb{R}$, one gets a covariant representation π_λ as above by setting

$$U_t v_\lambda = v_\lambda \quad \text{and} \quad U_t v_\mu = e^{it\chi(\mu - \lambda)} v_\mu$$

for any $t \in \mathbb{R}$, $\mu \in \mathcal{P}_\lambda$ and any μ-weight vector $v_\mu \in \mathcal{H}_\lambda$, where $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$ is the character induced by the assignment $\epsilon_j \mapsto d_j$. Comparing this situation with Theorem A, we see that the decomposition $\chi = \chi_{\min} + \chi_{\text{sum}}$ in this theorem corresponds to a decomposition $A = A_{\min} + A_{\text{sum}}$ of A as a sum of two commuting (simultaneously diagonalisable) operators $A_{\min}, A_{\text{sum}} \in B(\mathcal{H})$ such that $iA_{\text{sum}} \in \mathfrak{u}_1(\mathcal{H})$, and such that $A_{\min}$ yields a *minimal energy representation* π_λ, in the sense that the corresponding Hamiltonian $H_{\min}$ is non-negative (with eigenvalue 0 on the highest weight vector v_λ). Writing $\alpha^{\min}$ and α^{sum} for the $\mathbb{R}$-actions on G induced by $A_{\min}$ and A_{sum} respectively, this implies that α_t differs only from $\alpha_t^{\min}$ by an inner automorphism α_t^{sum} of G commuting with α_t and $\alpha_t^{\min}$ ($t \in \mathbb{R}$): in this case, we will say that the corresponding covariant representations of $G \rtimes_\alpha \mathbb{R}$ and $G \rtimes_{\alpha^{\min}} \mathbb{R}$ are *similar*. Thus Theorem A has the following corollary:

Corollary B. *Let $G = U_1(\mathcal{H})$ be endowed with a continuous $\mathbb{R}$-action $\alpha \colon \mathbb{R} \to \operatorname{Aut}(G)$, given by $\alpha_t(g) = e^{itA} g e^{-itA}$ for some self-adjoint operator $A \in B(\mathcal{H})$. Assume that A is diagonalisable. Then every positive energy covariant unitary representation $\pi_\lambda \colon G \rtimes_\alpha \mathbb{R} \to U(\mathcal{H}_\lambda)$ of $(G, \mathbb{R}, \alpha)$ as defined above is similar to a minimal energy representation.*

Finally, one can show that the highest weight representation $\widehat{\rho}_\lambda \colon U_1(\mathcal{H}) \to U(\mathcal{H}_\lambda)$ extends to a projective unitary representation $U_2(\mathcal{H}) \to PU(\mathcal{H}_\lambda)$ of the Hilbert–Lie group

$$U_2(\mathcal{H}) = \operatorname{GL}(\mathcal{H}) \cap (\mathbf{1} + \mathfrak{u}_2(\mathcal{H})),$$

and hence to a unitary representation on $\mathcal{H}_\lambda$ of a central extension $\widehat{U}_2(\mathcal{H})$ of $U_2(\mathcal{H})$, given at the Lie algebra level by the cocycle $\omega(X, Y) = i\lambda([X, Y])$ for $X, Y \in \mathfrak{u}_2(\mathcal{H})$, where we extend λ in a natural way to a continuous linear functional on $\mathfrak{u}_1(\mathcal{H})$ which contains $[X, Y]$ (see [Nee15]). Since the continuous $\mathbb{R}$-action (1.1) on $U_2(\mathcal{H})$ lifts canonically to the central extension (by construction of this

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[2]This follows from [Nee14b, Theorem 1.15] when $G = U_2(\mathcal{H})$, and [dlH72, Proposition 9 on p.48] ensures the extendability from $U_1(\mathcal{H})$ to $U_2(\mathcal{H})$.

central extension), the corresponding notion of positive energy for the associated projective covariant unitary representations of $U_2(\mathcal{H})$ is also described by Corollary B.

The situation discussed in this paper is a model case in which a rather detailed analysis of the positive energy condition can be carried out. For every triple $(G, \mathbb{R}, \alpha)$, where G is a Lie group and $\alpha \colon \mathbb{R} \to \mathrm{Aut}(G)$ defines a continuous action, it is a challenging natural problem to determine the irreducible positive energy representations $(\pi, \mathcal{H})$ of the topological group $G^\sharp := G \rtimes_\alpha \mathbb{R}$. As a consequence of the Borchers–Arveson Theorem ([BR87, Theorem 3.2.46]), for any such representation, the restriction $\rho := \pi|_G$ is irreducible (see [Nee14a, Theorem 2.5]) and the Hamiltonian of the extension to $G^\sharp$ is uniquely determined up to an additive constant determining the minimal energy level. Given α, the set of irreducible positive energy representations of $G^\sharp$ can therefore be considered as a subset $\widehat{G}_\alpha$ of the set $\widehat{G}$ of equivalence classes of irreducible unitary representations of G and one would like to determine this subset as explicitly as possible. In this paper this task is carried out for the subset $\widehat{G}_{hw}$ of "highest weight representations" of $G = U_1(\mathcal{H})$ in the case where α is given by conjugation with diagonal operators. Here Corollary B achieves in the Lie algebra context something similar as the Borchers–Arveson Theorem which also reduces the study of positive energy representations to minimal energy representations.

One can even show that the representation of the centrally extended group $\widehat{U}_2(\mathcal{H})$ extends to a group $\widehat{U}_{\mathrm{res}}$ containing a copy of the centraliser D of the diagonal operator $\mathrm{diag}((\lambda_j)_{j \in J})$ in $U(\mathcal{H})$ (see [Nee04, Theorem VII.18] for the A_J-case). Here D is a finite product $\prod_{m \in \mathbb{Z}} U(\ell^2(J_m))$ of full unitary groups where the factors correspond to the subsets $J_m := \{j \in J \mid \lambda_j = m\}$. The corresponding extension $\widehat{\pi}$ to $\widehat{U}_{\mathrm{res}}$ is a unitary Lie group representation for which one would like to understand the convex cone of all elements $X \in \mathfrak{u}_{\mathrm{res}}$ in the Lie algebra for which the operator $-i d\widehat{\pi}(X)$ is positive. If X is diagonal, this problem is solved by Theorem A if we put $\chi(\epsilon_j) := X_{jj}$, but the general case requires refined information on convex hulls of adjoint orbits (see [Nee10] for similar problems). We plan to address this issue in a separate paper because it is of a functional analytic flavour, whereas the present paper is purely algebraic.

Finally, this paper can be used as a basis for a similar study of positive energy highest weight representations of *affinisations* of simple Hilbert–Lie algebras $\mathfrak{k}$, obtained as double extensions of (twisted) loop algebras over $\mathfrak{k}$ (in the same way as one obtains affine Kac–Moody algebras from finite-dimensional simple Lie algebras): this is carried on in the papers [MN15a] and [MN15b].

Notation. Throughout this paper, we denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of positive natural numbers.

2. Preliminaries

2.1. **Locally finite root systems.** Let J be an infinite set and let $V := \mathbb{R}^{(J)} \subseteq \overline{V} := \mathbb{R}^J$ be the free vector space over J, with canonical basis $\{e_j \mid j \in J\}$ and standard scalar product $(e_j, e_k) = \delta_{jk}$. In the dual space $V^* \cong \mathbb{R}^J$, we consider the linearly independent system $\{\epsilon_j := e_j^* \mid j \in J\}$ defined by $\epsilon_j(e_k) = \delta_{jk}$.

Any infinite irreducible (possibly non-reduced) locally finite root system Δ can be described inside V^* for some suitable set J, and is of one of the following types ([LN04, §8]):

$$A_J := \{\epsilon_j - \epsilon_k \mid j, k \in J, \ j \neq k\},$$
$$B_J := \{\pm\epsilon_j, \pm(\epsilon_j \pm \epsilon_k) \mid j, k \in J, \ j \neq k\},$$
$$C_J := \{\pm 2\epsilon_j, \pm(\epsilon_j \pm \epsilon_k) \mid j, k \in J, \ j \neq k\},$$
$$D_J := \{\pm(\epsilon_j \pm \epsilon_k) \mid j, k \in J, \ j \neq k\},$$
$$BC_J := \{\pm\epsilon_j, \pm 2\epsilon_j, \pm(\epsilon_j \pm \epsilon_k) \mid j, k \in J, \ j \neq k\}.$$

For $X \in \{A, B, C, D, BC\}$, we will write $\Delta(X_J)$ for the above locally finite root system of type X_J. Note that the root systems of type A_J, B_J, C_J and D_J are reduced, whereas $\Delta(BC_J)$ is non-reduced.

2.2. **The Weyl group of Δ.** Let S_J denote the symmetric group on J, which we view as a subgroup of $\mathrm{GL}(\overline{V})$ with $w \in S_J$ acting as $w(e_j) := e_{w(j)}$. Given a permutation $w \in S_J$ with fixed-point set

$I \subseteq J$, we call the set $J \setminus I$ the *support* of w. We denote by $S_{(J)} \leq S_J$ the subgroup of restricted permutations, namely, the set of $w \in S_J$ with finite support. Note that $S_{(J)} \leq \mathrm{GL}(\overline{V})$ stabilises V; we will also view $S_{(J)}$ as a subgroup of $\mathrm{GL}(V)$.

We next view $\{\pm 1\}^J \subset \mathbb{R}^J$ as a subgroup of $\mathrm{GL}(\overline{V})$, acting by (componentwise) left multiplication: $\sigma(e_j) = \sigma_j e_j$ for $\sigma = (\sigma_j)_{j \in J} \in \{\pm 1\}^J$. Given some $\sigma = (\sigma_j)_{j \in J} \in \{\pm 1\}^J$, we call the subset $I = \{j \in J \mid \sigma_j = -1\}$ of J the *support* of σ. We denote by $\{\pm 1\}^{(J)}$ the set of all $\sigma \in \{\pm 1\}^J$ with finite support. Again, we may also view $\{\pm 1\}^{(J)}$ as a subgroup of $\mathrm{GL}(V)$. Finally, we let $\{\pm 1\}_2^{(J)}$ denote the index 2 subgroup of $\{\pm 1\}^{(J)}$, whose elements have a support of even cardinality.

Let $X \in \{A, B, C, D, BC\}$. We denote by $W = W(X_J)$ the Weyl group corresponding to $\Delta(X_J)$, which we view as a subgroup of $\mathrm{GL}(V)$ or $\mathrm{GL}(\overline{V})$. We then have the following descriptions ([LN04, §9]):

$$W(A_J) = S_{(J)},$$
$$W(B_J) = W(C_J) = W(BC_J) = S_{(J)} \ltimes \{\pm 1\}^{(J)},$$
$$W(D_J) = S_{(J)} \ltimes \{\pm 1\}_2^{(J)}.$$

2.3. **The positive energy condition.** Let $X \in \{A, B, C, D, BC\}$ and set $W = W(X_J)$. Fix some tuples $\lambda = (\lambda_j)_{j \in J} \in \mathbb{R}^J$ and $\chi = (d_j)_{j \in J} \in \mathbb{R}^J$.

Definition 2.1. We say that the triple (J, λ, χ) *satisfies the positive energy condition (PEC) for W* if the set $\lambda(W.\chi - \chi)$ is bounded from below. Here, we view λ as the linear functional

$$\lambda \colon V \to \mathbb{R} : e_j \mapsto \lambda_j,$$

and $W.\chi - \chi$ as a subset of V, by writing χ as $\chi = \sum_{j \in J} d_j e_j \in \overline{V}$. More precisely, recall from §2.2 that any element of W may be written as a product σw^{-1} for some $\sigma \in \{\pm 1\}^{(J)}$ and some $w \in S_{(J)}$. Then

$$\sigma w^{-1}.\chi - \chi = \sum_{j \in J} (\sigma_{w^{-1}(j)} d_j e_{w^{-1}(j)} - d_j e_j) = \sum_{j \in J} (\sigma_j d_{w(j)} - d_j) e_j \in V.$$

Note that for any $\sigma \in \{\pm 1\}^{(J)}$ and $w \in S_{(J)}$, we have

$$\lambda(\sigma w^{-1}.\chi - \chi) = \sum_{j \in J} \lambda_j (\sigma_j d_{w(j)} - d_j). \tag{2.1}$$

In particular, given two disjoint finite subsets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_k\}$ of J, the product w of the k transpositions $\tau_1, \ldots, \tau_k \in S_{(J)}$, where τ_s interchanges i_s and j_s ($s = 1, \ldots, k$), is an element of $S_{(J)}$ and we have

$$\lambda(w.\chi - \chi) = \lambda(w^{-1}.\chi - \chi) = \sum_{s=1}^{k} (\lambda_{j_s} - \lambda_{i_s})(d_{i_s} - d_{j_s}). \tag{2.2}$$

We record for future reference the following so-called *rearrangement inequality*.

Lemma 2.2. *Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be two non-decreasing sequences of real numbers. Let also $(c_1, \ldots, c_n)$ be a permutation of $(b_1, \ldots, b_n)$. Then*

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i c_i \geq \sum_{i=1}^{n} a_i b_{n+1-i}.$$

3. A FEW DEFINITIONS AND NOTATIONS

Fix some set J, as well as two tuples $\lambda = (\lambda_j)_{j \in J} \in \mathbb{R}^J$ and $\chi = (d_j)_{j \in J} \in \mathbb{R}^J$. We define the functions

$$D \colon J \to \mathbb{R} : j \mapsto d_j \quad \text{and} \quad \Lambda \colon J \to \mathbb{R} : j \mapsto \lambda_j.$$

We set

$$m_{\min} = \inf \Lambda(J) \in \mathbb{R} \cup \{-\infty\} \quad \text{and} \quad m_{\max} = \sup \Lambda(J) \in \mathbb{R} \cup \{\infty\}.$$

We call the triple (J, λ, χ) *nontrivial* if λ is non-constant, so that $m_{\min} \neq m_{\max}$.

For each $n \in \mathbb{R}$, we set
$$J_n := \Lambda^{-1}(n),$$
so that $J = \bigcup_n J_n$. Given some $r \in \mathbb{R}$, we also define the sets
$$J_n^{>r} = \{j \in J_n \mid d_j > r\} \quad \text{and} \quad J_n^{<r} = \{j \in J_n \mid d_j < r\}.$$
Finally, we let $\Lambda^\infty(J)$ denote the set of all $m \in \Lambda(J)$ such that J_m is infinite, and we set
$$\overline{\Lambda^\infty}(J) := \left(\Lambda^\infty(J) \cup \{m_{\min}, m_{\max}\}\right) \cap \mathbb{R}.$$

Definition 3.1. We call (J, λ, χ) *essentially bounded* if for all $m \in \Lambda(J)$, the following two conditions are satisfied:

(1) If $m \neq m_{\max}$, then $D(J_m)$ is bounded below.
(2) If $m \neq m_{\min}$, then $D(J_m)$ is bounded above.

Definition 3.2. Given a subset I of J, we call $r \in \mathbb{R}$ an *accumulation point for I* if either r is an accumulation point for $D(I)$, or if $D(I') = \{r\}$ for some infinite subset $I' \subseteq I$.

Definition 3.3. Assume that (J, λ, χ) is essentially bounded and nontrivial. Let $m \in \overline{\Lambda^\infty}(J)$.

(1) If $m \neq m_{\min}, m_{\max}$, then $D(J_m)$ is bounded, and hence J_m possesses an accumulation point. In this case, we let $r_m^{\min}$ and $r_m^{\max}$ respectively denote the minimal and maximal accumulation points of J_m.
(2) If $m = m_{\min}$, then $D(J_m)$ is bounded below. If J_m has an accumulation point, we let $r_m^{\min}$ denote the minimal one. Otherwise, we set $r_m^{\min} = \infty$.
(3) If $m = m_{\max}$, then $D(J_m)$ is bounded above. If J_m has an accumulation point, we let $r_m^{\max}$ denote the maximal one. Otherwise, we set $r_m^{\max} = -\infty$.

The following lemma, which easily follows from the definitions, provides the geometric picture to be kept in mind for the rest of this paper.

Lemma 3.4. *Assume that (J, λ, χ) is essentially bounded and nontrivial. Let $m \in \overline{\Lambda^\infty}(J)$.*

(1) *If $m \neq m_{\min}, m_{\max}$, then for each $\epsilon > 0$, there is some finite subset $I_\epsilon \subset J_m$ such that*
$$D(J_m \setminus I_\epsilon) \subseteq [r_m^{\min} - \epsilon, r_m^{\max} + \epsilon].$$
(2) *If $m = m_{\min}$ and $r_m^{\min} \neq \infty$, then for each $\epsilon > 0$, there is some finite subset $I_\epsilon \subset J_m$ such that*
$$D(J_m \setminus I_\epsilon) \subseteq [r_m^{\min} - \epsilon, \infty[.$$
(3) *If $m = m_{\max}$ and $r_m^{\max} \neq -\infty$, then for each $\epsilon > 0$, there is some finite subset $I_\epsilon \subset J_m$ such that*
$$D(J_m \setminus I_\epsilon) \subseteq]-\infty, r_m^{\max} + \epsilon].$$

Before proceeding with the study of the PEC, we need to introduce one more concept.

Definition 3.5. Given $r \in \mathbb{R}$, we call a subset I_r of J of the form $J_n^{<r}$ or $J_n^{>r}$ *summable* (with respect to (J, λ, χ)) if
$$\Sigma(I_r) := \sum_{j \in I_r} |d_j - r| < \infty.$$

4. Consequences of the PEC for $W(A_J)$

In this section, we fix some infinite set J and some tuples $\lambda = (\lambda_j)_{j \in J} \in \mathbb{R}^J$ and $\chi = (d_j)_{j \in J} \in \mathbb{R}^J$, and we assume that (J, λ, χ) satisfies the PEC for $W = W(A_J)$.

Lemma 4.1. *Let $m, n \in \mathbb{R}$ with $m < n$.*

(1) *If $J_m^{<r}$ and $J_n^{>r}$ are both nonempty for some $r \in \mathbb{R}$, then $D(J_m^{<r})$ and $D(J_n^{>r})$ are bounded.*
(2) *If J_m and J_n are both nonempty, then $D(J_m)$ (resp. $D(J_n)$) is bounded below (resp. above).*

Proof. To prove (1), assume for a contradiction that $D(J_m^{<r})$ is unbounded, and for each $k \in \mathbb{N}$, let $j_k \in J_m^{<r}$ such that $d_{j_k} < -k$. Pick any $j \in J_n^{>r}$. Consider for each $k \in \mathbb{N}$ the transposition $\tau_k \in W$ exchanging j and j_k. It then follows from (2.2) that

$$\lambda(\tau_k.\chi - \chi) = (\lambda_j - \lambda_{j_k})(d_{j_k} - d_j) = (n-m)(d_{j_k} - d_j) < -(n-m)(k+d_j)$$

for all $k \in \mathbb{N}$, contradicting the PEC for W. The proof that $D(J_n^{>r})$ is bounded is similar.

We now turn to the proof of (2). Assume that J_m and J_n are nonempty. We prove that $D(J_m)$ is bounded below, the proof for $D(J_n)$ being similar. Let $r \in \mathbb{R}$ be such that $J_n^{>r}$ is nonempty. If $J_m^{<r}$ is empty, then $\inf D(J_m) \geq r$. If $J_m^{<r}$ is nonempty, then $D(J_m^{<r})$ is bounded by (1), and hence $D(J_m)$ is bounded below, as desired. □

Lemma 4.2. *Let $m, n \in \mathbb{R}$ with $m < n$, and let $r \in \mathbb{R}$. Assume that $J_n^{>r}$ is infinite. Then $J_m^{<t}$ is finite for all $t < r$.*

Proof. Assume for a contradiction that $J_m^{<t}$ is infinite for some $t < r$. Let $\{i_1, i_2, \dots\}$ (resp. $\{j_1, j_2, \dots\}$) be an infinite countable subset of $J_m^{<t}$ (resp. $J_n^{>r}$). For each $k \in \mathbb{N}$, let $w_k \in W$ be the product of the k transpositions $\tau_1, \dots, \tau_k$, where τ_s interchanges i_s and j_s ($s \in \mathbb{N}$). It then follows from (2.2) that

$$\lambda(w_k.\chi - \chi) = \sum_{s=1}^{k} (\lambda_{j_s} - \lambda_{i_s})(d_{i_s} - d_{j_s}) = (n-m) \sum_{s=1}^{k} (d_{i_s} - d_{j_s}) < -(n-m)(r-t)k$$

for all $k \in \mathbb{N}$, contradicting the PEC for W. □

Lemma 4.3. *Let $m, n \in \mathbb{R}$ with $m < n$. Then there exists at most one $r \in \mathbb{R}$ such that $J_m^{<r}$ and $J_n^{>r}$ are both infinite.*

Proof. Assume that $J_m^{<r}$ and $J_n^{>r}$ are both infinite for two different values of r, say $r_1 < r_2$. Using Lemma 4.2 with $r = r_2$ and $t = r_1$ then yields the desired contradiction. □

Lemma 4.4. *If (J, λ, χ) satisfies the PEC for $W(A_J)$, it is essentially bounded.*

Proof. Let $m \in \Lambda(J)$. If $m \notin \Lambda^\infty(J)$, then $D(J_m)$ is finite, hence bounded, and there is nothing to prove. Assume now that $m \in \Lambda^\infty(J)$. If $m \neq m_{\max}$, then there is some $n \in \Lambda(J)$ such that $m < n$. Since $J_n \neq \emptyset$, Lemma 4.1(2) implies that $D(J_m)$ is bounded below, as desired. Similarly, if $m \neq m_{\min}$, so that there exists some $n \in \Lambda(J)$ with $m > n$, Lemma 4.1(2) implies that $D(J_m)$ is bounded above, proving the claim. □

Proposition 4.5. *Assume that (J, λ, χ) satisfies the PEC for $W = W(A_J)$. Let $m, n \in \overline{\Lambda^\infty}(J)$ be such that $m < n$. Then one of the following assertions holds.*

(1) *$r_n^{\max} < r_m^{\min}$. In this case, there is some $r \in \mathbb{R}$ such that $J_m^{<r}$ and $J_n^{>r}$ are both finite.*
(2) *$r_n^{\max} = r_m^{\min}$. In this case, $J_m^{<r_m^{\min}}$ and $J_n^{>r_n^{\max}}$ are both summable.*

Proof. Note first that (J, λ, χ) is essentially bounded by Lemma 4.4 and nontrivial by hypothesis, so that $r_m^{\min}$ and $r_n^{\max}$ are defined.

If $m = m_{\min}$ and $r_m^{\min} = \infty$, so that $D(J_m)$ is bounded below and J_m has no accumulation point, then $J_m^{<r}$ is finite for any $r \in \mathbb{R}$. Since in addition $D(J_n)$ is bounded above, so that $J_n^{>r}$ is finite for some large enough r, the statement (1) is satisfied. Similarly, if $n = m_{\max}$ and $r_n^{\max} = -\infty$, the statement (1) is satisfied, and we may thus assume from now on that $r_m^{\min}, r_n^{\max} \in \mathbb{R}$.

We distinguish three cases.
Case 1: $r_n^{\max} < r_m^{\min}$.

It then follows from Lemma 3.4 that $J_m^{<r}$ and $J_n^{>r}$ are both finite for any $r \in \mathbb{R}$ with $r_n^{\max} < r < r_m^{\min}$. Hence (1) is satisfied in this case.
Case 2: $r_n^{\max} = r_m^{\min}$.

Set $r = r_n^{\max} = r_m^{\min}$. We now prove that $J_m^{<r}$ and $J_n^{>r}$ must be both summable, so that (2) is satisfied. By Lemma 3.4, the sets $J_m^{<r-1/k}$ and $J_n^{>r+1/k}$ are finite for each $k \in \mathbb{N}$, so that $J_m^{<r} = \bigcup_{k\in\mathbb{N}} J_m^{<r-1/k}$ and $J_n^{>r} = \bigcup_{k\in\mathbb{N}} J_n^{>r+1/k}$ are at most countable.

If $J_m^{<r}$ and $J_n^{>r}$ are both finite, there is nothing to prove. Assume now that at least one of $J_m^{<r}$ and $J_n^{>r}$ is infinite, say $J_m^{<r}$ (the other case being similar). Write $J_m^{<r} = \{i_1, i_2, \dots\}$. We distinguish two cases.

Assume first that $J_n^{>r}$ is infinite. Write $J_n^{>r} = \{j_1, j_2, \dots\}$. For each $k \in \mathbb{N}$, let $w_k \in W$ be the product of the transpositions $\tau_1, \dots, \tau_k$, where τ_s is the transposition exchanging i_s and j_s $(s \in \mathbb{N})$. It then follows from (2.2) that

$$\lambda(w_k.\chi - \chi) = \sum_{s=1}^{k} (\lambda_{i_s} - \lambda_{j_s})(d_{j_s} - d_{i_s}) = (m-n) \sum_{s=1}^{k} (d_{j_s} - d_{i_s}).$$

In particular,

$$\inf_{k\in\mathbb{N}} \{\lambda(w_k.\chi - \chi)\} = -(n-m) \sum_{s=1}^{\infty} (d_{j_s} - r + r - d_{i_s}) = -(n-m)(\Sigma(J_n^{>r}) + \Sigma(J_m^{<r})).$$

Hence $J_m^{<r}$ and $J_n^{>r}$ must be both summable, as desired.

Assume next that $J_n^{>r}$ is finite. In particular $J_n^{>r}$ is summable, and it thus remains to show that $J_m^{<r}$ is also summable. Fix some $\epsilon > 0$ and some sequence $(\epsilon_s)_{s\in\mathbb{N}}$ of positive real numbers such that $\sum_{s\in\mathbb{N}} \epsilon_s < \epsilon$. Since r is an accumulation point for J_n, there is some infinite countable subset $\{j_1, j_2, \dots\} \subseteq J_n$ such that $d_{j_s} \geq \max(d_{i_s}, r - \epsilon_s)$ for each $s \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $w_k \in W$ be the product of the transpositions $\tau_1, \dots, \tau_k$, where τ_s is the transposition exchanging i_s and j_s $(s \in \mathbb{N})$. It then follows from (2.2) that

$$\lambda(w_k.\chi - \chi) = \sum_{s=1}^{k} (\lambda_{i_s} - \lambda_{j_s})(d_{j_s} - d_{i_s}) = (m-n) \sum_{s=1}^{k} (d_{j_s} - d_{i_s}).$$

In particular,

$$\inf_{k\in\mathbb{N}} \{\lambda(w_k.\chi - \chi)\} = -(n-m) \sum_{s=1}^{\infty} (d_{j_s} - r + r - d_{i_s}) \leq -(n-m)(-\epsilon + \Sigma(J_m^{<r})).$$

Hence $J_m^{<r}$ must be summable, as desired.

Case 3: $r_n^{\max} > r_m^{\min}$.

Let $r \in \mathbb{R}$ be such that $r_m^{\min} < r < r_n^{\max}$. Since $r_m^{\min}$ is an accumulation point for J_m, the set $J_m^{<r}$ is infinite. Similarly, since $r_n^{\max}$ is an accumulation point for J_n, the set $J_n^{>r}$ is infinite. Since there are infinitely many $r \in \mathbb{R}$ with $r_m^{\min} < r < r_n^{\max}$, Lemma 4.3 then yields a contradiction in this case. This concludes the proof of the proposition. $\square$

5. CHARACTERISATION OF THE PEC FOR λ BOUNDED AND DISCRETE

In this section, we let J denote some infinite set and $\lambda = (\lambda_j)_{j\in J}$ and $\chi = (d_j)_{j\in J}$ some elements of $\mathbb{R}^J$. In order to characterise the PEC for the triple (J, λ, χ), we will need to make some finiteness assumption on λ.

Definition 5.1. We call λ *bounded* (resp. *discrete*) if the subset $\Lambda(J)$ of $\mathbb{R}$ is bounded (resp. discrete).

Definition 5.2. Given a set J and a tuple $\lambda = (\lambda_j)_{j\in J} \in \mathbb{R}^J$, we define the following cones in $\mathbb{R}^J$:

$$C_{\min}(\lambda, A_J) = \{(d_j)_{j\in J} \in \mathbb{R}^J \mid \forall i, j \in J : \lambda_i < \lambda_j \implies d_i \geq d_j\},$$

$$C_{\min}(\lambda, B_J) = \{(d_j)_{j\in J} \in \mathbb{R}^J \mid \forall j \in J : \lambda_j d_j \leq 0 \quad \text{and} \quad \forall i, j \in J : |\lambda_i| < |\lambda_j| \implies |d_i| \leq |d_j|\}.$$

We also define the vector subspace $\ell^1(J) = \{(d_j)_{j\in J} \in \mathbb{R}^J \mid \sum_{j\in J} |d_j| < \infty\}$ of $\mathbb{R}^J$.

Proposition 5.3. *Let $X \in \{A, B\}$ and set $W = W(X_J)$. For a triple (J, λ, χ), the following are equivalent:*

(1) $\inf \lambda(W.\chi - \chi) = 0$, *i.e.,* $\lambda(w.\chi - \chi) \geq 0$ *for all* $w \in W$.

(2) $\chi \in C_{\min}(\lambda, X_J)$.

Proof. We first deal with the case $X = A$. If $\chi \notin C_{\min}(\lambda, A_J)$, then there exist some $i, j \in J$ with $\lambda_i < \lambda_j$ and $d_i < d_j$. Denoting by $w \in W$ the transposition exchanging i and j, we deduce from (2.2) that

$$\lambda(w.\chi - \chi) = (\lambda_j - \lambda_i)(d_i - d_j) < 0,$$

so that $\inf \lambda(W.\chi - \chi) < 0$.

Assume conversely that $\chi \in C_{\min}(\lambda, A_J)$ and let $w \in W = S_{(J)}$. Let I be some finite subset of J containing the support of w. By assumption, we may then write $I = \{i_1, \ldots, i_k\}$ so that

$$\lambda_{i_1} \leq \cdots \leq \lambda_{i_k} \quad \text{and} \quad d_{i_1} \geq \cdots \geq d_{i_k}.$$

Together with (2.1), the rearrangement inequality then implies that

$$\lambda(w^{-1}.\chi - \chi) = \sum_{s=1}^{k} \lambda_{i_s}(d_{w(i_s)} - d_{i_s}) \geq \sum_{s=1}^{k} \lambda_{i_s}(d_{i_s} - d_{i_s}) = 0.$$

Hence $\inf \lambda(W.\chi - \chi) = 0$, as desired.

We next deal with the case $X = B$. If $\chi \notin C_{\min}(\lambda, B_J)$, then either there exists some $j \in J$ such that $\lambda_j d_j > 0$, or else $\lambda_k d_k \leq 0$ for all $k \in J$ and there exist some $i, j \in J$ with $|\lambda_i| < |\lambda_j|$ and $|d_i| > |d_j|$. In the first case, denoting by $\sigma \in W$ the element of $\{\pm 1\}^{(J)}$ with support $\{j\}$, we deduce from (2.1) that

$$\lambda(\sigma.\chi - \chi) = -2\lambda_j d_j < 0.$$

In the second case, denoting by $w \in S_{(J)}$ the transposition exchanging i and j, and by $\sigma = (\sigma_k)_{k \in J}$ an element of $\{\pm 1\}^{(J)}$ with support in $\{i, j\}$ satisfying $\sigma_i \lambda_i d_j = -|\lambda_i d_j|$ and $\sigma_j \lambda_j d_i = -|\lambda_j d_i|$, we deduce from (2.1) that

$$\lambda(\sigma w^{-1}.\chi - \chi) = \lambda_i(\sigma_i d_j - d_i) + \lambda_j(\sigma_j d_i - d_j) = (|\lambda_i| - |\lambda_j|)(|d_i| - |d_j|) < 0.$$

In both cases, we deduce that $\inf \lambda(W.\chi - \chi) < 0$.

Assume conversely that $\chi \in C_{\min}(\lambda, B_J)$ and let $w \in S_{(J)}$ and $\sigma = (\sigma_j)_{j \in J} \in \{\pm 1\}^{(J)}$. Let I be some finite subset of J containing the supports of w and σ. By assumption, we may then write $I = \{i_1, \ldots, i_k\}$ so that

$$|\lambda_{i_1}| \leq \cdots \leq |\lambda_{i_k}| \quad \text{and} \quad |d_{i_1}| \leq \cdots \leq |d_{i_k}|.$$

Since moreover $\lambda_{i_s} d_{i_s} = -|\lambda_{i_s} d_{i_s}|$ for all $s = 1, \ldots, k$, we deduce from (2.1) and the rearrangement inequality that

$$\lambda(\sigma w^{-1}.\chi - \chi) = \sum_{s=1}^{k} \lambda_{i_s}(\sigma_{i_s} d_{w(i_s)} - d_{i_s}) \geq \sum_{s=1}^{k} |\lambda_{i_s}|(-|d_{w(i_s)}| + |d_{i_s}|) \geq \sum_{s=1}^{k} |\lambda_{i_s}|(-|d_{i_s}| + |d_{i_s}|) = 0.$$

Hence $\inf \lambda(W.\chi - \chi) = 0$, as desired. $\square$

Lemma 5.4. *Let $X \in \{A, B\}$. Assume that λ is bounded and that $\chi \in \ell^1(J)$. Then (J, λ, χ) satisfies the PEC for $W(X_J)$.*

Proof. Write $\lambda = (\lambda_j)_{j \in J}$ and $\chi = (d_j)_{j \in J}$, so that $C := \sum_{j \in J} |d_j| < \infty$. Set $M := \sup_{j \in J} |\lambda_j| < \infty$. Then for all $\sigma = (\sigma_j)_{j \in J} \in \{\pm 1\}^{(J)}$ and $w \in S_{(J)}$, we have

$$\lambda(\sigma w^{-1}.\chi - \chi) = \sum_{j \in J} \lambda_j(\sigma_j d_{w(j)} - d_j) \geq -M \sum_{j \in J}(|d_{w(j)}| + |d_j|) = -2MC.$$

Since σ and w were arbitrary, this proves the claim. $\square$

Lemma 5.5. *Let $X \in \{A, B\}$. Assume that λ is bounded. Then for all $\chi' \in \ell^1(J)$, the triple (J, λ, χ) satisfies the PEC for $W(X_J)$ if and only if the triple $(J, \lambda, \chi + \chi')$ satisfies the PEC for $W(X_J)$.*

Proof. Set $W = W(X_J)$, and assume that (J, λ, χ) satisfies the PEC for W. Since

$$\inf \lambda\big(W.(\chi + \chi') - (\chi + \chi')\big) \geq \inf \lambda(W.\chi - \chi) + \inf \lambda(W.\chi' - \chi'),$$

the triple $(J, \lambda, \chi + \chi')$ then satisfies the PEC for W by Lemma 5.4. Replacing χ by $\chi + \chi'$ and χ' by $-\chi'$, the converse follows. $\qquad\square$

Note that, up to now, we only refered to the locally finite root systems of type A_J and B_J. We now prove that these are indeed the only cases to be considered.

Lemma 5.6. *Assume that (J, λ, χ) satisfies the PEC for $W(B_J)$ or $W(D_J)$. Then*

$$\sum_{j \in J_+} |\lambda_j d_j| < \infty, \quad where \ J_+ := \{j \in J \mid \lambda_j d_j > 0\}.$$

Proof. Note first that since $W(D_J) = S_{(J)} \ltimes \{\pm 1\}_2^{(J)} \leq W(B_J)$, the PEC for $W(B_J)$ implies the PEC for $W(D_J)$. We may thus assume that (J, λ, χ) satisfies the PEC for $W(D_J)$. If J_+ is finite, there is nothing to prove. Assume now that J_+ is infinite. For each finite subset $I \subset J^+$ of even cardinality, let $\sigma_I \in \{\pm 1\}_2^{(J)}$ with support I. Then

$$\inf \lambda(W(D_J).\chi - \chi) \leq \lambda(\sigma_I.\chi - \chi) = -2 \sum_{i \in I} \lambda_i d_i = -2 \sum_{i \in I} |\lambda_i d_i|.$$

Since $\sum_{j \in J^+} |\lambda_j d_j|$ is the supremum of all sums $\sum_{i \in I} |\lambda_i d_i|$ with I a finite subset of J^+ of even cardinality, the claim follows. $\qquad\square$

Lemma 5.7. *Assume that (J, λ, χ) satisfies the PEC for $W(D_J)$. Let $m, n \in \Lambda(J)$ with $|m| < |n|$. Then $D(J_m)$ is bounded.*

Proof. Assume for a contradiction that $D(J_m)$ is unbounded, and choose some infinite countable set $\{i_0, i_1, \dots\} \subseteq J_m$ such that $|d_{i_s}| \geq s$ for all $s \in \mathbb{N}$. Let also $j \in J_n$. Fix some $s \in \mathbb{N}$, and let $w \in S_{(J)}$ be the transposition exchanging j and i_s. Let also $\sigma = (\sigma_j)_{j \in J} \in \{\pm 1\}_2^{(J)}$ with support in $\{j, i_0, i_s\}$ be such that $m\sigma_{i_s} d_j = -|md_j|$ and $n\sigma_j d_{i_s} = -|nd_{i_s}|$. It then follows from (2.1) that

$$
\begin{aligned}
\lambda(\sigma w^{-1}.\chi - \chi) &= m(\sigma_{i_0} d_{i_0} - d_{i_0}) + m(\sigma_{i_s} d_j - d_{i_s}) + n(\sigma_j d_{i_s} - d_j) \\
&\leq 2|md_{i_0}| + |m| \cdot (-|d_j| + |d_{i_s}|) + |n| \cdot (-|d_{i_s}| + |d_j|) \\
&= 2|md_{i_0}| - (|n| - |m|)(|d_{i_s}| - |d_j|) \\
&\leq 2|md_{i_0}| - (|n| - |m|)(s - |d_j|).
\end{aligned}
$$

Hence

$$\inf \lambda(W(D_J).\chi - \chi) \leq 2|md_{i_0}| - (|n| - |m|)(s - |d_j|).$$

As $s \in \mathbb{N}$ was arbitrary, this contradicts the PEC for $W(D_J)$, as desired. $\qquad\square$

Lemma 5.8. *Let $X \in \{B, C, D, BC\}$, and assume that λ is bounded and discrete. Then (J, λ, χ) satisfies the PEC for $W(X_J)$ if and only if it satisfies the PEC for $W(B_J)$.*

Proof. For $X = B, C, BC$, there is nothing to prove. Since $W(D_J) \leq W(B_J)$, it is also clear that the PEC for $W(B_J)$ implies the PEC for $W(D_J)$. Assume now that (J, λ, χ) satisfies the PEC for $W(D_J) = S_{(J)} \ltimes \{\pm 1\}_2^{(J)}$, and let us prove that it also satisfies the PEC for $W(B_J) = S_{(J)} \ltimes \{\pm 1\}^{(J)}$.

If $\Lambda(J) = \{m\}$ is a one-element set and (J, λ, χ) satisfies the PEC for $W(D_J)$, then

$$\inf \lambda(W(D_J).\chi - \chi) = \inf_{|F| \text{ even}} \Big\{ -2m \sum_{j \in F} d_j \Big\} > -\infty$$

implies

$$\inf \lambda(W(B_J).\chi - \chi) = \inf_F \Big\{ -2m \sum_{j \in F} d_j \Big\} > -\infty,$$

where the above infima are taken over finite subsets F of J. We therefore assume that $|\Lambda(J)| \geq 2$.

We distinguish two cases. Assume first that J has no accumulation point. Since $\Lambda(J)$ is finite, there exists some $m \in \Lambda^\infty(J)$. Thus $D(J_m)$ is unbounded. Lemma 5.7 then implies that $|m| \geq |n|$ for all $n \in \Lambda(J)$. Similarly, if $n \in \Lambda^\infty(J)$, then $D(J_n)$ is unbounded and hence $|n| = |m|$. In other words, $\Lambda^\infty(J) \subseteq \{\pm m\}$ and $|m| = \max\{|m_{\min}|, |m_{\max}|\}$. Thus either $\Lambda^\infty(J) = \{m\} \subseteq \{m_{\min}, m_{\max}\}$ with $|m| = \max_{j \in J} |\lambda_j|$, or else $m \neq 0$ and $\Lambda^\infty(J) = \{\pm m\} = \{m_{\min}, m_{\max}\}$.

Since (J, λ, χ) is essentially bounded by Lemma 4.4 and $|\Lambda(J)| \geq 2$, the sets $D(J_{m_{\max}})$ and $D(J_{m_{\min}})$ are respectively bounded above and below. As J has no accumulation point, this implies that the sets $J^{>0}_{m_{\max}}$ and $J^{<0}_{m_{\min}}$ are finite. Note also that the set

$$S = \{j \in J \mid \lambda_j \neq m_{\min}, m_{\max}\}$$

is finite because $\Lambda^\infty(J) \subseteq \{m_{\min}, m_{\max}\}$ and $\Lambda(J)$ is finite. In particular, the tuple $\chi' = (d'_j)_{j \in J}$ defined by

$$d'_j = \begin{cases} d_j & \text{if } j \in S \cup J^{>0}_{m_{\max}} \cup J^{<0}_{m_{\min}}, \\ 0 & \text{otherwise} \end{cases}$$

belongs to $\ell^1(J)$. Thus by Lemma 5.5, we may replace without loss of generality χ by $\chi - \chi'$. In other words, we may assume that

$$D(J_{m_{\min}}) \subseteq [0, \infty), \quad D(J_{m_{\max}}) \subseteq (-\infty, 0], \quad \text{and} \quad D(J_n) = \{0\} \quad \text{for all } n \neq m_{\min}, m_{\max}.$$

If $J_{m_{\min}}$ (resp. $J_{m_{\max}}$) is finite, we may, by a similar argument, assume that $D(J_{m_{\min}}) = \{0\}$ (resp. $D(J_{m_{\max}}) = \{0\}$). On the other hand, if $J_{m_{\min}}$ (resp. $J_{m_{\max}}$) is infinite, then $D(J_{m_{\min}})$ (resp. $D(J_{m_{\max}})$) is unbounded by hypothesis, and hence $m_{\min} \leq 0$ (resp. $m_{\max} \geq 0$) by Lemma 5.6.

If $\Lambda^\infty(J) = \{m\}$ with $|m| = \max_{j \in J} |\lambda_j|$, we may thus assume that $mD(J_m) \subseteq (-\infty, 0]$ and that $D(J_n) = \{0\}$ for all $n \neq m$. If $\Lambda^\infty(J) = \{\pm m\} = \{m_{\min}, m_{\max}\}$ is a 2-element set, we may similarly assume that $\pm mD(J_{\pm m}) \subseteq (-\infty, 0]$ and that $D(J_n) = \{0\}$ for all $n \neq \pm m$. In both cases, $\chi \in C_{\min}(\lambda, B_J)$. Hence (J, λ, χ) satisfies the PEC for $W(B_J)$ by Proposition 5.3.

We next assume that J has some accumulation point $r \in \mathbb{R}$. Set $M = \sup_{j \in J} |\lambda_j|$. Let $\epsilon > 0$ and let S be an infinite subset of J such that $D(S) \subseteq [r - \epsilon, r + \epsilon]$. Let $\sigma = (\sigma_j)_{j \in J} \in \{\pm 1\}^{(J)}$ and $w \in S_{(J)}$, and let I be some finite subset of J containing the supports of σ and w. Pick any $i \in S \setminus I$, and let $\tau = (\tau_j)_{j \in J} \in \{\pm 1\}^{(J)}$ with support contained in $\{i\}$ be such that $\tau\sigma \in \{\pm 1\}_2^{(J)}$. It then follows from (2.1) that

$$\lambda(\sigma w^{-1}.\chi - \chi) = \sum_{j \in J} \lambda_j(\sigma_j d_{w(j)} - d_j) = \sum_{j \in J} \lambda_j(\tau_j \sigma_j d_{w(j)} - d_j) - \lambda_i(\tau_i d_i - d_i)$$

$$\geq \lambda(\tau\sigma w^{-1}.\chi - \chi) - 2M(|r| + \epsilon).$$

Hence

$$\inf \lambda(W(B_J).\chi - \chi) \geq \inf \lambda(W(D_J).\chi - \chi) - 2M(|r| + \epsilon).$$

This concludes the proof of the lemma. □

Remark 5.9. For each $X \in \{A, B, C, D, BC\}$, let $C_{\min}(\lambda, X_J)$ denote the set of $\chi \in \mathbb{R}^J$ such that $\inf \lambda(W(X_J).\chi - \chi) = 0$. Note that, by Proposition 5.3, this is consistent with Definition 5.2.

For $X \in \{C, BC\}$, we have $W(X_J) = W(B_J)$, and hence $C_{\min}(\lambda, X_J) = C_{\min}(\lambda, B_J)$. In order to determine $C_{\min}(\lambda, D_J)$, we associate to each $\chi = (d_j)_{j \in J} \in \mathbb{R}^J$ the (possibly empty) set

$$I^{\min}_{\lambda, \chi} := \left\{ i \in J \mid |\lambda_i| = \inf_{j \in J} |\lambda_j| \quad \text{and} \quad |d_i| = \inf_{j \in J} |d_j| \right\}.$$

Note that $C_{\min}(\lambda, B_J) \subseteq C_{\min}(\lambda, D_J)$ because $W(D_J) \leq W(B_J)$. One can then check, as in the proof of Proposition 5.3 (or directly from [HN12, Corollary 3.2]), that $\chi \in C_{\min}(\lambda, D_J)$ if and only if either $\chi \in C_{\min}(\lambda, B_J)$, or else $\sigma_i \chi \in C_{\min}(\lambda, B_J)$ for some $i \in I^{\min}_{\lambda, \chi}$, where $\sigma_i \in \{\pm 1\}^{(J)}$ has support $\{i\}$. As this fact will not be needed in our characterisation of the positive energy condition (see Lemma 5.8), we leave it as an exercise.

We first characterise the PEC for $W(A_J)$.

Theorem 5.10. *Let J be a set, and let $\lambda = (\lambda_j)_{j \in J}$ and $\chi = (d_j)_{j \in J}$ be elements of $\mathbb{R}^J$. Assume that λ is discrete and bounded. Then the following are equivalent:*

(1) *(J, λ, χ) satisfies the PEC for $W(A_J)$.*
(2) *$\chi \in C_{\min}(\lambda, A_J) + \ell^1(J)$.*

Proof. The implication (2) $\implies$ (1) readily follows from Proposition 5.3 and Lemma 5.5. Assume now that (J, λ, χ) satisfies the PEC for $W(A_J)$, and let us prove that, up to substracting from χ some element of $\ell^1(J)$, one has $\chi \in C_{\min}(\lambda, A_J)$. Since $\chi \in C_{\min}(\lambda, A_J)$ if λ is constant, we may assume that (J, λ, χ) is nontrivial, that is, $m_{\min} \neq m_{\max}$. Moreover, note that (J, λ, χ) is essentially bounded by Lemma 4.4.

By assumption, $\Lambda(J)$ is finite. Write $\overline{\Lambda^\infty}(J) = \Lambda^\infty(J) \cup \{m_{\min}, m_{\max}\} = \{n_0, n_1, \ldots, n_k\}$ so that

$$m_{\min} = n_0 < n_1 < \cdots < n_k = m_{\max}$$

for some $k \geq 1$. Proposition 4.5 then implies that

$$r_{n_0}^{\min} \geq r_{n_1}^{\max} \geq r_{n_1}^{\min} \geq \cdots \geq r_{n_{k-1}}^{\max} \geq r_{n_{k-1}}^{\min} \geq r_{n_k}^{\max}.$$

For each $t \in \{1, \ldots, k\}$, we set

$$a_t = \begin{cases} \frac{1}{2}(r_{n_{t-1}}^{\min} + r_{n_t}^{\max}) & \text{if } r_{n_{t-1}}^{\min}, r_{n_t}^{\max} \in \mathbb{R}, \\ r_{n_t}^{\max} + 1 & \text{if } t = 1, \, r_{n_0}^{\min} = \infty \text{ and } r_{n_1}^{\max} \in \mathbb{R}, \\ r_{n_{t-1}}^{\min} - 1 & \text{if } t = k, \, r_{n_k}^{\max} = -\infty \text{ and } r_{n_{k-1}}^{\min} \in \mathbb{R}, \\ 0 & \text{if } t = k = 1, \, r_{n_0}^{\min} = \infty \text{ and } r_{n_1}^{\max} = -\infty, \end{cases}$$

so that

$$r_{n_0}^{\min} \geq a_1 \geq r_{n_1}^{\max} \geq r_{n_1}^{\min} \geq a_2 \geq \cdots \geq a_{k-1} \geq r_{n_{k-1}}^{\max} \geq r_{n_{k-1}}^{\min} \geq a_k \geq r_{n_k}^{\max}.$$

We also set $a_0 := \infty$ and $a_{k+1} := -\infty$. Fix some $t \in \{0, 1, \ldots, k\}$. We claim that the tuple $\chi_t = (d'_j)_{j \in J}$ defined by

$$d'_j = \begin{cases} d_j - a_{t+1} & \text{if } j \in J_{n_t} \text{ and } d_j < a_{t+1}, \\ d_j - a_t & \text{if } j \in J_{n_t} \text{ and } d_j > a_t, \\ 0 & \text{otherwise} \end{cases}$$

is in $\ell^1(J)$.

Let I_+ (resp. I_-) denote the set of $j \in J_{n_t}$ such that $d_j > a_t$ (resp. $d_j < a_{t+1}$). We have to show that

$$\sum_{j \in I_+} |d_j - a_t| < \infty \quad \text{and} \quad \sum_{j \in I_-} |d_j - a_{t+1}| < \infty.$$

We prove this for I_+, the proof for I_- being similar. Since if $t = 0$, the set I_+ is empty, we may assume that $t \in \{1, \ldots, k\}$. Moreover, since $a_t \geq r_{n_t}^{\max}$, the set I_+ is finite as soon as $a_t > r_{n_t}^{\max}$. Note that this includes in particular the case where $t = k$ and $r_{n_k}^{\max} = -\infty$, in which case $D(J_{n_t})$ is bounded above and J_{n_t} has no accumulation point, as well as the case where $t = 1$, $r_{n_0}^{\min} = \infty$ and $r_{n_1}^{\max} \in \mathbb{R}$, in which case $a_t = r_{n_t}^{\max} + 1 > r_{n_t}^{\max}$. Hence we may also assume that $a_t = r_{n_t}^{\max} \in \mathbb{R}$ and that $r_{n_{t-1}}^{\min} \in \mathbb{R}$. But then $r_{n_{t-1}}^{\min} = a_t = r_{n_t}^{\max}$, and hence the conclusion follows from Proposition 4.5.

Thus $\sum_{t=0}^{k} \chi_t \in \ell^1(J)$. Hence, up to replacing χ by $\chi - \sum_{t=0}^{k} \chi_t$, we may assume that

(5.1) $$D(J_{n_t}) \subseteq [a_{t+1}, a_t] \quad \text{for all } t = 0, 1, \ldots, k.$$

We next define the tuple $\chi' = (d'_j)_{j \in J}$ by

$$d'_j = \begin{cases} d_j - a_t & \text{if } \lambda_j \in \Lambda(J) \setminus \overline{\Lambda^\infty}(J) \text{ and } n_{t-1} < \lambda_j < n_t, \\ 0 & \text{otherwise.} \end{cases}$$

Since the set of $j \in J$ with $\lambda_j \in \Lambda(J) \setminus \overline{\Lambda^\infty}(J)$ is finite, $\chi' \in \ell^1(J)$. Moreover, it follows from (5.1) that $\chi - \chi' \in C_{\min}(\lambda, A_J)$. This concludes the proof of the theorem. $\square$

Before characterising the PEC for $W(B_J)$, we need one more lemma. For a tuple $\nu = (\nu_j)_{j \in J} \in \mathbb{R}^J$, we put
$$|\nu| := (|\nu_j|)_{j \in J} \in \mathbb{R}^J.$$

Lemma 5.11. *Assume that (J, λ, χ) satisfies the PEC for $W(B_J)$. Then $(J, -|\lambda|, |\chi|)$ satisfies the PEC for $W(A_J)$.*

Proof. Let $w \in S_{(J)}$, and let I be some finite subset of J containing the support of w. Let $\sigma = (\sigma_j)_{j \in J} \in \{\pm 1\}^{(J)}$ with support in I be such that $\lambda_j \sigma_j d_{w(j)} = -|\lambda_j d_{w(j)}|$ for all $j \in I$. It then follows from (2.1) that

$$-|\lambda|(w^{-1}.|\chi| - |\chi|) = -\sum_{j \in I} |\lambda_j| \cdot (|d_{w(j)}| - |d_j|) \geq \sum_{j \in I} \lambda_j (\sigma_j d_{w(j)} - d_j) = \lambda(\sigma w^{-1}.\chi - \chi).$$

Hence
$$\inf\left(-|\lambda|(W(A_J).|\chi| - |\chi|)\right) \geq \inf \lambda(W(B_J).\chi - \chi),$$
as desired. $\qquad \square$

Theorem 5.12. *Let J be a set, and let $\lambda = (\lambda_j)_{j \in J}$ and $\chi = (d_j)_{j \in J}$ be elements of $\mathbb{R}^J$. Assume that λ is discrete and bounded. Then the following are equivalent:*

    (1) *(J, λ, χ) satisfies the PEC for $W(B_J)$.*
    (2) *$\chi \in C_{\min}(\lambda, B_J) + \ell^1(J)$.*

Proof. The implication (2) $\implies$ (1) readily follows from Proposition 5.3 and Lemma 5.5. Assume now that (J, λ, χ) satisfies the PEC for $W(B_J)$ and let us prove that, up to substracting from χ some element of $\ell^1(J)$, one has $\chi \in C_{\min}(\lambda, B_J)$.

Since $(J, -|\lambda|, |\chi|)$ satisfies the PEC for $W(A_J)$ by Lemma 5.11, we know from Theorem 5.10 that
$$|\chi| \in C_{\min}(-|\lambda|, A_J) + \ell^1(J).$$

Let $\sigma = (\sigma_j)_{j \in J} \in \{\pm 1\}^J$ be such that $\sigma_j d_j \geq 0$ for all $j \in J$. In other words, $|\chi| = \sigma\chi$. Hence $\chi \in \sigma C_{\min}(-|\lambda|, A_J) + \sigma\ell^1(J)$. Note that $\sigma\ell^1(J) = \ell^1(J)$. Up to substracting from χ some element of $\ell^1(J)$, we may thus assume without loss of generality that

$$(5.2) \qquad\qquad\qquad\qquad \chi \in \sigma C_{\min}(-|\lambda|, A_J).$$

Note that we may in addition assume that (J, λ, χ) satisfies the PEC for $W(B_J)$ by Lemma 5.5. We deduce from (5.2) that $|\chi| \in C_{\min}(-|\lambda|, A_J)$, so that

$$(5.3) \qquad\qquad\qquad \forall i, j \in J: |\lambda_i| < |\lambda_j| \implies |d_i| \leq |d_j|.$$

On the other hand, Lemma 5.6 implies that

$$\sum_{j \in J_+} |d_j| < \infty, \quad \text{where } J_+ := \{j \in J \mid \lambda_j d_j > 0\}.$$

In particular, the tuple $\chi' = (d'_j)_{j \in J}$ defined by

$$d'_j = \begin{cases} 2d_j & \text{if } j \in J_+, \\ 0 & \text{otherwise} \end{cases}$$

belongs to $\ell^1(J)$. Since $\chi - \chi' = \sigma'\chi$, where $\sigma' = (\sigma'_j)_{j \in J} \in \{\pm 1\}^J$ has support J_+, the tuple $\chi - \chi'$ still satisfies the condition (5.3), with d_j replaced by $d_j - d'_j$ for all $j \in J$. Therefore, up to substracting $\chi' \in \ell^1(J)$ from χ, we may assume that $\lambda_j d_j \leq 0$ for all $j \in J$ and that (5.3) holds, so that $\chi \in C_{\min}(\lambda, B_J)$, as desired. $\qquad \square$

6. Proof of Theorem A

By [LN04], the root system Δ of the locally finite split simple Lie algebra $\mathfrak{g}$ over $\mathbb{K} = \mathbb{C}$ is isomorphic to one of the root systems $\Delta = \Delta(X_J)$ for $X \in \{A, B, C, D\}$ described in §2.1, where the Cartan subalgebra $\mathfrak{h}$ (resp. a one-dimensional extension of $\mathfrak{h}$ if $X = A$) is identified with $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = \operatorname{span}_{\mathbb{C}}\{e_j \mid j \in J\}$. The identification of $V_{\mathbb{C}}$ with $\mathfrak{h}^*$ (resp. a one-dimensional extension of $\mathfrak{h}^*$) induced by the assignment $e_j \mapsto \epsilon_j$ $(j \in J)$ yields in turn an identification of the Weyl group $W \leq \mathrm{GL}(\mathfrak{h}^*)$ of $\mathfrak{g}$ with the Weyl group $W(X_J)$ defined in §2.2.

Let $\lambda \in \mathfrak{h}^*$ be discrete and bounded. Then the restriction of λ to V is real valued, and λ is determined by the tuple $(\lambda_j)_{j \in J} \in \mathbb{R}^J$ defined by $\lambda_j = \lambda(e_j)$, $j \in J$. Identifying λ with this tuple, λ is then discrete and bounded in the sense of Definition 5.1.

Similarly, the character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$ is the restriction of a $\mathbb{Z}$-linear map

$$\widetilde{\chi} \colon \operatorname{span}_{\mathbb{Z}}\{\epsilon_j \mid j \in J\} \to \mathbb{R} : \epsilon_j \mapsto d_j,$$

and is thus determined by the tuple $(d_j)_{j \in J} \in \mathbb{R}^J$. Note that $\mathbb{Z}[\Delta] = \operatorname{span}_{\mathbb{Z}}\{\epsilon_j \mid j \in J\}$ in all cases, except for $\Delta = \Delta(A_J)$, in which case $\mathbb{Z}[\Delta]$ is the corank 1 submodule $\{\sum_{j \in J} x_j \epsilon_j \mid \sum_{j \in J} x_j = 0\}$ of $\operatorname{span}_{\mathbb{Z}}\{\epsilon_j \mid j \in J\}$. Hence, either $\widetilde{\chi} = \chi$, or else $\Delta = \Delta(A_J)$ and $\widetilde{\chi}$ is determined by χ up to a constant. As $W(A_J) = S_{(J)}$, modifying $\widetilde{\chi}$ by a constant does not modify the value of the infimum of $\widetilde{\chi}(W(A_J).\lambda - \lambda)$. We may thus safely replace χ by $\widetilde{\chi}$, which we identify with the tuple $(d_j)_{j \in J} \in \mathbb{R}^J$.

Finally, with the above identifications, we have for all $\sigma \in \{\pm 1\}^{(J)}$ and $w \in S_{(J)}$ that

$$\chi(\sigma w.\lambda - \lambda) = \chi\Big(\sum_{j \in J} \lambda_j(\sigma_{w(j)}\epsilon_{w(j)} - \epsilon_j)\Big) = \sum_{j \in J} \lambda_j(\sigma_{w(j)}d_{w(j)} - d_j) = \sum_{j \in J} d_j(\sigma_j \lambda_{w^{-1}(j)} - \lambda_j)$$

$$= \lambda\Big(\sum_{j \in J} d_j(\sigma_j e_{w^{-1}(j)} - e_j)\Big) = \lambda\Big(w^{-1}\sigma.\sum_{j \in J} d_j e_j - \sum_{j \in J} d_j e_j\Big)$$

$$= \lambda((\sigma w)^{-1}.\chi - \chi),$$

and hence

$$\inf \chi(W.\lambda - \lambda) = \inf \lambda(W.\chi - \chi).$$

In particular, the condition $\inf \chi(W.\lambda - \lambda) > -\infty$ in the statement of Theorem A is equivalent to requiring the triple (J, λ, χ) with $\lambda = (\lambda_j)_{j \in J}$ and $\chi = (d_j)_{j \in J}$ to satisfy the PEC for $W = W(X_J)$ in the sense of Definition 2.1.

For $X = A$, Theorem A thus sums up Proposition 5.3 and Theorem 5.10. Since $C_{\min}(\lambda, B_J) \subseteq C_{\min}(\lambda, D_J)$ (see Remark 5.9) and since the PEC for $W(B_J)$, $W(C_J)$ and $W(D_J)$ are equivalent by Lemma 5.8, the conclusion of Theorem A for $X \in \{B, C, D\}$ follows from Proposition 5.3 and Theorem 5.12.   □

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