STABLE REFLECTION LENGTH IN COXETER GROUPS

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Abstract. We introduce stable reflection length in Coxeter groups, as a way to study the asymptotic behaviour of reflection length. This creates connections to other well-studied stable length functions in groups, namely stable commutator length and stable torsion length. As an application, we give a complete characterisation of elements whose reflection length is unbounded on powers.

Introduction

A natural geometric function associated to a Coxeter group W is its *reflection length*, denoted **rl**: the reflection length of an element $w \in W$ is the minimal number of hyperplane reflections needed to reflect the fundamental chamber C onto the chamber wC in the Coxeter complex of W. This statistic is well understood for finite Coxeter groups [4] and was initially investigated for affine Coxeter groups, where it was shown to be bounded [30]. For Coxeter groups that are a direct product of finite and affine reflection groups, there are formulas for **rl** [15], [8]. Together with the fact that **rl** is additive under products, its study reduces to the case of Coxeter groups of irreducible indefinite type (Lemma 1.12).

The interest shifted to asymptotic behaviours with the result of Duszenko that in a Coxeter group of indefinite type, **rl** is unbounded [20]. Since then, some work went into understanding to what extent this unboundedness could be witnessed by cyclic subgroups, in particular those generated by Coxeter elements. This was initially achieved by Drake and Peters for universal Coxeter groups [19] and then by the second author for Coxeter groups with sufficiently large labels [28].

Our main result gives a full characterisation for arbitrary elements in arbitrary Coxeter groups. This involves the notion of *straight part* from [29]. We refer the reader to Subsection 1.2 for the definition. For now let us just recall that an element w is *straight* if $\ell_S(w^n) = n \cdot \ell_S(w)$ for all $n \geq 1$, and point out that a straight element is equal to its own straight part.

Theorem A. *Let W be a Coxeter group and $w \in W$. Let $\mathrm{Pc}(w) = P_1 \times \cdots \times P_r$ be the decomposition of the parabolic closure of w into irreducible components. Write $w = w_1 \cdots w_r$ with $w_i \in P_i$. Then exactly one of the following holds:*

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(1) *For each $i \in \{1, \dots, r\}$ such that P_i is of indefinite type, the straight part of w_i is a product of two involutions.*
(2) *The reflection length $\mathbf{rl}(w^n)$ grows linearly, in particular it is unbounded.*

In earlier works, the focus was on finding sufficient conditions on the Coxeter graph of a Coxeter group that would ensure the unboundedness of $\mathbf{rl}(w^n)$ for every Coxeter element w [19, 28]. We obtain a full combinatorial characterisation of when this holds.

Theorem B (Theorem 4.1)**.** *Let (W, S) be a Coxeter system of irreducible indefinite type with Coxeter graph Γ.*

(1) *There exists a Coxeter element $w \in W$ such that $\{\mathbf{rl}(w^n)\}_{n\geq 1}$ is bounded, if and only if Γ is bipartite.*
(2) *$\{\mathbf{rl}(w^n)\}_{n\geq 1}$ is bounded for every Coxeter element $w \in W$, if and only if Γ is a tree.*

We only state this in the case of irreducible indefinite type, but the general case reduces to this one (Corollary 4.6). Note that this gives a wealth of examples of Coxeter groups of irreducible indefinite type with the property that $\{\mathbf{rl}(w^n)\}_{n\geq 1}$ is bounded for some Coxeter elements, and unbounded for others.

A novelty in our approach lies in shifting the focus from reflection length to its stabilisation. We call this the *stable reflection length*, denoted **srl**. Then $\mathbf{srl}(w) > 0$ if and only if $\mathbf{rl}(w^n)$ grows linearly, and in particular is unbounded. This is analogous to other stable length functions that have a rich theory, most importantly *stable commutator length* **scl** [10] and *stable torsion length* **stl** [2]. In fact, these quantities are intimately connected. For **srl** and **stl**, this takes the form of a bi-Lipschitz equivalence (Lemma 1.16). For **scl**, this is less direct, but there is still a strong connection in the generic case, which is a main step towards Theorem A.

Proposition C (Corollary 3.5)**.** *Let W be a Coxeter group of irreducible indefinite type. Then for all $w \in W$ with $\mathrm{Pc}(w) = W$, the following are equivalent.*

- $\mathbf{scl}(w) = 0$*;*
- $\mathbf{srl}(w) = 0$*;*
- *The straight part of w is a product of two involutions.*

Therefore one can interpret **srl** as a tool that creates a bridge between the rich literature on **scl** and the geometry of reflection length. We hope that this will be useful beyond the problem at hand. Below, we propose two motivating questions for future research (Subsection 1.5).

Outline. In Section 1 we go over some preliminaries on Coxeter groups and length functions. In Section 2 we reduce the positivity of **scl** to an algebraic property: *chirality*. In Section 3 we characterise this in terms of products of involutions, proving Theorem A and Proposition C. Finally, in Section 4 we focus on Coxeter elements, proving Theorem B.

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1. Preliminaries

1.1. **Coxeter groups.** The basic theory of Coxeter groups is treated in detail in e.g. [18].

Definition 1.1. Let $\Gamma_0 = (S, E)$ be a finite graph with vertex set $S = \{s_1, \dots, s_n\}$, edge set $E = \{\{u, v\} \subseteq S \mid u \neq v\}$ and an edge-labelling function $m : E \to \mathbb{N}_{\geq 2} \cup \{\infty\}$. We abbreviate $m(s_i, s_j)$ with m_{ij}. The corresponding *Coxeter group* W is given by the presentation

$$W = \langle S \mid \; s_i^2 = \mathbb{1} \;\; \forall i \in \{1, \dots, n\}, \\ (s_i s_j)^{m_{ij}} = \mathbb{1} \; \forall \, i \neq j \in \{1, \dots, n\} \text{ with } m_{ij} < \infty \rangle.$$

The pair (W, S) is called a *Coxeter system*. The graph Γ obtained from Γ_0 by omitting edges with label 2 is called the *Coxeter graph* of (W, S).

We denote by $\ell_S(w)$ the minimal length of a word in the generators S representing an element $w \in W$.

Given a subset $I \subset S$, the induced subgraph of Γ_0 with vertex set I defines a Coxeter system (P, I), and P is isomorphic to the subgroup W_I of W generated by I. A subgroup of W is called *parabolic* if it is conjugate to W_I for some $I \subset S$. For $w \in W$ there exists a smallest parabolic subgroup containing w, called the *parabolic closure* of w and denoted $\mathrm{Pc}(w)$.

The group W is called *irreducible* if Γ is connected. This is equivalent to W not decomposing as a direct product of two Coxeter groups defined on proper subgraphs. The irreducible Coxeter systems split into three families: *finite* type (when the group is finite), *affine* type (when the group is infinite and virtually abelian), and *indefinite* type (in all other cases). If W is finite (not necessarily irreducible) it is called *spherical*.

1.2. **Straight elements.** Let W be of irreducible indefinite type. An element $w \in W$ is called *straight* if $\ell_S(w^n) = n \cdot \ell_S(w)$. The geometry of straight elements is especially well-behaved, so it is useful to extract straight elements out of arbitrary elements.

Suppose that $\mathrm{Pc}(w) = W$. By [29, Theorem 9.6], there is a largest spherical parabolic subgroup $P_w^{\max}$ of W normalised by w. As in [29, Definition 9.21], we associate to w its *core* $w_c = \mathrm{core}(w)$, so that w has a unique decomposition of the form $w = a w_c^n$ with $n \geq 1$ and $a \in P_w^{\max}$. This decomposition is called the *core splitting* of w. The element $w_\infty := w_c^n$ is then called the *straight part* of w — this terminology is motivated by the fact that if w is straight then $w = w_\infty$, see [29, Remark 9.25]. More generally, w is straight if and only if $w = w_\infty$ and w is cyclically reduced, see [29, Corollary 8.11]. See also [29, Lemma 8.9] for a more geometric definition of the straight part.

We collect here a few properties of cores and core splittings from [29].

Lemma 1.2 ([29])**.** *Let $w \in W$ with $\mathrm{Pc}(w) = W$. Then:*

1. $\mathrm{Pc}(w_c) = \mathrm{Pc}(w^m) = \mathrm{Pc}(aw) = W$ *and* $P_w^{\max} = P_{w_c}^{\max} = P_{w^m}^{\max} = P_{aw}^{\max}$ *for all* $m \neq 0$ *and* $a \in P_w^{\max}$.
2. *There are some* $n, N \geq 1$ *such that* $w^N = w_c^{nN}$.
3. $\mathrm{core}(w^m) = \mathrm{core}(w)$ *for all* $m \geq 1$, *and* $\mathrm{core}(w^{-1}) = \mathrm{core}(w)^{-1}$.
4. *Write* $P_w^{\max} = v W_I v^{-1}$ *for some spherical subset* $I \subseteq S$ *and some* $v \in W$ *of minimal length in* vW_I. *Then* $\mathrm{core}(v^{-1} w v) = v^{-1} \mathrm{core}(w) v$.
5. *If* v *commutes with* w, *then* $v = a w_c^n$ *for some* $a \in P_w^{\max}$ *and* $n \in \mathbb{Z}$.

Proof. (1) follows from [29, Lemma 9.16(3)] and (2) from the core splitting $w = a w_c^n$ of w. The first part of (3) follows from [29, Lemma 9.23], and its second part from the definition of the core ([29, Definition 9.21]). Statement (4) follows from the first assertion of [29, Lemma 9.26] ([29, Lemma 9.26] is actually stated for w cyclically reduced, but this assumption is not used for the first assertion of that lemma). Finally, (5) is [29, Proposition 9.29(1)]. □

1.3. **Length functions.**

Definition 1.3. Let G be a group and $Y \subset G$ a symmetric subset. The corresponding *length function* ℓ_Y is defined as

$$\ell_Y : G \to \mathbb{N} \cup \{\infty\}\,; \qquad g \mapsto \min\{n \in \mathbb{N} \mid g \in Y^n\}$$

with $Y^n = \{y_1 \cdots y_n \in G \mid y_i \in Y\}$. The identity element $\mathbb{1}$ has length zero. Elements in $G \setminus \langle Y \rangle$ have length ∞.

In this paper, we will mostly be concerned with a *stabilisation* of the previous notion [13].

Definition 1.4. Let $Y \subset G$ be a symmetric subset, and let ℓ_Y be the corresponding length function. The *stable length function* $\mathrm{s}\ell_Y$ is defined as

$$\mathrm{s}\ell_Y(g) \coloneqq \lim_{n\to\infty} \frac{\ell_Y(g^n)}{n},$$

when $g \in \langle Y \rangle$. If there exists $k \geq 1$ such that $g^k \in \langle Y \rangle$, we set $\mathrm{s}\ell_Y(g) \coloneqq \frac{\mathrm{s}\ell_Y(g^k)}{k}$. Otherwise, we set $\mathrm{s}\ell_Y(g) \coloneqq \infty$.

The limit in the definition above exists by Fakete's Lemma. Moreover, for $g \in \langle Y \rangle$ and $k \geq 1$, we have $\mathrm{s}\ell_Y(g^k) = k \cdot \mathrm{s}\ell_Y(g)$, so the extension of the domain of $\mathrm{s}\ell_Y$ is well-defined. We record two general facts.

Lemma 1.5 ([2, Lemma 2.2])**.** *Let G, H be two groups with conjugacy-invariant symmetric subsets $Y \subseteq G$ and $Z \subseteq H$. Suppose $\varphi\colon G \to H$ is a group homomorphism with $\varphi(Y) \subseteq Z$. Then*

$$\ell_Z(\varphi(g)) \leq \ell_Y(g) \ \text{ and } \ \mathrm{s}\ell_Z(\varphi(g)) \leq \mathrm{s}\ell_Y(g)$$

for all $g \in \langle Y \rangle$.

Lemma 1.6. *Let $g \in \langle Y \rangle$, and suppose that there exists $N \geq 1$ such that $\{\ell_Y(g^{Nk})\}_{k\geq 1}$ is bounded. Then $\{\ell_Y(g^n)\}_{n\geq 1}$ is bounded.*

Proof. Writing $n = kN + r$ for $r < N$ we get

$$\ell_Y(g^n) \leq \ell_Y(g^{kN}) + \ell_Y(g^r) \leq \sup_{k\geq 1} \ell_Y(g^{kN}) + \max_{0\leq i<N} \ell_Y(g^i).$$

The first term is bounded by assumption, and the second term is bounded being a maximum over a finite set of finite values. □

The next two definitions are important examples of (stable) length functions.

Definition 1.7. Let G be a group and let $C \subseteq G$ be the set of commutators in G. The corresponding length function is called *commutator length* and denoted **cl**; its stabilisation is called *stable commutator length* and denoted **scl**.

Computing **cl** over free groups is an NP-complete problem [24]. On the other hand, there is an algorithm for computing **scl** over free groups [11], which is even implemented in practice [12]. In general, the theory of **scl** is much richer than that of **cl**: we refer the reader to Calegari's book [10], or the surveys [9, 25].

Definition 1.8. Let G be a group and let T be the set of all torsion elements in G. The corresponding length function is called *torsion length* and denoted **tl**; its stabilisation is called *stable torsion length* and denoted **stl**.

This latter notion was mainly studied by Avery and Chen [2], who proved several results parallel to the most celebrated ones on **scl**. For instance, there is an algorithm for computing **stl** over free products of finite groups.

Now, we move to the most important length function in this paper, which is defined specifically for Coxeter groups.

Definition 1.9. Let (W, S) be a Coxeter system. The conjugates of the standard generators in S are called *reflections*. The set of reflections R generates W. The corresponding length function is called *reflection length* and denoted **rl**; its stabilisation is called *stable reflection length* and denoted **srl**.

Remark 1.10. Although this definition really only makes sense for Coxeter groups, the reflection length coincides with the *cancellation length* with respect to the finite normal generating set S [21]. For the general framework of cancellation length on groups, and its asymptotic properties, we refer the reader to [7].

It is easy to see that all of these functions are additive under direct products, this is [30, Proposition 1.2] for **rl**, is established similarly for **tl** and **cl**, and implies the same for the stable versions.

Lemma 1.11. *Let W_1, W_2 be Coxeter groups. Then*

$$\mathbf{rl}_{W_1 \times W_2}(w_1, w_2) = \mathbf{rl}_{W_1}(w_1) + \mathbf{rl}_{W_2}(w_2),$$
$$\mathbf{srl}_{W_1 \times W_2}(w_1, w_2) = \mathbf{srl}_{W_1}(w_1) + \mathbf{srl}_{W_2}(w_2).$$

Moreover, for Coxeter groups of finite and affine type, formulas for **rl** are known (see [15] and [8]). In particular, **rl** is bounded on these groups, and therefore **srl** vanishes. We deduce:

Lemma 1.12. *Let W be a Coxeter group, which we decompose as $W_0 \times W_1 \times \cdots \times W_r$, where W_0 is the product of its finite and affine components, and $W_1, \ldots, W_r$ are its components of indefinite type. Let $w \in W$, written as $w = w_0 w_1 \cdots w_r$ accordingly. Then $\mathbf{srl}_W(w) > 0$ if and only if $\mathbf{srl}_{W_i}(w_i) > 0$ for some $i \in \{1, \ldots, r\}$, and $\{\mathbf{rl}_W(w^n)\}_{n \in \geq 1}$ is unbounded if and only if $\{\mathbf{rl}_{W_i}(w_i^n)\}_{n \geq 1}$ is unbounded for some $i \in \{1, \ldots, r\}$.*

An additional useful fact about **rl** is that its restriction to a parabolic subgroup coincides with the reflection length of that subgroup [21, Corollary 1.4]. This implies the same fact about **srl**.

Lemma 1.13. *Let $W' < W$ be a parabolic subgroup. Then for all $w \in W'$, $\mathbf{rl}_W(w) = \mathbf{rl}_{W'}(w)$ and hence $\mathbf{srl}_W(w) = \mathbf{srl}_{W'}(w)$.*

Remark 1.14. This is a very useful property that will play an important role in the proof of Theorem A. The situation for **scl** is different (except in the special case that W' is a retract), in fact it is an open question whether $\mathbf{scl}_{W'}(w) > 0$ implies $\mathbf{scl}_W(w) > 0$ [7, Remark 1.10].

1.4. **(Stable) reflection length vs (stable) torsion length.** Reflections are torsion elements. Hence Lemma 1.5 implies that $\mathbf{tl}(w) \leq \mathbf{rl}(w)$ and $\mathbf{stl}(w) \leq \mathbf{srl}(w)$. Combined with a known relationship between **scl** and **stl** [27, Proposition 1] we obtain:

Lemma 1.15. *Let W be a Coxeter group. Then for all $w \in W$:*

$$2\,\mathbf{scl}(w) \leq \mathbf{stl}(w) \leq \mathbf{srl}(w) < \infty.$$

In fact, the inequality between **stl** and **srl** is a bi-Lipschitz equivalence.

Lemma 1.16. *Let W be a Coxeter group. Then there exists a constant $C = C(W)$ such that for all $w \in W$*

$$\mathbf{tl}(w) \leq \mathbf{rl}(w) \leq C\,\mathbf{tl}(w),$$

and similarly for **stl** *and* **srl**.

Proof. Clearly the statement for **tl** and **rl** implies the one for the stable versions. By [1, Proposition 2.87], every torsion element of W is contained in a finite parabolic subgroup. It follows that there are finitely many conjugacy classes of torsion elements in W, let us choose representatives $t_1, \dots, t_n$. Letting $C := \max_i \mathbf{rl}(t_i)$ we obtain the result. □

Since we are interested in the asymptotic behaviour, from now on we will only focus on **scl** and **srl**. Moreover, we now know that positivity of **scl** implies positivity of **srl**.

1.5. **Two questions.** By analogy with common themes in **scl**, we propose two motivating questions for future research.

Question 1.17. Let W be a Coxeter group. Is there a *spectral gap* in **srl** over W? Namely, does there exist a constant $C = C(W) > 0$ such that for every $w \in W$ either $\mathbf{srl}(w) > C$ or $\mathbf{srl}(w) = 0$?

When W is a *right angled* Coxeter group (Definition 4.3), a positive answer follows from Lemma 1.15, Lemma 3.1 below, and the spectral gap for **scl** [17, Corollary 6.18]. It is unknown whether a spectral gap in **scl** holds for *all* Coxeter groups, but Question 1.17 could be more approachable.

Question 1.18. Let W be a Coxeter group. Is $\mathbf{srl}(w)$ rational, for all $w \in W$?

Rationality is a very powerful property for **scl** and **stl**, but it remains an open problem in Coxeter groups. If W is a *universal* Coxeter group, i.e. a free product of cyclic groups of order 2, then **scl** [16, Theorem A] and **stl** [2, Theorem B] are rational. In this case, all torsion elements are reflections, so $\mathbf{srl} = \mathbf{stl}$ is rational as well. More generally **srl** is rational for virtually free Coxeter groups [26]; such groups are characterised in [18, Proposition 8.8.5].

2. Positivity of stable commutator length

Thanks to Lemma 1.12, to understand (stable) reflection length, we may reduce to the case in which W is of irreducible indefinite type. Moreover, by Lemma 1.13, when studying the (stable) reflection length of an element $w \in W$, we may assume that $\mathrm{Pc}(w) = W$. In this section, we give a sufficient condition for $\mathbf{scl}_W(w) > 0$, which by Lemma 1.15 implies $\mathbf{srl}_W(w) > 0$.

Definition 2.1. An element $g \in G$ is called *achiral* if there exists $m \geq 1$ such that g^m is conjugate to g^{-m}. Otherwise, g is called *chiral*.

This is an algebraic terminology that is commonly used in the literature on **scl** (see e.g. [5]). For rank one elements in groups acting on CAT(0) spaces, it coincides with the more geometric notion of *irreversible* from [14]: see [14, Lemma 2.2(ii)].

Achirality is an obvious obstruction to the positivity of **scl**.

Lemma 2.2. *If $g \in G$ and $m \geq 1$ are such that g^m is conjugate to g^{-m}, then g^{2km} is a commutator for all $k \geq 1$; in particular $\mathbf{scl}(g) = 0$.*

Proof. Let $f \in G$ be such that $fg^{-m}f^{-1} = g^m$. Then $g^{2km} = g^{km}fg^{-km}f^{-1}$. □

The other direction is more interesting.

Theorem 2.3. *Let W be a Coxeter group of irreducible indefinite type. Let $w \in W$ be such that $\mathrm{Pc}(w) = W$. Then $\mathbf{scl}(w) > 0$ if and only if w is chiral.*

This was essentially achieved by Bestvina–Fujiwara [6, Main Theorem] and Caprace–Fujiwara [14, Theorem 1.8]. However their statements do not immediately give Theorem 2.3. Since this is essentially a known result, we only give a minimal proof citing the literature, and refer the reader to those papers for the relevant definitions.

Proof. One direction is given by Lemma 2.2. Consider the proper action of W on the Davis complex $\Sigma(W)$ [18], which equipped with an appropriate piecewise Euclidean metric is a proper CAT(0) space [31, Theorem A]. The hypothesis implies that w is a rank one element [14, Proposition 4.5]. In particular [34, Theorem 1.5] implies that W is acylindrically hyperbolic (cf. [35, Theorem 4.4]) and $w \in W$ is a generalised loxodromic element. By [32, Theorem 1.4], there is a non-elementary acylindrical (therefore WPD) action of W on a hyperbolic graph such that w is loxodromic. Since moreover w is assumed to be chiral, [23, Theorem 4.2] implies that there exists a homogeneous quasimorphism $\varphi \colon W \to \mathbb{R}$ such that $\varphi(w) > 0$. By Bavard duality [3], this implies that $\mathbf{scl}(w) > 0$. □

3. Chirality and products of involutions

In this section we characterise chirality in terms of products of involutions (elements which can be expressed as a product of at most two involutions are also known as *strongly real* elements). Let us first observe how this has strong consequences for **rl**.

Lemma 3.1. *Let W be a Coxeter group, let $w \in W$ and suppose that there exist $a, b \in W$ such that $a^2 = b^2 = \mathbb{1}$ and $w = ab$. Then $\{\mathbf{rl}(w^n)\}_{n \geq 1}$ is bounded, in particular $\mathbf{srl}(w) = 0$.*

Proof. More precisely, we will show that in this case

$$\mathbf{rl}(w^n) \leq \begin{cases} \mathbf{rl}(a) + \mathbf{rl}(b) \text{ if } n \text{ is odd;} \\ 2\min\{\mathbf{rl}(a), \mathbf{rl}(b)\} \text{ if } n \text{ is even.} \end{cases} \tag{3.1}$$

Suppose first that $n = 2j + 1$ is odd. Then we write

$$w^n = (ab)^{2j+1} = w^j a w^{-j} b;$$

so w^n is a product of a conjugate of a and b. Suppose now that n is even. Then we write

$$w^n = (ab \cdots ba)b(ab \cdots ba)^{-1}b;$$

so w^n is a product of two conjugates of b. Similarly, w^n is a product of two conjugates of a. □

Here is an equivalent property, which will arise more naturally in our arguments.

Lemma 3.2. *Let G be a group, and let $g \in G$. Then the following are equivalent.*

(1) *g and g^{-1} are conjugate by some $x \in G$ such that $x^2 = \mathbb{1}$.*
(2) *There exist $a, b \in G$ such that $a^2 = b^2 = \mathbb{1}$ and $g = ab$.*

Proof. If $xgx = g^{-1}$ for some $x \in G$ with $x^2 = \mathbb{1}$, then $(gx)^2 = x^2 = \mathbb{1}$ and (2) holds with $a = gx$ and $b = x$. Conversely, if $g = ab$ for some $a, b \in G$ with $a^2 = b^2 = \mathbb{1}$, then $aga = g^{-1}$. □

The next theorem is the key result, which interprets achirality in terms of products of involutions. We refer the reader to Subsection 1.2 for the relevant definitions.

Theorem 3.3. *Let W be a Coxeter group of irreducible indefinite type, and let $w \in W$ with $\mathrm{Pc}(w) = W$. If w is achiral, then w_c and w_c^{-1} are conjugate by an involution.*

Proof. Up to conjugating w, we may assume by Lemma 1.2(4) that $P_w^{\max} = W_I$ for some spherical subset $I \subseteq S$. Let $m \geq 1$ and $x \in W$ such that $x^{-1}w^m x = w^{-m}$. Up to replacing m by some multiple, we may further assume by Lemma 1.2(2) that $x^{-1}w_c^m x = w_c^{-m}$. Lemma 1.2(3) then yields

$$w_c^{-1} = \mathrm{core}(w_c^{-1}) = \mathrm{core}(w_c^{-m}) = \mathrm{core}(x^{-1}w_c^m x) = \mathrm{core}(x^{-1}w_c x).$$

Note that

$$x^{-1}P_w^{\max}x = x^{-1}P_{w^m}^{\max}x = P_{x^{-1}w^m x}^{\max} = P_{w^{-m}}^{\max} = P_w^{\max}$$

by Lemma 1.2(1), and hence x normalises $P_w^{\max} = W_I$. Write $x = x_I\overline{x}$ with $x_I \in W_I$ and $\overline{x}$ of minimal length in $xW_I = W_I x$. Then

$$x^{-1}w_c x = \overline{x}^{-1} \cdot x_I^{-1}w_c x_I \cdot \overline{x} = \overline{x}^{-1} \cdot (x_I^{-1}w_c x_I w_c^{-1})w_c \cdot \overline{x},$$

with $x_I^{-1}w_c x_I w_c^{-1} \in W_I$. In particular, $x_I^{-1}w_c x_I \in W_I w_c$ so that $\mathrm{core}(x_I^{-1}w_c x_I) = w_c$ by uniqueness of the core splitting. Lemma 1.2(4) then yields

$$\mathrm{core}(x^{-1}w_c x) = \overline{x}^{-1}\,\mathrm{core}(x_I^{-1}w_c x_I)\overline{x} = \overline{x}^{-1}w_c\overline{x},$$

and hence $\overline{x}^{-1}w_c\overline{x} = w_c^{-1}$.

In particular, $\overline{x}^2$ commutes with w_c. By Lemma 1.2(5), this implies that $\overline{x}^2 = aw_c^n$ for some $a \in W_I$ and $n \in \mathbb{Z}$. If $n \neq 0$, then $\mathrm{Pc}(\overline{x}^2) = W$ and $P_{\overline{x}^2}^{\max} = W_I$ by Lemma 1.2(1), and hence $\mathrm{core}(\overline{x}) = \mathrm{core}(\overline{x}^2) = w_c^{\varepsilon}$ by Lemma 1.2(3), where $\varepsilon \in \{\pm 1\}$ is the sign of n. Thus, in that case, $\overline{x}$ has core splitting $\overline{x} = bw_c^{\varepsilon r}$ for some $b \in W_I$ and $r \geq 1$, and hence $w_c^{-1} = \overline{x}^{-1}w_c\overline{x} = b'w_c$ for some $b' \in W_I$. Comparing cores yields $w_c^{-1} = w_c$, contradicting the fact that w_c has infinite order. Therefore, $n = 0$ and $\overline{x}^2 \in W_I$. Since $\overline{x}$ is the unique element of minimal length in $\overline{x}W_I$, it follows from $\overline{x}W_I = \overline{x}^{-1}W_I$ that $\overline{x}^2 = \mathbb{1}$, and we conclude. □

Remark 3.4. In [14, Lemma 4.8], the authors show that, if w is achiral, then w^k is a product of two involutions, where k is the index in W of a torsion free finite index normal subgroup W_0. Theorem 3.3 removes the passage to a power and recovers that result: if $w = aw_\infty$ for some $a \in P_w^{\max}$, then $w^k = a'w_\infty^k$ for some $a' \in P_w^{\max}$. As w_∞^k and w^k both belong to W_0, the torsion-freeness of W_0 implies that $a' = \mathbb{1}$ and hence that $w^k = w_\infty^k$.

Corollary 3.5. *Let W be a Coxeter group of irreducible indefinite type, and let $w \in W$ with $\mathrm{Pc}(w) = W$. Then the following are equivalent.*

(1) $\mathbf{srl}(w) = 0$*;*
(2) $\{\mathbf{rl}(w^n)\}_{n\geq 1}$ *is bounded;*
(3) $\mathbf{scl}(w) = 0$*;*
(4) w^{mk} *is a commutator, for some $m \geq 1$ and all $k \geq 1$;*
(5) *w is achiral;*
(6) *The core of w is a product of two involutions;*
(7) *The straight part of w is a product of two involutions.*

Note that we cannot formally state that $\{\mathbf{cl}(w^n)\}_{n\geq 1}$ is bounded, since if w is not in the commutator subgroup, then by definition this sequence will take the value ∞ infinitely many times.

Proof. If $\{\mathbf{rl}(w^n)\}_{n\geq 1}$ is bounded, then $\mathbf{srl}(w) = 0$. Then by Lemma 1.15 also $\mathbf{scl}(w) = 0$, which by Theorem 2.3 implies that w is achiral. Theorem 3.3 and Lemma 3.2 in turn imply that w_c is a product of two involutions. By Lemma 3.2, this property passes to powers, and thus the straight part is also a product of two involutions. Passing to a further power, by Lemma 1.2(2), there is some $N \geq 1$ such that w^N is a product of two involutions, and so $\{\mathbf{rl}(w^{Nk})\}_{k\geq 1}$ is bounded by Lemma 3.1. Then Lemma 1.6 implies that $\{\mathbf{rl}(w^n)\}_{n\geq 1}$ is also bounded. This gives the equivalence of all items, except for (4), but (5) $\Rightarrow$ (4) $\Rightarrow$ (3) by Lemma 2.2. □

Proof of Theorem A. By Lemma 1.13, we may assume that $\mathrm{Pc}(w) = W$, and by Lemma 1.12 we may assume that W is of irreducible indefinite type. Then the result follows from Corollary 3.5. □

4. Coxeter elements

Recall that a *Coxeter word* is a word in the alphabet S where every generator appears exactly once, and a *Coxeter element* is one represented by a Coxeter word. If $w \in W$ is a Coxeter element, then $\mathrm{Pc}(w) = W$ [14, Corollary 4.3].

Theorem 4.1. *Let (W, S) be a Coxeter system of irreducible indefinite type with Coxeter graph Γ.*

(1) *There exists a Coxeter element that is conjugate to its inverse if and only if Γ is bipartite.*
(2) *Every Coxeter element is conjugate to its inverse if and only if Γ is a tree.*

Since Coxeter elements are straight [36], and hence coincide with their straight part, Theorem B is a combination of Theorem 4.1 and Corollary 3.5(2)$\Leftrightarrow$(7). Recall that a graph is bipartite if and only if it has no odd cycle.

Remark 4.2. Let us stress that we use the Coxeter convention for Γ, where two generators are connected by an edge if they *do not* commute. This is the precise opposite of the convention used for right angled Coxeter groups in geometric group theory.

We start with a reduction to the right angled case.

Definition 4.3. A Coxeter group is *right angled* if all edges of the Coxeter graph are labeled by ∞, that is $m_{ij} \in \{2, \infty\}$ for all $i \neq j$. Given a Coxeter system (W, S), its *right angled cover* (W_r, S) is obtained by replacing all labels other than 2 with infinity. In other words, the group W_r is defined by the sub-presentation of W where we only retain the commuting relations. It comes with a canonical quotient map $W_r \to W$ that restricts to the identity on S. Note that the Coxeter graphs of W and W_r differ only by their labels.

Lemma 4.4. *Let (W, S) be a Coxeter system and let (W_r, S) be its right angled cover. Then the quotient $W_r \to W$ induces a bijection between Coxeter elements of W_r and Coxeter elements of W. If this bijection maps w_r to w, then w_r is conjugate to w_r^{-1} in W_r if and only if w is conjugate to w^{-1} in W.*

Proof. The surjection $W_r \to W$ restricts to a surjection from the Coxeter elements of W_r to those of W. To see injectivity: if two Coxeter words represent the same element in W, then by the solution to the word problem [37] this can be witnessed using only braid moves. Because every generator appears exactly once, the only braid moves that can be applied are commuting relations, which are already available in W_r. Finally, from the description of conjugacy classes of Coxeter elements [22], we see that whether a Coxeter element is conjugate to its inverse can be witnessed by only using the commuting relations, and therefore holds for $w_r \in W_r$ if and only if it holds for its image $w \in W$. □

The proof of Theorem 4.1 will reduce to the following special cases.

Lemma 4.5. *Let (W, S) be a Coxeter system, and let Γ be its Coxeter graph.*

(1) *Suppose that Γ is an odd cycle. Then no Coxeter element is conjugate to its inverse.*
(2) *Suppose that Γ is a cycle. Then there exists a Coxeter element that is not conjugate to its inverse.*

Proof. Suppose that the Coxeter graph Γ is a cycle. Given a Coxeter word $\mathtt{w} = s_1 \cdots s_n$, we define its *curl* to be the number of edges $s_i s_j$ such that $i < j$, minus the number of edges $s_i s_j$ such that $i > j$. Combining [33, Theorem 1.6] and [22, Theorem 1.1], we see that two Coxeter words represent conjugate elements if and only if they have the same curl. Moreover, $\mathtt{w}^{-1} = s_n \cdots s_1$ has the opposite curl as $\mathtt{w}$. So if $\mathtt{w}$ represents a Coxeter element that is conjugate to its inverse, then it must have curl 0.

(1) If the cycle is odd, then the curl of any Coxeter word is odd, in particular non-zero, so no Coxeter element is conjugate to its inverse.

(2) Let $\mathtt{w} = s_1 \cdots s_n$ be a Coxeter word oriented along the cycle. It has curl $n - 1 > 0$, so the Coxeter element it represents is not conjugate to its inverse. □

Proof of Theorem 4.1. (1) Suppose that Γ is bipartite. Choose a bipartition with parts $\{s_1, \ldots, s_i\}$ and $\{s_{i+1}, \ldots, s_n\}$. Then $w = (s_1 \cdots s_i)(s_{i+1} \cdots s_n)$ is a product of two involutions, and therefore conjugate to its inverse. Conversely, suppose that there exists a Coxeter element $w \in W$ that is conjugate to its inverse. By Lemma 4.4, we may assume that W is right angled. Suppose by contradiction that Γ is not bipartite. Pick a minimal odd cycle Δ in Γ, and let V be the corresponding parabolic subgroup. By minimality, Δ is the Coxeter graph of V (i.e. there are no chords). Moreover, the retraction $W \to V$ maps w to a Coxeter element in V that is conjugate to its inverse. This contradicts Lemma 4.5(1).

(2) We again reduce to the right angled case by Lemma 4.4. If Γ is a tree, then all Coxeter elements are conjugate [22, Proposition 2.3]. Conversely, suppose that Γ is not a tree. Let Δ be a minimal cycle in Γ, and let V be the corresponding parabolic subgroup. Again, Δ is the Coxeter graph of V, so considering the retraction $W \to V$ we conclude by Lemma 4.5(2). □

Corollary 4.6. *Let (W, S) be a Coxeter system.*

(1) *There exists a Coxeter element w such that $\{\mathbf{rl}(w^n)\}_{n\geq 1}$ is bounded if and only the Coxeter graph of every indefinite component is bipartite.*
(2) *$\{\mathbf{rl}(w^n)\}_{n\geq 1}$ is bounded for every Coxeter element w if and only if the Coxeter graph of every indefinite component is a tree.*

Proof. A Coxeter element in W is a product of Coxeter elements of each component. Combine Lemma 1.12, Corollary 3.5 and Theorem 4.1. □

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