CONJUGACY CLASSES AND STRAIGHT ELEMENTS IN COXETER GROUPS

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ABSTRACT. Let W be a Coxeter group. In this paper, we establish that, up to going to some finite index normal subgroup W_0 of W, any two cyclically reduced expressions of conjugate elements of W_0 only differ by a sequence of braid relations and cyclic shifts. This thus provides a very simple description of conjugacy classes in W_0. As a byproduct of our methods, we also obtain a characterisation of straight elements of W, namely of those elements $w \in W$ for which $\ell(w^n) = n\,\ell(w)$ for any $n \in \mathbb{Z}$. In particular, we generalise previous characterisations of straight elements within the class of so-called cyclically fully commutative (CFC) elements, and we give a shorter and more transparent proof that Coxeter elements are straight.

1. Introduction

Let (W, S) be a Coxeter system. By a classical result of J. Tits [Tit69] (also known as Matsumoto's theorem, see [Mat64]), any two reduced expressions of a given element of W only differ by a sequence of braid relations. This yields in particular a very simple solution to the word problem for Coxeter groups.

In attempting to find an analoguous solution for the conjugacy problem (see [Kra09] for a thorough study of this problem), one might investigate a cyclic version of this theorem, and ask *whether two cyclically reduced expressions of conjugate elements of W only differ by a sequence of braid relations and cyclic shifts* (see [BBE$^+$12]). We recall that $w \in W$ is **cyclically reduced** if every cyclic shift of any reduced expression for w is still reduced.

Although the answer to this question is "no" in general (see Remark 3 below), it is probably true within the class of elements of W whose parabolic closure has only infinite irreducible components (more generally, it might be true that two cyclically reduced conjugate elements can be obtained from one another by a sequence of braid relations and cyclic shifts if and only if they have the same parabolic closure). Up to now, this "cyclic version" of Matsumoto's theorem had only be shown to hold within the class of Coxeter elements of W (see [EE09]). Our first theorem is a dramatic strengthening of this result.

We say that an element $u \in W$ has property **(Cent)** if whenever u normalises a finite parabolic subgroup of W, it centralises this subgroup.

2010 *Mathematics Subject Classification.* 20F55, 20E45.

*F.R.S.-FNRS Research Fellow.

Theorem A. *Let u be a cyclically reduced element of W of infinite order. Assume that u has property* (Cent)*. Then any cyclically reduced conjugate of u can be obtained from u by a sequence of braid relations and cyclic shifts.*

This thus provides a very simple description of conjugacy classes of elements of infinite order of W with property (Cent). Given a word $\omega = s_1 \dots s_j s_{j+1} \dots s_k$ with each $s_i \in S$ and such that $s_j = s_{j+1}$, we say that the word $\omega' = s_1 \dots s_{j-1} s_{j+2} \dots s_k$ is obtained from ω by **ss-cancellation**.

Corollary B. *Let u_1, u_2 be two elements of infinite order of W and assume that u_1 has property* (Cent)*. Then u_1 and u_2 are conjugate if and only if there exists some $w \in W$ obtained from each of u_1 and u_2 by a sequence of braid relations, cyclic shifts, and ss-cancellations.*

The proof of these results is given in Section 3.

Remark 1. Note that the class of elements of infinite order with property (Cent) is very large. For example, it contains any torsion-free finite index normal subgroup W_0 of W (see [DJ99, Lemma 1]). Note that the existence of such a subgroup W_0 is ensured by Selberg's lemma.

Remark 2. One can in fact give a slightly more precise version of property (Cent) under which the conclusion of Theorem A still holds, by specifying the spherical parabolic subgroups that should be centralised (see Remark 3.7). This is property **(Cent')**, which is by definition satisfied by an element $u \in W$ if every conjugate w of u obtained from u by a sequence of braid relations and cyclic shifts has the following property: whenever w normalises a spherical parabolic subgroup of the form $w_I W_J w_I^{-1}$ for some spherical subsets $J \subseteq I \subseteq S$ and some element $w_I \in W_I$, it centralises $w_I W_J w_I^{-1}$.

Remark 3. Assume that W possesses two conjugate standard parabolic subgroups, say $W_T = \langle T \rangle$ and $W_U = \langle U \rangle$ for some distinct subsets T and U of S. Then by [Kra09, Proposition 3.1.6] these subsets are conjugate, say $xTx^{-1} = U$ for some $x \in W$. Then for any cyclically reduced element w whose (standard) parabolic closure is W_T, the element xwx^{-1} is a cyclically reduced conjugate of w that cannot be obtained from w by a sequence of braid relations and cyclic shifts. Thus, if one wants to describe the conjugacy classes of arbitrary elements of W, one should allow, besides braid relations and cyclic shifts, conjugations by elements that conjugate a subset of S to another one.

Note that the conjugate standard parabolic subgroups of W were completely described by Deodhar (see e.g. [Kra09, Theorem 3.1.3]) and that this phenomenon is purely spherical: if W_T is irreducible and conjugate to W_U for some distinct subsets T and U of S, then T is spherical.

Finally, we mention that a weaker version of Theorem A for torsion elements of W has already been studied in several papers, in connection with characters of Hecke algebras (see [GKP00, Theorem 2.6] and the references in that paper). More precisely, it is known that, given an element w in a finite Coxeter group W, one can obtain, using only braid relations, cyclic shifts and ss-cancellations, a conjugate u of w of minimal length in its conjugacy class. Moreover, for any two conjugate elements $u, u' \in W$ of minimal length in their conjugacy class, there is a sequence $u = u_0, u_1, \dots, u_k = u'$ and elements $x_1, \dots, x_k \in W$ such that $u_{i-1} \overset{x_i}{\sim} u_i$ for each $i = 1, \dots, k$, where for any $w, w', x \in W$, we write $w \overset{x}{\sim} w'$ if $w' = x^{-1}wx$ and if $\ell(x^{-1}w) = \ell(x) + \ell(w)$ or $\ell(wx) = \ell(x) + \ell(w)$. As a corollary of the first statement and of our methods, we deduce in particular the following non-obvious fact, whose proof can be found at the end of Section 3.

Corollary C. *Assume that $w \in W$ either has property* (Cent) *or has finite order. Then w is cyclically reduced if and only if it is of minimal length in its conjugacy class.*

A second related problem, which we investigate in this paper as well, is to characterise the straight elements of W where, following [Kra09], we call an element $w \in W$ **straight** if $\ell(w^n) = n\,\ell(w)$ for all $n \in \mathbb{N}$.

The question whether Coxeter elements are straight has been the object of several papers (see [Spe09] and references therein) and this question has only been settled in full generality in 2009 by D. Speyer (see *loc.cit.*).

More generally, one might try to find natural necessary and sufficient conditions for an arbitrary element $w \in W$ to be straight. This problem for instance motivated the paper [BBE+12], in which the authors characterise the straight elements in the class of so-called *cyclically fully commutative* (CFC) elements (which "generalises" in some sense the class of Coxeter elements), provided that W satisfies some additional technical condition.

An obvious requirement for an arbitrary element $w \in W$ to be straight is that it should be cyclically reduced (see e.g. Lemma 4.1 below). This is one of the two conditions considered in [BBE+12]; the second condition is what the authors of *loc.cit.* call *torsion-freeness*, namely, the fact for an element $w \in W$ to have a standard parabolic closure with only infinite irreducible components. This second condition is however too restrictive in general. In this paper, we give the following less restrictive definition of torsion-freeness (which coincides with the one given in *loc.cit.* within the class of CFC elements, see Lemma 4.7): an element $w \in W$ is **torsion-free** if w has no reduced decomposition of the form $w = w_I n_I$ for some spherical subset $I \subseteq S$, some $w_I \in W_I \setminus \{1\}$ and some n_I normalising W_I. This is indeed an obvious requirement for w to be straight (see Lemma 4.1).

Our second theorem states that these two conditions are the only possible obstructions to straightness, without any restriction on W or on the class of elements of W considered.

Theorem D. *Let (W, S) be a Coxeter system and let $u \in W$ be cyclically reduced. Then the following are equivalent:*

(1) *u is straight.*
(2) *Every cyclically reduced conjugate of u obtained from u by a sequence of braid relations and cyclic shifts is torsion-free.*

Example 1. Let W be the affine Coxeter group of type $\tilde{A}_2$, with set of generators $S = \{s, t, u\}$. Thus $(st)^3 = (tu)^3 = (su)^3 = 1$. Consider the cyclically reduced element $w \in W$ with reduced expression $w = tustuts$. Note then that $w^2 = (tus)^4$ and hence w is not straight. Note also that w is torsion-free as it does not normalise any nontrivial finite standard parabolic subgroup of W. However, its cyclic shift $sws = (stustu)t = t(stustu)$ is not torsion-free (with $I = \{t\}$).

As a corollary, we generalise the characterisation of straight CFC elements given in [BBE$^+$12, Corollary 7.2] by removing the technical assumption on W considered in that paper. We recall that an element $w \in W$ is **fully commutative** (FC) if any two reduced expressions for w can be obtained from one another by iterated commutations of commuting generators. It is moreover **cyclically fully commutative** (CFC) if every cyclic shift of any reduced expression of w is a reduced expression for a FC element. We also recall that the **standard parabolic closure** of an element $w \in W$ is the smallest standard parabolic subgroup of W containing w, or else the subgroup of W generated by the elements of S appearing in any reduced decomposition of w.

Corollary E. *Let w be a CFC element of W. Then the following are equivalent:*
(1) *w is straight.*
(2) *The standard parabolic closure of w has only infinite irreducible components.*

In particular, this yields a shorter and more transparent proof that Coxeter elements are straight.

Corollary F. *Let $u = s_1 \dots s_n$ be a Coxeter element in W. Then u is straight if and only if W has only infinite irreducible components.*

The proof of these results is given in Section 4.

2. Preliminaries

2.1. **Basic definitions.** Basics on Coxeter groups and complexes can be found in [AB08, Chapters 1–3].

Let (W, S) be a Coxeter system with associated Coxeter complex $\Sigma = \Sigma(W, S)$ and with set of roots (or half-spaces) Φ. We denote by $C_0 = \{1_W\}$ the fundamental chamber of Σ and by $\{\alpha_s | s \in S\}$ the set of simple roots. We write $\ell(w)$ for the length of an element $w \in W$, that is, the number of generators (from S) appearing in a reduced decomposition of w.

We recall that a **gallery** Γ between two chambers D and E of Σ is a sequence $D = D_0, D_1, \dots, D_k = E$ of chambers such that for each $i = 1, \dots, k$, the chambers D_{i-1} and D_i are s_i-adjacent for some $s_i \in S$, where adjacency is understood as adjacency in the Cayley graph of (W, S). The integer $k \geq 1$ is called the **length** of Γ, and the sequence $(s_1, s_2, \dots, s_k) \in S^k$ is its **type**. A **minimal gallery** between the chambers C and D is then a gallery between C and D of minimal length. Given two chambers vC_0 and wvC_0 $(v, w \in W)$, the function associating to a gallery its type establishes a bijective correspondance between minimal galleries from vC_0 to wvC_0 and reduced expressions for $v^{-1}wv \in W$. In particular, the length $\ell(w)$ of an element $w \in W$ is also the length of a minimal gallery from C_0 to wC_0, or else the number of walls crossed by such a gallery, that is, the number of walls separating C_0 from wC_0.

For a subset I of S, we let W_I denote the corresponding **standard parabolic subgroup**, that is, the subgroup of W generated by I. Conjugates in W of standard parabolic subgroups are called **parabolic subgroups** of W. It is a standard fact that any intersection of parabolic subgroups of W is again a parabolic subgroup, and it thus makes sense to define the **parabolic closure** of a subset H of W, which we denote by $\mathrm{Pc}(H)$. In case H is a singleton $\{w\}$, we will also write $\mathrm{Pc}(w) := \mathrm{Pc}(\{w\})$.

For a subset I of S, we also let Φ_I denote the subset of Φ consisting of the roots of the form $v\alpha_s$ for some $v \in W_I$ and $s \in I$. Finally, we write $N_W(W_I)$ for the normaliser of W_I in W and we let N_I denote the stabiliser in W of the set of roots $\{\alpha_s | s \in I\}$. The following lemma follows from [Lus77, Lemma 5.2].

Lemma 2.1. *Let I be a spherical subset of S. Then $N_W(W_I) = W_I \rtimes N_I$ and $\ell(w_I n_I) = \ell(w_I) + \ell(n_I)$ for any $w_I \in W_I$ and $n_I \in N_I$.*

2.2. **Davis complex.** Let $|\Sigma|$ be the standard geometric realisation of Σ. We denote by X the Davis complex of W. Thus X is a $\mathrm{CAT}(0)$ subcomplex of the barycentric subdivision of $|\Sigma|$ on which W acts by cellular isometries. Moreover, each point $x \in X$ determines a unique spherical simplex of Σ, called its **support**. In this paper, we identify the walls, roots and simplices in Σ with their (closed) realisation in X. More precisely, we will identify a simplex A of Σ with the set of $x \in X$ whose support is a face of A. In particular, we view chambers as closed subsets of X.

2.3. **Actions on CAT(0)-spaces.** Consider the W-action on X. For an element $w \in W$, we let

$$|w| := \inf\{d(x, wx) \mid x \in X\} \in [0, \infty)$$

denote its **translation length** and we set

$$\mathrm{Min}(w) := \{x \in X \mid d(x, wx) = |w|\}.$$

A standard result of M. Bridson (see [Bri99]) asserts that such an action is **semi-simple**, meaning that $\mathrm{Min}(w)$ is nonempty for any $w \in W$. If w has finite order, then w has a fixed point (see e.g. [AB08, Theorem 11.23]). If w has infinite order, then $|w| \neq 0$ and $\mathrm{Min}(w)$ is the union of all the w-axes, where a **w-axis** is a geodesic line stabilised by w (on which w then acts by translation). Basics on $\mathrm{CAT}(0)$ spaces may be found in [BH99].

2.4. **Walls.** Let $w \in W$ be of infinite order. Given a w-axis L, we say that a wall is **transverse** to L if it intersects L in a single point. We call a wall m **w-essential** if it is transverse to some w-axis.

We recall that the intersection of a wall and any geodesic segment in X that is not completely contained in that wall is either empty or consists of a single point (see [NV02, Lemma 3.4]). Given $x, y \in X$, we say that a wall m **separates** x from y if the intersection of m with the geodesic segment joining x to y consists of a single point.

3. A cyclic version of Matsumoto's theorem

This section is devoted to the proof of Theorem A and of its corollaries. Throughout this section, we fix a Coxeter system (W, S).

Let $w \in W$. Given two chambers C, D of Σ, we denote by $\Gamma(C, D)$ their **convex hull**, namely, the reunion of all minimal galleries from C to D, viewed as a subcomplex of Σ. As mentioned in Section 2.2, we will also view $\Gamma(C, D)$ as a closed subcomplex of X. Let $\mathcal{C}_0$ be the set of chambers of Σ of the form $w^k C_0$ for some $k \in \mathbb{Z}$, and define inductively

$$\mathcal{C}_{i+1} := \bigcup_{D \in \mathcal{C}_i} \Gamma(D, wD) \subseteq X$$

for each $i \geq 0$. Set $\mathcal{C}_w := \bigcup_{i \geq 0} \mathcal{C}_i \subseteq X$.

Before describing the basic properties of $\mathcal{C}_w$, we need the following technical lemma.

Lemma 3.1. *Let C and D be two chambers of Σ, and pick points $x \in C \subset X$ and $y \in D \subset X$. Then there exists a minimal gallery from C to D containing the geodesic segment from x to y.*

Proof. If x and y lie in the interior of C and D respectively, then the lemma follows from [AB08, Proposition 12.25]. In the general case, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ (resp. $(y_n)_{n \in \mathbb{N}}$) of points in the interior of C (resp. D) that converges to x (resp. y). Thus for each $n \in \mathbb{N}$, the geodesic segment $[x_n, y_n]$ from x_n to y_n is entirely contained in some minimal gallery from C to D. As there are only finitely many such galleries, we may assume, up to choosing a subsequence, that $[x_n, y_n]$ is entirely contained in a given minimal gallery Γ between C and D for all large enough n. As Γ is closed, it then also contains the geodesic segment $[x, y]$ from x to y, as desired. $\square$

Lemma 3.2. *We have the following:*

1. *Each $\mathcal{C}_i$, $i \in \mathbb{N}$, is w-stable. In particular, $\mathcal{C}_w$ is w-stable.*
2. *$\mathcal{C}_w$ is a closed subset of X.*
3. *Given a point $x \in \mathcal{C}_w \subseteq X$, the geodesic from x to wx is entirely contained in $\mathcal{C}_w$.*

Proof. The first statement follows from a straightforward induction on i.

To see that $\mathcal{C}_w$ is closed, note that any sequence of points of $\mathcal{C}_w$ that converges in X is bounded, hence contained in finitely many chambers of $\mathcal{C}_w$. As the reunion of these chambers is a closed subset of $\mathcal{C}_w$, the conclusion follows.

We now prove (3). Let $x \in \mathcal{C}_w \subseteq X$. Pick some $i \geq 0$ and some chamber $D \in \mathcal{C}_i$ containing x. Then by Lemma 3.1, the geodesic from x to wx is contained in some minimal gallery from D to wD, and hence in $\mathcal{C}_{i+1} \subseteq \mathcal{C}_w$, as desired. $\square$

We say that $u \in W$ is **elementary related** to $v \in W$ if v admits a decomposition that is a cyclic shift of some reduced decomposition of u, that is, there is some reduced decomposition $u = s_1 \dots s_n$ ($s_i \in S$) of u and some $k \in \{1, \dots, n\}$ such that $v = s_k \dots s_n s_1 \dots s_{k-1}$. We say that $u \in W$ is **κ-related** to $v \in W$, which we denote by $u \sim_\kappa v$, if there exists a sequence $u = u_0, u_1, \dots, u_k = v$ for some $k \geq 1$ such that u_i is elementary related to

u_{i+1} for all $i = 0, \dots, k-1$. Note that if $u \sim_\kappa v$, then v is conjugate to u and $\ell(u) \geq \ell(v)$. Note also that the relation $\sim_\kappa$ is in general not symmetric, as one might have $u \sim_\kappa v$ with $\ell(u) > \ell(v)$. However, within the class of cyclically reduced elements of W, $\sim_\kappa$ is an equivalence relation.

Lemma 3.3. *Let D be a chamber of $\mathcal{C}_w$, say $D = vC_0$ for some $v \in W$. Then $w \sim_\kappa v^{-1}wv$.*

Proof. Let $i \in \mathbb{N}$ be such that $D \in \mathcal{C}_i$. We prove the claim by induction on i.

If $i = 0$, then $v = w^k$ for some $k \in \mathbb{Z}$ and there is nothing to prove.

Assume now that $i > 0$. Let E be a chamber of $\mathcal{C}_{i-1}$ such that D is on a minimal gallery Γ from E to wE, say $E = uC_0$ for some $u \in W$. Hence the type of Γ is obtained from a reduced expression of $u^{-1}wu$. Thus $v = uu_1$ for some $u_1 \in W$ appearing as an initial subword in some reduced expression of $u^{-1}wu$. In particular, $u^{-1}wu$ is elementary related to $v^{-1}wv$. As $w \sim_\kappa u^{-1}wu$ by induction hypothesis, the conclusion follows. □

Proposition 3.4. *Let $w \in W$. Then $\mathcal{C}_w$ contains a point of* $\mathrm{Min}(w)$*. In particular, if w has infinite order, then $\mathcal{C}_w$ contains a w-axis.*

Proof. For each $u \in W$, consider the continuous function $f_u \colon X \to \mathbb{R} : x \mapsto \mathrm{d}(x, ux)$. Let $u_1C_0, \dots, u_kC_0$ ($u_i \in W$) be the chambers at (gallery) distance at most $\ell(w)$ from C_0. For $i = 1, \dots, k$, let $a_i \in \mathbb{R}$ be the minimum of the function f_{u_i} over C_0.

For each chamber D of $\mathcal{C}_w$, let x_D be a point of D where f_w attains its minimum over D. Note then that if $D = vC_0$ for some $v \in W$, the function $f_{v^{-1}wv}$ attains its minimum over C_0 at $v^{-1}x_D$. Moreover, as $\ell(v^{-1}wv) \leq \ell(w)$ by Lemma 3.3, there is some $i \in \{1, \dots, k\}$ such that $v^{-1}wv = u_i$. Then $f_w(x_D) = \mathrm{d}(x_D, wx_D) = f_{u_i}(v^{-1}x_D) = a_i$. We have thus shown that the set $\{f_w(x_D) \mid D \text{ is a chamber of } \mathcal{C}_w\}$ is contained in $\{a_i \mid i = 1, \dots, k\}$ and is therefore finite. In particular, f_w attains its minimum over $\mathcal{C}_w$, say at $x \in \mathcal{C}_w$.

We claim that $\mathrm{d}(w^{-1}x, wx) = 2\,\mathrm{d}(x, wx)$, and hence that $x \in \mathrm{Min}(w)$ (see e.g. [BH99, Chapter II, Proposition 1.4 (2)]). Indeed, suppose for a contradiction that $\mathrm{d}(w^{-1}x, wx) < 2\,\mathrm{d}(x, wx)$. Let y be the midpoint of the geodesic from $w^{-1}x$ to x. Then $f_w(y) = \mathrm{d}(y, wy) < \mathrm{d}(y, x) + \mathrm{d}(x, wy) = \mathrm{d}(x, wx) = f_w(x)$. As $w^{-1}x$ and hence y belong to $\mathcal{C}_w$ by Lemma 3.2 (1) and (3), we get the desired contradiction.

The second statement follows from the fact that, if w has infinite order, then any w-axis intersecting $\mathcal{C}_w$ is entirely contained in $\mathcal{C}_w$ by Lemma 3.2 (1) and (3). □

Corollary 3.5. *Let $u \in W$ be of infinite order. Then there is some $w \in W$ with $u \sim_\kappa w$ such that* $\mathrm{Min}(w) \cap C_0$ *is nonempty and not contained in any w-essential wall.*

Proof. By Proposition 3.4, we know that $\mathcal{C}_u$ contains a u-axis. Let x be a point on that axis that is not contained in any u-essential wall. Let D be a chamber of $\mathcal{C}_u$ containing x, say $D = vC_0$ for some $v \in W$. Then Lemma 3.3 implies that $u \sim_\kappa v^{-1}uv$. The conclusion follows with $w = v^{-1}uv$. □

Here is a restatement of Theorem A.

Theorem 3.6. *Let $w \in W$ be cyclically reduced and of infinite order. Assume that w has property* (Cent)*. Then any cyclically reduced conjugate of w is κ-related to w.*

Proof. Let u be a cyclically reduced conjugate of w. By Corollary 3.5, there is no loss of generality in assuming that w and u both possess an axis through C_0, say $x_w \in \mathrm{Min}(w) \cap C_0$ and $x_u \in \mathrm{Min}(u) \cap C_0$. Choose $v \in W$ of minimal length so that $u = v^{-1}wv$. Note then that $vx_u \in \mathrm{Min}(w)$. We claim that vC_0 is a chamber of $\mathcal{C}_w$, which would imply the theorem by Lemma 3.3.

Indeed, assume for a contradiction that vC_0 is not a chamber of $\mathcal{C}_w$, and let $C_0, C_1, \dots, C_k = vC_0$ $(k \geq 1)$ be a minimal gallery from C_0 to vC_0 containing the geodesic segment $[x_w, vx_u]$ between x_w and vx_u (such a gallery exists by Lemma 3.1). Let $i \in \{0, \dots k-1\}$ be such that $D := C_i$ is a chamber of $\mathcal{C}_w$ but $E := C_{i+1}$ is not. Let m be the wall separating D from E. Then m intersects the geodesic $[x_w, vx_u] \subseteq \mathrm{Min}(w)$. Let $y \in (m \cap D) \cap [x_w, vx_u] \subseteq (m \cap D) \cap \mathrm{Min}(w)$, and let L be the w-axis through y. If L is transverse to m, then for some $\epsilon \in \{\pm 1\}$, the chambers D and $w^\epsilon D$ lie on different sides of m, and hence there is a minimal gallery from D to $w^\epsilon D$ passing through E, contradicting the fact that E is not a chamber of $\mathcal{C}_w$. Thus L is containd in m. As by assumption w centralises the finite parabolic subgroup generated by the reflections whose wall contain L, we deduce that w commutes with r_m. In particular, $u = (r_m v)^{-1} w (r_m v)$. Since $\ell(r_m v) < \ell(v)$, this contradicts the minimality assumption on v, as desired. □

Remark 3.7. Note that Remark 2 from the introduction readily follows from the proof of Theorem 3.6. Indeed, keeping the notations of the above proof, let $v_1 \in W$ be such that $D = v_1 C_0$, so that $w \sim_K v_1^{-1} w v_1$ by Lemma 3.3. Let also $A = v_1 W_I$ be the support of y, for some spherical subset I of S. Assume as in the proof that the w-axis L through y is contained in m. Let P be the spherical parabolic subgroup which is the parabolic closure of the set of reflections whose wall contains L. Thus w normalises P and $P \leq v_1 W_I v_1^{-1}$. Hence $v_1^{-1} w v_1$ normalises $v_1^{-1} P v_1 \leq W_I$ and $v_1^{-1} P v_1$ is of the form $w_I W_J w_I^{-1}$ for some $w_I \in W_I$ and some subset $J \subseteq I$. The conclusion follows.

Proof of Corollary B. Assume that u_1 and u_2 are conjugate. For $i = 1, 2$, let $w_i \in W$ be cyclically reduced and such that $u_i \sim_\kappa w_i$. It then follows from Theorem 3.6 that $w_1 \sim_\kappa w_2$. The conclusion follows. □

We conclude this section by proving Corollary C. To facilitate the exposition, we will call an element $u \in W$ *strongly cyclically reduced in $H < W$* (**SCR in** H for short) if $\ell(u) = \min\{\ell(v^{-1}uv) \mid v \in H\}$. Thus $u \in W$ is of minimal length in its conjugacy class if and only if it is SCR in W.

Lemma 3.8. *Let $w \in W$ be such that* $\mathrm{Pc}(w)$ *is standard. Then there exists some $v \in \mathrm{Pc}(w)$ such that vwv^{-1} is SCR in W.*

Proof. Let $T \subseteq S$ be such that $\mathrm{Pc}(w) = W_T$. Let also $v_1 \in W$ be such that $v_1 w v_1^{-1}$ is SCR in W. In particular, $\mathrm{Pc}(v_1 w v_1^{-1})$ is standard (see e.g. [CF10, Proposition 4.2]) and hence $v_1 W_T v_1^{-1} = \mathrm{Pc}(v_1 w v_1^{-1}) = W_U$ for some $U \subseteq S$. It then follows from [Kra09, Proposition 3.1.6] that there is some $u \in W_U$ such that $v_1 T v_1^{-1} = uUu^{-1}$. Set $x = u^{-1}v_1$, so that $x^{-1}Ux = T$.

Set also $v = x^{-1}ux \in W_T$. Note that $u(xwx^{-1})u^{-1} \in W_U$ and hence $vwv^{-1} = x^{-1}u(xwx^{-1})u^{-1}x$ has length $\ell(u(xwx^{-1})u^{-1}) = \ell(v_1wv_1^{-1})$. Thus vwv^{-1} is SCR in W, as desired. $\square$

Proof of Corollary C. Clearly, if w is SCR in W, then it is cyclically reduced. Conversely, assume that w is cyclically reduced and let us prove that it is SCR in W. If w has property (Cent), this readily follows from Theorem A, and we may thus assume that w has finite order. We will prove that there is some $w' \in W$ with $w \sim_\kappa w'$ and such that w' is SCR in W, yielding the claim.

By Lemma 3.3 together with Proposition 3.4, one can find some $w_1 \in W$ with $w \sim_\kappa w_1$ such that w_1 fixes a face of C_0, that is, $w_1 \in W_I$ for some spherical subset I of S. It then follows from [GKP00, Theorem 2.6] that there is some $w' \in W_I$ which is SCR in W_I and such that $w_1 \sim_\kappa w'$. In particular, it follows from [CF10, Proposition 4.2] that the parabolic closure of w' in W_I (or equivalently, in W) is standard. We then deduce from Lemma 3.8 that there is some $v \in \mathrm{Pc}(w') \leq W_I$ such that $v^{-1}wv$ is SCR in W. But as w' is SCR in W_I, we know that $\ell(w') = \ell(v^{-1}wv)$, and hence w' is in fact SCR in W, as desired. $\square$

4. Straight elements in Coxeter groups

This section is devoted to the proof of Theorem D and Corollaries E and F. Throughout this section, we fix a Coxeter system (W, S).

Lemma 4.1. *Let $w \in W$ be straight. Then w is cyclically reduced (more precisely, $\ell(w) = \min\{\ell(vwv^{-1}) \mid v \in W\}$) and torsion-free.*

Proof. Let $w \in W$ be straight. Assume first for a contradiction that it is not cyclically reduced: $\ell(w) \geq \ell(vwv^{-1}) + 1$ for some $v \in W$. Then for all $n \in \mathbb{N}$,

$$\ell(vw^nv^{-1}) + 2\ell(v) \geq \ell(w^n) = n\ell(w) \geq n\ell(vwv^{-1}) + n \geq \ell(vw^nv^{-1}) + n,$$

a contradiction.

Assume next for a contradiction that w is not torsion-free: there is a reduced decomposition $w = w_In_I$ for some spherical subset $I \subseteq S$, some $w_I \in W_I \setminus \{1\}$ and some n_I normalising W_I. Then for each $k \in \mathbb{N}$ there is some $w_k \in W_I$ such that $w^k = w_kn_I^k$. As W_I is finite, $w_k = w_l$ for some $k < l$ and hence $w^K = n_I^K$ for $K = l - k \in \mathbb{N}^*$. Thus $K\ell(w) = \ell(w^K) = \ell(n_I^K) \leq K\ell(n_I) < K\ell(w)$, a contradiction. $\square$

Lemma 4.2. *Let $v, w \in W$ be such that $\ell(w) = \ell(vwv^{-1})$. Then w is straight if and only if vwv^{-1} is straight.*

Proof. Suppose that w is straight and assume for a contradiction that vwv^{-1} is not, that is, there exists some $K \in \mathbb{N}$ such that $\ell((vwv^{-1})^K) < K\ell(vwv^{-1})$. As $\ell(w) = \ell(vwv^{-1})$, it follows that $K\ell(w) - 1 \geq \ell(vw^Kv^{-1})$. Hence

$$n(K\ell(w)-1) \geq n\ell(vw^Kv^{-1}) \geq \ell(vw^{Kn}v^{-1}) \geq \ell(w^{Kn})-2\ell(v) = Kn\ell(w)-2\ell(v)$$

for all $n \in \mathbb{N}$, yielding the desired contradiction for $n > 2\ell(v)$. $\square$

Lemma 4.3. *Let w be an infinite order element of W possessing an axis through C_0 (that is, such that* $\mathrm{Min}(w) \cap C_0 \neq \emptyset$*). If the walls separating C_0 from wC_0 are all w-essential, then w is straight.*

Proof. Let D be a w-axis through C_0. Assume for a contradiction that w is not straight. Then there exists a wall m separating C_0 from wC_0 and $w^n C_0$ from $w^{n+1}C_0$ for some nonzero $n \in \mathbb{N}$. In particular, m intersects D in at least two points, hence contains D. But this contradicts the assumption that m is w-essential. □

Remark 4.4. Note that if w is an infinite order element of W possessing an axis through the interior $\mathrm{int}(C_0)$ of C_0, then it is straight as it satisfies the hypotheses of Lemma 4.3: indeed, in that case, the walls separating C_0 from wC_0 are also the walls separating a point $x \in \mathrm{Min}(w) \cap \mathrm{int}(C_0)$ from wx and such an x is not contained in any wall.

Lemma 4.5. *Let $w \in W$ be of infinite order. Assume that* $\mathrm{Min}(w) \cap C_0$ *is nonempty and not contained in any w-essential wall. Then w is straight if and only if it is torsion-free.*

Proof. Pick $x \in \mathrm{Min}(w) \cap C_0$ such that x does not lie on any w-essential wall, and let L be the w-axis through x. Let $A \in \Sigma$ be the support of x, say $A = W_I$ for some spherical subset $I \subseteq S$. Then the walls containing x (that is, the walls containing A, or else the walls with associated roots in Φ_I) also contain L. It follows that w stabilises Φ_I and hence normalises W_I. Consider the decomposition $N_W(W_I) = W_I \rtimes N_I$ provided by Lemma 2.1 and write $w = w_I n_I$ accordingly, with $w_I \in W_I$ and $n_I \in N_I$.

Notice that L is also an axis for n_I as $n_I x = n_I(n_I^{-1} w_I n_I)x = w_I n_I x = wx \in L$ and $n_I^{-1}x = n_I^{-1} w_I^{-1} x = w^{-1}x \in L$. Moreover, no wall containing L separates C_0 from $n_I C_0$. Indeed, as C_0 is contained in the intersection of all roots α_s with $s \in I$, so is $n_I C_0$, yielding the claim. Hence n_I is straight by Lemma 4.3.

Now, if w is torsion-free, then $w_I = 1$ and $w = n_I$ is straight. The converse was established in Lemma 4.1. □

Lemma 4.6. *Let $u \in W$ be cyclically reduced. Then there is some $w \in W$ with $u \sim_\kappa w$ such that u is straight if and only if w is torsion-free.*

Proof. Let w be as in Corollary 3.5, so that in particular $\ell(u) = \ell(w)$. As u is straight if and only if w is straight by Lemma 4.2, the claim readily follows from Lemma 4.5. □

Proof of Theorem D. Let $u \in W$ be cyclically reduced. The implication (1) $\Rightarrow$ (2) follows from Lemma 4.2 together with Lemma 4.1. Assume now that (2) holds. Let $w \in W$ with $u \sim_\kappa w$ be as in Lemma 4.6. Then w is torsion-free by assumption, and hence u is straight, as desired. □

To prove Corollary E, we need one more technical lemma.

Lemma 4.7. *Let $w \in W$ be an FC element. Assume that w is not torsion-free. Then the standard parabolic closure of w possesses a nontrivial spherical irreducible component.*

Proof. Write $w = w_I n_I$ for some spherical $I \subseteq S$, some $w_I \in W_I \setminus \{1_W\}$ and some $n_I \in W$ normalising W_I. Let S_1 (resp. S_2) denote the set of generators of S appearing in a reduced decomposition of n_I (resp. w_I). Thus the standard parabolic closure of w coincides with $W_{S_1 \cup S_2}$.

Note that if $\ell(w_I) = 1$, say $w_I = s \in I$, then as $sn_I = n_I s'$ for some $s' \in I$, the FC condition implies that $s = s'$ and that s commutes with every generator of n_I. Moreover, given an FC element u and a reduced decomposition of the form $u = u_1 u_2 u_3$, the element u_2 is still FC. An easy induction on $\ell(w_I)$ thus yields that every generator in S_1 commutes with every generator in S_2.

Let now $s \in S_2$ and let W_T be the irreducible component of $W_{S_1 \cup S_2}$ containing s. Write $T_i = T \cap S_i$ for $i = 1, 2$, so that $T = T_1 \cup T_2$. Set also $T_1' = T_1 \setminus (T_1 \cap T_2)$. Then T is the disjoint union of T_1' and T_2, and every generator of T_1' commutes with every generator of T_2. As W_T is irreducible and T_2 contains s, we deduce that $T_1' = \emptyset$, that is, $T_1 \subseteq T_2$. Hence $T = T_2 \subseteq I$ and $W_T \subseteq W_I$ is spherical, as desired. □

Proof of Corollary E. The implication (1) ⇒ (2) follows from Lemma 4.1. Let now u be a CFC element whose standard parabolic closure has only infinite irreducible components. Note that the CFC condition implies that u is cyclically reduced. Let w be the conjugate of u provided by Lemma 4.6, so that in particular the standard parabolic closures of w and u coincide. As w is FC (since every cyclic shift of a CFC element is again CFC, see e.g. [BBE+12, Proposition 4.1]), it follows from Lemma 4.7 that it is torsion-free. Thus u is straight by Lemma 4.6, as desired. □

Proof of Corollary F. As Coxeter elements are CFC elements, this readily follows from Corollary E. □

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