Joint PhD position KU Leuven – UCLouvain

Project title: Correlations and Gap Probabilities for Random Tilings

Promotors: Tom Claeys (UCLouvain) and Arno Kuijlaars (KU Leuven)

A 4 years PhD position is opened jointly at UCLouvain and KU Leuven in Belgium. The PhD student will spend two years at the Institute for Research in Mathematics and Physics of UCLouvain followed by two years at the Department of Mathematics of KU Leuven. The PhD student is expected to start in October 2024 or soon after that date. The net monthly salary lies around 2500 €. Interested candidates are invited to apply before the end of June 2024, by sending a single pdf-file to tom.claeys@uclouvain.be and arno.kuijlaars@kuleuven.be, containing:

- a detailed curriculum vitae;
- a motivation letter and description of research interests;
- the name and e-mail address of at least one academic referee.

Candidates belonging to underrepresented groups are particularly encouraged to apply.

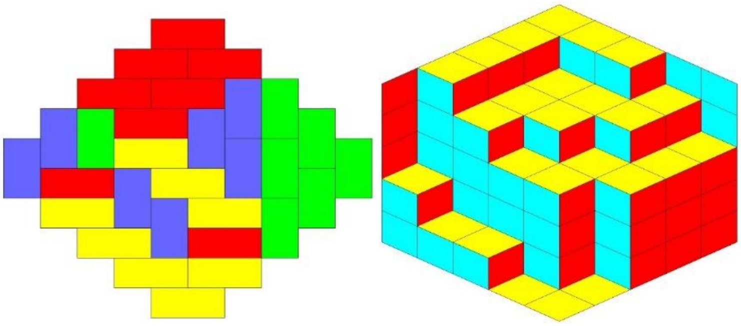
Project description

General context

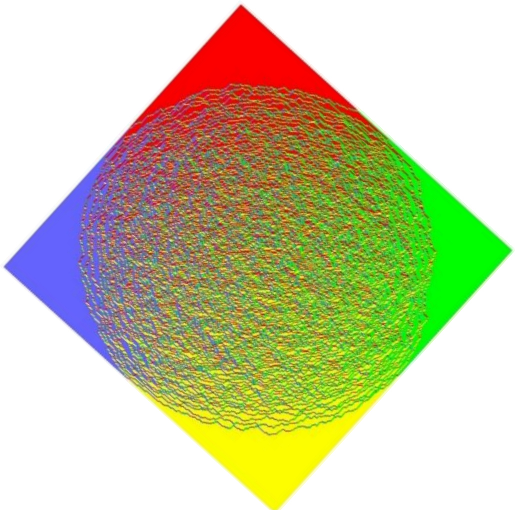
Random tiling models are important models in statistical mechanics and integrable probability. They are simple to define in elementary terms, but at the same time they have a rich combinatorial structure, and they show **remarkable asymptotic phenomena**. They are connected to other fundamental models in statistical mechanics, like dimer models, vertex models, non-intersecting path ensembles, random growth models, and random matrix models. Many of their asymptotic properties are **universal** in the sense that they are shared by a large class of models. This makes it possible to explore universal phenomena through the study of relatively simple prototype models which allow for exact expressions amenable to asymptotic analysis. The asymptotic analysis of these models requires a variety of **techniques from different areas of mathematics and mathematical physics**, such as combinatorics, Riemann surfaces, complex analysis and Riemann-Hilbert problems, Fredholm determinants and operator theory, and aspects of probability theory and stochastic processes. We refer the interested reader to [Go21] for a recent monograph on random tilings and related models.

Tiling models

In general, a planar tiling model consists of a two-dimensional domain and a number of given shapes of tiles. One then considers all possible tilings of the domain. Randomness comes into play by defining a probability measure on the space of all tilings, the simplest one being the uniform measure which assigns equal probability to each tiling. This project deals with two prototypical cases of random tiling models. First there are the **domino tilings of the so-called Aztec diamond** as shown in the figure below on the left, and second the **lozenge tilings of a hexagon** as shown below on the right.



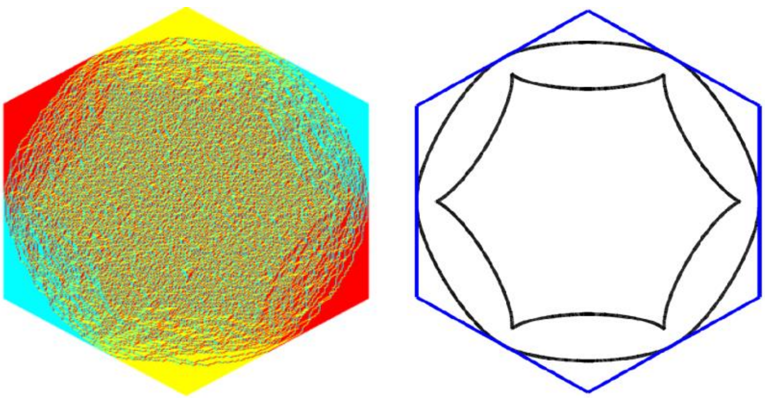
These random tilings show fascinating behaviors when the size of the system tends to infinity. The models exhibit several asymptotic phases that are separated by certain curves. In the case of a large uniform random tiling of the Aztec diamond, one observes a so-called **arctic curve** or arctic circle as shown in the figure below on the right. The striking feature, known as the *Arctic Circle Theorem [Jo05],* is that near each of the four corners a region appears with only one type of domino. These are called the **solid regions.** The solid regions are separated from the middle of the Aztec diamond, where all types of dominos are likely to occur. This middle region is called the **rough region**.



This phenomenon is by now very well understood for the **uniform model**. Similar arctic phenomena arise also in other models (6-vertex model, alternating sign matrices, fermionic systems) and fluctuations around the arctic curves show remarkable universal features. Major progress has been made recently to understand the shapes of curves separating these different asymptotic phases, the fluctuations around these curves, and the structure of correlations between tiles in the rough region, in a variety of models.

Weighted tilings

In a non-uniform model, not all tilings have the same probability. One may assign a weight to each tile in a tiling, depending on its location and on its shape. The product of the tile weights gives a weight for each tiling, and the non-uniformity appears by assigning higher probability to tilings with higher weight. The case of a **non-uniform model with doubly periodic weightings** is of primary interest, because on the one hand it allows for exact expressions of correlations between tiles, while on the other hand, a new phase can be observed. Besides the solid and the rough region, a third asymptotic phase may appear that is called the **smooth region**. The figure below shows a sample of the doubly periodic hexagon with period three on the left, and a phase diagram with two arctic curves on the right. The smooth region distinguishes itself from the rough region by a different decay scale of correlation between tiles. In the smooth region, correlations between tiles decay exponentially with the distance, in contrast to the correlations in the rough region that have a power-law decay.



Determinantal structure

Hexagon tilings and Aztec diamond tilings have many features in common. They can both be viewed as dimer models, and also as non-intersecting path models. Both points of view give rise to explicit formulas for the correlations between tiles, both in the uniform and non-uniform weighted models. These explicit formulas show that the correlations have a determinantal structure, and that the two systems are examples of **determinantal point processes**. The correlation kernel is either given by an inverse Kasteleyn matrix, or in terms of transition matrices in a form known as the Eynard-Mehta formula.

The **correlation kernel** is the fundamental object needed to understand finer asymptotic properties of the tiling models, such as correlations between tiles, or probabilities to observe specific patterns of tiles. In particular, m-point correlation functions of the model can be expressed as m x m determinants of a matrix involving the correlation kernel, and the probability to have a region without tiles of a given type can be expressed as a Fredholm determinant, which is the generalization of the matrix determinant to a class of integral operators.

In the uniform models, there is an exact and explicit **double contour integral expression** for the correlation kernel, which can be analysed asymptotically via saddle point methods. In the doubly periodic case, there still is an exact double integral expression for the correlation kernel, but as shown in [DK21], the integrand then involves **matrix-valued orthogonal polynomials** and the solution to a Riemann-Hilbert problem.

Probability of rare events

In the first part of the project, we want to understand the likelihood to observe certain **specific patterns of tiles**. For instance, we know from the above discussion that the solid region will typically end near the arctic curve, but how unlikely is it that the solid region extends beyond the arctic curve? Thanks to the interpretation of the tiling models as non-intersecting path ensembles, and thanks to the determinantal structure of correlations, the above question can be equivalently formulated in terms of gap probabilities, i.e. the probability that a random configuration of a determinantal point process contains no points in a given region.

It is a classical fact that gap probabilities can be expressed as **Fredholm determinants** of trace class integral operators depending on the region and on the correlation kernel. Such Fredholm determinant expressions are convenient for studying gap probabilities of typical events. In particular, they have been used to show that the fluctuations of the solid region in a random tiling around the arctic curve is governed by the **Tracy-Widom distribution** well known in random matrix theory.

Quantifying the probability of a rare event, like the probability that the solid-to-rough boundary in a specific configuration of tiles exhibits **large deviations from the arctic curve**, is a much harder task. In other models, like random matrix models, such questions have been investigated since many years, see [Fo14] for a review of the history on this topic. Many questions about probabilities of unlikely events have been answered using Hankel or Toeplitz determinants and Riemann-Hilbert techniques, see e.g. [DIK08, CG21] for questions about large gap asymptotics, and [ABB17, CFLW21] for questions about rigidity of eigenvalues of random matrices. Such questions are very natural for random tiling models too, but up to this point, there are no sharp results of this kind available in the literature around random tiling models, not even in the simplest case of uniform weightings.

We expect that the **double integral structure** of the correlation kernels can be exploited to characterize gap probabilities in terms of Riemann-Hilbert problems, using a method developed in [CGS17]. In turn, there is an extensive toolbox of techniques to describe the asymptotic behavior of Riemann-Hilbert problems. In the uniform models, we expect the Riemann-Hilbert problem to be explicit and amenable to asymptotic analysis. In the non-uniform models, the Riemann-Hilbert problem will involve matrix-valued orthogonal polynomials, and asymptotic analysis may be considerably more challenging. The first objective in the project is to study large gap asymptotics for Aztec diamond and hexagon tilings, first in the case of uniform weights, and if successful also in models with doubly periodic weightings.

Arctic curves for the hexagon

In recent years there has been a lot of activity to describe the arctic curve and the rough region for domino tilings of the Aztec diamond with **doubly periodic weights**, see e.g. [BD23], [CJ16], [DK21], [JM23], culminating in the very recent work of Berggren and Borodin [BB23+] that gives an almost complete treatment. The aim of this part of the project is to study the **doubly periodic hexagon**.

Although the two models are similar, the techniques that were successful for the Aztec diamond cannot be transferred directly to the hexagon. The main reason is that the matrix-valued orthogonal polynomials are significantly simpler in the case of the Aztec diamond, as they allow for explicit formulas at finite size. Moreover, the double integrals for the correlation kernels can be rewritten in a simplified form which is crucial for the asymptotic analysis in [BB23+]. These simplifications do not happen for the doubly periodic hexagon in case the periods are at least three.

To deal with the more complicated orthogonality a number of observations were made in [BGK23]. The starting observation is that the matrix-valued orthogonality can be interpreted as scalar orthogonality on a contour on a **compact Riemann surface**. The Riemann surface has the special property of being a **Harnack curve**, which means that the amoeba map is at most 2-to-1 on the curve, see [KO06], [KOS06]. This structure is also exploited in [BB23+]. The genus of the Riemann surface is equal to the number of bounded components in the complement of the amoeba. In the case of the hexagon with generic weights with period r, the genus is (r-1)(r-2)/2.

The scalar orthogonality is non-hermitian and the contour on which the orthogonality takes place is only defined up to homotopy. An ideal contour should have the so-called S-property in an external field, and this contour should also attract the zeros of the orthogonal polynomials on the Riemann surface, in the limit when their degree tends to infinity. A program to characterize the contours with the S-property was described in [BGK23]. Here notions of **logarithmic potential theory** that are well known in the complex plane, see [ST97], are transferred to the Riemann surface. A suitably defined energy functional acts on **probability measures on the Riemann surface**. Given an admissible contour the equilibrium measure on that contour is the probability measure with minimal energy. The ideal contour with the S-property maximizes the minimal energy.

It is the aim of this part of the project to carry out this program in detail for an as wide class of doubly periodic hexagon cases as possible. It is however reasonable to start with the period three case and to focus first on the description of the arctic curves in these models. The presence of additional symmetries in the assignment of the weights will help to determine the location of the S-contour, which in turn will help to find the arctic curves.

Suitability as a PhD project

Summarizing, the project consists of two major objectives.

- **Objective 1:** Study the **probability of rare events** in uniform tiling models, via Fredholm determinants and Riemann-Hilbert problems; if feasible, extend this to the doubly periodic case.
- **Objective 2:** Describe the arctic curves for the **doubly periodic hexagon**.

Both of these objectives are challenging and can generate a considerable impact on the research area. At the same time, as explained above, there is a clear path and methodology to be followed for each of the objectives, with both principal investigators being well qualified to provide supervision and support. This makes the project realistic for and **accessible to a beginning PhD student** with general background in mathematics and mathematical physics.

Moreover, we believe the project is **attractive for prospective PhD students** because of several reasons. First, the models under consideration are visually appealing, and the main objectives can be easily understood, at least on a non-technical heuristic level, without much prior knowledge. Secondly, an exciting feature of the methodology is that it relies on a combination of techniques from different areas of mathematics. To reach the first objective, aspects of operator theory and Fredholm determinants are needed, combined with the asymptotic and complex analysis of Riemann-Hilbert problems, along with notions of probability and stochastic processes. For the second objective, the geometry of Riemann surfaces will come into play, together with the study of matrix-valued orthogonal polynomials.

Work plan
The PhD student will start by studying relevant **literature** around tiling models, determinantal point process, and Riemann-Hilbert problems. The PhD student will also follow **courses** at UCLouvain and/or KU Leuven to broaden and deepen their knowledge and background.
During the **first two years of the PhD**, the research will be concerned with Objective 1, the study of gap probabilities, such that the PhD student will be able to benefit from the expertise of UCLouvain P.I. Tom Claeys on that topic. We expect that the main steps to achieve Objective 1 will be taken after two years, such that the results can be finalized and published at the latest during the third year. During the second year, the PhD student will in parallel initiate the research for Objective 2. This will facilitate a smooth transition from UCLouvain to KU Leuven after the second year.
During the **third and fourth year**, the research will be oriented towards Objective 2, facilitated by KU Leuven's P.I. Arno Kuijlaars' expertise of the methodology to be utilized there. Throughout the project, the PhD student will be encouraged to participate to seminars, international research schools, workshops and conferences, and to present their research both at local seminars and at international scientific events.

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